# Propagation of dipole solitons in inhomogeneous highly dispersive optical-fiber media

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We consider ultrashort light pulse propagation through an inhomogeneous monomodal optical fiber exhibiting higher-order dispersive effects. Wave propagation is governed by a generalized nonlinear Schrödinger equation with varying second-, third-, and fourth-order dispersions, cubic nonlinearity, and linear gain or loss. We construct a type of exact self-similar soliton solution that takes the structure of a dipole via a similarity transformation connected to the related constant-coefficients one. The conditions on the optical-fiber parameters for the existence of these self-similar structures are also given. The results show that the contribution of all orders of dispersion is an important feature to form this kind of self-similar dipole pulse shape. The dynamic behaviors of the self-similar dipole solitons in a periodic distributed amplification system are analyzed. The significance of the obtained self-similar pulses is also discussed. By performing numerical simulations, the self-similar soliton solutions are found to be stable under slight disturbance of the constraint conditions and the initial perturbation of white noise.

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## I. INTRODUCTION

Light pulses in optical fibers are referred to generically as solitons and are usually described within the framework of the cubic nonlinear Schrödinger equation (NLSE) that includes only basic effects on waves such as group velocity dispersion (GVD) and self-phase modulation (SPM) [1–3]. Depending on the anomalous dispersion or the normal dispersion, the NLSE model allows for either bright or dark solitons, respectively [4]. The formation of these solitons results from the exact balance between the GVD and SPM of the material. Due to the robust and stable nature of solitons, such wave packets have been successfully used as the information carriers (optical bits) to transmit digital signals over long propagation distances. However, many applications in various areas such as ultrahigh-bit-rate optical communication systems, infrared time-resolved spectroscopy, ultrafast physical processes, and optical sampling systems require ultrashort (femtosecond) pulses [3,5], which leads to the appearance of different higher-order effects in the optical material. For instance, the third-order dispersion plays a significant role in propagation if short pulses whose widths are nearly 50 fs have to be injected [6,7]. The fourth-order dispersion is also important when pulses are shorter than 10 fs [6,7]. In such a situation, the wave dynamics can be described by the higher-order NLSE incorporating the contribution of various physical phenomena on short-pulse propagation and generation. One notes that additional higher-order effects introduce several new important phenomena into the system dynamics that are absent in nonlinear media of the Kerr type.

However, for describing more realistic phenomena, the inhomogeneities present in nonlinear dispersive media should also be considered. Taking into account the inhomogeneities in the optical fiber, for example, the description of the optical pulse propagation is generally based on the generalized NLSE with distributed coefficients (i.e., allowing nonlinearity, dispersion, and gain profiles to change with the distance along the direction of propagation) [8,9]. Such a generalized model possesses a rich variety of exact self-similar solutions that are characterized by a strictly linear chirp [8–10]. These self-similar pulses (also called "similaritons") have attracted considerable interest in recent years because of their extensive applications in photonics and fiber-optic telecommunications [11]. Importantly, these similaritons can maintain the overall shape while allowing their parameters such as amplitude and width to change with the modulation of system parameters [12].

Recently, propagation of self-similar waves has drawn much interest and many important results have been presented, which is an essential prerequisite for understanding the dynamical processes and mechanisms of the complicated phenomena in different inhomogeneous media [13–28]. For example, bright and dark solitonlike similaritons [13,14] have been theoretically investigated in graded-index waveguide amplifiers with gain or loss and in the presence of spatial inhomogeneities. In addition, significant results have been obtained by studying self-similarity in Bragg gratings [15], stimulated Raman scattering [16], self-written waveguides [17], and fractal formation in optical materials that support spatial solitons [18]. Moreover, the self-similar nature of optical wave collapse in Kerr-like systems of higher dimensionality has been also elucidated in Ref. [19]. The existence of self-similar optical structures of the kink type has been also shown in a wide class of resonant nonlinear media [20]. Furthermore, stable spatial similaritons have been demonstrated in pure quintic nonlinear media doped with resonant impurities [21]. Recently, Dai *et al.* [22] investigated the dynamic behaviors of spatial similaritons in inhomogeneous nonlinear cubic-quintic media. They also discussed controllable optical rogue waves in the femtosecond regime [23]. In Ref. [24], the authors constructed explicit chirped and chirp-free self-similar solitary wave and cnoidal wave solutions of the generalized cubic-quintic NLSE by applying the similarity transformation method. Very recently, Choudhuri *et al.* [25] derived the exact self-similar localized pulse solutions for the NLSE with distributed cubic-quintic nonlinearities. Triki *et al.* [26] investigated the propagation of self-similar optical solitons on a continuous-wave background in a quadratic-cubic noncentrosymmetric waveguide. Liu *et al.* [27] constructed a variety of spatiotemporal self-similar wave solutions for the $(3+1)$-dimensional variable-coefficients NLSE with cubic-quintic nonlinearities. Serkin *et al.* [28] discovered solitary nonlinear Bloch waves of the bright and dark types in dispersion-managed fiber systems and soliton lasers.

However, most investigations on the propagation of self-similar solitons in inhomogeneous fiber systems have been focused on bright, dark, and kink-type self-similar solitary waves or solitons, as well as rogue and cnoidal waves. But many novel localized structures including, for example, dipole solitons, vortex solitons, and soliton trains have been demonstrated experimentally and theoretically in both one- and two-dimensional nonlinear media [29–34]. To our knowledge, no exact self-similar "dipole" soliton solutions have been previously reported within the framework of the variable-coefficient NLSE models. Moreover, the control of self-similar localized pulses under the combined influence of second-, third-, and fourth-order dispersions as well as Kerr nonlinearity management has not been widespread. In this paper, we demonstrate the existence of self-similar pulses that takes a dipole structure in inhomogeneous highly dispersive optical fibers and investigate their propagation dynamics for different parameters.

Our results are presented as follows. Section II presents the method used for obtaining traveling wave solutions of the extended NLSE that describes the propagation of extremely short pulses inside a highly dispersive optical fiber medium. In Sec. III, we derive the analytical quartic and dipole soliton solutions of the model and their characteristics. In Sec. IV, the variation of fiber dispersions, nonlinearity, and gain or loss is considered and a similarity transformation to reduce the generalized nonlinear Schrödinger equation with varying coefficients to the related constant-coefficients one is presented. Self-similar dipole structures of the generalized NLSE and their dynamical behaviors in a periodic distributed amplification system are reported in Sec. V. In Sec. VI, the analytical stability analysis of soliton solutions based on the theory of optical nonlinear dispersive waves is presented. In Sec. VII, the stability of the solutions is discussed numerically. In Sec. VIII, we present a physical discussion and some applications of the theoretical model under consideration. Finally, we summarize our work in Sec. IX.

## II. TRAVELING WAVES

In this section, we reduce the extended NLSE to an ordinary differential equation. The extended NLSE is derived for the assumptions of slowly varying envelope, instantaneous nonlinear response, and no higher-order nonlinearities [35]. This nonlinear equation has the next form for the optical pulse envelope $E(z,\tau)$,

$$iE_z = \alpha E_{\tau\tau} + i\sigma E_{\tau\tau\tau} - \epsilon E_{\tau\tau\tau\tau} - \gamma |E|^2 E, \tag{1}$$

where $z$ is the longitudinal coordinate, $\tau = t - \beta_1 z$ is the retarded time, and $\alpha = \beta_2/2$, $\sigma = \beta_3/6$, $\epsilon = \beta_4/24$, and $\gamma$ is the nonlinear parameter. The parameters $\beta_k = (d^k\beta/d\omega^k)_{\omega=\omega_0}$ are the $k$-order dispersion of the optical fiber and $\beta(\omega)$ is the propagation constant depending on the optical frequency.

