

Infinite invariant density in a semi-Markov process with continuous state variablesTakuma Akimoto^{1,*}, Eli Barkai² and Günter Radons³¹*Department of Physics, Tokyo University of Science, Noda, Chiba 278-8510, Japan*²*Department of Physics, Bar-Ilan University, Ramat-Gan*³*Institute of Physics, Chemnitz University of Technology, 09107 Chemnitz, Germany*

(Received 24 August 2019; revised manuscript received 7 April 2020; accepted 9 April 2020; published 12 May 2020)

We report on a fundamental role of a non-normalized formal steady state, i.e., an infinite invariant density, in a semi-Markov process where the state is determined by the interevent time of successive renewals. The state describes certain observables found in models of anomalous diffusion, e.g., the velocity in the generalized Lévy walk model and the energy of a particle in the trap model. In our model, the interevent-time distribution follows a fat-tailed distribution, which makes the state value more likely to be zero because long interevent times imply small state values. We find two scaling laws describing the density for the state value, which accumulates in the vicinity of zero in the long-time limit. These laws provide universal behaviors in the accumulation process and give the exact expression of the infinite invariant density. Moreover, we provide two distributional limit theorems for time-averaged observables in these nonstationary processes. We show that the infinite invariant density plays an important role in determining the distribution of time averages.

DOI: [10.1103/PhysRevE.101.052112](https://doi.org/10.1103/PhysRevE.101.052112)**I. INTRODUCTION**

There is a growing number of studies on applications of infinite invariant densities in physical literature, ranging from deterministic dynamics describing intermittency [1–5], models of laser cooling [6–9], anomalous diffusion [10–14], fractal-time renewal processes [15], and non-normalized Boltzmann states [16]. Infinite invariant densities are non-normalized formal steady states of systems and were studied in dynamical systems exhibiting intermittency [17–23]. The corresponding ergodic theory is known as infinite ergodic theory, which is based on Markovian stochastic processes [24,25], and states that time averages of some observables do not converge to the corresponding ensemble averages but become random variables in the long-time limit [21,26–31]. Thus, time averages cannot be replaced by ensemble averages even in the long-time limit. This striking feature is different from usual ergodic systems. Therefore, finding unexpected links between infinite ergodic theory and nonequilibrium phenomena attracts a significant interest in statistical physics [1–3,9–12,16,32–35].

In equilibrium systems, time averages of an observable converge to a constant, which is given by the ensemble average with respect to the invariant probability measure, i.e., the equilibrium distribution. However, in nonequilibrium processes, this ergodic property sometimes does not hold. In particular, distributional behaviors of time-averaged observables have been experimentally unveiled. Examples are the intensity of fluorescence in quantum dots [36,37], diffusion coefficients of a diffusing biomolecule in living cells [38–41], and interface fluctuations in Kardar-Parisi-Zhang universality

class [42], where time averages of an observable, obtained from trajectories under the same experimental setup, do not converge to a constant but remain random. These distributional behaviors of time averages of some observables have been investigated by several stochastic models describing anomalous diffusion processes [12,43–53].

While several works have considered applications of infinite ergodic theory to anomalous dynamics, one cannot apply infinite ergodic theory straightforwardly to stochastic processes. Therefore, our goal is to provide a deeper understanding of infinite ergodic theory in nonstationary stochastic processes. To this end, we derive an exact form of the infinite invariant density and expose the role of the non-normalized steady state in a minimal model for nonequilibrium nonstationary processes. In particular, we unravel how the infinite invariant density plays a vital role in a semi-Markov process (SMP), which characterizes the velocity of the generalized Lévy walk (GLW) [53,54].

Our work addresses three issues. First, what is the propagator of the state variable? In particular, we will show its relation to the mean number of renewals in the state variable. Second, we derive the exact form of the infinite invariant density, which is obtained from a formal steady state of the propagator found in the first part. Finally, we investigate distributional limit theorems of some time-averaged observables and discuss the role of the infinite invariant density. We end the paper with a summary.

II. INFINITE ERGODIC THEORY IN BROWNIAN MOTION

Before describing our stochastic model, we provide the infinite invariant density and its role in one of the simplest models of diffusion, i.e., Brownian motion. Statistical properties of equilibrium systems or nonequilibrium systems with

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steady states are described by a normalized density describing the steady state. On the other hand, a formal steady state sometimes cannot be normalized in nonequilibrium processes, where nonstationarity is essential [6–12,16]. Let us consider a free one-dimensional (1D) Brownian motion in infinite space. The formal steady state is a uniform distribution, which cannot be normalized in infinite space. To see this, consider the diffusion equation, $\partial_t P(x, t) = D\partial_x^2 P(x, t)$, where $P(x, t)$ is the density. Then, setting the left-hand side to zero yields a formal steady state, i.e., the uniform distribution. This is the simplest example of an infinite invariant density in nonequilibrium stochastic processes, where the system never reaches the equilibrium. Although the propagator of Brownian motion is known exactly, the role of the infinite invariant density is not so well-known. Here, we will demonstrate its use. Later, we will see parallels and differences to the results for our SMP.

First, we consider the occupation time statistics. The classical arcsine law states that the ratio between the occupation time that a 1D Brownian particle spends on the positive side and the total measurement time follows the arcsine distribution [55], which means that the ratio does not converge to a constant even in the long-time limit and remains a random variable. Moreover, the ratio between the occupation time that a 1D Brownian particle spends on a region with a finite length and the total measurement time does not converge to a constant. Instead, the normalized ratio exhibits intrinsic trajectory-to-trajectory fluctuations and the distribution function follows a half-Gaussian, which is a special case of the Mittag-Leffler distribution known from the occupation time distribution for Markov chains [24].

These two laws are distributional limit theorems for time-averaged observables because the occupation time can be represented by a sum of indicator functions. To see this, consider the Heaviside step function, i.e., $\theta(x) = 1$ if $x > 0$, otherwise zero. The occupation time on the positive side can be represented by $\int_0^t \theta(B_{t'}) dt'$, where B_t is a trajectory of a Brownian motion. The integral of $\theta(x)$ with respect to the infinite invariant density, i.e., $\int_{-\infty}^{\infty} \theta(x) dx$, is clearly diverging. On the other hand, if we consider $f(x) = \theta(x - x_a)\theta(x_b - x)$, i.e.; it is one for $x_a < x < x_b$, and zero otherwise, the integral of $f(x)$ with respect to the infinite invariant density remains finite. Therefore, the observable for the arcsine law is not integrable with respect to the infinite density while that for the latter case is integrable. Therefore, the integrability of the observable discriminates the two distributional limit theorems in occupation time statistics.

For nonstationary processes, the propagator never reaches a steady state, i.e., equilibrium state. However, a formal steady state exists and is described by the infinite invariant density for many cases. This infinite invariant density will characterize distributional behaviors of time-averaged observables. For Brownian motion, this steady state is trivial (uniform) and in some sense noninteresting. However, we will show that this integrability condition is rather general as in infinite ergodic theory of dynamical systems.

III. SEMI-MARKOV PROCESS

Here, we introduce an SMP, which couples a renewal process to an observable. A renewal process is a point pro-

cess where an interevent time of two successive renewal points is an independent and identically distributed random variable [56]. In SMPs, a state changes at renewal points. More precisely, a state remains constant in between successive renewals. In what follows, we consider continuous state variables. In particular, the state is characterized by a continuous scalar variable and the scalar value is determined by the interevent time. In this sense, the continuous-time random walk and a dichotomous process are SMPs [43,57]. Moreover, time series of magnitudes or distances of earthquakes can be described by an SMP because there is a correlation between the magnitude and the interevent time [58]. In the trap model [59] a random walker is trapped in random energy landscapes. Because escape times from a trap are independent and identically distributed random variables depending on the trap and its mean escape time is given by the energy depth of the trap, the value of the energy depth is also described by an SMP. Therefore, a state variable, in a different context, can have many meanings (see also Ref. [60]).

As a typical physical example of this process, we consider the GLW [54]. This system can be applied to many physical systems such as turbulence dynamics and subrecoil laser cooling [6–8,53,61–64], where the state is considered to be velocity or momentum. In the GLW a walker moves with constant velocities v_n over time segments of lengths τ_n between turning points occurring at times t_n , i.e., $\tau_n = t_n - t_{n-1}$, where flight durations τ_n are independent and identically distributed random variables. Thus, the displacement X_n in time segment $[t_{n-1}, t_n]$ is given by $X_n = v_n \tau_n$. A coupling between v_n and τ_n is given by joint probability density function (PDF) $\psi(v, \tau)$. As a specific coupling which we consider in this paper, the absolute values $|v_n|$ of the velocities and flight durations τ_n in elementary flight events are coupled deterministically via

$$|v_n| = \tau_n^{\nu-1}, \quad (1)$$

or equivalently via

$$\tau_n = |v_n|^{\frac{1}{\nu-1}}. \quad (2)$$

The quantity $\nu > 0$ is an important parameter characterizing a given GLW. This nonlinear coupling was also considered in Refs. [35,63,65]. The standard Lévy walk corresponds to case $\nu = 1$, implying that the velocity does not depend on the flight duration. In what follows, we focus on case $0 < \nu < 1$. Importantly, if $\tau \rightarrow \infty$, in this regime $|v_n| \rightarrow 0$. Thus, we will find accumulation of density in the vicinity of $v = 0$. This is because we assume a power-law distribution for flight durations, that favors long flight durations. Some investigations such as Refs. [53,54] concentrated on the behavior in coordinate space, where a trajectory $x(t)$ is a piecewise linear function of time t .

