Cyclization in bipartite random graphs

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In this paper the time evolution of a finite bipartite graph initially comprising two sorts of isolated vertices is considered. The graph is assumed to evolve by adding edges, one at a time. Each new edge connects either two linked components and forms a new component of a larger order (coalescence of graphs) or increases (by one) the number of edges in a given linked component (cycling). Any state of the graph is thus characterized by the set of occupation numbers (the numbers of linked components comprising a given numbers of vertexes of the both sorts and a given number of edges. Once the rate of appearance of an extra edge in the graph being known, the master equation governing the time evolution of the probability to find the random graph in a given state is reformulated in terms of the functional generating the probability to find the evolving graph in a given state. The exact solution of the evolution equation for the generating functional applies for analyzing the average population numbers of linked components. In the limit of large order of the graph the distribution factorizes into two multipliers, one of which is just the spectrum of linked components in the infinite bipartite graph, The second multiplier includes the dependence on the total size of the graph. Both these multipliers contain information on the emergence of the giant component that forms at a critical time.

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I. INTRODUCTION

Two finite sets of points (vertexes) connected with lines (edges) form a bipartite graph. Figure 1 displays such a graph where the vertexes are located on two circles [1–5]. The graph is characterized by its total order—the pair $(M,\ N)$—the numbers of vertexes of two sorts and the number of edges ν_{total}. In what follows we consider graphs of constant order having the random number of edges. We assume that the edges are added to the graph at random moments of time and the rate of their production is known.

Figure 1 explains what is this the random bipartite graph. Two sorts of vertexes, M hearts and N diamonds, are randomly connected with edges. Each graph comprises linked components—the subgraphs where each vertex is connected with all others by paths—the collections of edges belonging to this subgraph. In what follows we consider the graph where the edges connecting the vertices of the same sort are forbidden. The issue considered here is to find the average number of linked components containing exactly m, n vertexes and ν edges as the function of time t (the order-connectivity spectrum) assuming the edges to be randomly added to the graph one at a time. The initial graph is empty (no edges) and comprises M and N vertexes.

Random graphs already attracted the attention of the scientific community six decades ago [6–9]. The main interest in this problem was associated with the emergence of a giant linked component in graphs of large order and sufficiently high connectivity (large numbers of vertexes and edges) [6]. The order of this component occurs comparable to the total order of the graph. The undoubted similarity of random graphs with the structures met in Nature and technical devices has generated interest in studying the dynamics of changes in random graphs as they are filled with random connections [7–9].

Although now much is known about the statistical properties of unipartite random graphs, less attention was given to multipartite graphs that are also of great importance because they allow us to model the evolution of complex natural and artificial systems. Among them are mixed polymers, the Internet (a complex network of routers and computers), and interacting spins in ordered and disordered lattices. The dynamics of many important physical systems like Ising magnets can also be formulated in terms of graphs.

The random graphs display a very remarkable property: a sufficiently large graph always contains a giant linked component [6]. This component appears suddenly as the number of edges in the graph exceeds a critical value. This phenomenon is similar to the phase transition. The most familiar example of the transition perhaps is percolation in disordered electric chains (see Ref. [10]).

There exist two approaches to studying random graphs: (1) the static one that fixes the numnbers of vertexes and edges and then uses the combinatorial analysis for finding the distribution of linked components [7–9] and (2) the kinetic approach that considers the time evolution of the random graph with the edges being randomly added one at a time [11–22]. The second approach is especially efficient because it applies the Smoluchowskii equation for describing the time evolution of the order-connectivity spectra in thermodynamically large graphs ($M, N \to \infty$) rather than much more complex combinatorics.

In the present paper the approach developed in my fairly recent work [17] (see also Ref. [19]) applies for considering the dynamics of evolving finite multipartite graph. This approach relies upon the Marcus-Lushnikov scheme [11–13]. Although

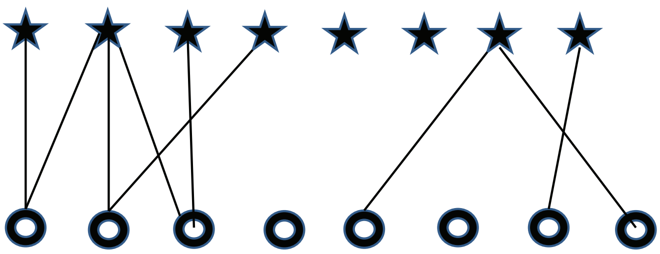

FIG. 1. A bipartite graph with a collection of linked components.

I had mentioned a couple of times the bipartite graphs in Refs. [18,19], the details of the solution were never published. In this paper the analysis is based on the Riddell theorem [23] applied for the first time for the analysis of unicyclic graphs in my recent work [18].

The remainder of the paper is divided as follows. In Sec. II the basic equations governing the time evolution of multipartite graphs are formulated and critical properties of the evolving graphs are discussed The exact expression for the order-connectivity spectrum is shown to factorized into two multipliers, the first of which depends on the total size of the graph while the second one is asymptotically independent of the graph size. This multiplier includes the numbers of linked components containing a given number of edges (Sec. III). In Sec. IV the Riddell theorem [23] for bipartite graphs applies for the derivation of the set of nonlinear integral equations for the generating functions for the numbers of graphs of given order and the connectivity. A recurrence procedure solving this set of equation is developed. Since the number of vertices in the tree is strictly fixed, adding one more edge without changing the order of the graph leads to the appearance of a cycle. The expression for the size-complexity spectrum through the generating functions X and Y is derived and the equation formulated in the preceding section are solved. Exact expressions for the spectrum of trees and single-cycled graphs are derived. The asymptotic form of the spectrum is also analyzed in Sec. V. The concluding Sec. VI connects the emergence of the giant component with the cyclization process in graphs.

II. BASIC EQUATIONS

A finite bipartite graph is a sum of linked subgraphs (components). No linked component contains isolated vertexes and is thus characterized by three integers, the numbers of vertices of two sorts, $1 \leqslant m, n < \infty$, and the number of edges $m + n - 1 \leqslant \nu \leqslant mn$. Each state of the bipartite random graph is given by the set of occupation numbers,

$$Q = (n_{1,0;0}, n_{0,1;0}, n_{1,1;1}, n_{2,1;2}\dots) = \{n_{m,n:\nu}\}, \tag{1}$$

where $n_{m,n:\nu}$ is the numbers of linked components in the graph.

