Boltzmann-type collision operators for Bogoliubov excitations of Bose-Einstein condensates: A unified framework

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Starting from the Bogoliubov diagonalization for the Hamiltonian of a weakly interacting Bose gas under the presence of a Bose-Einstein condensate, we derive the kinetic equation for the Bogoliubov excitations. Without dropping any of the commutators, we find three collisional processes. One of them describes the $1 \leftrightarrow 2$ interactions between the condensate and the excited atoms. The other two describe the $2 \leftrightarrow 2$ and $1 \leftrightarrow 3$ interactions between the excited atoms themselves.

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I. INTRODUCTION

Classical Boltzmann kinetic theory is the way to connect the macroscopic properties of gases of many particles to the fundamental interaction by collisions between those particles. Shortly after the establishment of quantum mechanics for bosons and fermions, Nordheim (cf. [1]) wrote the kinetic equations for dilute gases of quantum particles, which takes into account statistical effects linked to the possibility or not of overlaps of wave functions after a two-body interaction, relying on the assumption that the strength of the interaction is small. The resulting Boltzmann-Nordheim kinetic theory is correct in principle for describing dilute quantum gases. However, it has to be changed for Bose gases at low temperatures to include a condensate. As being shown long ago by Bogoliubov [2], the existence of this condensate changes the fundamental notion of what a particle is. Because of the interaction with the condensate, the notion of particles has to be modified by one of the quasiparticles, as guessed by Landau before. Even at equilibrium (see Ref. [3]) a fully coherent theory of quasiparticles is already a fairly nontrivial issue. Of course, it is even harder to derive a valid kinetic equation for quasiparticles. However, this kinetic theory is not too strongly changed if the difference between particles and quasiparticles is restricted to a relatively small population of those particles, which is the case if the kinetic energy per particle is still much larger than its interaction energy with the condensate. If the average kinetic energy per particle is of the order or less than the interaction with the condensate, we must take fully into account the Bogoliubov renormalization of the particle energy, that brings a lot of terms in the kinetic equation. A major issue then is to derive what was called by Gust and Reichl [4] the $1 \leftrightarrow 3$ interactions between the excited atoms, which is the main purpose of the present work.

In the works [5,6], Kirkpatrick and Dorfman started to tackle the complex problem of writing the kinetic equation for the gas of particles out of the condensate, coupled with those inside the condensate, something that began with Refs. [7,8]. In this work, the authors derived a mean-field kinetic equation for a dilute condensed Bose gas that describes the relaxation in terms of "collisions" between excitations. The work of Kirkpatrick and Dorfman was then extended by Zaremba *et al.* [9], where they introduced the full coupling system of a quantum Boltzmann equation for the density function of the normal fluid and thermal cloud and a Gross-Pitaevskii equation for the wave function of the Bose-Einstein condensate (BEC). Independently, the same model was also derived by Pomeau *et al.* [10], using the quantum Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy argument. In a series of papers [11–13], Gardiner and collaborators introduced a different model, which, at the limits, becomes the model of Zaremba *et al.* and Pomeau *et al.* We refer to [3,14] for further discussions on the topic. In all of these kinetic equations, there are two types of collisional processes:

(1) The C_{12} collision operator describes the $1 \leftrightarrow 2$ interactions between the condensate and the excited atoms.

(2) The C_{22} collision operator describes the $2 \leftrightarrow 2$ interactions between the excited atoms themselves.

In [4], Gust and Reichl proposed the third, previously missing, collisional process, which takes into account $1 \leftrightarrow 3$ type collisions between the excitations, in addition to the $1 \leftrightarrow 2$ and $2 \leftrightarrow 2$ type collisions already known to occur. They called it the collision operator C_{31}.

However, the derivation of the new collision operator C_{31} is very complicated, since the process generates around 40 000 individual terms and one will need to do a combinatorics problem for all of them. As a result, a concise mathematical justification for the existence of the missing collision operator C_{31} remains to be a challenging open problem over the years.

The aim of our work is to verify the validity of the collision operators C_{12}, C_{22}, C_{31} by a fairly simple framework. To this

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end, we focus only on the spatial homogeneous system. Our spatial homogeneous kinetic equation for the evolution of the density function $f(t, p)$ of the thermal cloud takes the form

$$\partial_t f(p) = C_{12}[f](p) + C_{22}[f](p) + C_{31}[f](p), \tag{1}$$

and the forms of C_{12}, C_{22}, C_{31} are given explicitly below:

$$\begin{aligned} C_{12}[f](t,p) = {} & 4\pi \frac{g^2 n}{V} \sum_{p_1,p_2,p_3 \neq 0} [\delta(p-p_1) - \delta(p-p_2) \\ & - \delta(p-p_3)]\delta[\omega(p_1) - \omega(p_2) - \omega(p_3)] \\ & \times \left(K^{1,2}_{1,2,3}\right)^2 \delta(p_1 - p_2 - p_3)\{f(p_2)f(p_3) \\ & \times [f(p_1)+1] - f(p_1)[f(p_2)+1] \\ & \times [f(p_3)+1]\}, \end{aligned} \tag{2}$$

$$\begin{aligned} C_{22}[f](t,p) = {} & \frac{g^2\pi}{V^2} \sum_{p_1,p_2,p_3,p_4 \neq 0} [\delta(p-p_1) + \delta(p-p_2) \\ & - \delta(p-p_3) - \delta(p-p_4)]\left(K^{2,2}_{1,2,3,4}\right)^2 \\ & \times \delta(p_1 + p_2 - p_3 - p_4)\delta[\omega(p_1) + \omega(p_2) \\ & - \omega(p_3) - \omega(p_4)]\{f(p_3)f(p_4)[f(p_2)+1] \\ & \times [f(p_1)+1] - f(p_1)f(p_2)[f(p_3)+1] \\ & \times [f(p_4)+1]\}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} C_{31}[f](t,p) = {} & \frac{3g^2\pi}{V} \sum_{p_1,p_2,p_3,p_4 \neq 0} [\delta(p-p_1) - \delta(p-p_2) \\ & - \delta(p-p_3) - \delta(p-p_4)] \\ & \times \left(K^{3,1}_{1,2,3,4}\right)^2 \delta(p_1 - p_2 - p_3 - p_4) \\ & \times \delta[\omega(p_1) - \omega(p_2) - \omega(p_3) - \omega(p_4)] \\ & \times \{f(p_3)f(p_4)f(p_2)[f(p_1)+1] \\ & - f(p_1)[f(p_2)+1][f(p_3)+1][f(p_4)+1]\}, \end{aligned} \tag{4}$$

in which n is the density of the condensate, $t \in \mathbb{R}_+$ is the time variable, $p \in \mathbb{R}^d \backslash \{O\}$ is the d-dimensional nonzero momentum variable, V is proportional to the volume of the periodic box $[-\frac{L}{2}, \frac{L}{2}]^d$, ω is the Bogoliubov dispersion relation defined in (24), and g is the interacting constant. We have normalized the Planck constant to be 1.

