Comment on "Shallow-water soliton dynamics beyond the Korteweg–de Vries equation"

G. I. Burde [*]
The Jacob Blaustein Institutes for Desert Research, Ben-Gurion University of the Negev, Sede Boqer Campus, 84990, Israel

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The authors of the paper "Shallow-water soliton dynamics beyond the Korteweg—de Vries equation" [1] write that they derived a new nonlinear equation describing shallow water gravity waves for an uneven bottom in the form of the higher (fifth)-order Korteweg—de Vries equation for surface elevation. The equation has been obtained by applying a perturbation method [2] for specific relations between the orders of the three small parameters of the problem $\alpha = O(\beta)$ and $\delta = O(\beta)$ up to the second order in β. In this comment, it is shown that the derivation presented in [1] is inconsistent because of an oversight concerning the orders of terms in equations of the Boussinesq system. Therefore the results, in particular, the new evolution equation and the dynamics that it describes, bear no relation to the problem under consideration. A consistent derivation is presented, and also results of applying the perturbation procedure with some other orderings between the small parameters are given to provide a broader view of the problem. Several new nonlinear evolution equations governing small amplitude shallow water waves for an uneven bottom have been derived.

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I. OUTLINE OF THE PROCEDURE

In order to demonstrate the inconsistency of the analysis of [1] and to provide a consistent derivation of the evolution equations describing shallow water waves for a nonflat bottom, we need to outline application of the perturbation method [2] to the nonflat bottom problem. To make the results more generally applicable, the case of nonzero surface tension (gravity-capillary waves) is considered, although the analysis of [1] is restricted to the pure gravity waves.

The standard system of equations and boundary conditions describing the two-dimensional irrotational wave motion of an inviscid incompressible fluid in a channel with the free surface and rigid bottom under the influence of gravity as well as surface tension, after an appropriate choice of nondimensional variables, can be reduced to the following:

$$\beta\phi_{xx} + \phi_{zz} = 0, \qquad 0 \leqslant z \leqslant 1 + \alpha\eta \tag{1}$$

$$\phi_z = \beta\delta\, h_x\phi_x, \qquad z = \delta\, h(x) \tag{2}$$

$$\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_z = 0, \qquad z = 1 + \alpha\eta \tag{3}$$

$$\phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_z^2 + \eta - \tau\beta\frac{\eta_{xx}}{\left(1+\alpha^2\beta\eta_x^2\right)^{3/2}} = 0, \quad z = 1 + \alpha\eta, \tag{4}$$

where t is time, x, z are respectively horizontal and vertical coordinates, with $z = \delta\, h(x)$ being the bottom, $\phi(x, z, t)$ is the velocity potential, $\eta(x, t)$ is the surface elevation, and $h(x)$ is the bottom variation function. The subscripts denote partial derivatives with respect to the corresponding variables, i.e., $\phi_{xx} = \phi_{2x} = \frac{\partial^2\phi}{\partial x^2}$ and so on. Equations (1)–(4) contain four nondimensional parameters: the amplitude parameter $\alpha = \frac{a}{H}$ and the wavelength parameter $\beta = \frac{H^2}{L^2}$, where H is the mean depth of the undisturbed stream far upstream, where the bottom is flat, and a and L are typical values of the amplitude and of the wavelength of the waves, the Bond number $\tau = \frac{T}{\rho g H^2}$, where T is the surface tension coefficient, ρ is the density of water and g is the acceleration due to gravity, and the variable bottom parameter $\delta = \frac{a_h}{H}$, where a_h is the amplitude of bottom variation.

The parameters α, β, and δ are assumed to be small. In order to apply a perturbation method, the relations between orders of the parameters are to be specified. In [1], the following relations are accepted (although the authors do not state it explicitly): $\alpha = O(\beta)$ and $\delta = O(\beta)$. It is convenient to deal with one small parameter β by introducing the relations

$$\alpha = A\beta, \quad \delta = q\beta, \tag{5}$$

where the constants A and q are trace parameters (one of them can be removed from all the relations by scaling).