Let us begin to add the edges to the initially empty bipartite graph. This process gives rise either to a coalescence of two linked components [bare vertexes are also considered as linked component (1, 0; 0) and (0, 1; 0)],

$$(m, n; \nu) + (k, l; \lambda) \longrightarrow (m + k, n + l; \nu + \lambda + 1), \tag{2}$$

or to filling a given linked component with one extra edge,

$$(m, n, \nu) \longrightarrow (m, n; \nu + 1). \tag{3}$$

The time evolution of the graph goes along the routes

$$Q^- \to Q \to Q^+ \quad \text{or} \quad \tilde{Q}^- \to Q \to \tilde{Q}^+.$$

The coalescence process changes three occupation numbers in the pair of the preceding states $Q^-, \tilde{Q}^-$,

$$n_{m,n;\nu}(Q^-) = n_{m,n;\nu}(Q) + 1, \tag{4}$$

$$n_{k,l;\nu}(Q^-) = n_{k,l;\nu}(Q) + 1, \tag{5}$$

$$n_{m+k,n+l;\nu+\lambda+1}(Q^-) = n_{m+k,n+l;\nu+\lambda+1}(Q) - 1. \tag{6}$$

If an edge is added to a linked component, then only two its occupation numbers change,

$$n_{m,n;\nu-1}(Q^-) = n_{m,n;\nu}(Q) - 1, \tag{7}$$

$$n_{m,n;\nu}(Q^-) = n_{m,n;\nu}(Q) + 1. \tag{8}$$

Let us introduce the time-dependent probability $W(Q, t)$ to find the graph in the state Q at time t and write the master equation for W,

$$\begin{aligned}\frac{dW(Q,t)}{dt} &= \sum_{Q^-} A(Q, Q^-)W(Q^-, t) - \sum_{Q^+} A(Q^+, Q)W(Q, t)\\ &\quad + \sum_{\tilde{Q}^-} B(Q, \tilde{Q}^-)W(\tilde{Q}^-, t) - \sum_{\tilde{Q}^+} B(\tilde{Q}^+, Q)W(Q, t).\end{aligned} \tag{9}$$

The transition rates A and B have the form

$$A(Q, Q^-) = \frac{1}{2T}(ml + nk)n_{m,n;\nu}(Q^-) \times [n_{k,l;\lambda}(Q^-) - \delta_{m,k}\delta_{n,l}\delta_{\nu,\lambda}] \tag{10}$$

and

$$B(Q, \tilde{Q}^-) = \frac{1}{T}(mn - \nu + 1)n_{m,n;\nu-1}(\tilde{Q}^-), \tag{11}$$

where the value of T defines the timescale of the coalescence process and $\delta_{\alpha,\beta}$ is the Kroneker delta.

It is more convenient to deal with the generating functional $\Psi(X, t)$,

$$\Psi(X, t) = \sum_Q W(Q, t) \prod_{m,n;\nu} x_{m,n;\nu}^{n(m,n;\nu)(Q)}. \tag{12}$$

The functional Ψ obeys the equation

$$T\frac{\partial \Psi}{\partial t} = (\hat{L}_f + \hat{L}_c)\Psi. \tag{13}$$

The right-hand side of this equation contains two differential operators, $\hat{L}_f$ and $\hat{L}_c$. The operator $\hat{L}_f$ adds the extra edges to linked components and does not change their orders,

$$\hat{L}_f = \sum_{m,n,\gamma}\left[(mn - \nu + 1)x_{m,n;\nu}\frac{\partial}{\partial x_{m,n;\nu-1}} - (mn - \nu)x_{m,n;\nu}\frac{\partial}{\partial x_{m,n;\nu}}\right]. \tag{14}$$

The operator $\hat{\mathcal{L}}_c$ responsible for the coalescence of a couple of linked components is

$$\hat{\mathcal{L}}_c = \frac{1}{2}\sum_{l,\lambda;m,\nu}(lm+nk)(x_{l+m,\lambda+\nu+1} - x_{l,\lambda}x_{m,\nu})\frac{\partial^2}{\partial x_{l,\lambda}\partial x_{m,\nu}}. \tag{15}$$

After some simple algebra the equation for Ψ acquires the form of Eq. (13) with

$$\hat{\mathcal{L}}_f = \sum_{m,n;\nu}(mn-\nu)(x_{m,n;\,\nu+1} - x_{m,n;\,\nu})\frac{\partial}{\partial x_{m,n;\,\nu}} \tag{16}$$

and

$$\hat{\mathcal{L}}_c = \frac{1}{2}\sum(ml+nk)(x_{m+k,n+l:\,\nu+\lambda+1} - x_{m,n;\,\nu}x_{k,l;\,\lambda}) \times \frac{\partial^2}{\partial x_{m,n;\,\nu}\partial x_{k,l;\,\lambda}} - \hat{N}M - N\hat{M}. \tag{17}$$

Here

$$\hat{M} = \sum_{m,n;\nu} m x_{m,n;\nu}\frac{\partial}{\partial x_{m,n;\nu}}, \quad \hat{N} = \sum_{m,n;\nu} n x_{m,n;\nu}\frac{\partial}{\partial x_{m,n;\nu}} \tag{18}$$

are the operators of the total masses of the first and the second components, respectively. The summation in Eq. (17) goes over all indexes $m, n; \nu$ and $k, l; \lambda$. The nonnegative integers M and N (the total numbers of the monomers of the first and the second components, respectively) are the eigenvalues of these operators. The evolution operator $\hat{\mathcal{L}}$ commutes with $\hat{M}$ and $\hat{N}$, which means that the functional Ψ can be chosen as an eigenfunctional of these operators. This fact is of great significance, because the above two equations make linear the right-hand side of Eq. (13) with respect to the differential operators $\hat{N}$ and $\hat{M}$ and thus allows for finding the solution, the details of which can be found in Ref. [24]. Here we write the final expression for the average composition spectrum (the details of derivation are given in the Appendix),

$$\bar{n}_{m,n;\,\nu}(t) = \binom{M}{m}\binom{N}{n} \times e^{-\mathcal{S}_{M,N)(m,n)}t/T}(e^{t/T}-1)^{\nu}C_{m,n;\,\nu}, \tag{19}$$

where

$$\mathcal{S}_{M,N}(m,n) = mn - mN - nM = MN - (M-m)(N-n) \tag{20}$$

is the total number of vertexes accessible for filling them by extra edges, and $C_{m,n;\,\nu}$ is the number of linked component of order m, n containing exactly ν edges. Equation (20) is similar to the result [17] for unipartite graphs.

III. FACTORIZATION

In this section we will demonstrate how the expression for the order-complexity spectrum $\bar{n}_{m,n;\nu}(t)$ [Eq. (19)] of linked components can be split into two multipliers,

$$\bar{n}_{m,n;\nu} = u_{M,N}u_{\infty}, \tag{21}$$

one of which ($u_{M,N}$) depends on the total orders M and N of the graph and another one (u_{∞}) does not.