This equation has been intensively studied for its importance from various viewpoints [35–39]. In particular, the modulational instability phenomena of Eq. (1) were analyzed in the region of the minimum group-velocity dispersion in Ref. [35]. In a very recent work, Kruglov and Harvey [36] presented an exact solitary wave solution having the functional form of "sech$^2$" for the NLSE (1) including second-, third-, and fourth-order dispersion effects. Moreover, Karpman and Shagalov [37,38] have studied the time behavior of the amplitudes, velocities, and other parameters of radiating solitons. Shagalov [39] has investigated the effect of the third- and fourth-order dispersions on the modulational instability. Roy *et al.* [40] have also studied the role of third- and fourth-order dispersions in the radiation emitted by fundamental soliton pulses in the form of dispersive waves within the framework of the dimensionless form of the NLSE (1).

We consider the solution of the generalized NLSE in the form,

$$E(z,\tau) = u(x)\exp[i(\kappa z - \delta\tau + \theta)], \tag{2}$$

where $u(x)$ is a real function depending on the variable $x = \tau - qz$, and $q = v^{-1}$ is the inverse velocity. Also, $\kappa$ and $\delta$ are the respective real parameters describing the wave number and frequency shift, while $\theta$ represents the phase of the pulse at $z = 0$.

Equations (1) and (2) lead to the next system of the ordinary differential equations:

$$(\sigma + 4\epsilon\delta)\frac{d^3u}{dx^3} + (q - 2\alpha\delta - 3\sigma\delta^2 - 4\epsilon\delta^3)\frac{du}{dx} = 0, \tag{3}$$

$$\epsilon\frac{d^4u}{dx^4} - (\alpha + 3\sigma\delta + 6\epsilon\delta^2)\frac{d^2u}{dx^2} + \gamma u^3 - (\kappa - \alpha\delta^2 - \sigma\delta^3 - \epsilon\delta^4)u = 0. \tag{4}$$

In the general case the system of Eqs. (3) and (4) is overdetermined because we have two differential equations for the function $u(x)$. However, if some constraints for the parameters in Eq. (3) are fulfilled the system of Eqs. (3) and (4) has nontrivial solutions. We refer to the solution of the extended NLSE where the function $E(z,\tau)$ is given by Eq. (2) with $u(x) \neq \text{const}$ as a non-plain-wave or traveling-wave solution.

The system of Eqs. (3) and (4) with $\epsilon \neq 0$ yields the non-plain-wave solutions if and only if the next relations are satisfied:

$$q = 2\alpha\delta + 3\sigma\delta^2 + 4\epsilon\delta^3, \quad \delta = -\frac{\sigma}{4\epsilon}. \tag{5}$$

The system of Eqs. (3) and (4) with $\epsilon = 0$ has the non-plain-wave solutions only when the parameter $\sigma = 0$. Note that

Eq. (3) is satisfied for an arbitrary function $u(x)$ according to the conditions in Eq. (5) with $\epsilon \neq 0$. The relations in Eq. (5) lead to the next expression for the velocity $v = 1/q$ defined in the retarded frame,

$$v = \frac{8\epsilon^2}{\sigma(\sigma^2 - 4\alpha\epsilon)}. \quad (6)$$

The relations in Eq. (5) reduce the system of Eqs. (3) and (4) to the ordinary nonlinear differential equation,

$$\epsilon \frac{d^4u}{dx^4} + b\frac{d^2u}{dx^2} - cu + \gamma u^3 = 0, \quad (7)$$

where the parameters $b$ and $c$ are

$$b = \frac{3\sigma^2}{8\epsilon} - \alpha, \quad c = \kappa + \frac{\sigma^2}{16\epsilon^2}\left(\frac{3\sigma^2}{16\epsilon} - \alpha\right). \quad (8)$$

By analytically solving Eq. (7), we obtain the soliton structures that can propagate in the highly dispersive fiber medium. However, it would be very difficult to find the closed form solutions of such ordinary nonlinear differential equations in which two even-order derivative terms coexist. Obtaining solutions in analytic form is of great interest since these are useful, for instance, to compare experimental results with theory. In the following, families of soliton solutions having the functional form of $\mathrm{sech}^2(\cdot)$ and $\mathrm{sech}(\cdot)\mathrm{th}(\cdot)$ are derived in the presence of all physical parameters.

### III. QUARTIC AND DIPOLE SOLITONS IN HIGHLY DISPERSIVE OPTICAL FIBER

In this section, we consider the traveling-wave solutions of Eq. (7) in the form

$$u(x) = \frac{F_N(x)}{G_M(x)} = \frac{\sum_{n=-N}^{N} A_n \exp[-nw(x-\eta)]}{\sum_{n=-M}^{M} B_n \exp[-nw(x-\eta)]}. \quad (9)$$

The quartic dark soliton solution of this form is given by

$$u(x) = A + B\,\mathrm{th}^2[w(x-\eta)] = D - \frac{B}{\mathrm{ch}^2[w(x-\eta)]}, \quad (10)$$

where $D = A + B$ and $D \neq 0$. In the case when $D = 0$ we have the $\mathrm{sech}^2$ solitary wave.

Substituting the function (10) into Eq. (7) and setting the coefficients of independent terms equal to zero, we obtain the following equations:

$$cD = \gamma D^3, \quad c = 16\epsilon w^4 + 4bw^2 + 3\gamma D^2, \quad (11)$$

$$40\epsilon w^4 + 2bw^2 + \gamma DB = 0, \quad 120\epsilon w^4 + \gamma B^2 = 0. \quad (12)$$

We now discuss solutions to these parametric equations for two cases: (1) $D = 0$ and (2) $D \neq 0$. In case 1 with $D = 0$ and $E_0 = -B$ we have

$$w = \frac{1}{4}\sqrt{\frac{8\alpha\epsilon - 3\sigma^2}{10\epsilon^2}}, \quad E_0 = \pm\sqrt{\frac{-3}{10\gamma\epsilon}}\left(\frac{3\sigma^2}{8\epsilon} - \alpha\right). \quad (13)$$

Furthermore, we get from Eqs. (11) and (12) a condition on the parameter $c$ as

$$c = -\frac{4}{25\epsilon}\left(\frac{3\sigma^2}{8\epsilon} - \alpha\right)^2. \quad (14)$$

Incorporating these results into Eq. (10), we obtain the following soliton solution to the extended NLSE (1) [36]:

$$E(z,\tau) = E_0\,\mathrm{sech}^2(w\xi)\exp[i(\kappa z - \delta\tau + \theta)], \quad (15)$$

where $\xi = \tau - v^{-1}z - \eta$, with $\eta$ being the position of the pulse at $z = 0$. The wave number $\kappa$ follows from Eqs. (8) and (14) as

$$\kappa = -\frac{4}{25\epsilon^3}\left(\frac{3\sigma^2}{8} - \alpha\epsilon\right)^2 - \frac{\sigma^2}{16\epsilon^3}\left(\frac{3\sigma^2}{16} - \alpha\epsilon\right). \quad (16)$$

The velocity $v$ and frequency shift $\delta$ in this soliton solution are given by Eqs. (5) and (6).

