Stock markets: A view from soft matter

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Different attempts to describe financial markets, and stock prices in particular, with the tools of statistical mechanics can be found in the literature, although a general framework has not been achieved yet. In this paper we use the physics of many-particle systems and the typical concepts of soft matter to study two sets of US and European stocks, comprising the biggest and most stable companies in terms of stock price and trading. Upon correcting for the center-of-mass motion, the structure and dynamics of the systems are studied (in the European set, the structure is studied for the UK subset only). The pair distribution of the stocks, corrected to account for the nonuniform distribution of prices, is close to 1, indicating that there is no direct interaction between stocks, similar to an ideal gas of particles. The dynamics is studied with the mean-squared price displacement (MSPD); the price correlation function, equivalent to the intermediate scattering function; the price fluctuation distribution; and two parameters for collective motions. The MSPD grows linearly and the velocity autocorrelation function is zero, as for isolated Brownian particles. However, the intermediate scattering function follows a stretched exponential decay, the fluctuation distributions deviate from the Gaussian shape, and strong collective motions are identified. These results indicate that the dynamics is much more complex than an ideal gas of Brownian particles, and similar, to some extent, to that of undercooled systems. Finally, two physical systems are discussed to aid in the understanding of these results: a low density colloidal gel, and a dense system of ideal, infinitely thin stars. The former reproduces the dynamical properties of stocks, linear mean-squared displacement (MSD), non-Gaussian fluctuation distribution, and collective motions, but also has strong structural correlations, whereas the latter undergoes a glass transition with the structure of an ideal gas, but the MSD has the typical two-step growth of undercooled systems.

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I. INTRODUCTION

Nonequilibrium statistical physics has been widely employed in studying different classes of financial markets, as they display hallmarks characteristic to this physical field. For instance, market price distributions are usually featured by long tails [1], which cannot be described through a Gaussian approach [2,3], or exhibit fractal properties [4], similar to many other natural systems formed in out-of-equilibrium processes [5,6], such as river networks [7], rock structure [8], or aggregation of mesoscopic particles [9,10]. Here, the state of the art has advanced significantly in the last decade [11,12]. Furthermore, the physical approach to financial markets provided by the physicists have contributed to the understanding of financial markets. Relevant contributions have been made by Lillo and Mantegna [13], Gabaix *et al.* [14], Brock and Hommes [15], and Plerou *et al.* [16], to cite just a few.

Following the initial empirical analysis and development of models within pure economy, the advent of physicists to finance started in the 1990s, bringing the techniques and models of statistical physics [17,18]. Whereas some works try to describe the mechanisms for the evolution of prices or indices within economic reasonings, such as the popular agent-based models [19–22], many models have exploited the similarities with some physical systems. For instance, the appearance of power laws in the distributions (of price fluctuations, of returns as a function of company size, etc.) is reminiscent of critical phenomena [23,24]; the concept of microscopic entropy has been used as an information measure for financial time series [25,26]; models inspired on the Ising model have been proposed to reproduce the dynamics of financial markets [27,28]; random matrix theory, initially developed to analyze the spectra of nuclei, has also been applied to describe the correlations between stocks or agents [29,30]. However, a unified framework in which markets are regarded as a nonequilibrium system, inherent to this physical discipline, is still pursued [31], in contrast to other, well-established particle systems, such as colloidal, atomic, or molecular systems.

The dynamics of many financial systems (stocks, indices, currency exchange, etc.) typically shows non-Brownian diffusion, with a linear increase of the mean-squared price difference, but a non-Gaussian fluctuation distribution [2,32]. This feature has attracted attention over decades, starting from

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simple diffusion, which however showed important deviations with respect to the experimental distribution [2]. The Levy distribution, with several modifications [33–36], and recently, the q-Gaussian diffusion, based on the Tsallis entropy model [37–39], are among the most widely used models to describe it. Other models have tried to establish a firm connection with well-known physical models, such as the financial Brownian particle model, where the equivalent of a microscopic equation of motion for Brownian motion (Langevin and Boltzmann equations) is obtained from the order book (the time-ordered requests for buy and sell orders) [40–42]. Furthermore, because anomalous diffusion with long tails in the displacement distribution is also observed in the complex dynamics of soft matter [43], recently, we applied a model borrowed from soft condensed matter to describe the distribution of fluctuations in the exchange rate of many different currencies [44,45]. The model describes the dynamics of the system as a combination of the oscillation within a basin of the free energy, and long-range hops from basin to basin. In the present work, we aim to extend this approach and use the observables and interpretation of soft condensed matter to propose a unified view between finance, markets, and a natural nonequilibrium physical system.

Two datasets composed of US and European stocks have been studied as multiparticle soft-matter systems. Stocks from the European market belong to different national floors from economy drivers of the European nations. In both cases, the stocks price from 2011 to 2018, with a time resolution of 1 day, are used, with the motion of the center of mass corrected prior to the analysis. The structure is studied using the log-price distribution and the pair distribution function, considering the nonhomogeneous price distribution. For the dynamics, the mean-squared price difference (MSPD), and its Fourier transform, equivalent to the intermediate scattering function (ISF) in soft matter, are the main observables, but we have also studied the log-price fluctuation distribution, and two parameters to probe the collective motion.

The pair distribution function, which is constant and equal to 1, indicates that there are no structural correlations, and the MSPD grows linearly in the whole period, in agreement with previous results. Furthermore, the velocity autocorrelation function is zero at all positive times. These results apparently show that the stocks diffuse freely, as noninteracting Brownian particles. However, the ISF shows a nonexponential decay, and its decay time decreases as $\sim q^{-1.8}$ with the wave number q, and the price fluctuation distribution deviates strongly from Gaussian (which is well known for other financial markets or indices). Also, cooperative motions in stocks, which have been long discussed [32,46], have been studied with two parameters for collective motion, borrowed from soft matter. These results are characteristic to undercooled particle systems, close to the glass transition; indeed the fluctuation distribution can be correctly reproduced with a model initially designed for glasses. Thus, we seek a physical system with a similar behavior. The analysis of colloidal gels or clusters produces results in some agreement with those of the stocks sets; the mean-squared displacement grows linearly, whereas collective motions and non-Gaussian position fluctuations are observed. However, gels or clusters are also characterized by strong structural correlations, which are absent in stocks. We quote also previous results for a dense system of stars with infinitely thin arms, which behave as an ideal gas. This system undergoes a glass transition without structural correlations, but has the typical dynamical signatures of undercooled systems, namely, a two-step growth in the MSD [47].