In the above collision operators, the kernels are defined as follows:

$$\begin{aligned} K^{1,2}_{1,2,3} = {} & u_{p_1}u_{p_2}u_{p_3} - v_{p_1}v_{p_2}v_{p_3} - u_{p_1}u_{p_2}v_{p_3} \\ & + v_{p_1}v_{p_2}u_{p_3} - u_{p_1}v_{p_2}u_{p_3} + v_{p_1}u_{p_2}v_{p_3}, \end{aligned} \tag{5}$$

$$\begin{aligned} K^{2,2}_{1,2,3,4} = {} & u_{p_1}u_{p_2}u_{p_3}u_{p_4} + u_{p_1}v_{p_2}u_{p_3}v_{p_4} + u_{p_1}v_{p_2}v_{p_3}u_{p_4} \\ & + v_{p_1}u_{p_2}v_{p_3}u_{p_4} + v_{p_1}u_{p_2}u_{p_3}v_{p_4} + v_{p_1}v_{p_2}v_{p_3}v_{p_4}, \end{aligned} \tag{6}$$

and

$$K^{3,1}_{1,2,3,4} = 2\left[u_{p_1}u_{p_2}v_{p_3}u_{p_4} + v_{p_1}v_{p_2}u_{p_3}v_{p_4}\right], \tag{7}$$

with u_p and v_p being defined later in (23).

To derive (1), we start with the Bogoliubov diagonalization process for the Hamiltonian of a weakly interacting Bose gas under the presence of a BEC, then focus on the derivation of the kinetic equation for the Bogoliubov excitations. In this process, we compute all of the commutators of the Bogoliubov excitations and do not drop any of them. We discover special mathematical structures of the commutators that allow us to reduce significantly the number of terms and the amount of computations. In particular, the computations of C_{31} reduce from 40 000 to only around 30 terms. Therefore, the combinatorics problem can simply be done and checked by hand.

Moreover, our framework provides a unified point of view for the different models, as it gives a simple explanation for the origins of the different collision operators based on the Bogoliubov diagonalization. To see this, we note that after the Bogoliubov transformation, the nonlinearity $\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$ of the Hamiltonian of the quantum system contains several types of nonlinearities including the following three special ones: (i) $\hat{b}^\dagger\hat{b}^\dagger\hat{b}$ and $\hat{b}^\dagger\hat{b}\hat{b}$; (ii) $\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}$; and (iii) $\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}$ and $\hat{b}^\dagger\hat{b}\hat{b}\hat{b}$, where $\hat{a}^\dagger, \hat{a}$ are bosonic creation and annihilation operators and $\hat{b}^\dagger, \hat{b}$ are their Bogoliubov transformations. The three types of collision operators then appear naturally as combinations of commutators of each type as follows.

(1) The C_{12} collision operator arises from commutators of the type $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}\hat{b}]]$ and $[\hat{b}^\dagger\hat{b}\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}^\dagger\hat{b}]]$.

(2) The C_{22} (Boltzmann-Nordheim and Uehling-Ulenbeck) collision operator arises from commutators of the type $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}]]$.

(3) The C_{31} collision operator arises from commutators of the types $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}\hat{b}\hat{b}]]$ and $[\hat{b}^\dagger\hat{b}\hat{b}\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}]]$.

The above argument provides a concise mathematical confirmation of the existence of C_{31}. For the experimental confirmations of C_{31}, we refer the readers to [3,15].

To conclude the introductory section, we remark that, when the temperature of the system is lower but close to the Bose-Einstein condensation transition temperature, the Bogoliubov dispersion relation can be approximated by the Hatree-Fock energy. In this case, $u_p \backsim 1$ and $v_p \backsim 0$. As a result, the kernel $K^{1,2}_{1,2,3} \backsim 1$ and the kernel $K^{2,2}_{1,2,3,4} \backsim 1$. On the other hand, the kernel $K^{3,1}_{1,2,3,4} \backsim 0$. As a result, in this temperature regime, the two collision operators C_{12} and C_{22} dominate the collisional processes. The contribution of the third collision operator C_{31} becomes significant when both u_p and v_p are large, corresponding to lower temperature regimes.

II. THE QUANTUM SYSTEM AND THE BOGOLIUBOV TRANSFORMATION

To begin our quantum description, since we are studying an interacting many-body quantum system, in which, dealing with the wave function for each individual particle becomes cumbersome, we introduce the boson field operator $\hat{\Psi}(x)$, and its conjugate $\hat{\Psi}^\dagger(x)$. These operators satisfy the the commutation relation

$$\begin{aligned} & [\hat{\Psi}(x), \hat{\Psi}(x')] = [\hat{\Psi}^\dagger(x), \hat{\Psi}^\dagger(x')] = 0 \\ & \text{and} \quad [\hat{\Psi}(x), \hat{\Psi}^\dagger(x')] = \delta(x - x'). \end{aligned} \tag{8}$$

The Hamiltonian of the system is now written

$$\hat{H} = \int_{\mathbb{T}_L^d} dx\, \hat{\Psi}^\dagger(x)\Big[-\frac{\hbar^2}{2m}\nabla^2 + U(x) + \frac{1}{2}\int_{\mathbb{T}_L^d} dx' \hat{\Psi}^\dagger(x)\mathcal{V}(x,x')\hat{\Psi}(x')\Big]\hat{\Psi}(x), \tag{9}$$

where $\mathbb{T}_L^d$ is the d-dimensional periodic torus $[-\frac{L}{2}, \frac{L}{2}]^d$; $\hbar$ is the Planck constant, m is the mass of the particle, U is an externally applied potential, and $\mathcal{V}(x,x')$ is the interaction potential between two particles at locations x, x'. To simplify our settings, we will not discuss particles in an external trapping potential, and set $U = 0$. We also take $\mathcal{V}(x,x') = g\delta(x-x')$, where g is the interacting constant. Inserting these two forms for the external and interaction potentials into (9), we find

$$\hat{H} = \int_{\mathbb{T}_L^d} dx\Big[-\frac{\hbar^2}{2m}\hat{\Psi}^\dagger(x)\nabla^2\hat{\Psi}(x) + \frac{g}{2}\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x)\hat{\Psi}(x)\hat{\Psi}(x)\Big]. \tag{10}$$