In the following, we denote the state variable as velocity and investigate the velocity distribution at time t , where a trajectory of velocity $v(t)$ is a piecewise constant function of t . An SMP consists of a sequence $\{\mathcal{E}_1, \mathcal{E}_2, \dots\}$ of elementary flight events $\mathcal{E}_n = (v_n, \tau_n)$. We note that this sequence \mathcal{E}_n ($n = 1, \dots$) is an independent and identically distributed random vector variable. Thus, the velocity process of a GLW is characterized by the joint PDF of velocity v and flight duration

τ in an elementary flight event:

$$\phi(v, \tau) = \langle \delta(v - v_i) \delta(\tau - \tau_i) \rangle. \quad (3)$$

The symbol $\delta(\cdot)$ denotes the Dirac δ function. PDF $\psi(\tau)$ of the flight durations is defined through the marginal density of the joint PDF $\psi(v, \tau)$:

$$\psi(\tau) = \int_{-\infty}^{+\infty} \phi(v, \tau) dv = \langle \delta(\tau - \tau_i) \rangle. \quad (4)$$

Similarly, one can get PDF $\chi(v)$ for the velocities of an elementary event as

$$\chi(v) = \int_0^{+\infty} \phi(v, \tau) d\tau = \langle \delta(v - v_i) \rangle. \quad (5)$$

In Lévy walk treatments usually $\psi(\tau)$ is prescribed and chosen as a slowly decaying function with a power-law tail:

$$\psi(\tau) \sim \frac{c}{|\Gamma(-\gamma)|} \tau^{-1-\gamma} \quad (\tau \rightarrow \infty) \quad (6)$$

with the parameter $\gamma > 0$ characterizing the algebraic decay and c being a scale parameter. A pair of parameters v and γ determines the essential properties of the GLW and the asymptotic behavior in the velocity space. Of special interest is the regime $0 < \gamma < 1$. There the sequence of renewal points $\{t_n, n = 0, 1, 2, \dots\}$, at which velocity $v(t)$ changes, i.e.,

$$t_n = \sum_{i=1}^n \tau_i \quad (7)$$

with $t_0 = 0$, is a nonstationary process in the sense that the rate of change is not constant but varies with time [6,43]. This is because the mean flight duration diverges, i.e., $\langle \tau_i \rangle = \int_0^\infty \tau \psi(\tau) d\tau = \infty$. To determine the last velocity $v(t)$ at time t , one needs to know the time interval straddling t , which is defined as $\tau \equiv t_{n+1} - t_n$ with $t_n < t < t_{n+1}$ and was discussed in Refs. [49,66]. In other words, to determine the distribution of the velocity at time t , one needs to know the time when the first renewal occurs after time t and the time for the last renewal event before t .

IV. GENERAL EXPRESSION FOR THE PROPAGATOR

A. Standard derivation

We are interested in the propagator $p(v, t)$, which is the PDF of finding a velocity v at time t , given that the process started at $t = 0$ with $v = 0$. To derive an expression for $p(v, t)$, we note that at every renewal time t_{n-1} the process starts anew with velocity v_n until $t < t_n$. So one needs the probability $R(t)dt$ of finding some renewal event in $[t, t + dt)$. This quantity is called sprinkling density in the literature [6] and it is closely related to the renewal density in renewal theory [56].

It is obtained from a recursion relation for the PDF $R_n(t) = \langle \delta(t - t_n) \rangle$ that the n th renewal point t_n occurs exactly at time t . Using the PDF $\psi(\tau)$, we get the iteration rule

$$R_{n+1}(t) = \int_0^t dt' \psi(t - t') R_n(t') \quad (8)$$

with the initial condition $R_0(t) = \delta(t)$, which means that we assume a renewal occurs at $t = 0$, i.e., *ordinary renewal*

process [56]. Summing both sides from $n = 0$ to infinity, one gets the equation of $R(t) \equiv \sum_{n=0}^\infty R_n(t)$ for $t > 0$, i.e.,

$$R(t) = \int_0^t dt' \psi(t - t') R(t') + R_0(t). \quad (9)$$

Equation (9) is known as the renewal equation. The solution of this equation is easily obtained in Laplace space as

$$\tilde{R}(s) = \frac{1}{1 - \tilde{\psi}(s)}, \quad (10)$$

where $\tilde{R}(s) = \int_0^\infty R(t) \exp(-st) dt$. The integral of $R(t)$ is related to the expected number of renewal events $\langle N(t) \rangle$ occurring up to time t , i.e.,

$$\langle N(t) \rangle = \int_0^t R(t') dt'. \quad (11)$$

Note that here the event at $t = 0$ is also counted while the event at $t = 0$ is often excluded in renewal theory.

With knowledge of $R(t)$, which in principle can be obtained by Laplace inversion of Eq. (10), one can formulate the solution of the propagator as

$$p(v, t) = \int_0^t dt' W(v, t - t') R(t'), \quad (12)$$

where $W(v, t - t')$ takes into account the last incompleting flight event, starting at the last renewal time t' , provided that the flight duration is longer than $t - t'$ with velocity v . Thus, $W(v, t)$ is given by

$$W(v, t) \equiv \int_t^\infty d\tau \phi(v, \tau). \quad (13)$$

Integrating this over all velocities leads to the survival probability $\Psi(t)$ of the sojourn time, i.e., the probability that an event lasts longer than a given time t

$$\Psi(t) = \int_{-\infty}^{+\infty} W(v, t) dv = \int_t^\infty d\tau \psi(\tau). \quad (14)$$

Using Eqs. (5), (10), and (13) one can write down the propagator in Laplace space

$$\tilde{p}(v, s) = \tilde{W}(v, s) \tilde{R}(s) = \frac{1}{s} \frac{\chi(v) - \tilde{\phi}(v, s)}{1 - \tilde{\psi}(s)}. \quad (15)$$

This is a general expression of the propagator and an analog of the Montroll-Weiss equation of the continuous-time random walk [67]. Recalling $\int dv [\chi(v) - \tilde{\phi}(v, s)] = 1 - \tilde{\psi}(s)$ gives $\int dv \tilde{p}(v, s) = 1/s$, implying that propagator $p(v, t)$ in the form of Eq. (15) is correctly normalized $\int dv p(v, t) = 1$.

In what follows, as a specific example, we consider a deterministic coupling between τ_i and $|v_i|$. The joint PDF $\phi(v, \tau)$ is specified as follows: Flight duration τ_i is chosen randomly from the PDF $\psi(\tau)$, and the corresponding absolute value of the velocity $|v_i|$ is deterministically given by $|v_i| = \tau_i^{\nu-1}$. Finally, the sign of v_i is determined with equal probability, implying that

$$\phi(v, \tau) = \frac{1}{2} [\delta(v - \tau^{\nu-1}) + \delta(v + \tau^{\nu-1})] \psi(\tau) \quad (16)$$

with $\phi(v, \tau) = \phi(-v, \tau)$. Alternatively, one can specify the velocity first using the PDF $\chi(v) = \chi(-v)$. Then, one can

express the joint PDF $\phi(v, \tau)$ also as

$$\phi(v, \tau) = \delta(\tau - |v|^{\frac{1}{\gamma-1}}) \chi(v). \quad (17)$$

Although Eqs. (16) and (17) are equivalent, the latter suggests a different interpretation of selecting an elementary event, e.g., the velocity is selected from $\chi(v)$ first and then this velocity state lasts for duration $\tau_i = |v_i|^{\frac{1}{\gamma-1}}$. Obviously prescribing $\psi(\tau)$ determines $\chi(v)$ [via Eqs. (5) and (16)] and vice versa [via Eqs. (4) and (17)]. From Eq. (13), one gets

$$W(v, t) = \chi(v) \theta(|v|^{\frac{1}{\gamma-1}} - t), \quad (18)$$

where $\theta(x)$ is the Heaviside step function.

Before deriving our main results, we give the equilibrium distribution of the propagator for $\gamma > 1$. Although we assumed $\gamma < 1$ in Eq. (6), the general expression for the propagator, Eq. (15), is exact also for $\gamma > 1$. For $\gamma > 1$, the mean flight duration $\langle \tau \rangle$ is finite and we have

$$p_{\text{eq}}(v) = \lim_{s \rightarrow 0} s \tilde{p}(v, s) = \frac{\int_0^\infty \phi(v, \tau) \tau d\tau}{\langle \tau \rangle}. \quad (19)$$

Therefore, for $\gamma > 1$, as expected, the equilibrium distribution exists; i.e., the propagator reaches a steady state:

$$p(v, t) \rightarrow p_{\text{eq}}(v) = \frac{\chi(v) |v|^{\frac{1}{\gamma-1}}}{\langle \tau \rangle} \quad (20)$$

for $t \rightarrow \infty$. Here we note that the equilibrium distribution has a different form for the decoupled case, i.e., $\phi(v, \tau) = \chi(v) \psi(\tau)$. In this case, it is easily obtained as $p_{\text{eq}}(v) = \chi(v)$.

Because the integration of $R(t)$ gives $\langle N(t) \rangle$, we get an exact expression for the propagator

$$p(v, t) = \chi(v) \{ \langle N(t) \rangle - \langle N[t - t_c(v)] \rangle \}, \quad (21)$$

where $t_c(v) \equiv |v|^{\frac{1}{\gamma-1}}$. We note that $\langle N(t) \rangle = 0$ when $t < 0$. In particular, one can express $p(v, t)$ as

$$p(v, t) = \chi(v) \begin{cases} \langle N(t) \rangle & \text{for } t < t_c(v) \\ \langle N(t) \rangle - \langle N[t - t_c(v)] \rangle & \text{for } t > t_c(v) \end{cases}. \quad (22)$$

Since we have made no approximation, the solution is formally exact, while the remaining difficulty is to obtain $\langle N(t) \rangle$. This is the central result of this section.

The mean number $\langle N(t) \rangle$ of renewals up to time t increases monotonically from $\langle N(t \rightarrow 0) \rangle = 1$ because the first jump is at $t_0 = 0+$, which implies that $\lim_{t \rightarrow 0} p(v, t) = \chi(v)$, which is the velocity distribution of the elementary event as given by Eq. (5). For a given velocity v satisfying $t < t_c(v)$, the function $p(v, t)$ increases until t reaches $t_c(v)$ because $\langle N(t) \rangle$ is a monotonically increasing function. Thereafter $p(v, t)$ stays constant or decreases because $\langle N(t) \rangle - \langle N[t - t_c(v)] \rangle$ stays constant or decreases depending on whether the renewal sequences $\{t_n, n = 0, 1, 2, \dots\}$ are equilibrium sequences or not [56,68–70]. This in turn depends on the shape of $\psi(\tau)$, more precisely on the decay of $\psi(\tau)$ for large τ , as detailed below.