At large T it is convenient to slightly regroup the multipliers in Eq. (22) for $\bar{n}_{m,n;\gamma}$. We borrow the multiplier T^{ν} from $(e^{t/T}-1)^{\nu}$ and input it to $u_{M,N}$. The result is

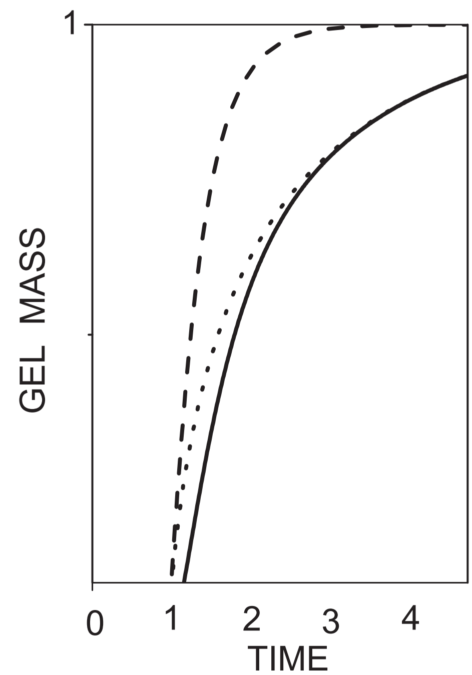

FIG. 2. The composition of the giant component vs time. Shown (dash and dot lines) are the time dependencies of the concentrations x_c and y_c of stars and circles in the giant component as the functions of time (in units of T). Solid line displays the rate of production of unicyclic graphs.

$$u_{M,N} = \frac{M!N!T^{-\nu}}{(M-m)!(N-n)!}e^{tmn/T}. \tag{22}$$

Let us apply the Stirling formula to approximate the factorials,

$$u_{M,N} \approx M^m N^n T^{-\nu}\exp[-\sqrt{MN}R(\mathrm{x},\mathrm{y})], \tag{23}$$

where $\mathrm{x} = m/M$, $\mathrm{y} = n/N$,

$$R(\mathrm{x},\mathrm{y}) = \mu[(1-\mathrm{x})\ln(1-\mathrm{x})+\mathrm{y}] + \nu[(1-\mathrm{y})\ln(1-\mathrm{y})+\mathrm{y}] - \frac{\mathrm{xy}t}{\tau},$$

and $\mu = \sqrt{M/N}$, $\nu = \sqrt{N/M}$, and $\tau = T/\sqrt{MN}$.

Differentiating $u_{M,N}$ over x and y we discover that $u_{M,N}$ has a sharp maximum at $\mathrm{x} = \mathrm{x}_c(t)$, and $\mathrm{y} = \mathrm{y}_c(t)$ with $\mathrm{x}_c(t)$, $\mathrm{y}_c(t)$ being the roots of the set of two equations,

$$t = \frac{T}{M\mathrm{x}_c}\ln\frac{1}{(1-\mathrm{y}_c)}, \quad t = \frac{T}{N\mathrm{y}_c}\ln\frac{1}{(1-\mathrm{x}_c)}. \tag{24}$$

A nonzero solution to this set exists at $t > t_c = T/\sqrt{MN}$.

One recognizes the equation for the composition $[\mathrm{x}_c(t)\cdot \mathrm{y}_c(t)]$ of the giant component in the bipartite graph [18]. The x dependence of u_M is presented in Fig. 2 for the pre- and postcritical periods. Below the transition point $t < t_c$ the factor $u_{M,N}$ does not have a hump.

The second term in Eq. (21),

$$u_{\infty} = \frac{1}{m!n!}e^{-(mN+nM)t/T}(Te^{t/T}-1)^{\nu}C_{m,n;\,\nu}, \tag{25}$$

is independent of M and N in the limit of large $T\to M,N$. Indeed, in the limit of large T the multiplier $(Te^{t/T}-1)^{\nu}$ converts to t^{ν}. The ratios M/T and N/T also become of order of unity.

IV. GENERATING FUNCTIONS

Let us introduce the polynomials $P_{m-1,n-1}(z)$ (see Ref. [24]) as the generating functions for the numbers $C_{m,n;\nu}$ of linked labeled bipartite graphs of order (m,n),

$$z^{m+n-1}P_{m-1,n-1}(z)=\sum_{\nu=m+n-1}^{mn}C_{m,n;\nu}z^{\nu}. \qquad (26)$$

The limits of summation in this equation are just a minimal number of edges in a tree of order m,n (lower limit) and the maximal number of edges mn in the complete graph (upper limit). Next, we introduce the bivariate exponential generating function for the number of linked bipartite components,

$$\begin{aligned} w(\xi,\eta;z)&=\sum_{m,n=1}^{\infty}\frac{\xi^m\eta^n}{m!n!}\sum_{\nu=m+n-1}^{mn}C_{m,n;\nu}z^{\nu}\\ &=\sum_{m,n=1}^{\infty}\frac{\xi^m\eta^n}{m!n!}z^{m+n-1}P_{m-1.n-1}(z). \end{aligned} \qquad (27)$$

According to the Riddell theorem [23] we have

$$w(\xi,\eta;z)=\ln\mathcal{W}(\xi,\eta;z), \qquad (28)$$

with $\mathcal{W}(\xi,\eta;z)$ being the exponential generating function of all labeled bipartite graphs. The latter is readily found. Indeed, the number of ways to connect m,n vertexes by ν edges is $\binom{mn}{\nu}$. Hence, the polynomial $(1+z)^{mn}$ generates the numbers of graphs of order m,n having exactly ν edges. The exponential generating function for these graphs is thus

$$\mathcal{W}(\xi,\eta;z)=\sum_{m,n=0}^{\infty}\frac{\xi^m\eta^n}{m!n!}(1+z)^{mn}. \qquad (29)$$

Here the summation goes over all nonnegative integers m,n. Hence,

$$\ln\sum_{m,n=0}^{\infty}\frac{\xi^m\eta^n}{m!n!}(1+z)^{mn}=\sum_{m,n=1}^{\infty}\frac{\xi^m\eta^n}{m!n!}z^{m+n-1}P_{m-1,n-1}(z). \qquad (30)$$

Equation (28) yields

$$\mathcal{W}\partial_\xi w=\partial_\xi\mathcal{W}\quad\text{and}\quad\mathcal{W}\partial_\eta w=\partial_\eta\mathcal{W}. \qquad (31)$$

Next, from Eq. (29) one finds that

$$\begin{aligned} \partial_\xi\mathcal{W}(\xi,\eta;z)&=\mathcal{W}(\xi,(1+z)\eta;z),\\ \partial_\eta\mathcal{W}(\xi,\eta;z)&=\mathcal{W}((1+z)\xi,\eta;z). \end{aligned} \qquad (32)$$

On combining Eqs. (28), (31), and (32) gives the set of equations for w,

$$\begin{aligned} \partial_\xi w(\xi,\eta,z)&=e^{w(\xi,(1+z)\eta;z)-w(\xi,\eta;z)},\\ \partial_\eta w(\xi,\eta,z)&=e^{w((1+z)\xi,\eta;z)-w(\xi,\eta;z)}. \end{aligned} \qquad (33)$$