Writing the wave function Ψ in terms of annihilation and creation operators, we obtain

$$\hat{a}_p(t) = \frac{1}{(2\pi L)^{d/2}}\int_{\mathbb{T}_L^d} dx\, e^{-ipx}\hat{\Psi}(t,x) \tag{11}$$

and

$$\hat{\Psi}(t,x) = \frac{1}{(2\pi L)^{d/2}}\sum_{p\in\mathbb{Z}_L^d} e^{ipx}\hat{a}_p(t), \tag{12}$$

where $\mathbb{Z}_L^d = (\mathbb{Z}/L)^d$. For the sake of simplicity, we employ the shorthand notations

$$\int_{\mathbb{T}_L^d} = \int, \quad \sum_{p\in\mathbb{Z}_L^d} = \sum_p, \quad \text{and} \quad V = (2\pi L)^d. \tag{13}$$

The annihilation and creation operators $\hat{a}_p$ and $\hat{a}_p^\dagger$ then satisfy the commutation relations

$$[\hat{a}_p, \hat{a}_{p'}] = [\hat{a}_p^\dagger, \hat{a}_{p'}^\dagger] = 0 \quad \text{and} \quad [\hat{a}_p, \hat{a}_{p'}^\dagger] = \delta(p-p'). \tag{14}$$

The Hamiltonian of the above system is then

$$H = \sum_p \epsilon_p \hat{a}_p^\dagger\hat{a}_p + \frac{g}{2V}\sum_{p,p_1,p_2,p_3} \times \delta(p+p_1-p_2-p_3)\hat{a}_p^\dagger\hat{a}_{p_1}^\dagger\hat{a}_{p_2}\hat{a}_{p_3}, \tag{15}$$

in which $\epsilon_p = \frac{p^2}{2m}$ and the function $\delta(p+p_1-p_2-p_3)$ means that we sum over $p, p_1, p_2, p_3 \in \mathbb{Z}_L^d$ such that $p+p_1 = p_2+p_3$. We set $\hbar = 1$, for the sake of simplicity.

The Bose-Einstein condensation occurs when a large number of cold bosons enter the same quantum state having zero momentum. According to the Bogoliubov theory [2], since the lowest energy state is occupied by a macroscopic number of particles in the condensate, one can neglect the quantum fluctuation of this state and replace its annihilation operator with a c number $\sqrt{N}$, with N being the number of condensate atoms:

$$\hat{a}_0 = \sqrt{N}. \tag{16}$$

We now split $\hat{a}_0$ and $\hat{a}_p$ $(p \neq 0)$ and decompose the Hamiltonian $\hat{H}$ as

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3, \tag{17}$$

with

$$\hat{H}_1 = \sum_{p\neq 0}\epsilon_p\hat{a}_p^\dagger\hat{a}_p + \frac{g}{2V}\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0 + \frac{g}{2V}\sum_{p\neq 0}[4\hat{a}_0^\dagger\hat{a}_p^\dagger\hat{a}_0\hat{a}_p + \hat{a}_p^\dagger\hat{a}_{-p}^\dagger\hat{a}_0\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_p\hat{a}_{-p}], \tag{18}$$

$$\hat{H}_2 = \frac{g\sqrt{N}}{V}\sum_{p_1,p_2,p_3\neq 0}\big[\delta(p_1-p_2-p_3)\hat{a}_{p_1}^\dagger\hat{a}_{p_2}\hat{a}_{p_3} + \delta(p_1+p_2-p_3)\hat{a}_{p_1}^\dagger\hat{a}_{p_2}^\dagger\hat{a}_{p_3}\big], \tag{19}$$

$$\hat{H}_3 = \frac{g}{2V}\sum_{p_1,p_2,p_3,p_4\neq 0}\delta(p_1+p_2-p_3-p_4)\hat{a}_{p_1}^\dagger\hat{a}_{p_2}^\dagger\hat{a}_{p_3}\hat{a}_{p_4}. \tag{20}$$

Defining the density $n = \frac{N}{V}$, we then find

$$\hat{H}_1 = \sum_{p\neq 0}\epsilon_p\hat{a}_p^\dagger\hat{a}_p + \frac{gnN}{2} + \frac{gn}{2}\sum_{p\neq 0}[2\hat{a}_p^\dagger\hat{a}_p + \hat{a}_p^\dagger\hat{a}_{-p}^\dagger + \hat{a}_p\hat{a}_{-p}], \tag{21}$$

which can be diagonalized using the Bogoliubov transformation

$$\hat{a}_p = u_p\hat{b}_p - v_p\hat{b}_{-p}^\dagger, \qquad \hat{a}_p^\dagger = u_p\hat{b}_p^\dagger - v_p\hat{b}_{-p}, \tag{22}$$

with

$$u_p, v_p = \left(\frac{\epsilon_p + gn}{2\omega_p} \pm \frac{1}{2}\right)^{1/2}, \tag{23}$$

where ω_p is the Bogoliubov dispersion relation

$$\omega_p = \left[\frac{gn}{m}p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2}. \tag{24}$$

After being diagonalized, H_1 takes the form

$$\hat{H}_1 = \sum_{p\neq 0}\omega_p\hat{b}^\dagger\hat{b}_p + E_0, \tag{25}$$

with

$$E_0 = \frac{gnN}{2} + \frac{1}{2}\sum_{p\neq 0}\left[\omega_p - gn - \frac{p^2}{2m} + \frac{m(gn)^2}{p^2}\right]. \tag{26}$$