The paper is organized as follows: In Sec. II, the details of the portfolios selection are given, and its overall (center-of-mass) behavior is studied. Section III presents the analysis of the structure and dynamics of the system, and a discussion in terms of physical systems that reproduce the behavior of stocks. The relevant conclusions are compiled in Sec. IV.

II. STOCK PORTFOLIOS

The prices of three sets of stocks are used in this work, all of them spanning from March 2, 2010 to November 1, 2018, with a time resolution of 1 day. The sets are composed of US stocks, UK stocks, and European (including UK) stocks, respectively. The US and UK portfolios comprise $N = 1095$ and $N = 89$ stocks, respectively, corresponding to the companies with available data along the time span considered and with the greatest average dollar volume. The European portfolio, on the other hand, contains $N = 240$ stocks of companies with available data along the period which are part of the national indices of the UK (FTSE100), Germany (DAX30), France (CAC40), Spain (IBEX35), Switzerland (SMI), Italy (FTSE MIB), Portugal (PSI20), and Holland (AEX). This selection (in the three sets) guarantees that only big and stable companies are considered in this analysis. All sets of data have been taken from Yahoo! Finance.

The selection of European stocks involves markets working with different currencies: UK pounds, Euros, and Swiss francs. Nevertheless, it was decided to study this heterogeneous portfolio to increase its size and improve the statistics, and to make the size of the overall markets more similar (the US stock market represents 40.6% of the global market, whereas the European stock markets considered here add to a total of 19.6%) [48]. However, the different working currencies inhibit the study of cross correlations, and thus the UK market has been taken as a proxy of the European one.

In all cases, we have used the logarithm of the price (henceforth bare log price) instead of the bare price $p(t)$, $x^o(t) = \ln[p(t)]$. In this way, differences in $x^o(t)$ are dimensionless, and allows one to compare stocks with prices in different currencies. In economy, differences in $x^o(t)$ are known as the log returns, normally used in the studies of financial markets, and reflect the relative increase or decrease of the price.

In the following, we analyze the structure and dynamics of the portfolios using the methods typical of particulate soft-matter systems, with the log price as particle position. For the structure, we use the log-price distributions and the pair distribution function, and for the dynamics we use several correlation functions such as the mean-squared log return (or log-price difference), Van Hove functions, and observables to capture the collective dynamics. As further discussed below, the average of the log price for all stocks in the portfolio, $\bar{x}(t) = 1/N \sum_i x_i^o(t)$, called the index, is not stable in the time

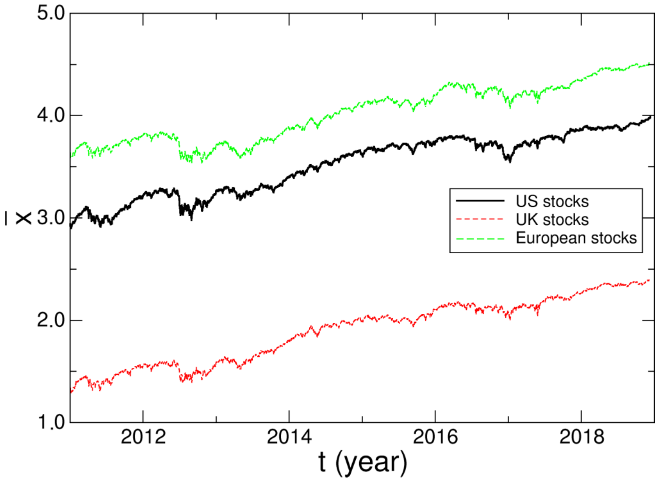

FIG. 1. Daily evolution of the indices of the three portfolios, as labeled, for the whole period considered.

interval studied here (note that the index is only an ensemble average, but not a time average). Thus, the effect of this drift is subtracted from the behavior of single stocks by considering

$$x_i(t) = x_i^o(t) - \bar{x}(t), \quad (1)$$

where the $x_i^o(t)$ is the bare log price of stock i at time t.

III. RESULTS AND DISCUSSION

The evolution of the three indices in the 9 years studied here is presented in Fig. 1. It is interesting to note that the three of them display very similar changes. The overall trend is an increase of one unit approximately, corresponding to a relative increase of ~170%. This implies an important change in the overall log-price distribution, which needs to be corrected. Thus, the log-price with the index subtracted, as given in Eq. (1), is used in all the subsequent analyses.

We study first the structure of the three systems, using the methods and techniques typical of particle systems, and then move to the analysis of the dynamics. The results indicate that the systems have no structure at the two-particle level, while the dynamics differs from Brownian motion. In the final part of this section, we seek a physical system that shows the properties of the portfolios studied here. Although all the properties cannot be reproduced, we conclude the section by showing that colloidal gels share some similarities with stock portfolios.

A. Structure

The log-price distribution of the three sets of stocks is presented in Fig. 2, averaged over the whole time interval considered in this work. Recall that the average is subtracted from the stock log prices, as shown by Eq. (1), and thus the average of this distribution is zero. Still, the three distributions are very different. The US stock log prices follow a bell-shaped distribution with a width of 0.82, while the distribution of the European stocks is more heterogeneous, of width 2.26. The heterogeneity of the European stocks is caused only partly by the different origin of the stocks composing this portfolio,

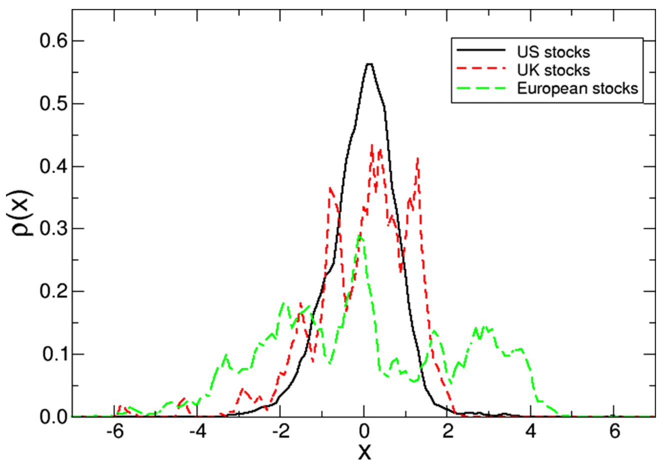

FIG. 2. Log-price distribution for all sets, as labeled.

as one would naively guess. Indeed, the UK stocks, the biggest component of the portfolio, is also more heterogeneous than the US portfolio, despite it containing only 89 stocks.