Let us introduce the generation functions $X(\xi,\eta;z)$ and $Y(\xi,\eta;z)$ for the polynomials $P_{m+1,n}(z)$ and $P_{m,n+1}(z)$ playing the same role as the polynomials $P_m(z)$ for unipartite graphs considered in Ref. [17]:

$$\begin{aligned} X(\xi,\eta;z)&=\sum_{m,n=1}^{\infty}\frac{\xi^m\eta^n}{m!n!}P_{m.n-1}(z),\\ Y(\xi,\eta;z)&=\sum_{m,n=1}^{\infty}\frac{\xi^m\eta^n}{m!n!}P_{m-1.n}(z). \end{aligned} \qquad (34)$$

It is easy to see from Eq. (27) that

$$\begin{aligned} \partial_\xi w(\xi,\eta,z)&=X(z\xi,z\eta:z),\\ \partial_\eta w(\xi,\eta,z)&=Y(z\xi,z\eta:z). \end{aligned} \qquad (35)$$

Then

$$\begin{aligned} \ln X(\xi z,\eta z;z)&=w[\xi,(1+z)\eta;z]-w(\xi,\eta;z),\\ \ln Y(\xi z,\eta z;z)&=w(1+z)\xi,\eta;z]-w(\xi,\eta;z). \end{aligned} \qquad (36)$$

On replacing the variables $\zeta=\xi z$, $\theta=\eta z$, $x=\xi(1+uz)/z$, and $x=\theta(1+uz)/z$ yields

$$\begin{aligned} \ln X(\xi,\eta;z)&=\eta\int_0^1 Y[\xi,(1+uz)\eta;z]\,du,\\ \ln Y(\xi,\eta;z)&=\xi\int_0^1 X[(1+uz)\xi,\eta;z]\,du. \end{aligned} \qquad (37)$$

The functions $X(\xi,\eta;z)$ and $Y(\xi,\eta;z)$ generate polynomials $P_{m+1,n}(z)$ and $P_{m,n+1}(z)$, respectively [see Eq. (34)].

V. CYCLED COMPONENTS

In what follows we will omit the arguments of X and Y, e.g., X stands for $X(\xi,\eta;z)$.

The linked components of order (m,n) having a minimal possible number of edges $\nu=m+n-1$ are trees, i.e., they have no cycles. According to Eq. (26) the number of linked components having exactly k cycles is

$$C_{m,n;m+n+k-1}=\frac{1}{k!}\left.\frac{d^kP_{m-1,n-1}}{dz^k}\right|_{z=0}. \qquad (38)$$

Thus the number $U_k(g,t)$ of linked components with k cycles at the moment t in the graph is [see Eqs. (21) and (26)],

$$\begin{aligned} U_k(m,n,t)&=\bar{n}_{m,n;m+n+k}(t)\\ &=\binom{M}{m}\binom{N}{n}e^{(mn-Mn-Nm)t/2T}(e^{t/T}-1)^{m+n+k-1}\\ &\quad\times\frac{1}{k!}\partial_\xi^m\partial_\eta^n\left.\frac{\partial^k w(\xi,\eta;z)}{\partial z^k}\right|_{z=0}. \end{aligned} \qquad (39)$$

A. Trees

The number of trees (linked components without cycles, $k=0$) is thus expressed through $P_{m,n}(0)$. Let us introduce the special notation,

$$x=X(\xi,\eta;0)=X^o\quad\text{and}\quad y=Y(\xi,\eta;0)=Y^o. \qquad (40)$$

Here and below the superscript o marks the functions at $z = 0$. Of course, x and y remain the functions of ξ and η.

At $z = 0$ Eq. (37) reduces to the set of two transcendent equations,

$$\ln x = \eta y, \quad \ln y = \xi x. \tag{41}$$

Then $P_{m,n}(0)$ is found from the following chain of equalities:

$$\begin{aligned} P_{m,n}(0) &= m!(n+1)!\mathrm{Coef}_{\xi,\eta}\frac{x}{\xi^{m+1}\eta^{n+2}} \\ &= m!(n+1)!\mathrm{Coef}_{u,v}\frac{e^{u(m+2)+v(n+2)}}{v^{m+1}u^{n+2}}e^{-(u+v)}(1-uv) \end{aligned} \tag{42}$$

where

$$u = \ln x \quad \text{and} \quad v = \ln y \tag{43}$$

and the operation Coef is defined as follows (see Ref. [25]):

$$\mathrm{Coef}_{u,v}\sum a_{k,l}u^k v^l = a_{-1,-1}. \tag{44}$$

The multiplier $e^{-(u+v)}(1-uv)$ in the last term is just the Jacobian appearing in replacing the variables $\xi, \eta \to u, v$. Rather simple algebra finally yields

$$P_{m,n}(0) = (m+1)^n(n+1)^m. \tag{45}$$

The number of trees is thus

$$\begin{aligned} U_0(m,n,t) &= \bar{n}_{m,n;m+n-1}(t) \\ &= \binom{M}{m}\binom{N}{n}e^{(mn-Mn-Nm)t/2T} \\ &\quad \times (e^{t/T}-1)^{m+n-1}m^{n-1}n^{m-1}. \end{aligned} \tag{46}$$

B. Unicyclic components

In order to find the spectrum of unicyclic component we should find the first derivative of polynomials $P_{m,n}(z)$ over z. To this end we introduced the notation,

$$\begin{aligned} Q &= \eta\int_0^1 Y(\xi, \eta(1+uz); z)\, du, \\ R &= \xi\int_0^1 X(\xi(1+uz), \eta; z)\, du, \end{aligned} \tag{47}$$

and differentiate Eq. (37) over z:

$$X_z = Q_z X, \quad Y_z = R_z Y \tag{48}$$

Let then expand $X(\xi(1+uz), \eta; z)$ and $Y(\xi, \eta(1+uz); z)$ in the integrands over uz:

$$\begin{aligned} X(\xi(1+uz), \eta; z) &= X + X_\xi \xi z u + X_{\xi\xi}\frac{\xi^2 z^2 u^2}{2!} + \cdots, \\ Y(\xi, \eta(1+uz); z) &= Y + Y_\eta \eta z u + Y_{\eta\eta}\frac{\eta^2 z^2 u^2}{2!} + \cdots. \end{aligned} \tag{49}$$

We also expand Q and R. On substituting Eq. (49) into Eq. (47) and integrating over u yield

$$\begin{aligned} Q &= \eta Y + Y_\eta\frac{\eta^2 z}{2!} + Y_{\eta\eta}\frac{\eta^3 z^2}{3!} + \ldots, \\ R &= \xi X + X_\xi\frac{\xi^2 z}{2!} + X_{\xi\xi}\frac{\xi^3 z^2}{3!} + \ldots. \end{aligned} \tag{50}$$

Their derivatives at $z = 0$ are then readily found:

$$\begin{aligned} Q_z^o &= \eta Y_z^o + Y_\eta^o\frac{\eta^2}{2!}, \\ R_z^o &= \xi X_z^o + X_\xi^o\frac{\xi^2}{2!}. \end{aligned} \tag{51}$$