For a discussion of the velocity profile $p(v, t)$ for a fixed time t , it is more convenient to rewrite Eq. (21) for $v > 0$ as

$$p(v, t) = \chi(v) \{ \langle N(t) \rangle - \langle N[t - t_c(v)] \rangle \theta[v - v_c(t)] \}, \quad (23)$$

where we introduced the critical velocity $v_c(t) = t^{\gamma-1}$ and $v_c(t)$ is monotonically decreasing as function of t because $0 < \gamma < 1$. For negative v , $p(v, t)$ follows from the symmetry $p(v, t) = p(-v, t)$. Thus, for a fixed t and $|v| < v_c(t)$, $p(v, t)$ is the same as $\chi(v)$, enlarged by the velocity-independent factor $\langle N(t) \rangle$, whereas for $|v| > v_c(t)$ it has a nontrivial v dependence due to the v dependence of $\langle N[t - t_c(v)] \rangle$. Note that at velocity $v = v_c(t)$ the profile of v jumps by the value

$$\delta p = \lim_{\varepsilon \rightarrow 0} \{ p[v_c(t) - \varepsilon, t] - p[v_c(t) + \varepsilon, t] \} = \chi[v_c(t)] \quad (24)$$

at the critical velocity $v = v_c(t)$ because we assume $\langle N(0) \rangle = 1$.

B. Another derivation of Eq. (21)

Here we give another derivation of the propagator, i.e., Eq. (21). The joint PDF of $v(t)$ with t satisfying $t_n < t < t_{n+1}$, denoted by $p_n(v, t)$, can be written as

$$p_n(v, t) = \langle \delta(\tau_{n+1} - |v|^{\frac{1}{\gamma-1}}) I(t_n < t < t_{n+1}) \rangle, \quad (25)$$

where $I(\cdot) = 1$ if the condition in the bracket is satisfied and 0 otherwise. It follows that the propagator can be obtained as a sum over the number of renewals n :

$$p(v, t) = \sum_{n=0}^{\infty} p_n(v, t). \quad (26)$$

Using $\chi(v)$ and $t_{n+1} = t_n + \tau_{n+1}$, we have

$$p(v, t) = \chi(v) \sum_{n=0}^{\infty} \langle I(t_n < t < t_n + |v|^{\frac{1}{\gamma-1}}) \rangle, \quad (27)$$

where we note that $\langle I(t_n < t) \rangle$ gives a probability:

$$\langle I(t_n < t) \rangle = \Pr[N(t) > n]. \quad (28)$$

The mean of $N(t)$ can be written as

$$\langle N(t) \rangle = \sum_{n=1}^{\infty} n \Pr[N(t) = n] = \sum_{n=0}^{\infty} \Pr[N(t) > n], \quad (29)$$

where we used identity $\Pr[N(t) = n] = \Pr[N(t) > n - 1] - \Pr[N(t) > n]$. Therefore, we have Eq. (21).

V. INFINITE INVARIANT DENSITY IN A SEMI-MARKOV PROCESS

A. Infinite invariant density

To proceed with the discussion of Eq. (23), we use Eq. (6) for the flight-duration PDF and consider $\gamma < 1$. The PDF of velocities $\chi(v)$ in an elementary event can be obtained by Eqs. (5) and (16):

$$\chi(v) = \frac{1}{2} \psi(|v|^{\frac{1}{\gamma-1}}) \frac{|v|^{-1-\frac{1}{\gamma-1}}}{1-v}. \quad (30)$$

For the specific choice for $\psi(\tau)$ given in Eq. (6) the asymptotic form with $0 < \gamma < 1$ yields

$$\chi(v) \sim \frac{c}{2(1-\gamma)\Gamma(-\gamma)} |v|^{-1+\frac{\gamma}{\gamma-1}} \text{ for } v \rightarrow 0. \quad (31)$$

First, we give the asymptotic behavior of $\langle N(t) \rangle$ for $t \rightarrow \infty$. Because the Laplace transform of $\psi(\tau)$ is given by $\tilde{\psi}(s) =$

$1 - cs^\gamma + o(s^\gamma)$ for $s \rightarrow 0$, Eqs. (10) and (11) yields the well-known result:

$$\langle N(t) \rangle \sim \frac{1}{c\Gamma(1+\gamma)} t^\gamma \text{ for } t \rightarrow \infty. \quad (32)$$

However, for our purposes, we need to go beyond this limit as shown below. The renewal function gives the exact form of the propagator [see Eq. (21)]. There are two regimes in the propagator as seen in Eq. (22). For $t < t_c(v)$, or equivalently $v < v_c(t)$, the propagator is given by

$$p(v, t) = \langle N(t) \rangle \chi(v). \quad (33)$$

In this regime the propagator is an increasing function of t because $\langle N(t) \rangle$ is a monotonically increasing function whose asymptotic behavior is given by Eq. (32), whereas the support $(-v_c(t), v_c(t))$ will shrink because $v_c(t) = t^{-(1-\nu)} \rightarrow 0$ as $t \rightarrow \infty$. For $t \gg 1$ and $v < v_c(t)$, implying $v < v_c(t) \ll 1$, the propagator becomes

$$p(v, t) \sim \frac{\sin(\pi\gamma)t^\gamma}{2\pi(1-\nu)} |v|^{-1+\frac{\gamma}{1-\nu}}. \quad (34)$$

This is a universal law in the sense that the asymptotic form does not depend on the detailed form of the flight-duration PDF such as scale parameter c .

For $t > t_c(v)$, or equivalently $v > v_c(t)$, the propagator is given through $\langle N(t) \rangle - \langle N[t - t_c(v)] \rangle$. For $t \gg 1$ and $v_c(v) < v \ll 1$,

$$\langle N(t) \rangle - \langle N[t - t_c(v)] \rangle \cong \frac{t^\gamma - [t - t_c(v)]^\gamma}{c\Gamma(\gamma + 1)}, \quad (35)$$

$$\sim \frac{t^{\gamma-1} t_c(v)}{c\Gamma(\gamma)}. \quad (36)$$

Therefore, the asymptotic behavior of the propagator becomes

$$p(v, t) \sim \chi(v) |v|^{\frac{1}{1-\nu}} \frac{t^{\gamma-1}}{c\Gamma(\gamma)} \text{ for } t \rightarrow \infty, \quad (37)$$

which can also be obtained simply by changing $\langle \tau \rangle$ in Eq. (20) into $\int_0^t \tau \psi(\tau) d\tau$ except for the proportional constant. Here one can define a formal steady state $I_\infty(v)$ using Eq. (37) as follows:

$$I_\infty(v) \equiv \lim_{t \rightarrow \infty} t^{1-\gamma} p(v, t) = \frac{\chi(v) |v|^{\frac{1}{1-\nu}}}{c\Gamma(\gamma)}. \quad (38)$$

This does not depend on t in the long-time limit and is a natural extension of the steady state for $\gamma > 1$, i.e., Eq. (20). In this sense, $I_\infty(x)$ is a formal steady state of the system. However, $I_\infty(v)$ is not normalizable and thus it is sometimes called *infinite invariant density*. Using Eq. (31), the asymptotic form of the infinite density for $v \ll 1$ becomes

$$I_\infty(v) \sim \frac{\gamma \sin(\pi\gamma)}{2\pi(1-\nu)} |v|^{-1-\frac{\gamma}{1-\nu}}. \quad (39)$$

We note that the infinite density describes the propagator only for $v > v_c(t)$. While $v_c(t) \rightarrow 0$ in the long-time limit, the propagator for $v \ll 1$ is composed of two parts, i.e., Eqs. (34) and (37). These behaviors are illustrated in Fig. 1, where the support of the propagator is restricted to $|v| < 1$. In particular, the accumulation at zero velocity for $v < v_c(t)$ and a trace of the infinite density for $v > v_c(t)$ are clearly shown. In general,

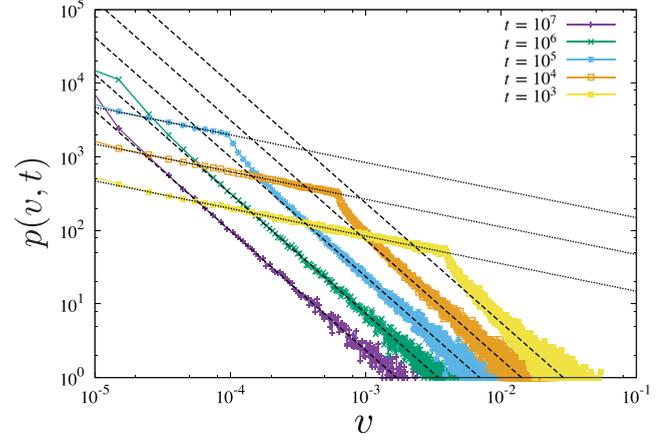


FIG. 1. Time evolution of the propagator for different times ($\gamma = 0.5$ and $\nu = 0.2$). Symbols with lines are the results of numerical simulations. Dashed and dotted lines are the theories, i.e., Eqs. (37) and (34), respectively. Infinite density can be observed for $v > v_c(t)$ while the propagator follows a different scaling, i.e., Eq. (34), for $v < v_c(t)$. We used the PDF $\psi(\tau) = \gamma\tau^{-1-\gamma}$ for $\tau \geq 1$ and $\psi(\tau) = 0$ for $\tau < 1$ as the flight-duration PDF.

the propagator for $v \gg 1$ is described by the small- τ behavior of the flight-duration PDF through Eq. (22).