In order to study the internal structure of the portfolios, we construct the pair distribution function for stocks, $g(w)$, with w the log-price difference. Because the distribution of stock log prices is not homogeneous, as shown previously, the standard definition of the pair distribution function needs to be generalized:

$$g(\omega)d\omega = \frac{\text{Number of pairs with } \omega_{ij} \in [\omega, \omega + d\omega]}{N \int_{-\infty}^{\infty} dx\, \rho(x)\rho(x+\omega)}, \quad (2)$$

where ω_{ij} is the log-price difference of the pair of stocks i and j. With this definition, one ensures that a system without correlations between the stocks is identified by $g(\omega) = 1$, irrespective of the overall distribution of particle positions.

The results for the US and UK stock markets are presented in Fig. 3. Because this function compares the log prices of different stocks, we do not include here the European portfolio, as this contains stocks from different national markets.

Both sets of stocks clearly show $g(\omega) \approx 1$, up to log-price separations comparable to the width of the log-price

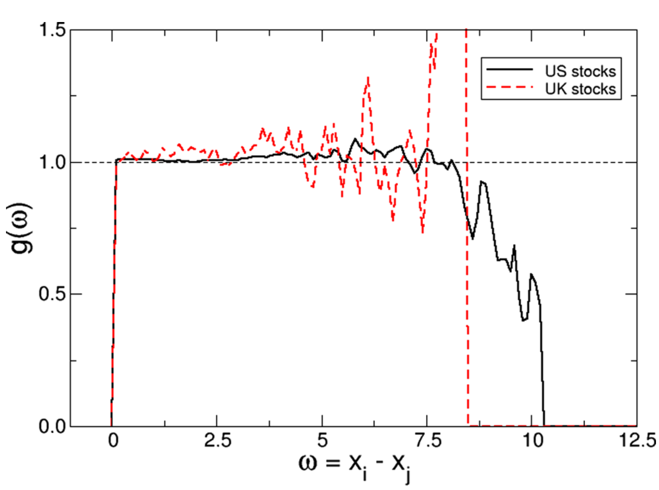

FIG. 3. Distribution of log-price differences for the US and UK stocks, as labeled.

distribution. Due to the small number of stocks in the UK portfolio, the statistical noise in the $g(\omega)$ of this is larger. In physical terms, $g(\omega) \approx 1$ indicates the absence of structural correlations, i.e., the stocks show the same structure as an ideal gas, albeit confined to produce a nonhomogeneous density. This result is not surprising, as no direct interaction between the stocks is expected, although previous results have identified correlations in the price evolution [29,30], as further discussed below. The pair distribution functions of Fig. 3 show that these correlations cannot be established at a structural level.

It must be mentioned that in the calculations of $g(w)$ shown in the previous figure, the density has been averaged over the full period (2011–2018). Because the index (and thus the distribution) changes notably during this time interval, the subtraction of the index in the calculation of the distribution has an important effect. In particular, this affects the denominator of Eq. (2). If the stock log prices are not corrected with the mean log price, the calculated $g(w)$ is larger than 1 for short log-price differences, and lower than 1 at larger distances. This apparent interaction is, however, artificial, and caused by the drift of the portfolio index.

One more aspect that needs to be discussed is that in Fig. 3 the whole systems are analyzed, which probably hides some structural correlations between specific pairs of stocks. This would correspond to a system of different species, with heterogeneous interactions between the constituents. The present results, however, indicate that these specific interactions, if they exist, are subdominant, and do not show up in the overall structure of the system.

B. Dynamics

We move now to study the dynamics of the portfolios, bearing in mind that the analysis of the structure yields the picture of the portfolios as a confined ideal gas. Let us recall that our data has a time resolution of 1 day, i.e., we focus on the daily evolution of the market, missing the intraday dynamics. The equivalent to the MSD in particle systems is the MSPD:

$$\langle[\delta x(\tau)]^2\rangle = \left\langle \frac{1}{N}\sum_{j=1}^{N}[x_j(t+\tau) - x_j(t)]^2 \right\rangle, \qquad (3)$$

where the summation runs over all stocks in the portfolio and the average implies different time origins, t. The MSPD for all sets of stocks are presented in Fig. 4, as well as the mean-squared displacement of the index. The dispersion of the data is also included in the graph for selected times as error bars, signaling from the MSPD of the 10% slowest stocks to the 10% fastest ones (or in the cumulative distribution, the range from 0.1 to 0.9). Despite the large dispersion of the data, the MSPDs grow almost linearly with time for the whole period studied, with little difference between them. This is confirmed by studying the power spectrum (see inset), which follows the $1/\omega^2$ behavior, characteristic of Brownian motion. This linear growth of the MSPD with time has been reported previously for market indices, currency exchange prices, etc. [32,46].

It is interesting to note that the growth of the MSPDs is below 0.1, so that the confinement of the stock log price,

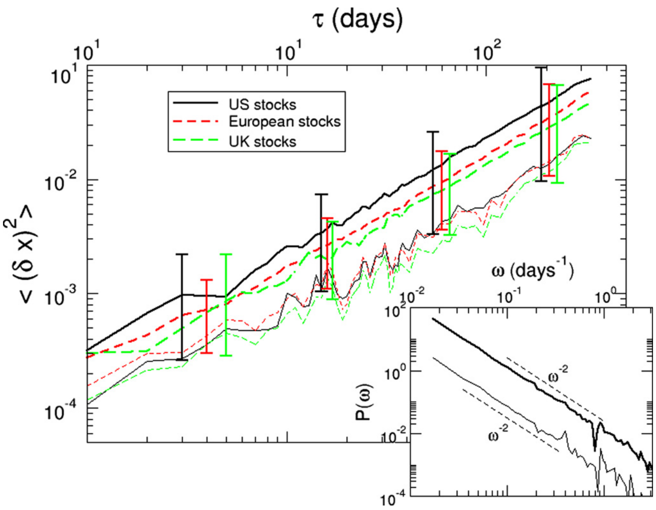

FIG. 4. Mean-squared log-price difference for the three sets of stocks, as labeled (thick lines), with the error bars signaling from the smallest 10% stocks to the fastest 10% ones. The thin lines present the mean-squared displacement of the index, with the same color code. The inset presents the power spectrum of the US stocks (thick line) and the index of this set (thin line).

shown in Fig. 2, does not affect the motion in the time range studied in the figure. On the other hand, the three indexes also grow linearly, with little difference between them. Diffusion of the center of mass of a system is also observed in colloidal systems due to the Brownian forces acting on the particles, but the precise relation depends on the structure of the system (individual particles, clusters, etc.). The diffusion coefficient for the index, obtained from the slope of the MSD vs time, $\langle(\delta x)^2\rangle = 2D\tau$, is smaller than the average for all stocks. Table I compiles the diffusion coefficients for the stocks and index of the three portfolios.