Physically, Eq. (15) describes a solitary pulse with amplitude $E_0$ and inverse temporal width $w$ depending on all order of dispersion as well as nonlinearity. It follows from Eq. (13) that this $\mathrm{sech}^2$ solitary wave exists when the next two conditions are satisfied: $\gamma\epsilon < 0$ and $8\alpha\epsilon > 3\sigma^2$.

In case 2 with $D \neq 0$, Eqs. (11) and (12) lead to the complex values for parameters $B$ and $D$. However, the function $u(x)$ in Eq. (7) is real which contradicts such complex parameters. Thus, the extended NLSE given by Eq. (1) does not have a quartic dark soliton solution.

We have also found that Eq. (7) admits an exact dipole soliton solution of the form

$$u(x) = E_0\frac{\mathrm{sh}[w(x-\eta)]}{\mathrm{ch}^2[w(x-\eta)]}. \quad (17)$$

Inserting this solution into Eq. (7) and equating the coefficients of independent terms, one obtains

$$c = \epsilon w^4 + bw^2, \quad 120\epsilon w^4 = \gamma E_0^2, \quad (18)$$

$$60\epsilon w^4 + 6bw^2 - \gamma E_0^2 = 0. \quad (19)$$

These equations yield the dipole soliton solution as

$$E(z,\tau) = E_0\,\mathrm{sech}(w\xi)\,\mathrm{th}(w\xi)\exp[i(\kappa z - \delta\tau + \theta)], \quad (20)$$

where $\xi = \tau - v^{-1}z - \eta$. Thus, we have the following relations for the pulse inverse width $w$ and amplitude $E_0$:

$$w = \frac{1}{4}\sqrt{\frac{3\sigma^2 - 8\alpha\epsilon}{5\epsilon^2}}, \quad E_0 = \pm\sqrt{\frac{6}{5\gamma\epsilon}}\left(\frac{3\sigma^2}{8\epsilon} - \alpha\right). \quad (21)$$

Equations (18) and (19) yield the parameter $c = 11b^2/100\epsilon$. Hence, it follows from Eq. (8) that the wave number $\kappa$ in the dipole soliton solution is given by

$$\kappa = \frac{11}{100\epsilon^3}\left(\frac{3\sigma^2}{8} - \alpha\epsilon\right)^2 - \frac{\sigma^2}{16\epsilon^3}\left(\frac{3\sigma^2}{16} - \alpha\epsilon\right). \quad (22)$$

The velocity $v$ and frequency shift $\delta$ in this dipole soliton solution are given by Eqs. (5) and (6), respectively. It follows from Eq. (21) that the dipole soliton solution exists when the next two conditions are satisfied: $\gamma\epsilon > 0$ and $8\alpha\epsilon < 3\sigma^2$.

The corresponding energy $\mathcal{E}$ of the dipole solitons is given by

$$\mathcal{E} = \int_{-\infty}^{+\infty} |E(z,\tau)|^2\, d\tau = \frac{(3\sigma^2 - 8\alpha\epsilon)^{3/2}}{4\gamma\epsilon\sqrt{5\epsilon^2}}. \quad (23)$$

Note that the energy of the pulse, $\mathcal{E}$, is the integral of motion ($d\mathcal{E}/dz = 0$) of the NLSE (1) for any pulses satisfying the boundary condition $E(z,\tau) \to 0$ for $\tau \to \pm\infty$ .

## IV. TRANSFORMATION TO GENERALIZED NLSE WITH VARIABLE COEFFICIENTS

We consider in this section the variations of fiber dispersion, nonlinearity, and gain or loss. For our purpose, the dynamics of pulses is described by the following generalized NLSE with distributed coefficients:

$$iU_s = D(s)U_{tt} + iP(s)U_{ttt} - Q(s)U_{tttt} - R(s)|U|^2U + i\Gamma(s)U, \tag{24}$$

where $D(s)$, $P(s)$, and $Q(s)$ are the variable parameters of second-, third-, and fourth-order dispersions, respectively. The function $R(s)$ stands for the varying Kerr nonlinearity parameter, while $\Gamma(s)$ denotes the amplification ($\Gamma(s) > 0$) or absorption parameter ($\Gamma(s) < 0$).

In the simplest case, when all the coefficients are constants and $\Gamma(s) = 0$, then Eq. (24) can be transformed into the constant-coefficient NLSE (1). It is of interest to control optical solitons in communication systems when all orders of dispersion, nonlinearity, and gain or loss are varied as described by the NLSE (24). In following, we first search for exact self-similar soliton solutions of the variable-coefficient NLSE (24) by employing the similarity transformation method and then discuss their propagation behaviors in a specified soliton control system.

We first construct the transformation [22–24]

$$U(s,t) = A(s)E[z(s), \tau(s,t)]e^{i\phi(s,t)}, \tag{25}$$

where $A(s)$ and $\phi(s,t)$ are both real functions, describing the amplitude and phase of the pulse, respectively, while $z = z(s)$ and $\tau = \tau(s,t)$ are two unknown functions to be determined.

Substituting Eq. (25) into Eq. (24) leads to Eq. (1), but now we must have the following set of equations:

$$A_s - \Gamma A - DA\phi_{tt} + 3PA\phi_t\phi_{tt} + QA\phi_{tttt} - 6QA\phi_t^2\phi_{tt} = 0, \tag{26}$$

$$\tau_s - 2D\tau_t\phi_t + 3P\tau_t\phi_t^2 - P\tau_{ttt} + 4Q\tau_{ttt}\phi_t + 6Q\tau_{tt}\phi_{tt} - 4Q\tau_t\phi_t^3 + 4Q\tau_t\phi_{ttt} = 0, \tag{27}$$

$$\phi_s - D\phi_t^2 - P\phi_{ttt} + P\phi_t^3 + 4Q\phi_t\phi_{ttt} - Q\phi_t^4 + 3Q\phi_{tt}^2 = 0, \tag{28}$$

$$(3P\phi_t - D)\tau_{tt} + 3P\tau_t\phi_{tt} + Q\tau_{tttt} - 12Q\tau_t\phi_t\phi_{tt} - 6Q\tau_{tt}\phi_t^2 = 0, \tag{29}$$

$$-3P\tau_t\tau_{tt} + 12Q\tau_t\tau_{tt}\phi_t + 6Q\tau_t^2\phi_{tt} = 0, \tag{30}$$

$$RA^2 = \gamma z_s, \tag{31}$$

$$(D - 3P\phi_t)\tau_t^2 - 4Q\tau_t\tau_{ttt} + 6Q\tau_t^2\phi_t^2 - 3Q\tau_{tt}^2 = \alpha z_s, \tag{32}$$

$$-P\tau_t^3 + \sigma z_s + 4Q\tau_t^3\phi_t = 0, \tag{33}$$

$$Q\tau_t^4 = \epsilon z_s, \tag{34}$$

$$\tau_{tt} = 0. \tag{35}$$

Solving these equations self-consistently allows us to find the following parameters that characterize the self-similar pulse:

$$A(s) = A_0 \exp\left[\int_0^s \Gamma(\zeta)\, d\zeta\right], \tag{36}$$

$$\tau(s,t) = k\left[t + p\left(4p^2 + \frac{2\alpha k^2 + 3p\sigma k}{\epsilon}\right)\int_0^s Q(\zeta)\, d\zeta\right] + t_0, \tag{37}$$

$$z(s) = \frac{k^4}{\epsilon}\int_0^s Q(\zeta)\, d\zeta, \tag{38}$$

$$\phi(s,t) = p\left[t + p\left(3p^2 + \frac{\alpha k^2 + 2p\sigma k)}{\epsilon}\right)\int_0^s Q(\zeta)\, d\zeta\right] + \phi_0, \tag{39}$$

where $k$ and $p$ are parameters relative to pulse width and phase shift, respectively, $\tau(s,t)$ is the mapping variable, and $z(s)$ represents the effective propagation distance. Here the subscript 0 denotes the initial values of the corresponding parameters at distance $s = 0$. Furthermore, the constraint conditions on the inhomogeneous fiber parameters are given as

$$D(s) = \left(6p^2 + \frac{\alpha k^2 + 3p\sigma k}{\epsilon}\right)Q(s), \tag{40}$$

$$R(s) = \frac{\gamma k^4}{\epsilon A_0^2}\exp\left[-2\int_0^s \Gamma(\zeta)\,d\zeta\right]Q(s), \tag{41}$$

$$P(s) = \left(4p + \frac{\sigma k}{\epsilon}\right)Q(s). \tag{42}$$

Equations (38) and (39) show that the effective propagation distance and phase are strongly dependent on the fourth-order dispersion parameter $Q(s)$. The latter influences the GVD parameter $D(s)$, Kerr nonlinearity parameter $R(s)$, and third-order dispersion parameter $P(s)$ as seen from the preceding constraints. Hence, one can control the dynamics of propagating self-similar dipole solitons in the fiber medium by selecting the profile of this parameter suitably.

Thus, the general form of self-similar solutions of the generalized NLSE (24) is of the form

$$U(s,t) = A_0 E\left\{\frac{k^4}{\epsilon}\int_0^s Q(\zeta)\,d\zeta,\, k\left[t + p\left(4p^2 + \frac{2\alpha k^2 + 3p\sigma k}{\epsilon}\right)\int_0^s Q(\zeta)\,d\zeta\right] + t_0\right\}$$
$$\times \exp\left[\int_0^s \Gamma(\zeta)\,d\zeta + i\phi(s,t)\right], \tag{43}$$

where the phase function $\phi(s,t)$ is given by Eq. (39) and $E(z,\tau)$ are the exact solutions of Eq. (1).

Therefore, exact self-similar solutions to Eq. (24) can be constructed by using the exact solutions of Eq. (1) via the transformation (43). One thus needs to use the closed form solutions of the constant-coefficient NLSE (1) presented above.

## V. SELF-SIMILAR DIPOLE SOLITON SOLUTIONS

Making use of the exact solution given in Eq. (20) of the extended constant-coefficient NLSE (1), the transformations in Eq. (43), and Eqs. (36)–(39), we can construct the self-similar solutions of the generalized NLSE with varying coefficients (24). The self-similar dipole soliton solution of Eq. (24) is then given by

$$U(s,t) = A_0E_0\exp\left[\int_0^s \Gamma(\zeta)\,d\zeta\right]\operatorname{sech}(w\xi)\,\operatorname{th}(w\xi)$$
$$\times \exp[i\Phi(s,t)], \tag{44}$$

where the traveling coordinate $\xi$ is given by

$$\xi(s,t) = kt - \eta + \left\{kp\left(4p^2 + \frac{2\alpha k^2 + 3p\sigma k}{\epsilon}\right) - \frac{k^4}{v\epsilon}\right\}$$
$$\times \int_0^s Q(\zeta)\,d\zeta + t_0, \tag{45}$$

and the phase of the field, $\Phi$, has the form

$$\Phi(z,t) = \kappa z - \delta\tau + \theta + \phi(s,t), \tag{46}$$

where $\tau$ and $z$ are given by Eqs. (37) and (38), respectively, while the phase $\phi(s,t)$ is given by Eq. (39).

From the results obtained above, we see that the contribution of all orders of dispersion is necessary for the existence of self-similar dipole soliton solutions for the generalized NLSE with distributed coefficients (24). This is markedly different from the dipole structures of many constant-coefficient NLSE models describing femtosecond pulse dynamics in homogeneous fibers [30–32], which exist only when second-, third-, and fourth-order dispersions are compensated. It should be noted that the simultaneous compensation of various orders of dispersion is generally more difficult in optical systems [30]. Therefore, our results could be of importance in applications of dipole-type solitons in optical-fiber systems exhibiting dispersive effects up to the fourth order.

To examine the dynamical evolution of the obtained self-similar structure in the optical-fiber medium, it is worthwhile to consider a specific soliton control system. Here, we focus on studying the propagation of self-similar waves through a periodically distributed amplification system similar to that of Ref. [23]. In particular, we suppose that the fourth-order dispersion management takes the form of a cosinelike space-dependent rapidly varying function as [23] $Q(s) = d_4\cos(gs)$, while the gain function is given by $\Gamma(s) = \Gamma_0$. Here $d_4$ and $g$ are the parameters to describe fourth-order dispersion and $\Gamma_0$ represents the constant net gain or loss. From the practical point of view, the propagation with periodic dispersion is of great importance as it has application in enhancing the signal-to-noise ratio and reducing Gordon-Hauss time jitter and is also helpful in suppressing the phase matched condition for four-wave mixing [41,42]. Then, according to Eq. (38), the effective propagation distance can be obtained as $z(s) = \frac{d_4k^4}{\epsilon g}\sin(gs)$, implying that $z$ varies periodically with the propagation distance $s$. Furthermore, the amplitude can be calculated from Eq. (36) as $A(s) = A_0\exp(\Gamma_0 s)$. This means that the amplitude of the self-similar pulse will undergo increase ($\Gamma_0 > 0$) and decrease ($\Gamma_0 < 0$) along the propagation distance, while it remains a constant when the gain (loss) vanishes ($\Gamma_0 = 0$). As concerns the other parameters, they can be obtained exactly through Eqs. (40)–(42).

Consider first the most interesting case when the optical-fiber medium is not subject to the effect of the gain or loss effect (i.e., $\Gamma_0 = 0$). The evolution of the self-similar dipole soliton solution (44) calculated with the framework of the generalized NLSE (24) is shown in Figs. 1(a) and 1(b) with the parametric values $\alpha = -1$, $\gamma = -2$, $\sigma = 1.53$,

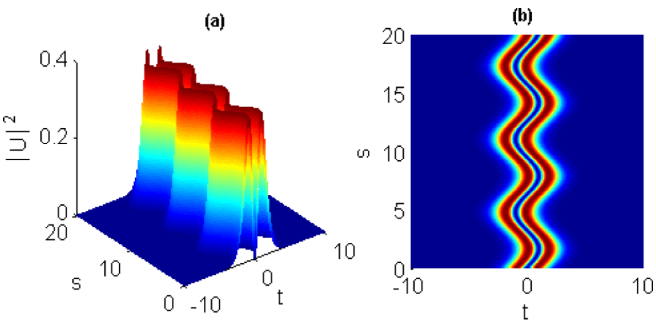


FIG. 1. Evolution of the dipole self-similar intensity wave profile $|U(s,t)|^2$ as computed from Eq. (44). The parameters are defined in the text.

and $\epsilon = -\frac{1}{4}$. Also we take $k = 1$, $p = 3.055$, $A_0 = 0.3098$, $d_4 = -0.01$, $\eta = 0$, $g = 1$, and $t_0 = 0$. From these figures, one can clearly see that the self-similar structure displays a snakelike behavior along the propagation distance due to the presence of the periodic distributed dispersion parameter $Q(s)$. For such an oscillatory trajectory, the self-similar pulse keeps no change in propagating along the optical medium although its position oscillates periodically (which is called "snakelike" in Ref. [43]).