III. NEW FORMS OF $\hat{H}_2$ AND $\hat{H}_3$

A. New form of $\hat{H}_2$

By the calculations to be detailed in Appendix A, we arrive at the following form of $\hat{H}_2$ in terms of the new operators $\hat{b}$ and $\hat{b}^\dagger$:

$$\hat{H}_2 = \hat{H}_{1,2} + \hat{H}_{3,0}, \tag{27}$$

where

$$\hat{H}_{1,2} = g\sqrt{\frac{n}{V}} \sum_{p_1,p_2,p_3\neq 0} \delta(p_1 - p_2 - p_3) K^{1,2}_{1,2,3} \times \left(\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3} + \hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\right), \tag{28}$$

$$K^{1,2}_{1,2,3} = u_{p_1}u_{p_2}u_{p_3} - v_{p_1}v_{p_2}v_{p_3} - u_{p_1}u_{p_2}v_{p_3} + v_{p_1}v_{p_2}u_{p_3} - u_{p_1}v_{p_2}u_{p_3} + v_{p_1}u_{p_2}v_{p_3},$$

and

$$\hat{H}_{3,0} = g\sqrt{\frac{n}{V}} \sum_{p_1,p_2,p_3\neq 0} \delta(p_1 + p_2 + p_3) \times \left[K^{3,0}_{1,2,3}(\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_1} + \hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3})\right], \tag{29}$$

$$K^{3,0}_{1,2,3} = u_{p_1}v_{p_2}v_{p_3} - v_{p_1}u_{p_2}u_{p_3}.$$

The Hamiltonian $\hat{H}_{1,2}$ contains strings of annihilation and creation operators of the types $\hat{b}^\dagger\hat{b}\hat{b}$ and $\hat{b}^\dagger\hat{b}^\dagger\hat{b}$, which indicate the processes of one/two Bogoliubov excitations being created while two/one Bogoliubov excitations are annihilated. On the other hand, the Hamiltonian $\hat{H}_{3,0}$ contains strings of annihilation and creation operators of the types $\hat{b}\hat{b}\hat{b}$ and $\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger$, representing the process of the creation or annihilation of three excitations simultaneously. From a physical point of view, we can see that $\hat{H}_{3,0}$ does not contribute to the collision integrals, while the main contribution comes from $\hat{H}_{1,2}$. We will show later in Sec. IV, by explicit computations, that this is indeed the case.

B. New form of $\hat{H}_3$

Similarly, we also find a new form for $\hat{H}_3$ (the details of this computation will be given in Appendix B):

$$\hat{H}_3 = \hat{H}_{2,2} + \hat{H}_{1,1} + \hat{H}'_{2,2} + \hat{H}_{3,1} + \hat{H}'_{3,1} + \hat{H}_{4,0}, \tag{30}$$

where

$$\hat{H}_{2,2} = \frac{g}{2V} \sum_{p_1,p_2,p_3,p_4\neq 0} \delta(p_1 + p_2 - p_3 - p_4) K^{2,2}_{1,2,3,4} \times \hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4},$$

$$K^{2,2}_{1,2,3,4} = u_{p_1}u_{p_2}u_{p_3}u_{p_4} + u_{p_1}v_{p_2}u_{p_3}v_{p_4} + u_{p_1}v_{p_2}v_{p_3}u_{p_4} + v_{p_1}u_{p_2}v_{p_3}u_{p_4} + v_{p_1}u_{p_2}u_{p_3}v_{p_4} + v_{p_1}v_{p_2}v_{p_3}v_{p_4}, \tag{31}$$

$$\hat{H}_{1,1} = \frac{g}{2V} \sum_{p_1,p_2\neq 0} K^{1,1}_{1,2}\hat{b}^\dagger_{p_1}\hat{b}_{p_1}, \tag{32}$$

$$K^{1,1}_{1,2} = 4v^2_{p_1}v^2_{p_2} + 4u^2_{p_1}v^2_{p_2} + 4u_{p_1}v_{p_1}u_{p_2}v_{p_2}, \tag{33}$$

$$\hat{H}'_{2,2} = \frac{g}{2V} \sum_{p_1,p_2\neq 0} \left[u_{p_1}v_{p_1}u_{p_2}v_{p_2} + 2v^2_{p_1}v^2_{p_2}\right], \tag{34}$$

$$\hat{H}_{3,1} = \frac{g}{2V} \sum_{p_1,p_2,p_3,p_4\neq 0} \delta(p_1 - p_2 - p_3 - p_4) K^{3,1}_{1,2,3,4} \times \left[\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4} + \hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\right], \tag{35}$$

$$K^{3,1}_{1,2,3,4} = 2\left[u_{p_1}u_{p_2}v_{p_3}u_{p_4} + v_{p_1}v_{p_2}u_{p_3}v_{p_4}\right],$$

$$\hat{H}'_{3,1} = \frac{g}{2V} \sum_{p_1,p_2\neq 0} \left[\hat{b}_{p_1}\hat{b}_{-p_1}K^{2,0}_{1,2} + \hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_1}K^{2,0}_{1,2}\right], \tag{36}$$

$$K^{2,0}_{1,2} = u^2_{p_1}u_{p_2}v_{p_2} + v^2_{p_1}u_{p_2}v_{p_2} + 4u_{p_1}v_{p_1}v^2_{p_2}, \tag{37}$$

and

$$\hat{H}_{4,0} = \frac{g}{2V} \sum_{p_1,p_2,p_3,p_4\neq 0} \delta(p_1 + p_2 + p_3 + p_4) \times K^{4,0}_{1,2,3,4}\left[\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_4} + \hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\right], \tag{38}$$

with

$$K^{4,0}_{1,2,3,4} = u_{p_1}u_{p_2}v_{p_3}v_{p_4}. \tag{39}$$

We remark that the Hamiltonian $\hat{H}_{2,2}$ contains strings of annihilation and creation operators of the types $\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}$, indicating the processes of two Bogoliubov excitations being created while two Bogoliubov excitations are annihilated. Similarly, the Hamiltonian $\hat{H}_{3,1}$ represents the processes of three/one Bogoliubov excitations being created while one/tree Bogoliubov excitations are annihilated. From a physical point of view, the main contribution to the collision integrals comes from $\hat{H}_{2,2}, \hat{H}_{3,1}$ since the effects of $\hat{H}_{4,0}, \hat{H}'_{2,2}, \hat{H}'_{3,1}, \hat{H}_{1,1}$ are similar with that of the Hamiltonian $\hat{H}_{3,0}$ discussed above and can be ignored. In Sec. IV, this prediction will be shown by a more precise mathematical argument.