Combining Eqs. (48) and (51) leads to the set of linear equations for X_z^o and Y_z^o:

$$\begin{aligned} X_z^o - x\eta Y_z^o &= y_\eta(\eta^2/2!)x, \\ -y\xi X_z^o + Y_z^o &= x_\xi(\xi^2/2!)y. \end{aligned} \tag{52}$$

Differentiating Eq. (41) over ξ and η yields

$$\begin{aligned} x_\xi &= \frac{\eta x^2 y}{\Delta}, \quad x_\eta = \frac{xy}{\Delta}, \\ y_\xi &= \frac{xy}{\Delta}, \quad y_\eta = \frac{\xi x y^2}{\Delta}, \end{aligned} \tag{53}$$

where

$$\Delta = 1 - \ln x \ln y.$$

On substituting this into Eq. (52) we finally get

$$\begin{aligned} X_z^o &= \frac{\eta^2\xi x^2 y^2}{\Delta^2}(1+\xi x), \\ Y_z^o &= \frac{\xi^2\eta x^2 y^2}{\Delta^2}(1+\eta y). \end{aligned} \tag{54}$$

C. *k*-cyclic components

Full expansions of X and Y over uz look as follows:

$$\begin{aligned} X(\xi(1+uz), \eta; z) &= \sum_{s=0}^{\infty}\frac{(uz)^s\xi^s}{s!}\partial_\xi^s X, \\ Y(\xi, \eta(1+uz); z) &= \sum_{s=0}^{\infty}\frac{(uz)^s\eta^s}{s!}\partial_\eta^s Y. \end{aligned} \tag{55}$$

On substituting this into Eq. (47) and integrating over u yields

$$\begin{aligned} Q(\xi, \eta; z) &= \sum_{s=0}^{\infty}\frac{\eta^{s+1}z^s}{(s+1)!}\partial_\eta^s Y, \\ R(\xi, \eta; z) &= \sum_{s=0}^{\infty}\frac{\xi^{s+1}z^s}{(s+1)!}\partial_\xi^s X. \end{aligned} \tag{56}$$

Let us differentiate l times over z both sides of Eq. (48). The result is

$$\begin{aligned} \partial_z^{l+1}X &= \sum_{k=0}^{l}\binom{l}{k}\partial_z^{k+1}Q\,\partial_z^{l-k}X, \\ \partial_z^{l+1}Y &= \sum_{k=0}^{l}\binom{l}{k}\partial_z^{k+1}R\,\partial_z^{l-k}Y. \end{aligned} \tag{57}$$

Equation (56) yields the $k+1$-th derivatives of Q and R,

$$\partial_z^{k+1} Q = \sum_{s=0}^{\infty}\sum_{p=0}^{k+1}\binom{k+1}{p}\frac{\eta^{s+1}}{(s+1)!}\left(\partial_z^p z^s\right)\partial_z^{k-p+1}\partial_\eta^s Y,$$

$$\partial_z^{k+1} R = \sum_{s=0}^{\infty}\sum_{p=0}^{k+1}\binom{k+1}{p}\frac{\xi^{s+1}}{(s+1)!}\left(\partial_z^p z^s\right)\partial_z^{k-p+1}\partial_\xi^s X. \quad (58)$$

Now we put $z=0$ in this set and obtain

$$\left(\partial_z^{k+1}Q\right)^o = \sum_{p=0}^{k+1}\binom{k+1}{p}\frac{\eta^{p+1}}{(p+1)}\left(\partial_z^{k-p+1}\partial_\eta^p Y\right)^o,$$

$$\left(\partial_z^{k+1}R\right)^o = \sum_{p=0}^{k+1}\binom{k+1}{p}\frac{\xi^{p+1}}{(p+1)}\left(\partial_z^{k-p+1}\partial_\xi^p X\right)^o. \quad (59)$$

On substituting this result into Eq. (59), and separating the terms with $p=0$ $k=l$ in the sums on the RHS of these equations, give the set of two equations for $(\partial_z^{(l+1)}X)^o$ and $(\partial_z^{(l+1)}Y)^o$,

$$\left(\partial_z^{l+1}X\right)^o - \eta x\left(\partial_z^{l+1}Y\right)^o = \Phi_l,$$
$$\left(\partial_z^{l+1}Y\right)^o - \xi y\left(\partial_z^{l+1}X\right)^o = \Psi_l, \quad (60)$$

where

$$\Phi_l = \sum_{k=0}^{l}\sum_{p=0}^{k+1} Z_l(k,p)\eta^{p+1}\left(\partial_z^{k-p+1}\partial_\eta^p Y\right)^o\left(\partial_z^{l-k}X\right)^o,$$

$$\Psi_l = \sum_{k=0}^{l}\sum_{p=0}^{k+1} Z_l(k,p)\xi^{p+1}\left(\partial_z^{k-p+1}\partial_\xi^p X\right)^o\left(\partial_z^{l-k}Y\right)^o, \quad (61)$$

and

$$Z_l(k,p) = \frac{1}{(p+1)}\binom{l}{k}\binom{k+1}{p}(1-\delta_{p,0}\delta_{k,l}). \quad (62)$$

The solution to this set is

$$\left(\partial_z^{l+1}X\right)^o = \frac{\eta x\Psi_l + \Phi_l}{\Delta}, \quad \left(\partial_z^{l+1}Y\right)^o = \frac{\xi y\Phi_l + \Psi_l}{\Delta}. \quad (63)$$

The initial functions $\partial_z X^o$ and $\partial_z Y^o$ are given by Eq. (54).

D. Leading approximation

We look for the solution to recurrences (60) in the form

$$\frac{1}{k!}\left(\partial_z^{(k)}X\right)^o = a_k(\eta\xi x^2y^2)^k/\Delta^{3k-1},$$
$$\frac{1}{k!}\left(\partial_z^{(k)}Y\right)^o = b_k(\eta\xi x^2y^2)^k/\Delta^{3k-1}. \quad (64)$$

According to Eq. (54) we have

$$a_1 = \eta(1+\xi x), \quad b_1 = \xi(1+\eta y). \quad (65)$$

Let us substitute Eq. (65) into Eq. (61) and neglect all terms containing $1/\Delta^K$ with $K < 3l-2$: i.e., we retain the most singular terms. The proportionality coefficients a_k and b_k are then determined from the set of recurrences,

$$a_{l+1} - \eta x b_{l+1} = \phi_l,$$
$$b_{l+1} + \xi y a_{l+1} = \psi_l, \quad (66)$$

where

$$\phi_l = \frac{\xi}{l+1}\sum_{k=0}^{l-1}(k+1)b_{k+1}a_{l-k} + \frac{(3l-1)}{2}\xi^2 b_l,$$

$$\psi_l = \frac{\eta}{l+1}\sum_{k=0}^{l-1}(k+1)a_{k+1}b_{l-k} + \frac{(3l-1)}{2}\eta^2 a_l. \quad (67)$$