When the self-similar dipole soliton is subjected to the action of a constant gain or loss, that is, $\Gamma(s) = \Gamma_0$, its intensity decreases when $\Gamma_0 < 0$ and increases when $\Gamma_0 > 0$, and the time shift and the group velocity of the soliton pulse are changing while the soliton keeps its shape in propagation along the fiber [Figs. 2(a) and 2(b)]. One readily concludes that the gain parameter affects only the evolution of the soliton peak and has no influence on the width or shape of the pulse.

Another interesting behavior appears when the gain or loss function is chosen to vary periodically with the propagation distance as $\Gamma(s) = \sin(s)$. This spatial profile of gain (loss) was first used in studying soliton management in inhomogeneous pure Kerr media [44]. The corresponding intensity profiles of self-similar dipole solitons are shown in Figs. 3(a) and 3(b) for the same values of parameters as those in Fig. 1, except $A_0 = 0.1$. As can be seen from this figure, in the presence of periodic gain, the dipole solitons emerge periodically in the inhomogeneous fiber system.

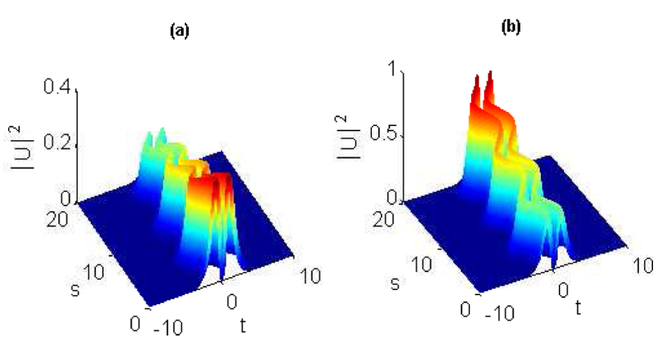


FIG. 2. Evolution of the dipole self-similar intensity wave profile $|U(s,t)|^2$ as computed from Eq. (44) when (a) $\Gamma_0 = -0.02$ and (b) $\Gamma_0 = 0.02$. The other parameters are the same as in Fig. 1.

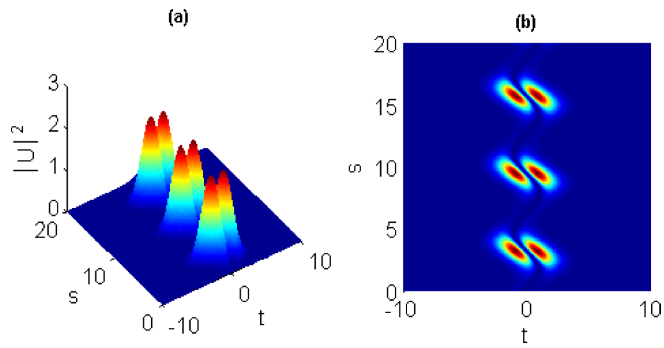


FIG. 3. Evolution of the dipole self-similar intensity wave profile $|U(s,t)|^2$ as computed from Eq. (44) when $\Gamma(s) = \sin(s)$. The other parameters are the same as in Fig. 1 except $A_0 = 0.1$.

Let us now investigate the propagation dynamics of self-similar dipole pulses in a distributed fiber system whose fourth-order dispersion and gain or loss parameters are distributed according to [44] $Q(s) = \tanh(s)$ and $\Gamma(s) = \sin(s)$. Figures 4(a) and 4(b) show the nonlinear evolution of the self-similar solution (44) for the same values of parameters as those in Fig. 1, except $A_0 = 0.1$. We observe an interesting periodic occurrence of dipole solitons appearing for this choice of dispersion and gain or loss management, as can be seen from Fig. 4.

## VI. NONLINEAR DISPERSIVE WAVES

We consider in this section the analytical stability analysis of soliton solutions of the generalized NLSE based on the theory of optical nonlinear dispersive waves. For our purpose, we develop the dynamics of nonlinear dispersive waves in the next form,

$$E(z,\tau) = F(\omega)\exp[i\Theta(z,\tau)], \tag{47}$$

where the amplitude $F(\omega)$ and phase $\Theta(z,\tau)$ are real functions. The wave number $k$ and frequency $\omega$ of nonlinear dispersive waves are given by the equations $k = \Theta_z$ and $\omega = -\Theta_\tau$. We assume that $\omega(z,\tau) = \tilde{\omega}(Z,T)$ and $k(z,\tau) = \tilde{k}(Z,T)$, where $Z = \varepsilon z$ and $T = \varepsilon\tau$ with $\varepsilon \ll 1$ are slow variables. Here $\tilde{\omega}(Z,T)$ and $\tilde{k}(Z,T)$ are slowly varying functions

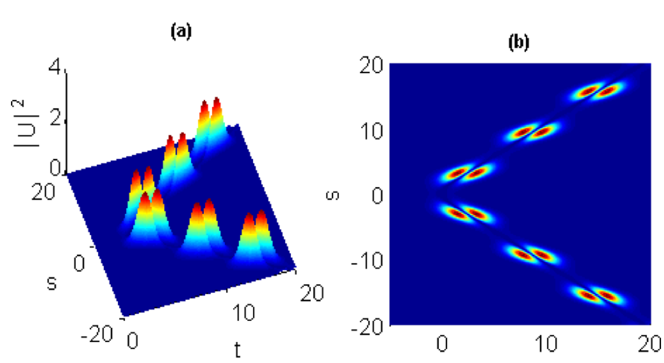


FIG. 4. Evolution of the dipole self-similar intensity wave profile $|U(s,t)|^2$ as computed from Eq. (44) when $\Gamma(s) = \sin(s)$ and $Q(s) = \tanh(s)$. The other parameters are the same as in Fig. 1 except $A_0 = 0.1$.