IV. THE QUANTUM LIOUVILLE EQUATION AND ASSUMPTIONS

The full state of the system is described by the full density matrix $\hat{\rho}(t)$ which obeys the quantum Liouville equation

$$\partial_t\hat{\rho} = -i[\hat{H}, \hat{\rho}]. \tag{40}$$

In order to derive the quantum kinetic equation, there are two key points:

(1) First, due to the uncertainty principle, we cannot specify exactly the number of particles at positions and momenta. We can only describe the number distribution of particles in a quantum state. As a consequence, the average number of quantum particles in quantum states with wave vectors can be considered to be analogous to the average number of classical particles with momenta.

(2) Second, in order to derive the quantum Boltzmann equation, we impose the Bogoliubov assumption that for a system that is out of equilibrium, the relaxation to equilibrium can occur in many different stages in which the stages' timescales are totally different from one stage to another. During the relaxation process, in each successive stage, the set of relevant parameters (expectation values and mean fields) used to describe the evolution is reduced. The Bogoliubov assumption is very similar to the *molecular chaos assumption* which implies that the system can be described by a reduced number of parameters, for example, the single-particle phase space distribution function.

Employing the standard elimination process (cf. [7,8]), we get the following spatial homogeneous equation for the single-particle phase space distribution

function $f(p) = \langle \hat{b}_p^\dagger \hat{b}_p \rangle = \mathrm{Tr}(\hat{\rho}_0 \hat{b}_p^\dagger \hat{b}_p)$:

$$\partial_t f = \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}, [f, \tilde{\hat{H}}(s)]]), \tag{41}$$

where $\tilde{\hat{H}}$ has exactly the same form with $H(s)$, except that all the operators $\hat{b}, \hat{b}^\dagger$ are replaced by $e^{is\omega}\hat{b}$ and $e^{-is\omega}\hat{b}^\dagger$. And following [7,8]

$$\hat{\rho}_0 = \exp\left(-\sum_p \xi_p \hat{b}_p^\dagger \hat{b}_p - \Omega\right), \tag{42}$$

with $\Omega = \log \mathrm{Tr}[\exp(-\sum_p \xi_p \hat{b}_p^\dagger b_p)]$.

We set $E_0 = 0$ and $H'_{2,2} = 0$ since they are constants, and approximate the right-hand side of (41) as

$$\begin{aligned}&\int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}(t), [f(s), \tilde{\hat{H}}(s)]]) \\ &\quad \approx \mathfrak{L}_{1,2} + \mathfrak{L}_{3,0} + \mathfrak{L}_{2,2} + \mathfrak{L}_{3,1} + \mathfrak{L}_{4,0} + \mathfrak{L}_{1,1} + \mathfrak{L}_{2,0}.\end{aligned} \tag{43}$$

Notice that in [6], the authors used an equivalent process, but only kept $\mathfrak{L}_{1,2}$ to get the $1 \leftrightarrow 2$ collision operator for the low-temperature regime, while we keep all of the seven terms.

The forms of $\mathfrak{L}_{1,2}$, $\mathfrak{L}_{3,0}$, $\mathfrak{L}_{2,2}$, $\mathfrak{L}_{3,1}$, $\mathfrak{L}_{4,0}$, $\mathfrak{L}_{1,1}$, and $\mathfrak{L}_{2,0}$ are computed as follows.

The form of $\mathfrak{L}_{1,2}$. This quantity comes from $\hat{H}_{1,2}$,

$$\mathfrak{L}_{1,2} = \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}_{1,2}, [f, \tilde{\hat{H}}_{1,2}(s)]]), \tag{44}$$

where $\tilde{\hat{H}}_{1,2}$ has exactly the same form with $\hat{H}_{1,2}(s)$, except that all the operator $\hat{b}, \hat{b}^\dagger$ are replaced by $e^{is\omega}\hat{b}$ and $e^{-is\omega}\hat{b}^\dagger$. Adapting the procedure in [6–8], we write

$$\begin{aligned}\mathfrak{L}_{1,2} \approx \frac{g^2 n}{V} &\sum_{p_1,p_2,p_3,p'_1,p'_2,p'_3 \neq 0} \int_{-\infty}^{0} ds\, e^{is[\omega(p_1)-\omega(p_2)-\omega(p_3)]} \\ &\times K^{1,2}_{1,2,3} K^{1,2}_{1',2',3'} \delta(p_1 - p_2 - p_3)\delta(p'_1 - p'_2 - p'_3) \\ &\times \mathrm{Tr}\big[\hat{\rho}_0\big(\big[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big]\big] \\ &+ \big[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\big]\big]\big)\big].\end{aligned} \tag{45}$$

$K^{1,2}_{1',2',3'}$ has the same formulation with $K^{1,2}_{1,2,3}$, in which p_1, p_2, p_3 are replaced by p'_1, p'_2, p'_3. We approximate (cf. [7,8])

$$\int_{-\infty}^{0} ds\, e^{is[\omega(p_1)-\omega(p_2)-\omega(p_3)]} \approx \pi\delta[\omega(p_1) - \omega(p_2) - \omega(p_3)], \tag{46}$$

and write

$$\begin{aligned}\mathfrak{L}_{1,2} \approx \frac{g^2 n\pi}{V} &\sum_{p_1,p_2,p_3,p'_1,p'_2,p'_3 \neq 0} K^{1,2}_{1,2,3} K^{1,2}_{1',2',3'} \\ &\times \delta[\omega(p_1) - \omega(p_2) - \omega(p_3)]\delta(p_1 - p_2 - p_3) \\ &\times \delta(p'_1 - p'_2 - p'_3)\mathrm{Tr}\big[\hat{\rho}_0\big(\big[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big]\big] \\ &+ \big[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\big]\big]\big)\big].\end{aligned} \tag{47}$$

The form of $\mathfrak{L}_{3,0}$. Similarly, this quantity comes from $\hat{H}_{3,0}$:

$$\begin{aligned}\mathfrak{L}_{3,0} &= \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}_{3,0}, [f, \tilde{\hat{H}}_{3,0}(s)]]) \\ &\approx \frac{g^2 n\pi}{V} \sum_{p_1,p_2,p_3,p'_1,p'_2,p'_3 \neq 0} K^{3,0}_{1,2,3} K^{3,0}_{1',2',3'} \\ &\quad \times \delta[\omega(p_1) + \omega(p_2) + \omega(p_3)] \\ &\quad \times \delta(p_1 + p_2 + p_3)\delta(p'_1 + p'_2 + p'_3) \\ &\quad \times \mathrm{Tr}\big[\hat{\rho}_0\big(\big[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}^\dagger_{p'_1}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big]\big] \\ &\quad + \big[\hat{b}_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_1}\big]\big]\big)\big].\end{aligned} \tag{48}$$

$K^{3,0}_{1',2',3'}$ has the same formulation with $K^{3,0}_{1,2,3}$, in which p_1, p_2, p_3 are replaced by p'_1, p'_2, p'_3.