It is customary in the analysis of fluctuating time series to subtract the local trend, so-called detrended analysis [49]. This allows the identification of the fluctuating part and its analysis. For every stock, to the value of the individual quotation at time t, we have subtracted a linear trend, fitted in a time interval around t. The MSPD of the corrected data shows a linear increase for short times, below the length of the interval for the fitting, and flattens for longer times. The latter behavior is indicative of fluctuations caused by a white noise, compatible with Brownian motion in colloidal particles.

In experiments on soft matter, the dynamics is usually studied with the density-density autocorrelation function, or ISF. For Brownian motion, the ISF decays exponentially to zero, whereas a nonexponential decay indicates the existence of interactions among the particles, polydispersity, etc., and in

TABLE I. Self-diffusion coefficients of the stocks in the three portfolios, D_s, and diffusion coefficient of the index, D_i.

Portfolio	D_s (day^{-1})	D_i (day^{-1})
US stocks	1.15×10^{-4}	3.65×10^{-5}
Eur. stocks	6.88×10^{-5}	3.20×10^{-5}
UK stocks	8.58×10^{-5}	3.66×10^{-5}

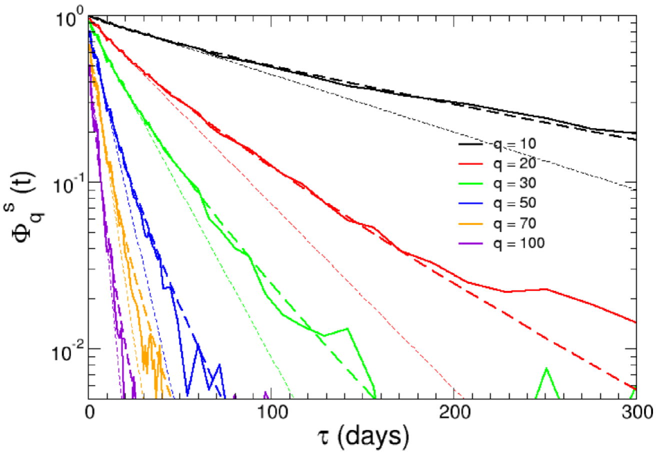

FIG. 5. Self part of the intermediate scattering function of the US stocks for different wave numbers, increasing from top to bottom as labeled. The thin lines show the exponential fitting of the initial decay, and the thick lines correspond to the fitting of the stretched exponential.

solids it decays to a positive value. We generalize the ISF to stock markets:

$$\Phi_q(\tau) = \left\langle \frac{1}{N} \sum e^{iq[x_j(t+\tau)-x_k(t)]} \right\rangle, \tag{4}$$

where q is the wave number, i the imaginary unit, the summation runs over all pairs of particles j and k, and the angular brackets imply average over the time origin t. This function requires a large statistics, but its general behavior is captured by its self part, $\Phi_q^s(\tau)$, setting $j = k$, which is the Fourier transform of the MSPD. The self-intermediate scattering function (sISF) for the US stocks is presented in Fig. 5 for different wave numbers. The correlation function decays to zero, as expected for a fluid state, but the decay is more stretched than a simple exponential (shown by the thin dashed lines). In fact, they can be well reproduced by the Khoulrausch stretched exponential:

$$\Phi_q^s(\tau) = A_q \exp\{-(\tau/\tau_q*)^{\beta_q}\}, \tag{5}$$

which is typically used to describe the correlation function in undercooled liquids. There, A_q represents the nonergodicity parameter, τ_q* is the time scale for the relaxation, and β_q is the stretching exponent. In our case, $\beta_q \approx 0.8$ for all wave numbers, i.e., the decay is only slightly stretched. Similar results are found for the other sets of stocks.

The timescale of the decay of the sISF as a function of the wave number is plotted in Fig. 6 for the three sets of stocks. For a Brownian particle, this timescale depends on the wave number as q^{-2}, while Fig. 6 shows that the timescale in stocks decays with an exponent close to, but different from -2. This analysis shows that the dynamics of the three sets of stocks cannot be simply described by Brownian motion but it presents slight deviations.

One more aspect that can be used to study the stock dynamics borrowed from particle systems is the velocity autocorrelation function, which is zero in Brownian dynamics for all (positive) times [42,50]. We have defined the daily stock velocity as $v_i(t) = x_i(t+1) - x_i(t)$, measured in units of day^{-1}. The velocity autocorrelation function is then calculated as

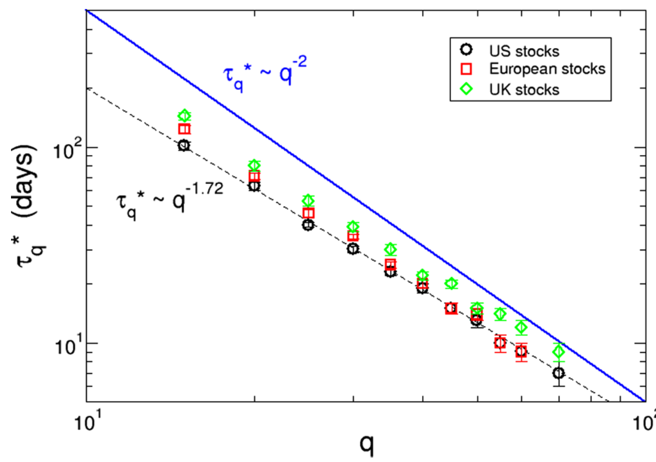

FIG. 6. Timescale of the decay of the sISF for the three sets of stocks. The dashed lines show the ω^{-2} behavior.

$$Z(\tau) = \left\langle \frac{1}{N} \sum v_i(t+\tau) v_i(t) \right\rangle, \tag{6}$$

where the average implies different time origins, as above. Figure 7 shows the velocity autocorrelation function for the European, US, and UK stocks. In agreement with Brownian motion, the correlation function is zero within the statistical noise for all times, except at $t = 0$. It can be argued that the decay of the velocity correlation function has occurred within the first day, and is observed for the price correlations [32], but this would be equivalent to using the Langevin description of Brownian motion, which is valid for timescales below the diffusion time.