of variables $Z$ and $T$. The substitution of Eq. (47) to NLSE (1) leads to the series of nonlinear equations. The equations in zero and first order to small parameter $\varepsilon$ are

$$k(\omega) = k_0(\omega) + \gamma F^2(\omega), \quad k_0(\omega) = \alpha\omega^2 + \sigma\omega^3 + \epsilon\omega^4, \tag{48}$$

$$\frac{\partial F}{\partial z} + k'(\omega)\frac{\partial F}{\partial \tau} = 2\gamma F F'\frac{\partial F}{\partial \tau} - \frac{1}{2}k_0''(\omega)F\frac{\partial \omega}{\partial \tau}, \tag{49}$$

with $k'(\omega) = dk(\omega)/d\omega$ and $F' = dF(\omega)/d\omega$. Thus, Eq. (48) is the nonlinear dispersion relation and Eq. (49) is the equation for the amplitude $F(\omega)$ of nonlinear waves defined in Eq. (47). Note that such forms of equations were found first in fluid mechanics [45]. We have by definition that $\Theta_{z\tau} = k_\tau$ and $\Theta_{\tau z} = -\omega_z$, which yield the equation $\omega_z + k_\tau = 0$. This equation for varying frequency $\omega(z, \tau)$ has the next form,

$$\frac{\partial \omega}{\partial z} + k'(\omega)\frac{\partial \omega}{\partial \tau} = 0, \tag{50}$$

where $k'(\omega) = 2\alpha\omega + 3\sigma\omega^2 + 4\epsilon\omega^3 + 2\gamma F(\omega)F'(\omega)$. The nature of the system of Eqs. (48)–(50) based on the method of slow variables can be hyperbolic or elliptic. We first consider the case when this system of equations is hyperbolic. The characteristics connected to the hyperbolic system of Eqs. (48)–(50) are given by

$$\frac{d\tau}{dz} = k'(\omega), \quad \frac{d\omega}{dz} = 0, \tag{51}$$

$$\frac{dF}{dz} = 2\gamma F(F')^2\frac{\partial \omega}{\partial \tau} - \frac{1}{2}k_0''(\omega)F\frac{\partial \omega}{\partial \tau}. \tag{52}$$

It follows from Eq. (51) that $dF/dz = F'(\omega)d\omega/dz = 0$. Thus, Eq. (52) yields the differential equation

$$\frac{dF(\omega)}{d\omega} = \pm\frac{1}{2}\sqrt{\frac{k_0''(\omega)}{\gamma}}, \quad k_0''(\omega) = 2\alpha + 6\sigma\omega + 12\epsilon\omega^2. \tag{53}$$

Equations (51) and (53) with $k'(\omega) = k_0'(\omega) + 2\gamma F(\omega)F'(\omega)$ yield the next characteristic equation,

$$\frac{d\tau}{dz} = k_0'(\omega) \pm \gamma F(\omega)\sqrt{\frac{k_0''(\omega)}{\gamma}}. \tag{54}$$

It follows from this equation that the system of Eqs. (48)–(50) is hyperbolic when the condition $k_0''(\omega)/\gamma \geqslant 0$ is satisfied, and the system of Eqs. (48)–(50) is elliptic for the next condition, $k_0''(\omega)/\gamma < 0$. Thus, the characteristic equations can be defined only in the hyperbolic case. The integration of Eq. (53) leads to the real amplitude $F(\omega)$ as

$$F(\omega) = \pm\frac{1}{2}\int\sqrt{\gamma^{-1}(2\alpha + 6\sigma\omega + 12\epsilon\omega^2)}d\omega + C, \tag{55}$$

where $C$ is an integration constant.

Equations (53) and (54) lead to apparent physical interpretation of stability for the soliton solutions of Eq. (1). We may suppose that the soliton is stable when it cannot radiate the nonlinear dispersive waves. One can show, using the linearized Eqs. (49) and (50), that outgoing nonlinear dispersive waves connected with such radiation processes exist only in the case when the above system of equations is hyperbolic. Moreover, in the case of elliptic equations the problem of optical pulse radiation is not correct from the mathematical point of view. This situation takes place in the case when $k_0''(\omega)/\gamma < 0$. This relation follows from Eqs. (53) and (54) because the outgoing nonlinear dispersive waves do not exist when the square root $\sqrt{k_0''(\omega)/\gamma}$ is imaginary and hence the system of Eqs. (47)–(50) is elliptic. Note that the inequality $k_0''(\omega)/\gamma < 0$ is satisfied in some domain $\mathcal{D}$ of parameters $\alpha$, $\sigma$, $\epsilon$, and $\gamma$. Thus, this physical interpretation of stability of the soliton solution yields the stability domain $\mathcal{D}_{\rm st}$ for a given type of soliton as $\mathcal{D}_{\rm st} = \mathcal{D}_{\rm sol}\bigcap\mathcal{D}$. Here $\mathcal{D}_{\rm sol}$ is the domain where the given type of soliton solution is defined.

We first consider this stability criterion for $\sigma = \epsilon = 0$. In this case we have the relation $k_0(\omega) = \alpha\omega^2$ and hence the amplitude $F(\omega)$ of a nonlinear dispersive wave is given by Eq. (55) as

$$F(\omega) = \pm\tfrac{1}{2}\omega\sqrt{2\alpha/\gamma} + C. \tag{56}$$

Thus, the characteristic in Eq. (54) has the form

$$\frac{d\tau}{dz} = 3\alpha\omega \pm \gamma C\sqrt{\frac{2\alpha}{\gamma}}. \tag{57}$$

It follows from this equation that the system of Eqs. (48)–(50) is elliptic when the condition $\gamma\alpha < 0$ is satisfied. It is well known that this is the stability condition for the soliton solution of the NLSE with second-order dispersion term only. Thus, the formulated above criterion $k_0''(\omega)/\gamma < 0$ of soliton stability leads to the correct result. Hence, the optical pulses cannot radiate the nonlinear dispersive waves when $\gamma\alpha < 0$ and hence the system of Eqs. (48)–(50) is elliptic.

Now we consider the stability condition for soliton solutions given in Eqs. (15) and (20). These quartic and dipole soliton solutions take place for the frequency $\delta = -\sigma/4\epsilon$. The functions $k_0'(\omega)$ and $k_0''(\omega)$ at $\omega = \delta$ are

$$k_0'(\delta) = \frac{\sigma(\sigma^2 - 4\alpha\epsilon)}{8\epsilon^2} = v^{-1}, \quad k_0''(\delta) = \frac{8\alpha\epsilon - 3\sigma^2}{4\epsilon}. \tag{58}$$

Note that the velocity $v$ in this equation coincides with the soliton velocity given in Eq. (6). Thus, Eq. (54) at $\omega = \delta$ has the form

$$\frac{d\tau}{dz} = v^{-1} \pm \frac{1}{2}\gamma F(\delta)\sqrt{\frac{8\alpha\epsilon - 3\sigma^2}{\gamma\epsilon}}. \tag{59}$$

The criterion $k_0''(\omega)/\gamma < 0$ of soliton stability at $\omega = \delta$ yields the next inequality,

$$\frac{8\alpha\epsilon - 3\sigma^2}{\gamma\epsilon} < 0. \tag{60}$$

It follows from Eq. (59) that the system of Eqs. (47)–(50) is elliptic in the domain of parameters $\alpha$, $\sigma$, $\epsilon$, and $\gamma$ given by Eq. (60). We have two different domains $\mathcal{D}_1$ and $\mathcal{D}_2$ where the inequality in Eq. (60) is satisfied. The first domain $\mathcal{D}_1$ is given by two conditions: $\gamma\epsilon < 0$ and $8\alpha\epsilon > 3\sigma^2$. This domain coincides with the domain of existence of the quartic soliton solution in Eq. (15). Hence, the quartic soliton is stable in the entire domain $\mathcal{D}_1$ of the soliton solution in Eq. (15). The second domain $\mathcal{D}_2$ is given by the next two conditions: $\gamma\epsilon > 0$ and $8\alpha\epsilon < 3\sigma^2$. This domain coincides with the domain of existence of the dipole soliton solution in Eq. (20). Thus, the dipole soliton is stable in the entire domain $\mathcal{D}_2$ of solution solution in Eq. (20). Note that $\mathcal{D} = \mathcal{D}_1 \bigcup \mathcal{D}_2$ is the domain where the inequality in Eq. (60) is satisfied. Moreover, these two domains $\mathcal{D}_1$ and $\mathcal{D}_2$ do not intersect: $\mathcal{D}_1 \bigcap \mathcal{D}_2 = \emptyset$.