B. Scaling function

Rescaling v by $v' = t^{1-\nu}v$ in the propagator, we find a scaling function. In particular, the rescaled propagator does not depend on time t and approaches the scaling function denoted by $\rho(v')$ in the long-time limit ($t \rightarrow \infty$):

$$p_{\text{res}}(v', t) \equiv p(v'/t^{1-\nu}, t) \left| \frac{dv}{dv'} \right| \rightarrow \rho(v'), \quad (40)$$

where $\rho(v')$ is the scaling function defined as

$$\rho(v') \equiv \begin{cases} \frac{\sin(\pi\gamma)}{2\pi(1-\nu)} |v'|^{-1+\frac{\gamma}{1-\nu}} & (|v'| < 1) \\ \frac{\sin(\pi\gamma)\{1 - (1 - |v'|^{\frac{1}{1-\nu}})^\gamma\}}{2\pi(1-\nu)} |v'|^{-1+\frac{\gamma}{1-\nu}} & (|v'| \geq 1) \end{cases}, \quad (41)$$

and we used Eq. (35). Note that $v'_c \equiv t^{1-\nu}v_c(t) = 1$. In the scaling function, the long-time limit is taken in advance. Thus, the scaling function describes only small- v behaviors of $p(v, t)$. In other words, large- v behaviors of $\rho(v)$ are matched with those of $I_\infty(v)$ while large- v behaviors of $\rho(v)$ are matched with small- v behaviors of $I_\infty(v)$. The scaling function is normalized and continuous at $v' = 1$, whereas $p(v, t)$ is not continuous at $v = v_c(t)$ for finite t because the jump in the propagator at $v = v_c$ is given by Eq. (24) and $\chi[v_c(t)] \rightarrow 0$ for $t \rightarrow \infty$. As shown in Fig. 2, rescaled propagators at different times t coincide with the scaling function for $t \gg 1$. Note that the scaling function describes the behavior of $p(v, t)$ for $v \ll 1$ in the long-time limit, which does not capture the behavior of $p(v, t)$ for $v > 1$. Large- v behaviors of $p(v, t)$ can be described by $I_\infty(v)$. Although $I_\infty(v)$ depends on the details of $\chi(v)$, the scaling function is

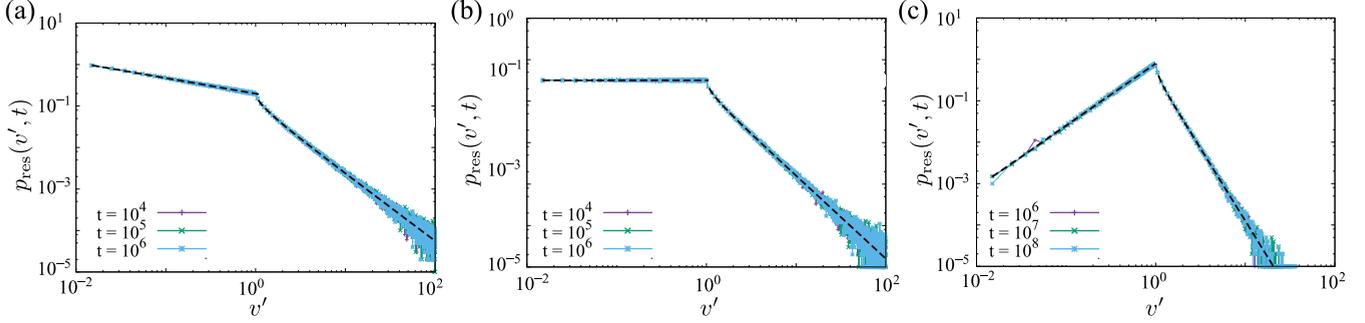


FIG. 2. Rescaled propagators for (a) $\nu = 0.2$, (b) $\nu = 0.5$, and (c) $\nu = 0.8$ ($\gamma = 0.5$). Symbols with lines are the results of numerical simulations for different times t . Dashed lines represent the scaling functions, i.e., Eq. (40). The flight-duration PDF is the same as that in Fig. 1.

not sensitive to all the details except for γ . In this sense, it is a general result.

C. Ensemble averages

The theory of infinite ergodic theory is a theory of observables. This means that we must classify different observables and define the limiting laws with which their respective ensemble averages are obtained in the long-time limit. We will soon consider also time averages. Consider the observable $f(v)$. The corresponding ensemble average is given by

$$\begin{aligned} \langle f[v(t)] \rangle &\equiv \int_{-\infty}^{\infty} f(v)p(v, t)dv \\ &= \int_{-v_c(t)}^{v_c(t)} p(v, t)f(v)dv + \int_{v_c(t)}^{\infty} p(v, t)f(v)dv \\ &\quad + \int_{-\infty}^{-v_c(t)} p(v, t)f(v)dv. \end{aligned} \quad (42)$$

If we take the time $t \rightarrow \infty$, then we have

$$\begin{aligned} \langle f[v(t)] \rangle &\cong \int_{-1}^1 \rho(v')f(v'/t^{1-\nu})dv' + t^{\gamma-1} \int_{v_c(t)}^{\infty} I_{\infty}(v)f(v)dv \\ &\quad + t^{\gamma-1} \int_{-\infty}^{-v_c(t)} I_{\infty}(v)f(v)dv, \end{aligned} \quad (43)$$

where we performed a change of variable and used the scaling function in the first term, and we also used $p(v, t) \cong t^{\gamma-1}I_{\infty}(v)$ for $|v| > v_c(t)$ in the second and third terms. Moreover, we assume that the second and third term in Eq. (42) does not diverge. In what follows, we consider $f(v) = |v|^{\alpha}$. When $f(v)$ is integrable with respect to $\rho(v)$, i.e., $\int_{-\infty}^{\infty} \rho(v)f(v)dv < \infty$, α satisfies the following inequality:

$$-\frac{\gamma}{1-\nu} < \alpha < \frac{1-\gamma}{1-\nu}. \quad (44)$$

In this case, the leading term of the asymptotic behavior of the ensemble average is given by the first term:

$$\langle f[v(t)] \rangle \sim t^{-\alpha(1-\nu)} \int_{-1}^1 \rho(v)f(v)dv \quad (t \rightarrow \infty), \quad (45)$$

where we used Eq. (40):

$$\int_{-1}^1 \rho(v)f(v/t^{1-\nu})dv \sim t^{-\alpha(1-\nu)} \int_{-1}^1 \rho(v)|v|^{\alpha}dv \quad (t \rightarrow \infty). \quad (46)$$

Thus, the ensemble average goes to zero and infinity in the long-time limit for $\alpha > 0$ and $\alpha < 0$, respectively. On the other hand, when $f(v)$ is integrable with respect to $I_{\infty}(v)$, i.e., $\int_{-\infty}^{\infty} I_{\infty}(v)f(v)dv < \infty$, where $f(v)$ satisfies $f(v) \sim v^{\alpha}$ with $\alpha > \frac{1-\gamma}{1-\nu} > 0$ for $v \rightarrow 0$, the sum of the second and third terms of Eq. (43) becomes

$$\begin{aligned} t^{\gamma-1} \int_{v_c(t)}^{\infty} I_{\infty}(v)f(v)dv + t^{\gamma-1} \int_{-\infty}^{-v_c(t)} I_{\infty}(v)f(v)dv \\ = t^{\gamma-1} \int_{-\infty}^{\infty} I_{\infty}(v)f(v)dv, \end{aligned} \quad (47)$$

Because the relation among α , ν , and γ satisfies $\alpha(1-\nu) > 1-\gamma$, the asymptotic behavior of the ensemble average is given by

$$\langle f[v(t)] \rangle \sim t^{\gamma-1} \int_{-\infty}^{\infty} I_{\infty}(v)f(v)dv \quad (t \rightarrow \infty). \quad (48)$$

Structure of Eqs. (45) and (48) is very similar to an ordinary equilibrium averaging in the sense that there is a time-independent average with respect to $\rho(v)$ or $I_{\infty}(v)$ on the right-hand side, where the choice of $\rho(v)$ or $I_{\infty}(v)$ depends on whether the observable is integrable with respect to $\rho(v)$ or $I_{\infty}(v)$. The beauty of infinite ergodic theory is that this can be extended to time averages, which as mentioned will be discussed below.

In the long-time limit, $p(v, t)$ behaves like a δ distribution in the following sense:

$$\begin{aligned} \int_{-\infty}^{\infty} \lim_{t \rightarrow \infty} p(v, t)f(v)dv = \int_{-\infty}^{\infty} \rho(x) \lim_{t \rightarrow \infty} f(x/t^{1-\nu})dx \\ = f(0). \end{aligned} \quad (49)$$

Equation (49) is clearly obtained when $f(v)$ is integrable with respect to $\rho(v)$, i.e., $\int_{-\infty}^{\infty} \rho(v)f(v)dv < \infty$. Even when $f(v)$ is not integrable with respect to $\rho(v)$, Eq. (49) is valid if $f(v)$ is integrable with respect to $I_{\infty}(v)$. In fact, the asymptotic behavior of the ensemble average $\langle f[v(t)] \rangle$ becomes $\langle f[v(t)] \rangle \rightarrow 0 = f(0)$ for $t \rightarrow \infty$, as shown above.

Therefore, Eq. (49) is valid in this case. When both integrals diverge, Eq. (49) is no longer valid. However, if there exists a positive constant ε such that $\psi(\tau) = 0$ for $\tau < \varepsilon$, Eq. (49) is always valid. In the long-time limit, the ensemble average is trivial in the sense that it simply gives the value of the observable at $v = 0$. At this stage, there is no replacement of a “steady state” concept. However, in general, the scaling function $\rho(v)$ describes the propagator near $v = 0$ while the infinite invariant density $I_\infty(v)$ describes the propagator for $|v| > 0$ including large- v behaviors. Therefore, as shown in Eqs. (45) and (48), both the scaling function and the infinite invariant density play an important role for the evaluation of certain ensemble averages at time t .

VI. DISTRIBUTIONAL LIMIT THEOREMS

When the system is stationary, a time average approaches a constant in the long-time limit, which implies ergodicity of the system. However, time averages of some observables may not converge to a constant but properly scaled time averages converge in distribution when the system is nonstationary as it is case for $\gamma < 1$. While we focus on regime $0 < \nu < 1$, the following theorems can be extended to regime $\nu > 1$.