So far, we have obtained conflicting conclusions about the stock markets. Whereas the linear growth of the MSPD and the absence of velocity correlations are comparable to the Brownian motion of independent particles, the self part of the ISF indicates the existence of weak interactions or polydispersity. In addition to this discussion, it is well known

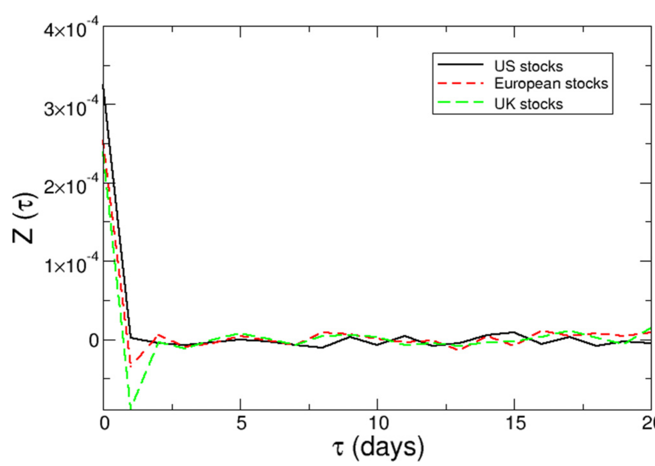

FIG. 7. Velocity autocorrelation function for the three sets of stocks, as labeled.

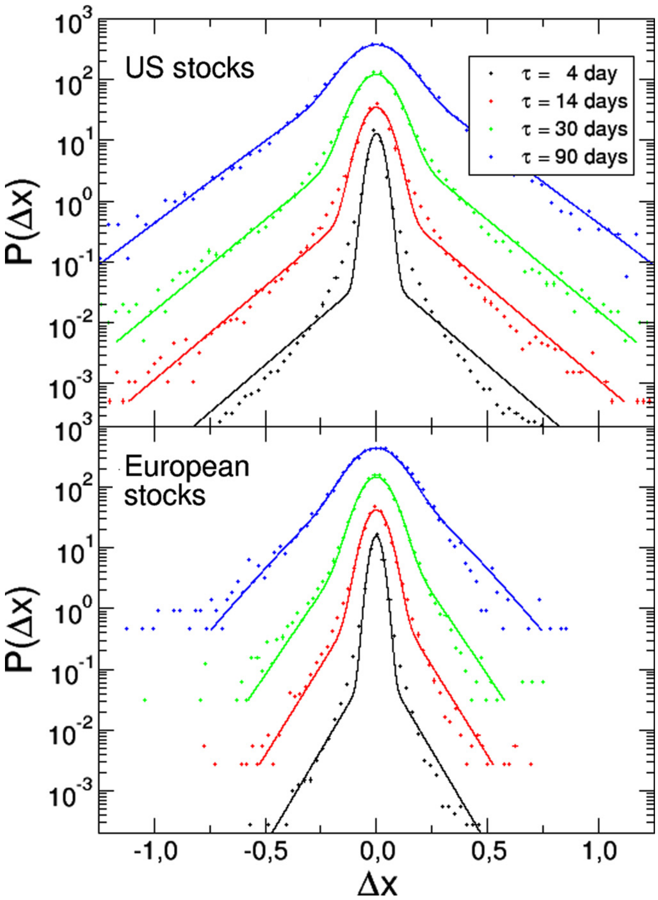

FIG. 8. Distribution of log-price differences for different times, increasing from bottom to top as labeled. The thin US stocks are presented in the upper panel, and European stocks in the bottom one.

that the distribution of log-price differences for a given time interval, which corresponds to the Van Hove function in particle motion, is not Gaussian. Within Brownian motion, this distribution should be Gaussian—its second moment being the MSD. However, it has been thoroughly discussed in the literature that this is not the case for financial markets [2,32,46].

Figure 8 shows the distributions of log-price variations, Δx, for different time lags, τ, both for the US and the European sets. The experimental distributions show a prominent peak at the origin, $\Delta x = 0$, with long tails for large differences, differing notably from a Gaussian distribution.

These tails are typical of different systems with slow or arrested dynamics: undercooled fluids (atomic or colloidal), close to a glass transition or granular matter [51,52], but also in other financial markets [13,39]. Note that we study all the stocks, different from other works where the index is used [53], although the shape of the distribution is very similar in all cases. Different models have been proposed to describe it within statistical physics, such as the q-Gaussian distribution function [39,52,54], but this requires typically fitting the whole set of parameters for every lag- time, τ.

We used recently a model derived originally for particle systems with slow dynamics to reproduce quantitatively the distribution of fluctuations of the Euro/USD exchange rate [44] and many other currency pairs [45] for all lag times. The physical model separates the rattling of particles inside the cages formed by their neighbors, and long jumps from cage to cage. Within the potential energy landscape picture of glasses, with many local shallow minima or basins (and possibly one deep narrow minimum corresponding to crystal), these two motions correspond to oscillations in the local basins and jumps from basin to basin. The dynamics of the particles within the cage of neighbors, or basins, is described by an Ornstein-Uhlenbeck process:

$$f_{\rm vib}(r,\tau) = \sqrt{\frac{\alpha}{2\pi D(1-e^{-2\alpha\tau})}}\,\exp\left\{-\frac{\alpha r^2}{2D(1-e^{-2\alpha\tau})}\right\} \quad (7)$$

with r the particle displacement, D the short-time diffusion coefficient, and $\alpha = D/l^2$, with l the size of the cage [55–57]. This corresponds to a particle describing Brownian motion with a linear central force pulling it towards its origin. Long-range jumps are possible, according to a distribution $f_{\rm jump}(r)$, and the time interval for a jump follows a distribution $\phi(\tau) = \tau_j^{-1}\exp(-\tau/\tau_j)$. In the model proposed by Chaudri *et al.* [51], two different timescales are used for the first and subsequent jumps, τ_1 and τ_2, respectively, with $\tau_2 < \tau_1$. The overall distribution, $G(r,\tau)$, gives the probability of a displacement r, at time lag τ, and it is calculated in the Fourier-Laplace domain, $G(q,s)$. $G(r,\tau)$ is recovered by back transforming to the space-time domain as

$$G(r,\tau) = \tau_1 f_{\rm vib}(r)\phi_1(\tau) + FT^{-1}\Bigg[\tilde{f}_{\rm vib}(q)\tilde{f}(q)\tau_2 \times \frac{\exp\{(\tilde{f}(q)-1)\tau/\tau_2\} - \exp(-\tau/\tau_1)}{\tau_2-\tau_1+\tilde{f}(q)\tau_1}\Bigg]. \quad (8)$$