To satisfy Eqs. (1) and (2) a substitution is made, as follows:

$$\phi = \sum_{m=0}^{\infty}\frac{(-1)^m\beta^m}{(2m)!}\frac{\partial^{2m}f(x,t)}{\partial x^{2m}}z^{2m} + \sum_{m=0}^{\infty}\frac{(-1)^m\beta^{m+1}}{(2m+1)!}\frac{\partial^{2m+1}F(x,t)}{\partial x^{2m+1}}z^{2m+1}, \tag{6}$$

where the part with odd powers of z is introduced in order to have the possibility to satisfy the boundary condition (2). Substituting (6) into (2) yields

$$G - \beta q(hf_x)_x - \tfrac{1}{2}\beta^3q^2(h^2G_x)_x + \tfrac{1}{6}\beta^4q^3(h^3f_{3x})_x + \tfrac{1}{24}\beta^6q^4(h^4G_{3x})_x - \tfrac{1}{120}\beta^7q^5(h^5f_{5x})_x + \cdots = 0, \quad G = F_x. \tag{7}$$

[*] georg@bgu.ac.il

Equation (7) defines the relation between the functions $F(x,t)$ and $f(x,t)$ which is to be used in the expansion (6). To explicitly define that relation, $G(x,t)$ is to be represented as series in β as follows:

$$G(x,t) = \beta G^{(1)}(x,t) + \beta^2 G^{(2)}(x,t) + \beta^3 G^{(3)}(x,t) + \beta^4 G^{(4)}(x,t) + \cdots . \quad (8)$$

Substituting (8) into (7) and collecting terms with equal powers of β yields

$$b^2[G^{(1)} - q(hf_x)_x] + b^3 G^{(2)} + b^4 G^{(3)} + b^5(G^{(4)} + \cdots) + \cdots = 0, \quad (9)$$

and then from the first three terms of (9) we obtain

$$G^{(1)} = q(hf_x)_x, \quad G^{(2)} = 0, \quad G^{(3)} = 0. \quad (10)$$

The corresponding expression for the potential is obtained by substituting (8), with $G^{(1)}$, $G^{(2)}$, and $G^{(3)}$ defined by (10), into (6) which yields

$$\begin{aligned}\phi = {} & f - \beta\frac{1}{2}z^2 f_{xx} + \beta^2\left(zG^{(1)} + \frac{1}{24}z^4 f_{4x}\right) \\ & + \beta^3\left(-\frac{1}{6}z^3 G^{(1)}_{2x} - \frac{1}{720}z^6 f_{6x}\right) \\ & + \beta^4\left(\frac{1}{120}z^5 G^{(1)}_{4x} + \frac{1}{40320}z^8 f_{8x}\right) \\ & + \beta^5(zG^{(4)} + \cdots), \quad G^{(1)} = (hf_x)_x. \end{aligned} \quad (11)$$

It is seen that both in Eq. (9) obtained from the bottom condition (2) and in the expression (11) for the potential, the correction $G^{(4)}$ does not appear in the terms up to the order of β^4. Thus, the expression for G in the form

$$G = \beta q(hf_x)_x \quad (12)$$

can be used in calculations up to that order.

To compare equations of the present comment with those of Ref. [1], one has to take into account that a counterpart of the substitution (6) for ϕ is written in [1] in the form $\phi = \sum_{m=0}^{\infty} z^m \phi^{(m)}$. Therefore, when comparing the equations of [1] with those of the present paper, one has to take

$$\phi^{(0)} = f, \quad \phi^{(1)} = \beta G. \quad (13)$$

Then the expression for ϕ given in [1] by Eq. $(7)_{\rm KRI}$ (in what follows, to avoid confusion, numbers of equations from [1] are supplied by the subscript) would be identical to (11) with the terms up to the third order retained (except for a misprint, the sign of the term $-\frac{1}{720}\beta^3 z^6 f_{6x}$ in Eq. $(7)_{\rm KRI}$ is wrong). For convenience of discussion below, we will write out the relation (12) in the notation of [1] [Eq. $(6)_{\rm KRI}$]:

$$\phi^{(1)} = \beta\delta\big(h\phi^{(0)}_x\big)_x. \quad (14)$$

Next, the expressions (6) are to be substituted into the surface conditions (3) and (4) which, upon differentiating (4) with respect to x, yields a system of two equations for the surface elevation $\eta(x,t)$ and the quantity $w(x,t) = f_x$ in the form of an infinite series with respect to β. Then, retaining the terms up to $O(\beta^n)$, we arrive at the nth-order Boussinesq system. In the zero order, the Boussinesq system reads $\eta_t + w_x = 0$, $w_t + \eta_x = 0$ so both w and η satisfy the linear wave equation $\zeta_{tt} - \zeta_{xx} = 0$, which describes waves traveling in two directions. A wave moving to the right corresponds in this order of approximation to $w = \eta$ and $\eta_t + \eta_x = 0$. To derive equations describing right-moving waves in higher orders in β, one can, applying the procedure developed in [2], reduce the system of equations for w and η to an asymptotically equivalent set of equations consisting of a relationship between the horizontal velocity w and the surface elevation η and an evolution equation for the elevation. To do this, it is set

$$w = \sum_{i=0}^{N_w} R_i\beta^i, \quad \eta_t = \sum_{i=0}^{N_\eta} S_i\beta^i, \quad (15)$$

where R_i and S_i depend on η and its x derivatives, and possibly some nonlocal variables, with $R_0 = \eta$ and $S_0 = -\eta_x$. The functions R_i and S_i are determined from the requirement of consistency of (15) with the Boussinesq system (N_w and N_η denote the number of terms needed for achieving this). An iterative procedure starting from the zero order of approximation and continuing to the higher orders is applied. In each order, the t derivatives of η are replaced by their expressions from the lower order equations.

The second-order Boussinesq system obtained using (11) in (3) and (4) takes the form

$$\begin{aligned}&\eta_t + w_x + \beta\big(A(w\eta)_x - \tfrac{1}{6}w_{3x} - q(wh)_x\big) \\ &\quad + \beta^2\big(-\tfrac{1}{2}A(\eta w_{2x})_x + \tfrac{1}{120}w_{5x} + \tfrac{1}{2}q(wh)_{3x}\big) = 0, \end{aligned} \quad (16)$$

$$\begin{aligned}&w_t + \eta_x + \beta\big(Aww_x - \tfrac{1}{2}w_{2xt} - \tau\eta_{3x}\big) \\ &\quad + \beta^2\big(-A(\eta w_{xt})_x + \tfrac{1}{2}Aw_x w_{2x} - \tfrac{1}{2}Aww_{3x} \\ &\quad + \tfrac{1}{24}w_{4xt} + q(hw_t)_{2x}\big) = 0. \end{aligned} \quad (17)$$

Here the terms with the coefficient q are those originated from the nonflatness of the bottom. [The second-order Boussinesq system in [1], given by Eqs. $(8)_{\rm KRI}$ and $(9)_{\rm KRI}$, after setting $\alpha = A\beta$, $\delta = q\beta$, and $\tau = 0$, coincides with the system (16), (17).] In the lowest (zero) order, the system (16), (17) and the asymptotically equivalent system (15) describing a right-moving wave are reduced to

$$\eta_t + w_x = 0, \quad w_t + \eta_x = 0; \quad w = \eta, \quad \eta_t + \eta_x = 0. \quad (18)$$

In the next order iteration, we look for a solution for w corrected to first order as

$$w = \eta + \beta Q^{(1)}, \quad (19)$$

where $Q^{(1)}$ is a function of η and its x derivatives, and possibly nonlocal variables. Substituting (19) into Eqs. (16) and (17), with the terms of order higher than $O(\beta)$ dropped, upon expressing the t derivatives of η in terms of its x derivatives using the zero-order equation $\eta_t + \eta_x = 0$ yields

$$\eta_t + \eta_x + \beta\Big(2A\eta\eta_x - \tfrac{1}{6}\eta_{3x} - q(h\eta)_x + Q^{(1)}_x\Big) = 0, \quad (20)$$

$$\eta_t + \eta_x + \beta\Big(A\eta\eta_x + \tfrac{1}{2}(1-2\tau)\eta_{3x} + Q^{(1)}_t\Big) = 0. \quad (21)$$

The function $Q^{(1)}$ is sought such that the two equations (20) and (21) agree (up to the first order in β), and here we arrive at the point where the authors of [1] introduce an inconsistency in order to obtain the desired result.