To obtain the distribution of these time averages, we consider the propagator of the integrals of these observables along a trajectory from 0 to t , denoted by $X(t)$, which are piecewise-linear functions of t and can be described by a *continuous accumulation process* (see Fig. 3) [31]. Time average of function $f(v)$ is defined by

$$\bar{f}(t) \equiv \frac{1}{t} \int_0^t f[v(t')] dt' = \frac{X(t)}{t}. \quad (50)$$

As specific examples, we will consider time averages of the absolute value of the velocity and the squared velocity, i.e., $f(v) = |v|$ or $f(v) = v^2$. Integrated value $X(t)$ can be represented by

$$X(t) = \sum_{n=1}^{N(t)-1} f(v_n) \tau_n + f[v_{N(t)}][t - t_{N(t)-1}]. \quad (51)$$

The stochastic process of $X(t)$ can be characterized by a recursion relation, which is the same as in the derivation of the velocity propagator. Let $R_f(x, t)$ be the PDF of $x = X(t)$ when a renewal occurs exactly at time t , then we have

$$R_f(x, t) = \int_0^x dx' \int_0^t dt' \phi_f(x', t') R_f(x - x', t - t') + R_f^0(x, t), \quad (52)$$

where $\phi_f(x, \tau) = \delta[x - f(\tau^{\nu-1})\tau] \psi(\tau)$ and $R_f^0(x, t) = \delta(x)\delta(t)$. Here, we assume that function $f(v)$ is an even function. We note that we use a deterministic coupling between τ and v , i.e., Eq. (1). The PDF of $X(t)$ at time t is given by

$$P_f(x, t) = \int_0^x dx' \int_0^t dt' \Phi_f(x', t') R_f(x - x', t - t'), \quad (53)$$

where

$$\Phi_f(x, t) = \int_t^\infty d\tau \psi(\tau) \delta[x - f(\tau^{\nu-1})t]. \quad (54)$$

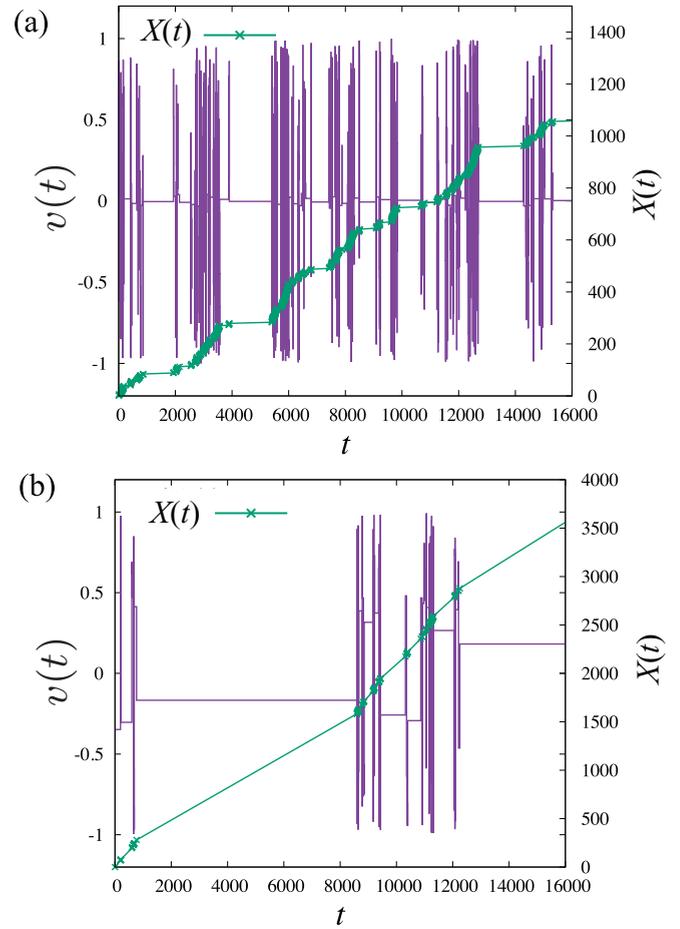


FIG. 3. Trajectories of $v(t)$ and the integral of a function $f(v) = |v|$, i.e., $X(t) = \int_0^t |v(t')| dt'$. Because velocity $v(t)$ is a piecewise constant function, $X(t)$ is a piecewise linear function of t . Parameter sets (γ, ν) are $(0.8, 0.2)$ and $(0.5, 0.8)$ for (a) and (b), respectively.

The double-Laplace transform with respect to x and t yields

$$\tilde{P}_f(k, s) = \frac{\tilde{\Phi}_f(k, s)}{1 - \tilde{\phi}_f(k, s)}, \quad (55)$$

where $\tilde{\phi}_f(k, s)$ and $\tilde{\Phi}_f(k, s)$ are the double-Laplace transforms of $\phi_f(x, \tau)$ and $\Phi_f(x, t)$ given by

$$\tilde{\phi}_f(k, s) = \int_0^\infty d\tau e^{-s\tau - kf(\tau^{\nu-1})\tau} \psi(\tau) \quad (56)$$

and

$$\tilde{\Phi}_f(k, s) = \int_0^\infty dt e^{-st} \int_t^\infty d\tau e^{-kf(\tau^{\nu-1})t} \psi(\tau), \quad (57)$$

respectively. Equation (55) is the exact form of the PDF of $X(t)$ in Laplace space.

Before considering a specific form of $f(v)$, we show that there are two different classes of distributional limit theorems of time averages. Expanding $e^{-kf(\tau^{\nu-1})\tau}$ in Eq. (56), we have

$$\tilde{\phi}_f(k, s) \cong \tilde{\psi}(s) - k \int_0^\infty d\tau f(\tau^{\nu-1})\tau \psi(\tau) e^{-s\tau} + O(k^2). \quad (58)$$

Using Eq. (30), one can write the second term with $s \rightarrow 0$ as

$$\begin{aligned} \int_0^\infty d\tau f(\tau^{\nu-1})\tau\psi(\tau) &= \frac{1}{1-\nu} \int_0^\infty dv f(v)v^{\frac{3-\nu}{\nu-1}}\psi(v^{\frac{1}{\nu-1}}) \\ &= 2 \int_0^\infty f(v)v^{\frac{1}{\nu-1}}\chi(v)dv = 2c\Gamma(\gamma) \\ &\quad \times \int_0^\infty f(v)I_\infty(v)dv. \end{aligned} \quad (59)$$

When $f(v)$ is integrable with respect to the infinite invariant density, i.e., $\int_0^\infty f(v)I_\infty(v)dv < \infty$, the second term is still finite for $s \rightarrow 0$. As shown below, we will see that the integrability gives a condition that determines the shape of the distribution function for the normalized time average, i.e., $\bar{f}(t)/\langle \bar{f}(t) \rangle$.

A. Time average of the absolute value of v

In this section, we show that there are two phases for distributional behaviors of time averages. The phase line is determined by a relation between γ and ν . As a specific choice of function $f(v)$, we consider the absolute value of the velocity, i.e., $f(v) = |v|$. Thus, $X(t)$ is given by

$$X(t) = \sum_{n=1}^{N(t)-1} \tau_n^\nu + \tau_{N(t)}^{\nu-1} [t - t_{N(t)-1}], \quad (60)$$

For $\nu < \gamma$, the moment $\langle \tau^\nu \rangle$ is finite, i.e., $\langle \tau^\nu \rangle < \infty$. This condition is equivalent to the following condition represented by the infinite density:

$$\langle f(v) \rangle_{\text{inf}} = \int_0^\infty f(v)I_\infty(v)dv < \infty. \quad (61)$$

The double Laplace transform $\tilde{P}_{|v|}(k, s)$ is calculated in Appendix C [see Eq. (C4)]. For $s \rightarrow 0$, the leading term of $-\frac{\partial \tilde{P}_{|v|}(k, s)}{\partial k} \Big|_{k=0}$ becomes

$$-\frac{\partial \tilde{P}_{|v|}(k, s)}{\partial k} \Big|_{k=0} \sim \frac{\langle \tau^\nu \rangle}{cs^{1+\gamma}}. \quad (62)$$

It follows that the mean of $X(t)$ for $t \rightarrow \infty$ becomes

$$\langle X(t) \rangle \sim \frac{\langle \tau^\nu \rangle}{c\Gamma(1+\gamma)} t^\gamma. \quad (63)$$

Since the mean of $X(t)$ increases with t^γ , we consider a situation where $k \sim s^\gamma$ for small k , $s \ll 1$ in the double-Laplace space. Thus, all the term $k/s^\nu (\ll 1)$ and $O(k^2/s^\gamma)$ in Eq. (C4) can be ignored. It follows that the asymptotic form of $\tilde{P}(k, s)$ is given by

$$\tilde{P}_{|v|}(k, s) = \frac{cs^{\gamma-1}/\langle \tau^\nu \rangle}{k + cs^\gamma/\langle \tau^\nu \rangle}. \quad (64)$$

This is the double Laplace transform of PDF $G_t'(\langle \tau^\nu \rangle x/c)$ [71], where

$$G_t(x) = 1 - L_\gamma(t/x^{1/\gamma}) \quad (65)$$

and $L_\gamma(x)$ is a one sided Lévy distribution; i.e., the Laplace transform of PDF $l_\gamma(x) \equiv L_\gamma'(x)$ is given by e^{-kx} . By a

straightforward calculation one obtain the asymptotic behavior of the second moment as follows:

$$\langle X(t)^2 \rangle \sim \frac{2\langle \tau^\nu \rangle^2 t^{2\gamma}}{c^2\Gamma(1+2\gamma)}. \quad (66)$$

Furthermore, the n th moment can be represented by

$$\langle X(t)^n \rangle \sim \frac{n!\Gamma(1+\gamma)^n}{\Gamma(1+n\gamma)} \langle X(t) \rangle^n \quad (67)$$

for $t \rightarrow \infty$. It follows that random variable $X(t)/\langle X(t) \rangle$ converges in distribution to a random variable M_γ whose PDF follows the *Mittag-Leffler distribution* of order γ , where

$$\langle e^{-zM_\gamma} \rangle \sim \sum_{n=0}^{\infty} \frac{\Gamma(1+\gamma)^n}{\Gamma(1+n\gamma)} (-z)^n. \quad (68)$$