Since $\omega(p) > 0$ for $p \neq 0$, the equation $\omega(p_1) + \omega(p_2) + \omega(p_3) = 0$ does not have any solution. The quantity $\mathfrak{L}_{3,0}$ is then 0.

The form of $\mathfrak{L}_{2,2}$. This quantity comes from $\hat{H}_{2,2}$:

$$\begin{aligned}\mathfrak{L}_{2,2} &= \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}_{2,2}, [f, \tilde{\hat{H}}_{2,2}(s)]]) \\ &\approx \frac{g^2 \pi}{4V^2} \sum_{p_1,p_2,p_3,p_4,p'_1,p'_2,p'_3,p'_4 \neq 0} K^{2,2}_{1,2,3,4} K^{2,2}_{1',2',3',4'} \\ &\quad \times \delta[\omega(p_1) + \omega(p_2) - \omega(p_3) - \omega(p_4)] \\ &\quad \times \delta(p_1 + p_2 - p_3 - p_4)\delta(p'_1 + p'_2 - p'_3 - p'_4) \\ &\quad \times \mathrm{Tr}\big(\hat{\rho}_0\big[\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\big]\big]\big).\end{aligned} \tag{49}$$

$K^{2,2}_{1',2',3',4'}$ has the same formulation with $K^{2,2}_{1,2,3,4}$, in which p_1, p_2, p_3, p_4 are replaced by p'_1, p'_2, p'_3, p'_4.

The form of $\mathfrak{L}_{3,1}$. This quantity comes from $\hat{H}_{3,1}$:

$$\begin{aligned}\mathfrak{L}_{3,1} &= \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}_{3,1}, [f, \tilde{\hat{H}}_{3,1}(s)]]) \\ &\approx \frac{g^2 \pi}{4V^2} \sum_{p_1,p_2,p_3,p_4,p'_1,p'_2,p'_3,p'_4 \neq 0} K^{3,1}_{1,2,3,4} K^{3,1}_{1',2',3',4'} \\ &\quad \times \delta[\omega(p_1) - \omega(p_2) - \omega(p_3) - \omega(p_4)] \\ &\quad \times \delta(p_1 - p_2 - p_3 - p_4)\delta(p'_1 - p'_2 - p'_3 - p'_4) \\ &\quad \times \mathrm{Tr}\big[\hat{\rho}_0\big(\big[\hat{b}^\dagger_{p'_4}\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\big]\big] \\ &\quad + \big[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4}, \big[\hat{b}^\dagger_p \hat{b}_p, \hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\big]\big]\big)\big].\end{aligned} \tag{50}$$

$K^{3,1}_{1',2',3',4'}$ has the same formulation with $K^{3,1}_{1,2,3,4}$, in which p_1, p_2, p_3, p_4 are replaced by p'_1, p'_2, p'_3, p'_4.

The form of $\mathfrak{L}_{4,0}$. This quantity comes from $\hat{H}_{4,0}$:

$$\begin{aligned}\mathfrak{L}_{4,0} &= \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}_{4,0}, [f, \tilde{\hat{H}}_{4,0}(s)]]) \\ &\approx \frac{g^2 \pi}{4V^2} \sum_{p_1,p_2,p_3,p_4,p'_1,p'_2,p'_3,p'_4 \neq 0} K^{4,0}_{1,2,3,4} K^{4,0}_{1',2',3',4'} \\ &\quad \times \delta[\omega(p_1) + \omega(p_2) + \omega(p_3) + \omega(p_4)] \\ &\quad \times \delta(p_1 + p_2 + p_3 + p_4)\delta(p'_1 + p'_2 + p'_3 + p'_4)\end{aligned}$$

$$\times \mathrm{Tr}\big[\hat{\rho}_0\big(\big[\hat{b}^\dagger_{p'_4}\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}^\dagger_{p'_1}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\big]\big] + \big[\hat{b}_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_1}\big]\big]\big)\big]. \quad (51)$$

$K^{4,0}_{1',2',3',4'}$ has the same formulation with $K^{4,0}_{1,2,3,4}$, in which p_1, p_2, p_3, p_4 are replaced by p'_1, p'_2, p'_3, p'_4. Since $\omega(p) > 0$ for $p \neq 0$, the equation $\omega(p_1) + \omega(p_2) + \omega(p_3) + \omega(p_4) = 0$ does not have any solution. The quantity $\mathfrak{L}_{4,0}$ is indeed 0.

The form of $\mathfrak{L}_{2,0}$. This quantity comes from $\hat{H}_{2,0}$:

$$\begin{aligned}\mathfrak{L}_{2,0} &= \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}_{2,0}, [f, \tilde{\hat{H}}_{2,0}(s)]]) \\ &\approx \frac{g^2\pi}{4V^2}\sum_{p_1,p_2,p'_1,p'_2\neq 0} K^{2,0}_{1,2}K^{2,0}_{1',2'}\delta[\omega(p_1)+\omega(-p_1)] \\ &\quad\times \mathrm{Tr}\big[\hat{\rho}_0\big(\big[\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{-p'_1}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}_{p_1}\hat{b}_{-p_1}\big]\big] \\ &\quad + \big[\hat{b}_{p'_1}\hat{b}_{-p'_1}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_1}\big]\big]\big)\big]. \quad (52)\end{aligned}$$

$K^{2,0}_{1',2'}$ has the same formulation with $K^{2,0}_{1,2}$, in which p_1, p_2 are replaced by p'_1, p'_2. Since $\omega(p) > 0$ for $p \neq 0$, the equation $\omega(p_1) + \omega(-p_1) = 0$ does not have any solution. The quantity $\mathfrak{L}_{2,0}$ is again 0.