VI. BACK TO SPECTRA

The solution to the set (67) is independent of Δ and is given by the sum of products $\xi^\alpha\eta\beta x^\gamma y^\delta$. We therefore begin by calculating the partial contributions of a single term,

$$H = \mathrm{Coef}_{\xi,\eta}\frac{\xi^\alpha\eta^\beta x^\gamma y^\delta}{\Delta^{3k-1}\xi^{m+1}\eta^{n+2}}, \quad (68)$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ are nonnegative integers. In order to evaluate H we introduce the variables

$$x = e^u, \quad y = e^v, \quad \xi = ve^{-u}, \quad \eta = ue^{-v}. \quad (69)$$

Taking into account that the Jacobian of this transformation $J = \partial(\xi,\eta)/\partial(u,v) = e^{-u-v}(1-uv)$ we have

$$H = \mathrm{Coef}_{u,v}\frac{e^{(m+c)u+(n+d)v}}{(1-uv)^{3k-2}u^{n+a+1}v^{m+b+1}} \quad (70)$$

with

$$a = 1-\beta,\ b = -\alpha,\ c = \gamma-\alpha,\ d = \delta-\beta+1. \quad (71)$$

Let us expand the factor $(1-uv)^{3k-2}$ in powers of uv,

$$\frac{1}{(1-uv)^{3k-2}} = \sum_{r=0}^{\infty}\binom{3k+r-3}{r}(uv)^r. \quad (72)$$

Finally we have

$$H = \frac{(m+c)^{m+c}(n+b)^{n+b}}{(m+c)!(n+b)!}\mathcal{Z}, \quad (73)$$

where

$$\mathcal{Z} = \sum_{r=0}^{r_m}\binom{3k+r-3}{r} \times \frac{(m+a)!(n+b)!(m+c)^{-r}(n+d)^{-r}}{(m+a-r)!(n+b-r)!}. \quad (74)$$

Here $r_m = \min(m+c, n+d)$.

A. Unicyclic graphs

From Eqs. (39) and (70) we have

$$\partial_z P_{m,n}(0) = m!(n+1)!\mathrm{Coef}_{\xi,\eta}\frac{\partial_z X(\xi,\eta;0)}{\xi^{m+1}\eta^{n+2}}$$
$$= m!(n+1)!(R+S), \quad (75)$$

where

$$R = \mathrm{Coef}_{u,v}\frac{e^{u(m+1)+v(n+1)}}{v^m u^n(1-uv)} \quad (76)$$

and

$$S = \mathrm{Coef}_{u,v}\frac{e^{u(m+1)+v(n+1)}}{v^m u^n(1-uv)}v. \quad (77)$$

We expand $1/(1-uv)$ in powers of uv and obtain

$$R=\sum_{r\leqslant r_{mn}}\frac{(m+1)^{n-r-1}(n+1)^{m-r-1}}{(m-r-1)!(n-r-1)!},\tag{78}$$

where $r_{mn}=\min(m-1,n-1)$ and

$$S=\sum_{r\leqslant r_{mn}}\frac{(m+1)^{n-r-1}(n+1)^{m-r-2}}{(n-r-1)!(m-r-2)!}.\tag{79}$$

After a simple but tedious algebra we have

$$\partial_z P_{m,n}(0)=m!n!\sum_{r\leqslant r_{mn}}\frac{(m+1)^{n-r-1}(n+1)^{m-r-1}}{(m-r-1)!(n-r-1)!}(m+n-r+1).\tag{80}$$

The RHS can be expressed through the truncated McDonald's function

$$I_\mu^G(x)=\sum_{k=0}^{G}\frac{(x/2)^{2k+\mu}}{k!(k+\mu)!}.\tag{81}$$

Indeed, at $m\leqslant n$ we replace $s=m-r-1$ and find

$$\partial_z P_{m,n}(0)=m!n!\sum_{s=0}^{m-1}\frac{(m+1)^{n-m+s}(n+1)^{s}}{s!(n-m+s)!}(n+s+2).\tag{82}$$

B. Asymptotic analysis

Two approximations,

$$\begin{aligned}\frac{m!}{(m-r-1)!}&=(m-r)(m-r-1)\cdots m\\&=m^{r+1}\left(1-\frac{r}{m}\right)\cdots\left(1-\frac{1}{m}\right)\\&\approx m^{r+1}\exp\left(-\sum_{s=1}^{r}\frac{s}{m}\right)\\&\approx m^{r+1}\exp(-r^2/2m)\end{aligned}\tag{83}$$

and

$$\binom{3k+r-3}{r}\approx\frac{r^{3k-3}}{(3k-3)!},\tag{84}$$

apply in our further asymptotic analysis. Equation (83) can be cast into the form

$$\begin{aligned}\partial_z P_{m,n}(0)&=(m+n)m^{(n-1))}n^{(m-1)},\\&\quad\times\int_0^\infty \exp[-(r^2/2m)+(r^2/2n)]\,dr\\&=\frac{1}{2}(m+n)m^{(n-1))}n^{(m-1)}\sqrt{\frac{\pi mn}{2(m+n)}}.\end{aligned}\tag{85}$$

Now we calculate $\mathcal{Z}$ [see Eqs. (73) and (74)]

$$\begin{aligned}\mathcal{Z}&=\frac{1}{(3k-3)!}\int_0^\infty r^k\exp[-(r^2/2m)+(r^2/2n)]\,dr\\&=\frac{1}{(3k-3)!}\left[\sqrt{\frac{\pi mn}{2(m+n)}}\right]^{k+1}\Gamma(k/2).\end{aligned}\tag{86}$$

It is important to note that the asymptotic expression for $\mathcal{Z}$ is independent of the parameters a, b, c, d [see Eq. (77)]. This fact allows us to write the asymptotic form of the spectrum of k-cyclic biparite graphs not even solving the set (68).

VII. RESULTS AND DISCUSSION

In this paper the kinetic approach of Ref. [18] applied for the analysis of the time evolution of initially empty bipartite random graph. Assuming that the M, N graph evolves by randomly adding edges one at a time, the expression for the average order-complexity spectrum was shown to factorize into two multipliers $\bar{n}_{M,N}=u_{M,N}u_\infty$, the first of which ($u_{M,N}$) describes the effects related to the finiteness of the graph, whereas the second one (u_∞) coincides with that found in the thermodynamic limit [24]. It is remarkable that both multipliers carry the information on the emergence of the giant component at the critical time $t=t_c=1/\sqrt{MN}$. The first multiplier $u_{M,N}$ has a sharp maximum at the critical time $t=t_c=1/\sqrt{MN}$. The second one has an algebraic form $c(m,n,t_c)\propto(mn)^{-3/2}$ with the second moment of u_∞ diverging as $1/(t_c-t)$ [24]. However, their singular behavior is suppressed with the finiteness of the graph. The algebraic part of the spectrum is modulated by an exponential factor depending now on M and N

The number of k-cycled graphs has been expressed in terms of k-derivatives of the polynomials $P_{m,n}(z)$ over z which are, in turn, serve as the generating functions for the number of bipartite graphs having exactly m and n vertexes and ν edges. The set of nonlinear integral equations (39) for the generating functions for the polynomials $P_{m,n}(z)$ has been derived by applying the Riddell theorem.