II. THE INCONSISTENT DERIVATION OF REF. [1]

The point is that the term $-\beta q(h\eta)_x$ in Eq. (20) makes Eqs. (20) and (21) incompatible if $Q^{(1)}$ is sought as a function of η and its x derivatives. If that term in Eq. (20) were absent, Eqs. (20) and (21) would be compatible and, with a proper choice of $Q^{(1)}$, both reduced to the standard Korteweg—de Vries (KdV) equation. Therefore the authors of [1], in order to have that desired situation, change the form of the Boussinesq system (16), (17) assuming the first-order term $-\beta q(wh)_x$, which is a source of the term $-\beta q(h\eta)_x$ in Eq. (20), to be of the second order. As a result, the term $-\beta q(h\eta)_x$, destroying the integrability of the first-order system (20) and (21), is moved to the second order. Note that in [1], it is not clear that with the first-order equation (16) including the term $-\beta q(wh)_x$ one can neither proceed to the second order nor have the KdV equation in the leading order. Instead, the following reasoning is provided (in the citation from [1], in order not to complicate matters, numbers of equations from [1] are replaced by numbers of the corresponding equations of the present comment; note also that it is set $\delta = q\beta$ in the present comment):

"In Eq. (16) there are two terms depending on the variable bottom, the first-order term $\delta(hw)_x$ and the second-order term $\frac{1}{2}\beta\delta(hw)_{3x}$, whereas Eq. (17) contains only the second-order term $\beta\delta(hw_t)_{2x}$. However, the bottom boundary condition Eq. (14), which is the source of these terms, is already second order in $\beta\delta$. Therefore, we will treat all these terms on the same footing, as second-order ones, i.e., replacing $\delta(hw)_x$ by $\beta\delta(hw)_x/b$, $b \neq 0$, during derivations and substituting $b = \beta$ in the final formulas."

However, it is not justified to introduce changes into equations of the Boussinesq system. The Boussinesq system (16), (17) has been obtained as the result of applying the algorithmic perturbation method to the original equations (1)–(4) of the problem, and so Eqs. (16) and (17) have the form that adheres to the problem formulation. In particular, if the first-order term is present in one of the equations and absent in the second equation, it is a feature specific for the present problem formulation. Therefore any change made in equations of the Boussinesq system leads to the results not matching the original problem.

The form of the bottom boundary condition (14) can in no way be considered as justification for changing the equations of the Boussinesq system. The bottom boundary condition in that form has been used for obtaining the expression (8) for the potential and that expression has been used in the derivation of the Boussinesq system (16), (17) so that the condition (14) is the source of all the terms originating from the nonflatness of the bottom. If the condition were incorrect and should be changed in some way it would influence all those terms, not only one of them. Moreover, even the statement, that "... the bottom boundary condition Eq. (14)... is already second order in $\beta\delta$," is incorrect. Such an impression may arise only if one treats $\phi^{(1)}$ and $\phi^{(0)}$ as the quantities of the same order of magnitude. But they are not of the same order: $\phi^{(1)} = O(\beta)$ and $\phi^{(0)} = O(1)$ [see (13)] and so the condition (14) is first order in β as it is explicitly seen from the form (12) that it takes in the variables f and G.

To conclude, the derivation of [1], based on the incorrect Boussinesq system without the term $-\beta q(wh)_x$ in the first order and with an additional term included into the second order, is inconsistent. Correspondingly, the results, in particular the new evolution equation and the dynamics that it describes, bear no relation to the considered problem.