In other words, the normalized time averages defined by $\langle \tau^\nu \rangle X(t)/(ct^\gamma)$ do not converge to a constant but the PDF converge to a nontrivial distribution (the *Mittag-Leffler distribution*). In particular, the PDF can be represented through the Lévy distribution:

$$G_1'(x) = \frac{1}{\gamma} x^{-\frac{1}{\gamma}-1} l_\gamma(x^{-1/\gamma}). \quad (69)$$

To quantify trajectory-to-trajectory fluctuations of the time averages, we consider the ergodicity breaking (EB) parameter [44] defined by

$$\text{EB}(t) \equiv \frac{\langle \bar{f}(t)^2 \rangle - \langle \bar{f}(t) \rangle^2}{\langle \bar{f}(t) \rangle^2}, \quad (70)$$

where $\langle \cdot \rangle$ implies the average with respect to the initial condition. When the system is ergodic, it goes to zero as $t \rightarrow \infty$. On the other hand, it converges to a nonzero constant when the trajectory-to-trajectory fluctuations are intrinsic. For $\nu < \gamma < 1$, the EB parameter becomes

$$\text{EB}(t) \rightarrow \text{ML}(\gamma) \equiv \frac{2\Gamma(1+\gamma)^2}{\Gamma(1+2\gamma)} - 1 \quad (t \rightarrow \infty), \quad (71)$$

which means that time averages do not converge to a constant but they become a random variable with a nonzero variance. For $\gamma > 1$, the EB parameter actually goes to zero in the long-time limit. Moreover, it also goes to zero as $\gamma \rightarrow 1$ in Eq. (71). We note that the condition (61) is general in a sense that the distribution of time averages of function $f(v)$ satisfying the condition (61) follows the Mittag-Leffler distribution, which is the same condition as in infinite ergodic theory [21].

For $\nu > \gamma$, $\langle \tau^\nu \rangle$ diverges and equivalently $\langle f(v) \rangle_{\text{inf}} = \infty$, which results in a distinct behavior of the time averages. Using Eq. (C6), we have

$$-\frac{\partial \tilde{P}_{|v|}(k, s)}{\partial k} \Big|_{k=0} \sim \frac{\gamma\Gamma(\nu-\gamma)}{(1+\gamma-\nu)\Gamma(1-\gamma)} \frac{1}{s^{1+\nu}} \quad (72)$$

for $s \rightarrow 0$. The inverse Laplace transform gives

$$\langle X(t) \rangle \sim \frac{\gamma|\Gamma(\nu-\gamma-1)|}{\Gamma(1-\gamma)\Gamma(1+\nu)} t^\nu \quad (73)$$

for $t \rightarrow \infty$. Therefore, $X(t)$ scales as t^ν , which means that all the terms of k/s^ν in Eq. (C5) cannot be ignored. These

terms give the higher-order moments. Performing the inverse Laplace transform of terms proportional to $1/s^{1+\nu}$ gives

$$\langle X(t)^n \rangle \propto t^{n\nu} \quad (74)$$

for $t \rightarrow \infty$. By Eq. (C8), the EB parameter becomes

$$\begin{aligned} \text{EB}(t) &\rightarrow A(\gamma, \nu) \\ &\equiv \frac{2(1 + \gamma - \nu)\Gamma(1 + \nu)^2}{\Gamma(1 + 2\nu)} \\ &\quad \times \left[\frac{(1 + \gamma - \nu)\Gamma(2\nu - \gamma)\Gamma(1 - \gamma)}{\gamma(2 - 2\nu + \gamma)\Gamma(\nu - \gamma)^2} + 1 \right] - 1 \end{aligned} \quad (75)$$

for $t \rightarrow \infty$. This EB parameter depends on γ as well as ν ($> \gamma$) and was found also in Ref. [52]. We note that $A(\gamma, \nu)$ is a decreasing function of ν . Therefore, trajectory-to-trajectory fluctuations of the time averages becomes insignificant for large ν . In particular, $A(\gamma, \nu)$ converges to $\text{ML}(\gamma)$ and 0 for $\nu \rightarrow \gamma + 0$ and $\nu \rightarrow 1 - 0$, respectively. In other words, the system becomes ergodic in the sense that the time averages converge to a constant in the limit of $\gamma \rightarrow 1$ (and $\nu \rightarrow 1$).

B. Time average of the squared velocity

For $f(v) = v^2$, $X(t)$ can be represented by

$$X(t) = \sum_{n=1}^{N(t)-1} \tau_n^{2\nu-1} + \tau_{N(t)}^{2\nu-2} [t - t_{N(t)-1}]. \quad (76)$$

By the same calculation as in the previous case, using $\phi_{v^2}(z, \tau) = \delta(z - \tau^{2\nu-1})\psi(\tau)$ and $\Phi_{v^2}(z, t) = \int_t^\infty d\tau \psi(\tau) \delta(z - \tau^{2\nu-2}t)$, one can express the double Laplace transform of $P(z, t)$ as

$$\tilde{P}_{v^2}(k, s) = \frac{\tilde{\Phi}_{v^2}(k, s)}{1 - \tilde{\Psi}_{v^2}(k, s)}. \quad (77)$$

Therefore, the limit distribution of $X(t)/\langle X(t) \rangle$ can be obtained in the same way as for the previous observable. In particular, the Mittag-Leffler distribution is a universal distribution of the normalized time average of v^2 if $2\nu - 1 < \gamma$, i.e., $f(v) = v^2$ is integrable with respect to the infinite invariant density. On the other hand, the distribution of normalized time averages $X(t)/\langle X(t) \rangle$ becomes another distribution for $t \rightarrow \infty$ if $2\nu - 1 > \gamma$ (see Appendix C). It follows that $\langle X(t) \rangle \propto t^{2\nu-1}$ for $t \rightarrow \infty$ and the EB parameter becomes

$$\text{EB}(t) \rightarrow A(\gamma, 2\nu - 1). \quad (78)$$

This expression was also obtained in Ref. [52]. The exponent $2\nu - 1$ in Eq. (78) is different from that found in the EB parameter for $f(v) = |v|$ with $\nu > \gamma$. Therefore, our distributional limit theorem is not universal but depends on the observable. On the other hand, the exponent γ in the EB parameter for $2\nu - 1 < \gamma$ is the same as that for $f(v) = |v|$ with $\nu < \gamma$.

Figure 4 shows that our theory works very well for both observables. For $f(v) = v^2$ with $\nu = 0.4$ and $\nu = 0.5$ ($\gamma = 0.3$), both of which satisfy $2\nu - 1 < \gamma$, the EB parameters do not depend on ν . Moreover, Fig. 4 shows that the EB parameter given by $A(\gamma, \nu)$ is a decreasing function of ν for $\nu < \gamma$.

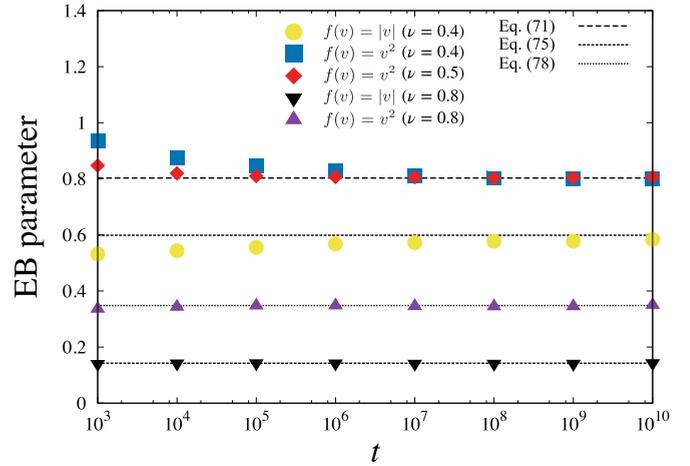


FIG. 4. Ergodicity breaking parameter as a function of the measurement time for $\nu = 0.4, 0.5$, and $\nu = 0.8$ ($\gamma = 0.3$). We note that $\nu = 0.4$ and 0.5 satisfy $2\nu - 1 < \gamma$ while $\nu = 0.8$ satisfies $2\nu - 1 > \gamma$. Symbols represent the results of numerical simulations. The long-dashed line represents Eq. (71), the two dotted lines represent Eq. (75), and the dotted line represent Eq. (78). The flight-duration PDF is the same as that in Fig. 1.

VII. CONCLUSION

We investigated the propagator in an SMP and provided its exact form, which is described by the mean number of renewals [see Eq. (23)]. We assumed that $\chi(v) = \chi(-v)$ and that this function has support at zero velocity. More specifically, the relation $v = \tau^{\nu-1}$ implies that long flight durations favor velocity close to zero since $0 < \nu < 1$ and this is the reason for an accumulation of probability in the vicinity of zero velocity in this model. We prove that the propagator accumulates in the vicinity of zero velocity in the long-time limit when the mean flight-duration diverges ($\gamma < 1$) and the coupling parameter fulfills $\nu < 1$. Taking a closer look at the vicinity of $v = 0$, we found universal behaviors in the asymptotic forms of the propagator. In particular the asymptotic behavior of the propagator for $\nu \ll 1$ follows two scaling laws, i.e., the infinite invariant density Eq. (38) and the scaling function Eq. (40). The scaling function describes a detailed structure of the propagator near $v = 0$ including zero velocity while the infinite invariant density describes the propagator outside $v_c = t^{\nu-1}$. Clearly $v_c \rightarrow 0$ when $t \rightarrow \infty$, and interestingly the asymptotic form outside v_c becomes a universal form that is unbounded at the origin and cannot be normalized, i.e., an *infinite invariant density*. One advantage of considering the topic with an SMP is that we can attain an explicit expression for the infinite invariant density Eq. (38). In contrast in general it is hard to find exact infinite invariant measures in deterministic dynamical systems, for example in the context of the Pomeau-Mannville map [17].