The main results of this article can be formulated as follows:

(1) It has been shown that in sufficiently large graphs the average order-complexity spectrum [Eq. (19)] factorizes into two multipliers $\bar{n}_{m,n;\nu}=$ one of which depends on the total order of graph while the second does not. It depends only on the order-complexity of its irreducible components $m, n; \nu$. This factorization repeats my result [17] found for ordinary graphs comprising one sort of vertexes. The multiplier $u_{M,N}$ has a maximum in the plane m, n the position of which defines the phase transition point $t=t_c$ and the time dependence of the composition of giant components [see Eq. (24) and Fig. 2]. The analytical expressions for the both multipliers are given by Eqs. (23) and (25).

(2) Equations (19) and (23) show that the k-cycled order-complexity spectrum contains the multiplier $T^{-(k-1)}$. The spectrum of trees is thus proportional to T, the spectrum of unicyclic graphs is proportional to 1. More complex graphs contain negative powers of T, which means that in the limit of large T the trees and the unicyclic components contribute to the time evolution process of the graph. This fact gives the answer to the questions, what is going on at the transition point and what is the gel. At first sight, the kinetics of coalescing trees should be described by the Smoluchowskii equation. It is not so in the presence of cyclization: the transition of trees by adding one extra edge that goes with the rate $T^{-1}(m-1)(l-1)\bar{n}_{l,m}$ [see Eq. (14)], where $\nu=m+n-1$ in trees) considerably modifies the spectrum of trees. The

point is that the cyclization is the first-order process (linear in the concentration of graphs) that does not contribute to the kinetics compared to the second-order process of coalescence of graphs. The ratio of the rates of these processes is $1/T \ll 1$ at $m, n \propto 1$. At large m, n the rates of cyclization and coalescence become comparable, and the giant component emerges. The number concentration C of trees then obeys the equation [Eq. (55) of Ref. [24]] $\dot{C} \propto (1 - \mathrm{x}_c\mathrm{y}_c)$.

(3) The average mass spectrum has been expressed in terms of the polynomials $P_{m,n}$ introduced in Ref. [24]. Here these polynomials have been introduced as the generating functions for the number of irreducible components of the graph with fixed degree of filling ν. This step allows one to apply the Riddell [23] theorem for deriving the set of integral equations (37).

(4) The general expression for the number of k–cycled components derived in Sec. V [Eq. (39)] has been used for deriving the exact expression for the spectrum of the unicyclic components [Eq. (46)].

(5) The asymptotic behavior of k-cycled components has been analyzed in Sec. VI. It has been shown that their spectra have the form $c_{m,n}^{(k)} \to [\pi mn/(m+n)]^k$ [Eq. (86)].

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APPENDIX

We construct the solution to Eq. (13) in the form

$$\Psi = M!N!\mathrm{Coef}_{u,v}\left\{u^{-M}v^{-N} \times \exp\left[\sum_{m,n;\nu} x_{m,n;\nu}a_{m,n;\nu}(t)u^m v^n\right]\right\}, \quad \text{(A1)}$$

where the coefficients $a_{m,n;\nu}(t)$ will be defined later [see Eq. (A5)]. Here the operation Coef (see Ref. [25]) is introduced as follows:

$$\mathrm{Coef}_{u,v}\left[\sum_{m,n} b_{m,n}u^m v^n\right] = b_{-1,-1}. \quad \text{(A2)}$$

The spectrum of linked components of order m, n with ν edges $\bar{n}_{m,n;\nu}(t)$ can be expressed in terms of $a_{m,n;\nu}(t)$ as follows:

$$\bar{n}_{m,n;\nu}(t) = \left.\frac{\partial\Psi(\{x_{m,n;\nu}\}, t)}{\partial x_{m,n;\nu}}\right|_{\{x_{m,n;\nu}\}=\{1\}} = M!N!a_{m,n;\nu}(t)\mathrm{Coef}_{u,v}\{u^{-M+m}v^{-N+n} \times \exp[G(u, v; 1|t)]\}. \quad \text{(A3)}$$

Here

$$G(u, v; \zeta|t) = \sum_{m,n;\nu} a_{m,n;\nu}(t)u^m v^n \zeta^\nu \quad \text{(A4)}$$

is the generating function for $a_{m,n;\nu}(t)$.

On substituting Ψ in the form (A1) into Eq. (13) with $\hat{\mathcal{L}}_f$ and $\hat{\mathcal{L}}_c$ defined by Eqs. (14) and (15) yields the set of equations for $a_{m,n;\nu}(t)$,

$$T\frac{da_{m,n;\nu}}{dt} = \sum_{k,l=0}^{m.n}\sum_{\mu=0}^{\nu-1}(m-l)ka_{m-l,n-k;\nu-\mu-1}(t)a_{l,k;\mu}(t) + mna_{m,n;\nu}(t),$$

$$-\frac{1}{2}(Mn+Nm)a_{m,n;\nu}(t) + (mn-\nu+1)a_{m,n;\nu-1} - (mn-\nu)a_{m,n;\nu}. \quad \text{(A5)}$$

This set is subject to the condition corresponding to the initially empty graph,

$$a_{m,n;\nu}(0) = (\delta_{m,1}\delta_{n,0} + \delta_{m,0}\delta_{n,1})\delta_{\nu,0}. \quad \text{(A6)}$$

It is easy to check that the condition (A6) corresponds to $\Psi|_{t=0} = x_{1,0;0}^M x_{0,1;0}^N$.

The equation for the generating function G can be readily derived from Eqs. (A4) and (A5),

$$T\frac{\partial G}{\partial t} = \zeta\left[u\frac{\partial G}{\partial u}v\frac{\partial G}{\partial v} + uv\frac{\partial^2 G}{\partial u\partial v}\right] - (\zeta-1)\zeta\frac{\partial G}{\partial\zeta} - \frac{1}{2}\left(Nu\frac{\partial G}{\partial u} + Mv\frac{\partial G}{\partial v}\right). \quad \text{(A7)}$$

The initial condition for this equation is

$$G(u, v; \zeta|0) = u + v. \quad \text{(A8)}$$

Now let

$$D(u, v; \zeta|t) = \exp[G(ue^{Nt/2T}, ve^{Mt/2T}; \zeta|t)]. \quad \text{(A9)}$$

Then, instead of Eq. (A7), we derive a linear equation for D,

$$T\frac{\partial D}{\partial t} = \zeta uv\frac{\partial^2 D}{\partial u\partial v} - (\zeta-1)\zeta\frac{\partial D}{\partial\zeta}. \quad \text{(A10)}$$