Here, $\tilde{f}(q) = \tilde{f}_{\rm vib}(q)\tilde{f}_{\rm jump}(q)$ and $\tilde{f}(q)$ is the Fourier transform of function $f(r)$, with q the conjugate variable of the displacement in the Fourier space and FT^{-1} denotes the inverse Fourier transform. This model, originally designed for particles, is used to fit financial data taking now r as the difference of log return, Δx, and noticing that this is a one-dimensional variable, instead of the three-dimensional particle displacement. Also, while the distribution for long-range jumps, $f_{\rm jump}(r)$, is Gaussian in particles, we assume here a Laplace distribution, which is more appropriate for stocks due to the mixture of many different Gaussian distributions [58].

The model, Eq. (8), is used to describe and rationalize the distribution of the fluctuations in the log return, using the diffusion coefficient, D, the distances l and d, and the timescales τ_1 and τ_2 as fitting parameters. It must be mentioned that these parameters have a simple physical interpretation, contrary to other models [54]. The optimization is achieved adjusting the absolute moments of order $O = \{0.1, 1, 2, 3, 4\}$, i.e., minimizing the parameter $\chi = \sum_{o\in O}\sum_{\tau\in T}\{\ln[mom_e(\tau,o)] - \ln[mom_t(\tau,o)]\}^2$, where $mom_e(\tau,o)$ and $mom_t(\tau,o)$ are the empirical and theoretical moments of order o at time τ, respectively. T is a selection of time lags, from 1 day to 3 years. Note that the moment of order 0.1 improves the fit around the mode.

The model can indeed fit the moments of both the experimental data of the US and European stocks, with some

TABLE II. Optimal parameters to fit the experimental distributions, obtained from the moments of the distribution (see text).

Portfolio	D (day^{-1})	l	d	τ_1 (day)	τ_2 (day)
US stocks	1.17×10^{-4}	0.173	0.195	200	169.9
Eur. stocks	8.13×10^{-5}	0.159	0.098	200	61.59

deviations noticed only for small times; the maximum relative difference between the experimental and fitted distributions is below 3% for long lag times, and between 3% and 5% for short times. The parameters that produce the optimal fitting are given in Table II. Note that in both cases, the rattling inside a cage of size ∼0.17 takes a long time (200 days), until a jump outside this basin takes the system to a different local minimum, not very far away.

The distributions resulting from these selections of parameters are presented as continuous lines, with the empirical ones (points) in Fig. 8 for time lags of 4, 14, 30, and 90 days. The overall agreement is good, capturing the two length and time scales in the experimental data, although some differences can be observed, particularly for short times in the US stocks. The fittings to the UK stocks (not shown) are of similar quality, although the experimental data is noisier, due to the smaller number of stocks.

Another important difference in the dynamics of stocks, compared to Brownian motion, is the existence of cooperative motion. In colloids, this has been discussed in connection with nonergodic transitions, either led by the steric hindrance (at high particle density), or by attractions, namely, gelation. Different parameters have been devised to identify and characterize these modes in particle systems, most of them studying the relative motion of two particles, so-called, four-point correlation functions [59,60]. The simplest parameter that can be defined is the difference of particle displacements, which has been used to study also the dynamic arrest in two dimensions [61]. This can be simply applied to a one-dimensional system, yielding

$$\gamma(\tau) = \left\langle \frac{1}{N} \sum_{i,j}{}' \left[\delta x_i^o(\tau) - \delta x_j^o(\tau)\right]^2 \right\rangle, \tag{9}$$

where N is the number of stocks, and the pair $i = j$ is excluded in the summation; the brackets indicate averaging over time origins. Note that the bare log prices, $x_i^o(\tau)$, are used instead of the corrected ones, and $\delta x_i^o(\tau) = x_i^o(t+\tau) - x_i^o(t)$. This is introduced to identify collective motions that drive the average. If all stocks move independently (Brownian motion), $\gamma(\tau) = 2\langle(\delta x^o)^2\rangle$, but if all stocks move in the same way, $\gamma(\tau) = 0$. Figure 9 presents the ratio $\gamma(\tau)/\langle(\delta x^o)^2\rangle$ for the three sets of stocks. Although this ratio fluctuates strongly, all sets present a similar value of the ratio, well below 2, indicating the presence of the cooperative motions of groups of stocks.

Alternatively, Muranaka and Hiwatari defined a different parameter, based on the MSD as well, to study cooperative motion [62]. In this case,

$$C(\tau) = \frac{1}{\langle(\delta x^o)^2\rangle} \left\langle \frac{1}{N^2} \sum_{i,j}{}' \delta x_i^o(\tau) \delta x_j^o(\tau) \right\rangle \tag{10}$$

is close to zero when all stocks move independently, while $C(\tau) \to 1$ indicates that all stocks move by the same amount. (Note that again the bare log prices are used). Figure 10 shows the results for the three of them. Again, all sets of stocks show similar degrees of cooperativeness in their dynamics.