Further, while the Mittag-Leffler distribution describing the distribution of time averages of integrable observables is well known, from the Aaronson-Darling-Kac theorem, we considered here also another distributional limit theorem [see Eqs. (75) and (78)] which describes the distribution of time averages of certain nonintegrable observables. Therefore, the integrability of the observable with respect to the

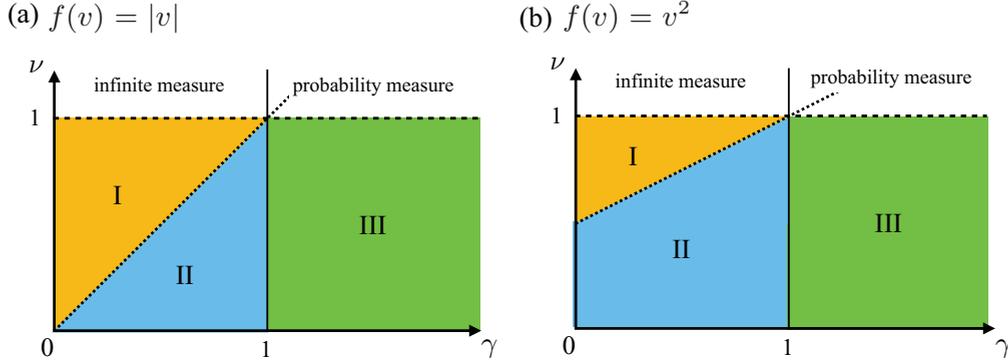


FIG. 5. Phase diagram of the parameter space (γ, ν) for (a) $f(v) = |v|$ and (b) $f(v) = v^2$. The solid line $\gamma = 1$ describes the boundary of the infinite measure. The dotted line represents the boundary that the average of the observable $f(v)$ with respect to the infinite or probability measure diverges. For $\int f(v)p_{\text{eq}}(v)dv < \infty$ and $\gamma > 1$ (region III), the time average converges to a constant, implying the EB parameter goes to zero. For $\int f(v)I_{\infty}(v)dv < \infty$ and $\gamma < 1$ (region II), the EB parameter becomes a nonzero constant given by Eq. (71), implying the time averages remain random variables. For $\int f(v)I_{\infty}(v)dv = \infty$ and $\gamma < 1$ (region I), the EB parameter becomes a nonzero constant given by Eqs. (75) or (78), implying the time averages remain random variables and it depends on γ as well as ν , which is different from case $\int f(v)I_{\infty}(v) < \infty$ and $\gamma < 1$.

infinite invariant density establishes a criterion on the type of distributional limit law, which is similar to findings in infinite ergodic theory. These results will pave the way for constructing physics of nonstationary processes. Finally, we summarize our results by the phase diagram shown in Fig. 5. The infinite invariant density is always observed for $\gamma < 1$. On the other hand, the boundary of the regions I and II depends on the observation function $f(v)$.

ACKNOWLEDGMENTS

This work was supported by JSPS KAKENHI Grants No. 16KT0021 and No. 18K03468 (T.A.). E.B. thanks the Israel Science Foundation and Humboldt Foundation for support.

APPENDIX A: EXACT FORM OF THE PROPAGATOR OUTSIDE $[-v_c(t), v_c(t)]$

Here we consider a specific form for the flight-duration PDF to obtain the exact form of the propagator outside $[-v_c(t), v_c(t)]$. As a specific form, we use

$$\psi(\tau) = -\frac{d}{dt}E_{\gamma}(-t^{\gamma}) = \frac{1}{\tau^{1-\gamma}} \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{n\gamma}}{\Gamma(\gamma n + \gamma)}, \quad (\text{A1})$$

where $E_{\gamma}(z)$ is the Mittag-Leffler function with parameter γ defined as [72]

$$E_{\gamma}(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n + 1)}. \quad (\text{A2})$$

In fact, the asymptotic behavior is given by a power law [72], i.e.,

$$\psi(\tau) \sim \frac{\Gamma(\gamma + 1) \sin(\gamma\pi)}{\pi} \tau^{-1-\gamma} \quad (\tau \rightarrow \infty). \quad (\text{A3})$$

Moreover, it is known that the Laplace transform of $\psi(\tau)$ is given by

$$\tilde{\psi}(s) = \frac{1}{1 + s^{\gamma}}. \quad (\text{A4})$$

Therefore, the Laplace transform of $\langle N(t) \rangle$ becomes

$$\frac{1}{s[1 - \tilde{\psi}(s)]} = \frac{1}{s^{1+\gamma}} + \frac{1}{s}, \quad (\text{A5})$$

and its inverse Laplace transform yields

$$\langle N(t) \rangle = \frac{1}{\Gamma(1 + \gamma)} t^{\gamma} + 1 \quad (\text{A6})$$

for any $t > 0$. For $\nu \ll 1$, $\chi(v)$ is given by

$$\chi(v) \sim \frac{1}{2(1 - \nu)|\Gamma(-\gamma)|} |v|^{-1 + \frac{\gamma}{1-\nu}}. \quad (\text{A7})$$

It follows that the propagator outside $[-v_c(t), v_c(t)]$ becomes

$$p(v, t) \sim \frac{t^{\gamma} - (t - |v|^{\frac{1}{1-\nu}})^{\gamma}}{2(1 - \nu) \sin(\gamma\pi)} \pi |v|^{-1 + \frac{\gamma}{1-\nu}}. \quad (\text{A8})$$

for $t \gg 1$ and $|v| > t^{\nu-1}$. As shown in Fig. 6, the propagator outside $[-v_c(t), v_c(t)]$ is described by Eq. (A8), whereas we did not use Eq. (A1).

APPENDIX B: ANOTHER PROOF OF THE ASYMPTOTIC BEHAVIOR OF THE PROPAGATOR OF v

To obtain the propagator, i.e., the PDF of velocity v at time t , it is almost equivalent to have the PDF $\psi_t(\tau)$ of time interval straddling t , i.e., $\tau_{N(t)-1}$, where $N(t) - 1$ is the number of renewals until t (not counting the one at $t_0 = 0$). In ordinary renewal processes, the double Laplace transform of the PDF with respect to τ and t is given by [66]

$$\tilde{\phi}(k, s) = \frac{\tilde{\psi}(k) - \tilde{\psi}(k + s)}{s[1 - \tilde{\psi}(s)]}. \quad (\text{B1})$$

For $\gamma < 1$, the asymptotic behavior of this inverse Laplace transform can be calculated using a technique from Ref. [43].

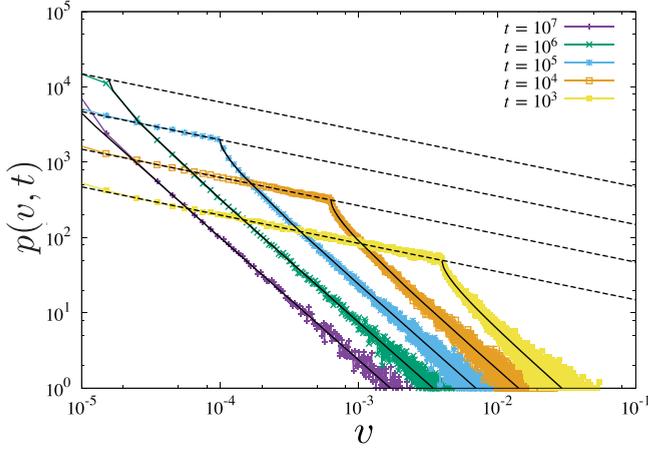


FIG. 6. Time evolution of the propagator for different times ($\gamma = 0.5$ and $\nu = 0.2$). Symbols with lines are the results of numerical simulations, which is the same as those in Fig. 1. Dashed and solid lines are the theories, i.e., Eqs. (37) and (A8), respectively. The flight-duration PDF is the same as that in Fig. 1.

For t and $\tau \gg 1$,

$$\psi_t(\tau) \sim \begin{cases} \frac{\sin \pi \gamma}{\pi} \frac{t^\gamma}{\tau^{1+\gamma}} \left[1 - \left(1 - \frac{\tau}{t} \right)^\gamma \right] & (\tau < t) \\ \frac{\sin \pi \gamma}{\pi} \frac{t^\gamma}{\tau^{1+\gamma}} & (\tau > t). \end{cases} \quad (\text{B2})$$

This is the asymptotic result, which does not depend on the details of the flight-duration PDF, i.e., different flight-duration PDFs give the same result if the power-law exponent γ is the same. On the other hand, detail forms of $\psi_t(\tau)$, e.g., the behavior for small t and τ , depend on details of the flight-duration PDF [15].

Here we consider a situation where the relation between the velocity and the flight duration is given by $|v| = \tau^{\nu-1}$. The PDF of velocity v at time t , i.e., the propagator, can be represented through the PDF $\psi_t(\tau)$:

$$p(v, t) = \frac{1}{2|v-1|} |v|^{\frac{1}{\nu-1}-1} \psi_t(|v|^{\frac{1}{\nu-1}}). \quad (\text{B3})$$

Note that $p(v, t)$ is symmetric with respect to $v = 0$. Using Eq. (B2) yields

$$p(v, t) \sim \begin{cases} \frac{\sin \pi \gamma}{2\pi|1-v|} t^\gamma |v|^{-1+\frac{\gamma}{1-\nu}} \left[1 - \left(1 - \frac{|v|^{\frac{1}{\nu-1}}}{t} \right)^\gamma \right] & (|v| > t^{\nu-1}) \\ \frac{\sin \pi \gamma}{2\pi|1-v|} t^\gamma |v|^{-1+\frac{\gamma}{1-\nu}} & (|v| < t^{\nu-1}). \end{cases} \quad (\text{B4})$$

The asymptotic form for $\nu < 1$ becomes

$$p(v, t) \sim \begin{cases} \frac{\sin \pi \gamma}{2\pi|1-v|} t^{\gamma-1} |v|^{-1+\frac{\gamma}{1-\nu}} & (|v| \ll t^{\nu-1}) \\ \frac{\gamma \sin \pi \gamma}{2\pi(\nu-1)} t^\gamma |v|^{-1+\frac{1-\gamma}{\nu-1}} & (t^{\nu-1} \ll |v|) \end{cases} \quad (\text{B5})$$

for $t \rightarrow \infty$. Therefore, this is consistent with the propagator we obtained in this paper, Eqs. (34) and (37).