The initial condition for this equation follows from Eq. (A8),

$$D(u, v; \zeta|0) = e^{u+v}. \quad \text{(A11)}$$

Equation (A10) is readily solved by separating variables. Let

$$D(u, v; \zeta|t) = \sum_{m,n,\kappa}\Theta_\kappa(t,)Z_{m,n,\kappa}(\zeta)u^m v^n.$$

Then $\Theta_\kappa(t) = e^{\kappa t/T}$ and

$$\kappa Z_{m,n;\kappa} = \zeta mnZ_{m,n;\kappa} + \zeta(1-\zeta)\frac{dZ_{m,n;\kappa}}{d\zeta}, \quad \text{(A12)}$$

where κ is a separation constant. The solution to this equation is

$$Z_{m,n;\kappa}(\zeta) = b_{m,n;\kappa}(1-\zeta)^{mn-\kappa}\zeta^\kappa. \quad \text{(A13)}$$

The function D should be analytical at $\zeta = 0$. Hence, $\kappa = s$, where s is a nonnegative integer. Next, the coefficients $b_{m,n;\kappa}$ should be chosen from the initial condition (A8). It is easy to see that

$$b_{m,n;s} = \frac{1}{m!n!}\binom{mn}{s}.$$

We then come to the result

$$D(u,v;\zeta|t)=\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}(1-\zeta+\zeta e^{t/T})^{mn}. \quad \text{(A14)}$$

In order to return to $a_{g,\nu}(t)$ we use Eq. (38),

$$\ln\sum_{m,n}\frac{u^m v^n}{m!n!}(1+\delta)^{mn}=\sum_{m,n=1}^{\infty}\frac{u^m v^n}{m!n!}\delta^{m+n-1}P_{m-1,n-1}(\delta), \quad \text{(A15)}$$

with $\delta=\zeta(e^t-1)$. Then Eqs. (A11) and (A15) allow us to restore

$$A_{m,n}(\zeta,t)=\sum_{\nu}a_{m,n;\nu}(t)\zeta^{\nu}.$$

We find

$$A_{m,n}(\zeta,t)=\frac{1}{m!n!}e^{-(mN+nM)t/2T)}\times(e^{t/T}-1)^{m+n-1}\zeta^{m+n-1}P_{m-1,n-1}[\zeta(e^{t/T}-1)], \quad \text{(A16)}$$

where the polynomial $P_{m,n}(\delta)$ is defined as the generating function for the numbers $C_{m,n;\nu}$ of a linked labeled bipartite graph of order m,n,

$$\delta^{m+n-1}P_{m-1,n-1}(\delta)=\sum_{\nu=m+n-1}^{mn}C_{m,n;\nu}\delta^{\nu}. \quad \text{(A17)}$$

Equation (A17) is readily applied for restoring $a_{m,n;\nu}(t)$. The result is

$$a_{m,n;\nu}(t)=\frac{1}{m!n!}e^{-(mN+nM)t/2T)}(e^{t/T}-1)^{\nu}C_{m,n;\nu}. \quad \text{(A18)}$$

Now we are ready to find the average number of linked components of order m,n with ν edges. From Eq. (A3) we have

$$\bar{n}_{m,n;\nu}(t)=M!N!a_{m,n;\nu}(t)\mathrm{Coef}_{u,v}u^{-M+m}v^{-N+n}D(u,v;1|t). \quad \text{(A19)}$$

At $\zeta=1$ Eq. (A14) gives

$$D(u,v;1|t)=\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}e^{mnt/T}. \quad \text{(A20)}$$

Hence,

$$\mathrm{Coef}_{\mathrm{u,v}}u^{-M+m}v^{-N+n}D(u,v;1|t)=\frac{\exp(-m\mathcal{N}t/2-n\mathcal{M}t/2+mnt/T)}{(M-m)!(N-n)!}. \quad \text{(A21)}$$

We thus come to the result

$$\bar{n}_{m,n;\nu}(t)=\binom{M}{m}\binom{N}{n}e^{(mn-mN-nM)t/T}(e^{t/T}-1)^{\nu}C_{m,n;\nu}. \quad \text{(A22)}$$

Because the function $D(u,v,1|t)$ coincides with the D–function corresponding to the spectrum of coagulating particles in the system with the kernel $K(m,n;k,l)=mk+nl$ considered in Ref. [19], we can easily derive the expression for the spectrum of linked components over their orders (numbers of vertices), $\bar{n}_{m,n}=\sum_{\nu}\bar{n}_{m,n;\nu}$. The result has the same form as the average particle mass spectrum in the coagulating system with the coagulation kernel $K(g,l)=mk+nl$,

$$\bar{n}_{m,n}(t)=\binom{M}{m}\binom{N}{n}e^{(mn-mN-nM)t/T}\times(e^{t/T}-1)^{m+n-1}P_{m,n}(e^{t/T}-1). \quad \text{(A23)}$$

[1] F. Harari, *Graph Theory* (Addison-Wesley, Reading. MA, 1969).
[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications* (Macmillan, New York, 1978).
[3] R. Diestel, *Graph Theory* (Springer-Verlag, New York, 1997).
[4] D. B. West, *Introduction to Graph Theory* (Prentice-Hall, New York, 1996).
[5] R. J. Wilson, *Introduction to Graph Theory*, 3rd ed. (Longman, New York, 1985).
[6] P. Erdős and A. Rényi, Pub. Math. (Debrecen) **6**, 250 (1959).
[7] B. Bollobás, *Random Graphs*, 2nd ed. (Cambridge University Press, New York, 2001).
[8] M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, Oxford, 2010).
[9] R. Cohen and S. Havlin, *Complex Networks: Structure, Robustness and Function* (Cambridge University Press, Cambridge, 2010).
[10] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 2018).
[11] A. H. Marcus, Technometrics **10**, 133 (1968).
[12] D. J. Aldous, Bernoulli **5**, 3 (1999).
[13] A. A. Lushnikov, J. Colloid Interface Sci. **65**, 276 (1978).
[14] J. L. Spouge, Can. J. Chem. **62**, 1262 (1984).
[15] J. L. Spouge, J. Stat. Phys. **38**, 573 (1985).
[16] F. Leyvraz, Phys. Rep. **383**, 95 (2003).
[17] A. A. Lushnikov, Phys. Rev. E **98**, 042135 (2018).
[18] A. A. Lushnikov, J. Phys. A **38**, L777 (2005).
[19] A. A. Lushnikov, Physica D **222**, 37 (2006).
[20] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, 2010).
[21] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **71**, 026129 (2005).
[22] E. Ben-Naim and P. L. Krapivsky, J. Phys. A **37**, L189 (2004).
[23] R. J. Riddell and G. E. Ulenbeck, J. Chem. Phys. **21**, 2056 (1953).
[24] A. A. Lushnikov, Phys. Rev. E **73**, 036111 (2006).
[25] G. P. Egorychev, *Integral Representation and Counting the Combinatorial Sums* (in Russian) (Nauka, Moscow, 1977).