To further explore the origin of this cooperative motion, the distribution functions of these two parameters have been studied. Figure 11 shows the distributions of both of them for a time lag equal to 30 days for the US stocks. For comparison, the corresponding distributions for a system of noninteracting Brownian particles are also presented. The distribution for $\gamma(\tau)$ has a prominent peak at $\gamma = 0$, and extends to large values of γ, while $C(\tau)$ presents positive and negative values. The comparison with the ideal system shows that the distribution of $\gamma(\tau)$ is less stretched in the case of the US stocks, resulting in a smaller average. The distribution of $C(\tau)$, on the

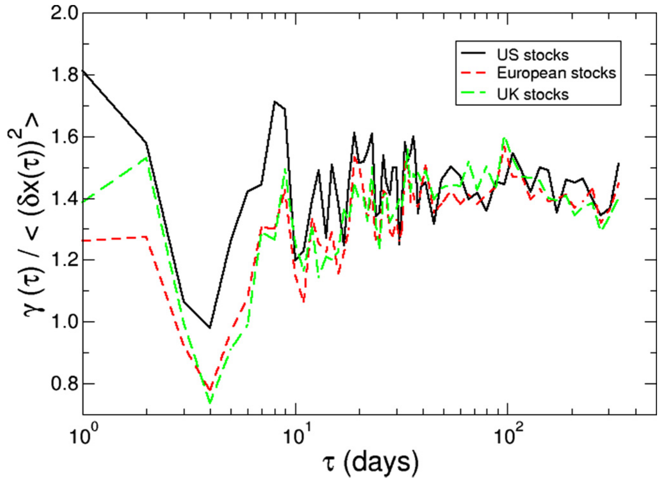

FIG. 9. $\gamma(\tau)/\langle(\delta x)^2\rangle$ for the US, European, and UK stocks, as labeled [see text for the definition of $\gamma(\tau)$].

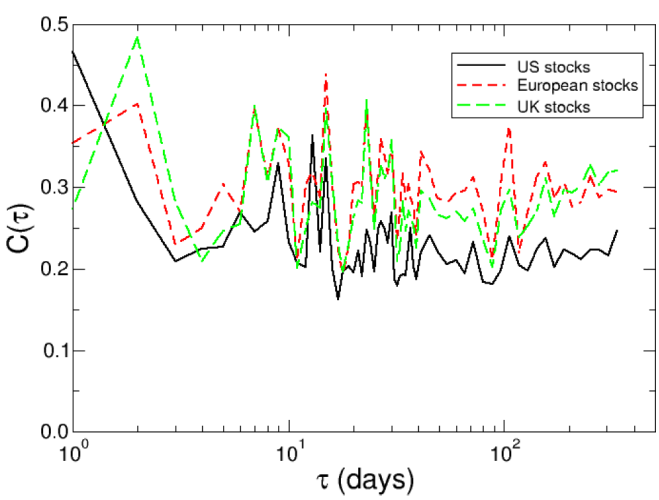

FIG. 10. $C(\tau)$ for all sets of stocks, as labeled.

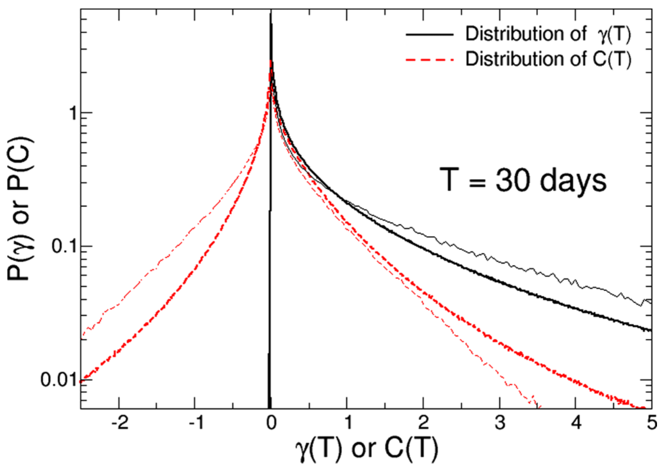

FIG. 11. Distribution functions of $C(\tau)$ and $\gamma(\tau)$, as labeled, for $\tau = 30$ days of the US stocks (thick lines). The thin lines represent the results for noninteracting colloidal particles.

other hand, is symmetric for the ideal gas, but it is skewed for the US portfolio to positive values, yielding a positive average, indicative of cooperative motions. Similar results are obtained for the UK stocks, although the reduced number of stocks results in an increased statistical noise.

The distributions of $\gamma(\tau)$ and $C(\tau)$ both deviate from the expected results for the ideal gas. However, these deviations do not allow us to identify clusters of stocks that move cooperatively.

C. Discussion

In this section we seek to rationalize the results presented so far, in terms of the properties of soft-matter systems. In short, we have shown above that stocks present some signatures of noninteracting particles but also of slow dynamics, typical of undercooled systems. In particular, the lack of internal structure, the linear growth of the MSPD, and the absence of velocity autocorrelation, are signatures of a noninteracting Brownian system, while the nonexponential decay of the sISF, the collective motion, and non-Gaussian Van Hove functions are typical of undercooled fluids. However, the latter show a two-step growth of the MSD (or two-step decay in the correlation functions), separated by an intermediate plateau [60,63]. The extension of the plateau increases upon approaching the glass transition, and ideally diverges at the glass point.

While this is the case for most glasses, in colloidal systems with short-range attractions, the situation is different. For small attraction strength, the particles form clusters, which percolate for larger strengths and finally a gel is formed when the structure does not relax. The low density of the system introduces diffusion of independent clusters or breathing modes without breaking the bonds between particles. These appear as new relaxation mechanisms for the structure at intermediate and strong attraction strengths. In this case, the particle dynamics is also apparently Brownian while other parameters show the signatures of undercooled fluids. These relaxation modes are possible due to the low density of particles in the system, and are not present in other glass-forming systems at high density.