We emphasize that the stability criterion formulated for Eq. (1) can be extended to more general nonlinear Schrödinger equations which have the Hamiltonian form. Thus, we suppose that the extended stability criterion is applicable to Hamiltonian systems. This requirement is connected with the conservation of energy and other integrals of motion for solitary wave solutions. We also note that the transformation of the generalized NLSE with variable coefficients developed in Sec. IV does not influence the stability of quartic and dipole solitons because the transformation in Eq. (25) is connected with scaling functions only. More rigorous analytical consideration of the stability of the quartic and dipole solitons will be presented elsewhere. We confirm in the next section the stability of quartic and dipole solitons and hence the stability criterion given in Eq. (60) by numerical simulations of Eq. (1).

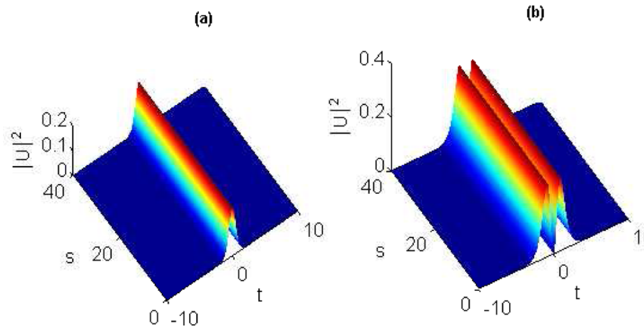


FIG. 5. The numerical evolution of (a) the quartic soliton solution obtained from inserting Eq. (15) into Eq. (43) and (b) the dipole soliton solution (44) under the perturbed constraint condition $D(s) = 0.9(6p^2 + \frac{\alpha k^2 + 3p\sigma k}{\epsilon})Q(s)$. Here the parameters adopted for the bright soliton solution are $\alpha = -2$, $\gamma = 0.3$, $\sigma = 1$, $\epsilon = -\frac{1}{4}$, $k = 3.162$, $A_0 = 1$, $d_4 = -0.0011$, $\eta = 0$ $g = p = 1$, and $t_0 = 0$. The parameters for the dipole solution (44) are the same as those used in Fig. 1.

## VII. NUMERICAL STABILITY ANALYSIS

In this section, we finally confirm the stability of the self-similar soliton solutions and discuss the constraint conditions by direct numerical simulation of Eq. (1). In the above results, we have determined the exact self-similar soliton solutions under the constraint conditions (40)–(42). These conditions present the strict balance among various types of dispersion, Kerr nonlinearity, and gain or loss. But it may be difficult to produce exactly such constraint conditions in real applications. Therefore, a study of the perturbations in the constraint conditions (40)–(42) is necessary. Here we take the self-similar quartic soliton solution obtained from inserting the exact solution (15) into the transformation (43) as well as the self-similar dipole soliton solution (44) and the soliton control system used in Fig. 1, and perturb the constraint conditions in the following way: $D(s) = 0.9(6p^2 + \frac{\alpha k^2 + 3p\sigma k}{\epsilon})Q(s)$. The other conditions are unchanged. The corresponding numerical evolution of the self-similar solitons is shown in Fig. 5. It is seen that the self-similar quartic soliton pulse is still stable after propagating over 40 dispersion lengths as shown in Fig. 5(a). We also see that the self-similar dipole soliton can propagate stably to the same long distance except that a slight shifting of the peak will appear, as demonstrated in Fig. 5(b). Besides this example case, we have performed more numerical simulations for the constraint conditions by perturbing other management parameters and the results show that the self-similar solitons remain stable during the propagation under the finite perturbed conditions. Thus, one can conclude that the perturbations in the parametric conditions do not affect the evolution of the self-similar structures. Consequently, we may infer that the soliton control system discussed here may relax the limitations to the constraint conditions. This may make conditions (40)–(42) more realistic and give an effective support for the realization of self-similar optical quartic and dipole solitons in experiment.

Now we analyze the stability of the self-similar wave solutions presented above with respect to finite initial perturbations. Here we performed direct simulations to demonstrate the stability of the solutions by adding a white noise in the pulse $U(t,0)$ so that the perturbed pulse reads [46,47] $U_{\text{pert}} = U(t,0)[1 + 0.1\,\text{random}(t)]$. The evolution plots of self-similar quartic and dipole soliton solutions under the perturbation of 10% white noise are shown in Figs. 6(a) and 6(b), respectively. We observe that the self-similar quartic soliton can propagate in a stable way under finite initial perturbations of the additive white noise [Fig. 6(a)]. It can be also seen that under white noise perturbations, the dipole structure is still stable. In addition, we have performed other numerical simulations to analyze the stability of the obtained solutions under 10% amplitude perturbation; the results reveal that the main character of the solutions is not influenced by finite initial perturbations such as amplitude.

We have also performed numerical simulations to analyze the stability of quartic and dipole soliton solutions (15) and (20) of the extended NLSE with constant coefficients given in Eq. (1). We have found by numerical simulations that these solitons are stable under initial small perturbations (such as

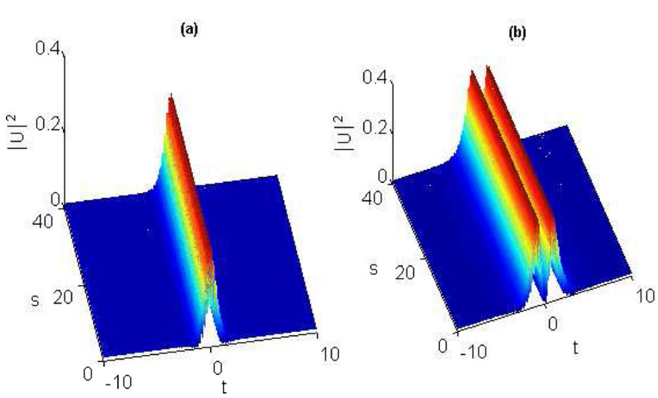


FIG. 6. The numerical evolution of (a) the quartic soliton solution obtained from inserting Eq. (15) into Eq. (43) and (b) the dipole soliton solution (44) under the perturbation of white noise whose maximal value is 0.1. The parameters are the same as in Figs. 5(a) and 5(b).

amplitude and white noise) and also under unideal parameter profiles. Thus, we can conclude that these quartic and dipole solitons are stable in both inhomogeneous and homogeneous optical fiber media.