The form of $\mathfrak{L}_{1,1}$. This quantity comes from $\hat{H}_{1,1}$:

$$\begin{aligned}\mathfrak{L}_{1,1} &= \int_{-\infty}^{0} ds\, \mathrm{Tr}(\hat{\rho}_0[\hat{H}_{1,1}, [f, \tilde{\hat{H}}_{1,1}(s)]]) \\ &\approx \frac{g^2\pi}{4V^2}\sum_{p_1,p_2,p'_1,p'_2\neq 0} K^{1,1}_{1,2}K^{1,1}_{1',2'} \\ &\quad\times \int_{-\infty}^{0} ds\, \mathrm{Tr}\big(\hat{\rho}_0\big[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_1}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_1}\big]\big]\big). \quad (53)\end{aligned}$$

$K^{1,1}_{1',2'}$ has the same formulation with $K^{1,1}_{1,2}$, in which p_1, p_2 are replaced by p'_1, p'_2. This quantity is also 0 due to the fact that

$$\big[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_1}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_1}\big]\big] = 0.$$

We finally obtain the spatial homogeneous equation

$$\begin{aligned}\partial_t f = \frac{g^2 n\pi}{V}&\sum_{p_1,p_2,p_3,p'_1,p'_2,p'_3\neq 0} K^{1,2}_{1,2,3}K^{1,2}_{1',2',3'} \\ &\times \delta(p_1-p_2-p_3)\delta(p'_1-p'_2-p'_3)\delta[\omega(p_1)-\omega(p_2) \\ &-\omega(p_3)]\big\langle\big[\big[\big[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\big]\big] \\ &+ \big[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\big]\big]\big]\big\rangle \\ &+ \frac{g^2\pi}{4V^2}\sum_{p_1,p_2,p_3,p_4,p'_1,p'_2,p'_3,p'_4\neq 0} K^{2,2}_{1,2,3,4}K^{2,2}_{1',2',3',4'} \\ &\times \delta(p_1+p_2-p_3-p_4)\delta(p'_1+p'_2-p'_3-p'_4) \\ &\times \delta[\omega(p_1)+\omega(p_2)-\omega(p_3)-\omega(p_4)] \\ &\times \big\langle\big[\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\big]\big]\big\rangle \\ &+ \frac{g^2\pi}{4V^2}\sum_{p_1,p_2,p_3,p_4,p'_1,p'_2,p'_3,p'_4\neq 0} K^{3,1}_{1,2,3,4}K^{3,1}_{1',2',3',4'} \\ &\times \delta[\omega(p_1)-\omega(p_2)-\omega(p_3)-\omega(p_4)] \\ &\times \delta(p_1-p_2-p_3-p_4)\delta(p'_1-p'_2-p'_3-p'_4) \\ &\times \big\langle\big[\big[\hat{b}^\dagger_{p'_4}\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\big]\big] \\ &+ \big[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4}, \big[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\big]\big]\big]\big\rangle. \quad (54)\end{aligned}$$

V. THE KINETIC EQUATION

The left-hand side of (54) is split into three terms; each contains special types of commutators. The first term includes commutators of the types $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}\hat{b}]]$ and $[\hat{b}^\dagger\hat{b}\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}^\dagger\hat{b}]]$. The computations in Appendix C show that this term is indeed the C_{12} collision operator. The second term has commutators of the type $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}]]$. This term can be shown to be the collision operator C_{22}. The explicit computations of this collision operator are postponed to Appendix D. The C_{31} collision operator comes from the last term, which involves commutators of the types $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}\hat{b}\hat{b}]]$ and $[\hat{b}^\dagger\hat{b}\hat{b}\hat{b}, [\hat{b}^\dagger\hat{b}, \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}]]$. These computations are given in detail in Appendix E. In conclusion, by computing explicitly the commutators on the right-hand side of (54), we finally arrive at the kinetic equation (1). We emphasize that the density of the thermal cloud $f(t, p)$ is defined only for $p \neq 0$ due to the fact that the condensate has been factored out in the Bogoliubov diagonalization (16).

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APPENDIX A

Let us expand $\delta(p_1 + p_2 - p_3)\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\hat{a}_{p_3}$ in terms of $\hat{b}^\dagger_{p_1}, \hat{b}_{p_1}, \hat{b}^\dagger_{p_2}, \hat{b}_{p_2}, \hat{b}^\dagger_{p_3}$, and $\hat{b}_{p_3}$:

$$\begin{aligned}\delta(p_1+p_2-p_3)\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\hat{a}_{p_3} &= \delta(p_1+p_2-p_3)\big[u_{p_1}\hat{b}^\dagger_{p_1} - v_{p_1}\hat{b}_{-p_1}\big]\big[u_{p_2}\hat{b}^\dagger_{p_2} - v_{p_2}\hat{b}_{-p_2}\big]\big[u_{p_3}\hat{b}_{p_3} - v_{p_3}\hat{b}^\dagger_{-p_3}\big] \\ &= \delta(p_1+p_2-p_3)\big[u_{p_1}u_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3} - v_{p_1}u_{p_2}u_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3} - u_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{-p_2}\hat{b}_{p_3} \\ &\quad + v_{p_1}v_{p_2}u_{p_3}\hat{b}_{-p_1}\hat{b}_{-p_2}\hat{b}_{p_3} - u_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_3} + v_{p_1}u_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_3} \\ &\quad + u_{p_1}v_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{-p_2}\hat{b}^\dagger_{-p_3} - v_{p_1}v_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}_{-p_2}\hat{b}^\dagger_{-p_3}\big]. \quad (A1)\end{aligned}$$

