Reply to "Comment on 'Shallow-water soliton dynamics beyond the Korteweg–de Vries equation'"

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We agree with the objection raised by Burde that the wave equation in our previous paper [Phys. Rev. E **90**, 012907 (2014)] was not derived in a consistent way. However, our paper contains an important result on the existence of soliton solutions to the extended Korteweg–de Vries equation.

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First, we agree that the derivation of Eq. (18) in our previous paper [1] for the case $\alpha = O(\beta)$, $\delta = O(\beta)$ was not consistent. This was the result of an unfortunate oversight.

We emphasize an unexpected and important result obtained in Ref. [1], however. We found an analytic single-soliton solution to the extended Korteweg–de Vries (KdV2) equation derived by Marchant and Smyth [2], despite nonintegrability of KdV2. This result is correct and very important. The KdV2 solitons have the same form as the Korteweg-de Vries (KdV) solitons but with slightly different coefficients. In consequence, this discovery inspired us to hypothesize that KdV2 may possess other analytic solutions of the same form as KdV solutions but with different coefficients. This hypothesis has been positively verified for periodic (cnoidal) solutions in Ref. [3] and for periodic "superposition" solutions in Ref. [4]. There is, however, a significant difference between analytic solutions to KdV and KdV2. For instance, both KdV and KdV2 solitons have the same form, $\eta(x,t) = A\,\mathrm{Sech}^2[B(x - vt)]$. However, KdV imposes two linearly independent conditions on the coefficients A, B, v, whereas KdV2 imposes three such conditions. For KdV, there is one parameter family of freedom. Consequently, solutions of different velocities (and amplitudes) can occur and multisoliton solutions exist. In the case of KdV2, for the given equation (given values of α, β), there exists only one KdV2 soliton, fixed by equation parameters. Therefore, multisoliton for KdV2 cannot exist [5].

The criticism raised in the early version of the Comment [6] inspired us to rethink the problem. It resulted in the consistent derivation of three KdV-type nonlinear wave equations for the uneven bottom. These equations generalize the KdV, the fifth-order KdV (KdV5), and the Gardner equations. In all these cases, the Boussinesq equations become compatible and can be reduced to a single wave equation only when the bottom function is piecewise linear. The details of the theory, as well as several examples of numerical simulations, are already published in Ref. [7].

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