## VIII. DISCUSSION

We discuss in this section some application of the NLSE model with higher-order dispersion considered in this study. For high-speed optical communication systems and large channel-handling capacity, it is necessary to transmit solitons at a high bit rate of ultrashort pulses [48]. With the development of optical sources generating light pulses in the femtosecond domain [49], the transmission of these solitons through optical fibers becomes possible. Moreover, since the first achievement of few-cycle pulses [50], it is now possible to reach extremely short light pulses. Usually, the dynamics of femtosecond optical pulses are described by families of NLSEs which incorporate the contributions of higher-order dispersive and nonlinear terms in addition to those of the cubic model. The major higher-order correction terms come from the dispersion slope, the dispersive effect of the Kerr coefficient, and the stimulated Raman scattering. A special case, which was extensively studied, is ultrashort pulse propagation in an optical fiber for which the carrier frequency of the signal, $\omega_0$, is located at a local minimum or maximum of the GVD, where the third-order dispersion is identically zero [51–55]. In this case, the fourth-order dispersion will play an important role and the propagation equation for solitons is the generalized NLSE including only $\beta_2$ and $\beta_4$ to describe the dispersion [51–55]. Such a model supports an exact soliton solution which is referred as a quartic soliton [56]. One notes that the term *quartic soliton* here refers to a solitary pulse which results from interplay between anomalous GVD and SPM but modified by the presence of fourth-order dispersion. Experimentally, silicon photonics offers a particularly attractive medium in which to generate waveguide structures exhibiting a wide range of dispersion profiles wherein the propagation of pulses is described by the NLSE. Particularly, the possibility of observing quartic soliton pulses has been recently demonstrated in specially designed silicon-based slot waveguides excited with quasitransverse magnetic polarized pulses for which the Raman effect is absent [56]. Experimental and numerical evidence for the so-called pure-quartic soliton originating purely from the interaction of negative fourth-order dispersion and SPM has also been reported [57]. But near the zero-dispersion wavelength, the third-order dispersion plays a crucial role and the pulse dynamics is described by the NLSE modified to include the third-order dispersion [58]. At the same time, it becomes important to take into account the effect of fourth-order dispersion since the development of mode-locked laser sources emitting pulses of duration below 10 fs [59,60].

In the Conclusion we discuss the possibility of the existence of $\mathrm{sech}^2$ solitons in spatially designed slot waveguides based on silicon and silica or silicon nanocrystals. The high refractive index of silicon, combined with the silicon-on-insulator (SOI) technology, allows a confinement of the optical modes and a consequent increase of the nonlinear coefficient of the waveguide, enabling efficient nonlinear optical interactions at low power levels and in relatively short length. The SOI waveguide fabricated along a special direction of the surface leads to a regime when the stimulated Raman scattering cannot occur for an input pulse exciting a quasi-transverse-magnetic (quasi-TM) mode of the waveguide. Moreover, it is shown numerically that the self-steepening term added to generalized NLSE (1) does not disturb the soliton solution [56]. In this work the input wavelength is $\lambda_0 \simeq 2.6\ \mu$m and the parameters of the SOI waveguide are $\beta_2 \simeq -0.05\ \mathrm{ps}^2\ \mathrm{m}^{-1}$, $\beta_3 \simeq 1.9 \times 10^{-5}\ \mathrm{ps}^3\ \mathrm{m}^{-1}$, and $\beta_4 \simeq -2.5 \times 10^{-5}\ \mathrm{ps}^4\ \mathrm{m}^{-1}$. The nonlinear coefficient for the silicon-based structure is $\gamma \simeq 42\ \mathrm{m}^{-1}\ \mathrm{W}^{-1}$. Note that these parameters satisfy the relations $\gamma\beta_4 < 0$ and $2\beta_2\beta_4 > \beta_3^2$, which are necessary and sufficient for the existence of the $\mathrm{sech}^2$ soliton solution. Note that these conditions are also satisfied when the third-order group velocity dispersion $\beta_3$ is much larger, for example, $\beta_3 \simeq 10^{-3}\ \mathrm{ps}^3\ \mathrm{m}^{-1}$. Such a value of the third-order dispersion can be achieved by changing the input wavelength under the condition that the above necessary constraints are satisfied. Note that this can be obtained in spatially designed slot waveguides based on silicon and silica or silicon nanocrystals. Moreover, the parameters of the SOI waveguide as $\beta_2 \simeq -0.05\ \mathrm{ps}^2\ \mathrm{m}^{-1}$, $\beta_3 \simeq 1.9 \times 10^{-5}\ \mathrm{ps}^3\ \mathrm{m}^{-1}$, and $\beta_4 \simeq 2.5 \times 10^{-5}\ \mathrm{ps}^4\ \mathrm{m}^{-1}$ with positive nonlinear coefficient, for example, $\gamma \simeq 42\ \mathrm{m}^{-1}\ \mathrm{W}^{-1}$, satisfy the relations $\gamma\beta_4 > 0$ and $2\beta_2\beta_4 < \beta_3^2$, which are necessary and sufficient for the existence of dipole solitons. We have found using the theory developed in Sec. VI and numerical simulations that the $\mathrm{sech}^2$ solitons and dipole solitons are robust when the self-steepening term is added to the generalized NLSE (1). However, the numerical simulations show that in this case the solitons undergo a small frequency shift. Thus, the absence of the Raman effect for TM-polarized pulses together with specially designed slot waveguides allows to be realized the necessary and sufficient conditions for the existence of $\mathrm{sech}^2$ soliton and dipole solutions of the nonlinear Schrödinger equation including higher-order dispersions.

## IX. CONCLUSION

In this paper, we have investigated the variable-coefficient nonlinear Schrödinger equation incorporating, at the highest order, a fourth-order dispersion, which governs the femtosecond optical pulse propagation in an inhomogeneous highly dispersive fiber media. We have first constructed the relation between this generalized wave equation and the related constant-coefficients one via a similarity transformation. Then, based on the obtained transformation, we have derived the exact self-similar dipole soliton solutions of the considered model. Conditions on the varying optical-fiber parameters for the existence of these self-similar structures are also presented. It is found that the existence of these self-similar dipole solitons in an inhomogeneous highly dispersive optical-fiber medium crucially depends, indeed, on all orders of dispersion. We have further discussed the propagation dynamics of self-similar waves in a fiber system with periodically changing dispersion. It is observed that the self-similar wave structure and dynamical behavior can be controlled by choosing appropriate parameters of fourth-order dispersion and gain or loss. The numerical simulation shows that the self-similar quartic and dipole solitons can propagate in a stable way under slight disturbance of the constraint conditions and the initial perturbation of white noise.

Finally, let us discuss the possible applications of the self-similar structures presented here. It is well known that the nonlinear-Schrödinger-type equations with distributed coefficients provide more realistic models than their constant-coefficient counterparts, because no optical fiber is homogeneous in reality due to manufacturing problems and long distance communication. The results given here show that an inhomogeneous fiber system with higher-order dispersion can support quartic and dipole solitons with a self-similar pulse shape. These self-similar solutions are exact solutions for all $z$ (unlike the asymptotically exact parabolic solutions) and may well find new applications in fiber-optic amplifiers and in the study of propagation of quartic and dipole soliton pulses in femtosecond fiber laser systems or in communication links with distributed dispersion and nonlinearity management and the presence of gain (loss). This can be readily achieved particularly in view of their stability under perturbations which may well lead to reduced noise in a generated pulse stream. Because of the stable propagation of these localized light pulses, they should be observable in optical fibers with higher-order dispersion. In general, the full experimental exploitation of these solutions requires optical fibers with tailor-made dispersion profiles and nonlinearity profiles. While this is clearly a technical challenge, such fibers may well become available, enabling the development of new types of pulsed and oscillatory light sources in the future.

## ACKNOWLEDGMENT

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