III. THE CONSISTENT DERIVATION

A condition for Eqs. (20) and (21) to be compatible is that an equation for $Q^{(1)}$ obtained by subtracting Eq. (20) from Eq. (21) as follows,

$$Q_t^{(1)} - Q_x^{(1)} - A\eta\eta_x + \left(\frac{2}{3} - \tau\right)\eta_{3x} + q(h\eta)_x = 0, \tag{22}$$

is satisfied. The analysis reveals that Eq. (22) cannot be satisfied if $Q^{(1)}$ is assumed to be a function of η, its x derivatives, and the bottom function $h(x)$, but it can be satisfied if $Q^{(1)}$ depends also on the nonlocal variable $p = \int \eta(x,t)dx$. (Limits of integration are not indicated, but here and in what follows, the integration from some x_0, where the elevation is zero, to a current value of x is implied.) Then substituting $Q^{(1)}$ into Eq. (22) eventually results in the relation which can be valid only if $h_{2x}(x) = 0$ or $h = kx$. In the second order in β, the expression for w is corrected by $\beta^2 Q^{(2)}$, and it is found that with $Q^{(2)}$ dependent on η, its x derivatives, and the nonlocal variable p, the two equations obtained from the Boussinesq system are incompatible.

The above results are related to derivation of an evolution equation for the surface elevation which does not include nonlocalities. If the analysis is not restricted to equations not containing nonlocal terms, an evolution equation for the surface elevation in the first order in β can be derived for arbitrary bottom function $h(x)$. To represent the results in a more concise form let us make a change of variables:

$$\eta(x,t) = R(x,\phi), \quad w(x,t) = U(x,\phi);$$
$$Q^{(a)}(x,t) = Q^{(c)}(x,\phi); \quad \phi = x + t. \tag{23}$$

Then integrating equation (22) expressed in the variables (23) yields

$$Q^{(1)} = -\frac{1}{4}AR(x,\phi)^2 + \frac{1}{24}(2-3\tau)R_{2x}(x,\phi) + \frac{1}{2}qh(x)R(x,\phi) + \frac{1}{2}q\int R(x,\phi)h'(x)dx. \tag{24}$$

It is readily verified that with $Q^{(1)}$ defined by (24), Eqs. (20) and (21) of the Boussinesq system, transformed to the variables (23), become identical and upon integrating by parts take the form

$$2R_\phi + R_x + \beta\left[\frac{3}{4}ARR_x + \frac{1}{48}(1-3\tau)R_{3x} - \frac{1}{4}q\left((hR)_x - \int h''(x)R(x,\phi)dx\right)\right] = 0. \tag{25}$$

It is evident from (25) that a necessary condition for derivation of an evolution equation not including nonlocal terms is $h''(x) = 0$ or $h(x) = kx$. In this case, making a change of variables inverse to Eq. (23) in Eq. (25) yields an evolution equation without nonlocalities. Equation (25) is a general evolution equation for the surface elevation governing shallow water waves on an uneven bottom in the first order in β derived under the assumption that $\alpha = O(\beta)$ and $\delta = O(\beta)$ for an arbitrary bottom relief $h(x)$. The presence of the nonlocal term does not prevent obtaining solutions for a specific bottom function $h(x)$ using numerical methods. Without restriction to evolution equations not containing nonlocalities, no obstacles exist to extend the procedure to the next orders, but a structure of nonlocal terms becomes more complicated.

IV. OTHER ORDERINGS

A. The case of $\alpha = O(\beta)$ and $\delta = O(\beta^2)$

The relations $\alpha = A\beta$ and $\delta = q\beta^2$ are used in order to deal with one small parameter β. It is evident that the same substitution (6) for satisfying Eq. (1) can be used, but the form of the relation (7), obtained upon substituting (6) into the boundary condition (2), changes. Again, retaining the two first terms in that relation is sufficient to provide a required accuracy, and then applying the procedure outlined in Sec. I yields the Boussinesq system. Further calculations proceed in the same way as un Sec. III. To be as concise as possible we will omit details. Note only that in the first order in β, an evolution for η is a common KdV equation with no traces of the bottom relief and that in the second order in β, again the restriction $h = kx$ arises. For arbitrary bottom relief $h(x)$, the Boussinesq system transformed to the variables (23) is satisfied up to the second order in β by the following:

$$U = R + \beta\frac{1}{24}[-6AR^2 + (2-3\tau)R_{2x}] + \beta^2\left[\frac{1}{8}A^2R^3 - \frac{1}{64}A(5-19\tau)R_x^2 + \frac{1}{4}A\tau RR_{2x} + \left(\frac{1}{360} + \frac{\tau}{192} - \frac{3\tau^2}{128}\right)R_{4x} + q\frac{1}{2}(hR + S_1)\right],$$
$$S_1(x,\phi) = \int h'(x)R(x,\phi)dx. \tag{26}$$

$$2R_\phi + R_x + \beta\left(\frac{3}{4}ARR_x + \frac{1-3\tau}{48}R_{3x}\right) + \beta^2\left(-\frac{3}{4}A^2R^2R_x - A\frac{1-24\tau}{48}R_xR_{2x} - A\frac{1-15\tau}{96}RR_{3x} + \frac{1+15\tau-45\tau^2}{2880}R_{5x} + q\frac{1}{4}(-(hR)_x + S_2)\right) = 0,$$
$$S_2(x,\phi) = \int h''(x)R(x,\phi)dx. \tag{27}$$

It is seen that $h = kx$ is a necessary condition for deriving an evolution equation without nonlocalities. Substituting $h(x) = kx$ into (27), upon making a change of variables inverse to (23), yields an equation not containing nonlocal terms.

B. The case of $\alpha = O(\beta^2)$, $\delta = O(\beta^2)$

It is set $\alpha = A\beta^2$ and $\delta = q\beta^2$, in what follows. Since the analysis proceeds along the lines of the two considered above, details are omitted to keep the presentation concise. For arbitrary $h(x)$, the relations solving the Boussinesq system in the variables (23) up to the second order in β are

$$U = R + \beta\frac{1}{24}(2-3\tau)R_{2x} + \beta^2\left[-\frac{1}{4}AR^2 + \left(\frac{1}{360} + \frac{\tau}{192} - \frac{3\tau^2}{128}\right)R_{4x} + q\frac{1}{2}(hR + S_1)\right],$$
$$S_1(x,\phi) = \int h'(x)R(x,\phi)dx, \tag{28}$$

$$2R_\phi + R_x + \beta\frac{1}{48}(1-3\tau)R_{3x} + \beta^2\left(\frac{3}{4}ARR_x + \frac{1+15\tau-45\tau^2}{2880}R_{5x} + q\frac{1}{4}(-(hR)_x + S_2)\right) = 0,$$
$$S_2(x,\phi) = \int h''(x)R(x,\phi)dx. \tag{29}$$

Again, the condition $h'' = 0$ is a necessary condition for derivation of an evolution equation not containing nonlocalities.

V. CONCLUDING REMARKS

The main incentive for writing the present comment was to stop referring to the equation derived in [1] as the equation describing shallow water gravity waves for an uneven bottom. Besides referring, extensions of the analysis of [1] appear in the literature. Properties of solutions of that equation are studied by the authors (e.g., [3]) and by other researchers. In [4], the analysis of [1] is extended to the third order in β. As the authors of [4] claim, “The main purpose of this article is to go beyond the new fifth-order KdV equation derived by Karczewska, Rozmej and Infeld.” Correspondingly, the authors of [4] make in the lower orders the same inconsistent assumption as that discussed in Sec. II of the present comment. There is no need to say that all the results of the aforementioned studies bear no relation to the problem of shallow water waves with a nonflat bottom.

The second purpose of the present analysis, evidently related to the first one, was to present a consistent analysis of the problem of deriving evolution equations describing shallow water waves for the case of a nonflat bottom with a small amplitude of the bottom height variation. Since there are three small parameters in the problem, imposing different relations between the orders of small parameters, as a matter of fact, results in different problems from the point of view of mathematics. (Note that the issue of ordering of small parameters is not mentioned in [1], and it is not indicated that all three parameters are assumed to be of the same order.) The results of the present study can be summarized as follows: If the analysis is restricted to constructing evolution equations not containing nonlocalities, then for all considered orderings, a consistent derivation is possible only for the bottom function of the form $h(x) = kx$. The analysis not restricted by a requirement of locality reveals that evolution equations governing small amplitude shallow water waves for an uneven bottom, for an arbitrary bottom relief, inevitably include a nonlocal term.

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