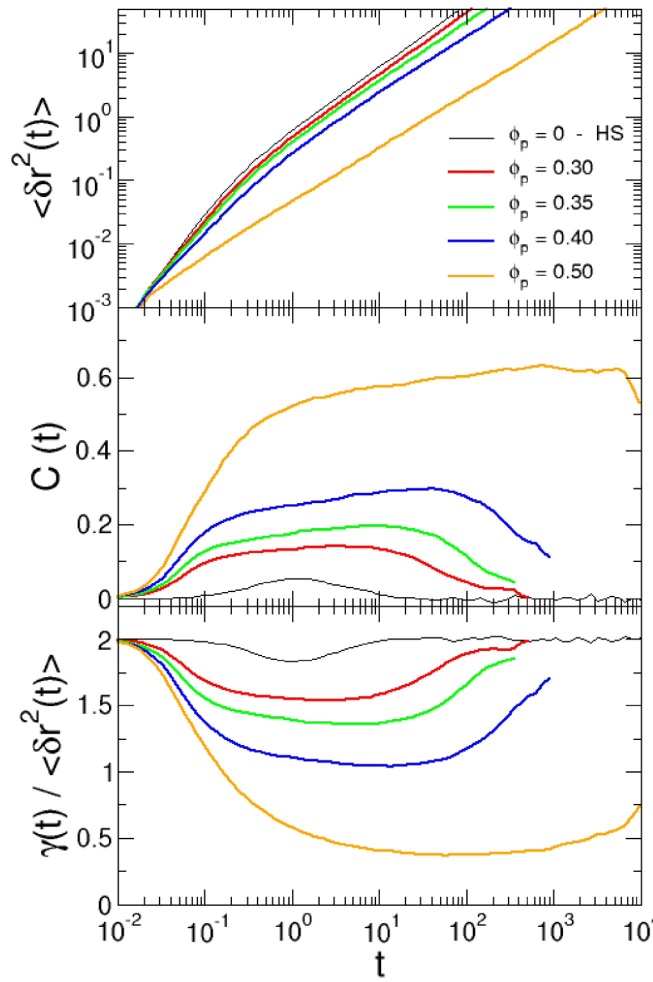

FIG. 12. Mean-squared displacement (top panel) and analysis of the cooperative motions in different states with low volume fraction, $\phi_c = 0.10$, for increasing attractions, from top to bottom as labeled. The system is composed of $N = 1000$ particles, with radii distributed according to a flat distribution of half-width $0.1a$, where a is the mean particle radius. The attractive interaction considers this polydispersity, and on average its range is $0.1a$, and its strength is given by $16\phi_p k_B T$.

Figure 12 shows the MSD and the parameters of collective motion used above, for different states with increasing attraction strength. These results have been obtained in simulations of quasihard polydisperse particles undergoing Langevin dynamics, to mimic Brownian motion. A short-range attraction is added to induce gelation and a long-range repulsive barrier to inhibit liquid-gas separation. Further details can be found in Puertas *et al.* [64].

For $\phi_p = 0.50$, the MSD grows linearly over five decades, although a significant slowing down of the dynamics is observed in the long-time diffusion coefficient. On the other hand, the collective parameters, equivalent to $\gamma(t)$ and $C(t)$, indicate that collective motions are indeed relevant, in qualitative agreement with the MSPD and collective motions in Figs. 4, 9, and 10. The collective motions in gelling systems originate from the motion of particles in clusters, or branches of the percolating structure, if the attraction is strong enough. A snapshot of the system with $\phi_p = 0.50$ is shown in Fig. 13.

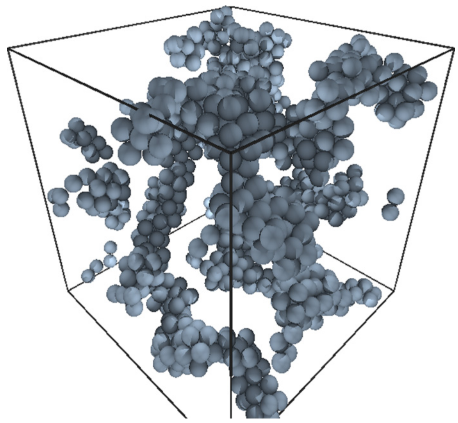

FIG. 13. Snapshot of the system with a volume fraction of colloidal particles of $\phi_c = 0.10$ and attraction strength $\phi_p = 0.50$.

A branched, percolating structure with large voids is observed, where long-time diffusion is caused by the slow restructuring of the network and the collective motions correspond to the movement of a branch. However, it must be stressed that colloidal systems with short-range attractions are characterized also by internal structuring, as shown in the snapshot, which are ultimately responsible for the dynamical behavior.

It must also be mentioned that special systems have also been devised previously to produce a glass transition without structural correlations [47,65]. In that system, infinitely thin hard needles [65] or stars [47] are considered, which structurally behave as ideal gases, whereas the impenetrability condition provokes a significant dynamic slowing down at large densities, although the glass transition is not reached. For states with a significant arrest, the non-Gaussian parameter grows, indicating that the distribution of position fluctuation deviates from the Gaussian shape.

IV. CONCLUSIONS

The daily dynamics of two sets of stocks from the US stock market (US stocks) and different national floors from economy drivers of European countries (EU stocks) in the period 2011–2018 have been analyzed in this work using the concepts and tools of soft matter. This is motivated by a previous work where a model, initially developed to describe the dynamics of undercooled fluids, glasses, and granular matter, was used in the analysis of the distribution of fluctuations in the exchange rates of different currency pairs. Here, a stock market is considered as a system comprised of particles, whose structure and dynamics is studied.

On the one hand, the equivalent of the MSD, namely, the MSPD, grows linearly, the velocity autocorrelation function is zero, and the pair distribution functions (for the US and UK markets) show no structural correlation between the stocks. On the other hand, the log-price correlation function, equivalent to the sISF, shows a nonexponential decay, the distributions of fluctuations of the stock prices are non-Gaussian, featured with wings, i.e., leptokurtic distribution, and a significant degree of collective motion is observed by parameters borrowed from the analysis of undercooled systems. Whereas the former results point to a noninteracting Brownian system, which can be easily described by free diffusion, the latter ones are characteristic of a strongly interacting system.

Although this dichotomy does not have an evident physical analog to our knowledge, we propose two physical systems which share some qualitative similarities with the stock market. The first one is a colloid with short-range attractions, which, upon increasing the attraction strength, forms reversible clusters, which percolate for strong enough interactions. Indeed, the MSD of this system grows linearly for strong attractions, where the bonds affect particle diffusion even at short times, but also present a high degree of cooperativeness. However, gels also present strong structural correlations, in stark difference with the case of stock markets. The second system which can be compared with stocks is an extension of the ideal gas, namely, it is composed of rigid infinitely thin stars, which has no structural correlations, but can show a significant slowing down for large density. We hope that a more appropriate physical model system can be found or devised that can aid in future studies of stock markets. In particular, introducing heterogeneous interactions between stocks, as some pairs are expected to be more correlated than others, appears a key ingredient of future models.

The work presented here aims to provide a method of studying and analyzing the stock markets, using the theoretical toolbox of statistical mechanics, particularly of soft condensed matter physics. Obvious extensions of this work are the consideration of other parameters in the analysis of the structure and dynamics, seeking correlations between them that can help in identifying interaction potentials between the assets.

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