For $\gamma > 1$, the PDF $\psi_t(\tau)$ has an equilibrium distribution, i.e., for $t \rightarrow \infty$ the PDF $\psi_t(\tau)$ is given by

$$\psi_t(\tau) \sim \frac{\tau \psi(\tau)}{\langle \tau \rangle}, \quad (\text{B6})$$

where $\langle \tau \rangle$ is the mean flight duration [49].

APPENDIX C: THE DOUBLE LAPLACE TRANSFORM $\tilde{P}(k, s)$ AND THE EXACT FORM OF THE SECOND MOMENT OF $X(t)$ FOR $\nu > \gamma$

Here, we represent the double Laplace transform $\tilde{P}(k, s)$ as an infinite series expansion. Expanding $e^{-k\tau^\nu}$ in Eqs. (56) and (57), we have

$$\tilde{\phi}_{|v|}(k, s) \cong \tilde{\psi}(s) - \langle \tau^\nu \rangle k + O(k^2) \quad (\text{C1})$$

and

$$\tilde{\phi}_{|v|}(k, s) \cong \tilde{\psi}(s) + \frac{cs^\gamma}{|\Gamma(-\gamma)|} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Gamma(n\nu - \gamma) \left(\frac{k}{s^\nu} \right)^n \quad (\text{C2})$$

for $\nu < \gamma$ and $\gamma < \nu$, respectively, where $\langle \tau^\nu \rangle \equiv \int_0^\infty \tau^\nu \psi(\tau) d\tau$. Moreover, we have

$$\tilde{\Phi}_{|v|}(k, s) \cong \frac{1 - \tilde{\psi}(s)}{s} + \frac{c}{|\Gamma(-\gamma)| s^{1-\gamma}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(n\nu - \gamma + 1)}{\gamma + (1-\nu)n} \left(\frac{k}{s^\nu} \right)^n \quad (\text{C3})$$

for $\gamma < 1$. Using Eq. (55), we have

$$\begin{aligned}\tilde{P}_{|v|}(k, s) &= \frac{cs^\gamma}{s} \left[1 + \frac{1}{|\Gamma(-\gamma)|} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\Gamma(nv - \gamma + 1)}{\gamma + (1 - \nu)n} \left(-\frac{k}{s^\nu}\right)^n \right] \frac{1}{cs^\gamma + \langle \tau^\nu \rangle k + O(k^2)} \\ &= \frac{1}{s} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\gamma \Gamma(nv - \gamma + 1)}{\Gamma(1 - \gamma) \{\gamma + (1 - \nu)n\}} \left(-\frac{k}{s^\nu}\right)^n \right] \left[1 + \frac{\langle \tau^\nu \rangle k}{c s^\gamma} + O(k^2/s^\gamma) \right]^{-1}\end{aligned}\quad (C4)$$

for $\nu < \gamma$ and

$$\begin{aligned}\tilde{P}_{|v|}(k, s) &= \frac{cs^\gamma}{s} \left[1 + \frac{1}{|\Gamma(-\gamma)|} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\Gamma(nv - \gamma + 1)}{\gamma + (1 - \nu)n} \left(-\frac{k}{s^\nu}\right)^n \right] \left[cs^\gamma - \frac{cs^\gamma}{|\Gamma(-\gamma)|} \sum_{n=1}^{\infty} \frac{\Gamma(nv - \gamma)}{n!} \left(-\frac{k}{s^\nu}\right)^n \right]^{-1} \\ &= \frac{1}{s} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\gamma \Gamma(nv - \gamma + 1)}{\Gamma(1 - \gamma) \{\gamma + (1 - \nu)n\}} \left(-\frac{k}{s^\nu}\right)^n \right] \left[1 - \sum_{n=1}^{\infty} \frac{\gamma \Gamma(nv - \gamma)}{n! \Gamma(1 - \gamma)} \left(-\frac{k}{s^\nu}\right)^n \right]^{-1} \\ &= \frac{1}{s} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\gamma \Gamma(nv - \gamma + 1)}{\Gamma(1 - \gamma) \{\gamma + (1 - \nu)n\}} \left(-\frac{k}{s^\nu}\right)^n \right] \left[1 + \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{\gamma \Gamma(nv - \gamma)}{n! \Gamma(1 - \gamma)} \left(-\frac{k}{s^\nu}\right)^n \right\}^m \right]\end{aligned}\quad (C5)$$

for $\nu > \gamma$.

The coefficient of the term proportional to k in Eq. (C5) is

$$\frac{-1}{s^{1+\nu}} \left[\frac{\gamma \Gamma(\nu - \gamma + 1)}{(\gamma + 1 - \nu) \Gamma(1 - \gamma)} + \frac{\gamma \Gamma(\nu - \gamma)}{\Gamma(1 - \gamma)} \right]. \quad (C6)$$

Moreover, by considering the coefficient of the term proportional to k^2 in Eq. (C5), the leading term of the second moment of $X(t)$ in the Laplace space ($s \rightarrow 0$) can be represented as

$$\left. \frac{\partial^2 \tilde{P}_{|v|}(k, s)}{\partial k^2} \right|_{k=0} \sim \frac{M_2(\nu, \gamma)}{s^{1+2\nu}}, \quad (C7)$$

where

$$M_2(\nu, \gamma) = \frac{2\gamma \Gamma(2\nu - \gamma)}{(2 - 2\nu + \gamma) \Gamma(1 - \gamma)} + \frac{2\gamma^2 \Gamma(\nu - \gamma)^2}{(1 + \gamma - \nu) \Gamma(1 - \gamma)^2}. \quad (C8)$$

It follows that the asymptotic behavior of $\langle X(t)^2 \rangle$ is given by Eq. (74) with $n = 2$.

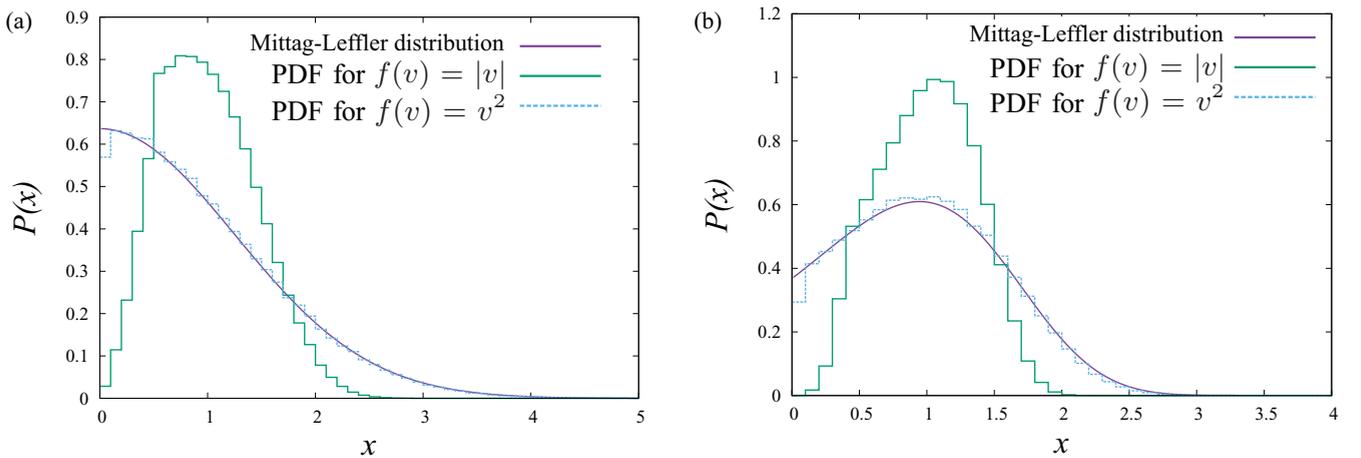


FIG. 7. Probability density function $P(x)$ of time averages for (a) $\gamma = 0.5$ and $\nu = 0.7$ and (b) $\gamma = 0.7$ and $\nu = 0.8$. The solid and the dotted histograms represent PDFs obtained by numerical simulations for $f(v) = |v|$ and $f(v) = v^2$, respectively. The solid line is the PDF of the Mittag-Leffler distribution. Note that the PDFs for $f(v) = v^2$ follow the Mittag-Leffler distribution with order $\gamma = 0.5$ and 0.7 in cases (a) and (b), respectively. On the other hand, the PDFs for $f(v) = |v|$ depend on the exponent γ as well as ν , implying that the PDFs are different from the Mittag-Leffler distribution. A similar distribution was found in distributional limit theorems of time-averaged observables in infinite ergodic theory [31].

Since $\langle X(t)^n \rangle$ grows as $\langle X(t)^n \rangle \propto t^{nv}$, one can define $M_n(\nu, \gamma)$ as

$$\langle X(t)^n \rangle \sim \frac{\Gamma(1 + \nu)^n M_n(\nu, \gamma)}{\Gamma(1 + n\nu) M_1(\nu, \gamma)^n} \langle X(t) \rangle^n. \quad (\text{C9})$$

It follows that the random variable $X(t)/\langle X(t) \rangle$ converges in distribution to a random variable $M_{\nu, \gamma}$ which depends on both ν and γ . More precisely, one obtains

$$\langle e^{-zM_{\nu, \gamma}} \rangle \sim \sum_{n=0}^{\infty} \frac{\Gamma(1 + \nu)^n M_n(\nu, \gamma)}{n! \Gamma(1 + n\nu) M_1(\nu, \gamma)^n} (-z)^n. \quad (\text{C10})$$

Therefore, the PDF of the normalized time average defined by $X(t)/t^\nu$ converges to a nontrivial distribution that is different from the Mittag-Leffler distribution (see Fig. 7).

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