Similarly, we expand $\delta(p_1 - p_2 - p_3)\hat{a}^\dagger_{p_1}\hat{a}_{p_2}\hat{a}_{p_3}$ in terms of $\hat{b}^\dagger_{p_1}$, $\hat{b}_{p_1}$, $\hat{b}^\dagger_{p_2}$, $\hat{b}_{p_2}$, $\hat{b}^\dagger_{p_3}$, and $\hat{b}_{p_3}$:

$$\begin{aligned}\delta(p_1 - p_2 - p_3)\hat{a}^\dagger_{p_1}\hat{a}_{p_2}\hat{a}_{p_3} &= \delta(p_1 - p_2 - p_3)\big[u_{p_1}\hat{b}^\dagger_{p_1} - v_{p_1}\hat{b}_{-p_1}\big]\big[u_{p_2}\hat{b}_{p_2} - v_{p_2}\hat{b}^\dagger_{-p_2}\big]\big[u_{p_3}\hat{b}_{p_3} - v_{p_3}\hat{b}^\dagger_{-p_3}\big]\\ &= \delta(p_1 - p_2 - p_3)\big[u_{p_1}u_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3} - v_{p_1}u_{p_2}u_{p_3}\hat{b}_{-p_1}\hat{b}_{p_2}\hat{b}_{p_3} - u_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_2}\hat{b}_{p_3}\\ &+ v_{p_1}v_{p_2}u_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{-p_2}\hat{b}_{p_3} - u_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}^\dagger_{-p_3} + v_{p_1}u_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}_{p_2}\hat{b}^\dagger_{-p_3}\\ &+ u_{p_1}v_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_2}\hat{b}^\dagger_{-p_3} - v_{p_1}v_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{-p_2}\hat{b}^\dagger_{-p_3}\big].\end{aligned} \tag{A2}$$

We perform the following change of variables, for the terms in (A1), taking into account the fact that $p_1, p_2, p_3 \neq 0$:

$$\begin{aligned}\delta(p_1 + p_2 - p_3)u_{p_1}u_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3} &\to \delta(p_1 - p_2 - p_3)u_{p_1}u_{p_2}u_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1},\\ -\delta(p_1 + p_2 - p_3)v_{p_1}u_{p_2}u_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3} &\to -\delta(p_1 - p_2 - p_3)u_{p_1}v_{p_2}u_{p_3}\hat{b}_{p_2}\hat{b}^\dagger_{p_1}\hat{b}_{p_3} = -\delta(p_1 - p_2 - p_3)u_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3},\\ -\delta(p_1 + p_2 - p_3)u_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{-p_2}\hat{b}_{p_3} &\to -\delta(p_1 - p_2 - p_3)u_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3},\\ \delta(p_1 + p_2 - p_3)v_{p_1}v_{p_2}u_{p_3}\hat{b}_{-p_1}\hat{b}_{-p_2}\hat{b}_{p_3} &\to \delta(p_1 - p_2 - p_3)u_{p_1}v_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}_{p_2}\hat{b}_{p_3},\\ \delta(p_1 + p_2 - p_3)v_{p_1}u_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_3} &\to \delta(p_1 - p_2 - p_3)v_{p_1}u_{p_2}v_{p_3}\hat{b}_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_3} = \delta(p_1 - p_2 - p_3)v_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1},\\ \delta(p_1 + p_2 - p_3)u_{p_1}v_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{-p_2}\hat{b}^\dagger_{-p_3} &\to \delta(p_1 - p_2 - p_3)v_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\hat{b}^\dagger_{p_3} = \delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1},\\ -\delta(p_1 + p_2 - p_3)v_{p_1}v_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}_{-p_2}\hat{b}^\dagger_{-p_3} &\to -\delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}v_{p_3}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}^\dagger_{p_1} = -\delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3},\\ -\delta(p_1 + p_2 - p_3)u_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_3} &\to -\delta(p_1 - p_2 - p_3)v_{p_1}u_{p_2}u_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_1}.\end{aligned} \tag{A3}$$

Notice that in the above computations, we used identities like $\delta(p_1 - p_2 - p_3)\hat{b}_{p_2}\hat{b}^\dagger_{p_1}\hat{b}_{p_3} = \delta(p_1 - p_2 - p_3)\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3} + \delta(p_1 - p_2 - p_3)\delta(p_1 - p_2)\hat{b}_{p_3} = \delta(p_1 - p_2 - p_3)\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}$ due to the fact that all p_1, p_2, p_3 are nonzero. Similar computations can also be carried out for the terms in (A2):

$$\begin{aligned}-\delta(p_1 - p_2 - p_3)u_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_2}\hat{b}_{p_3} &\to -\delta(p_1 - p_2 - p_3)u_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1},\\ \delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}u_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{-p_2}\hat{b}_{p_3} &\to \delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}u_{p_3}\hat{b}_{p_2}\hat{b}^\dagger_{p_1}\hat{b}_{p_3} = \delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}u_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3},\\ -\delta(p_1 - p_2 - p_3)u_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}^\dagger_{-p_3} &\to -\delta(p_1 - p_2 - p_3)u_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\hat{b}^\dagger_{p_3} = -\delta(p_1 - p_2 - p_3)u_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1},\\ \delta(p_1 - p_2 - p_3)v_{p_1}u_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}_{p_2}\hat{b}^\dagger_{-p_3} &\to \delta(p_1 - p_2 - p_3)v_{p_1}u_{p_2}v_{p_3}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}^\dagger_{p_1} = \delta(p_1 - p_2 - p_3)v_{p_1}u_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3},\\ \delta(p_1 - p_2 - p_3)u_{p_1}v_{p_2}v_{p_3}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_2}\hat{b}^\dagger_{-p_3} &\to \delta(p_1 - p_2 - p_3)u_{p_1}v_{p_2}v_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_1},\\ -\delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}v_{p_3}\hat{b}_{-p_1}\hat{b}^\dagger_{-p_2}\hat{b}^\dagger_{-p_3} &\to -\delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}v_{p_3}\hat{b}_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_3} = -\delta(p_1 - p_2 - p_3)v_{p_1}v_{p_2}v_{p_3}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}.\end{aligned} \tag{A4}$$

We finally do the change of variables

$$\begin{aligned}&\delta(p_1 - p_2 - p_3)\big[\big(u_{p_1}v_{p_2}v_{p_3} - v_{p_1}u_{p_2}u_{p_3}\big)\big(\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_1} + \hat{b}_{-p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big)\big]\\ &= \delta(p_1 + p_2 + p_3)\big[\big(u_{p_1}v_{p_2}v_{p_3} - v_{p_1}u_{p_2}u_{p_3}\big)\big(\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_1} + \hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big)\big].\end{aligned} \tag{A5}$$

Putting together all of the identities in (A1)–(A5) yields the new form (27) of $\hat{H}_2$.

APPENDIX B

Let us now expand $\delta(p_1 + p_2 - p_3 - p_4)\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\hat{a}_{p_3}\hat{a}_{p_4}$ in terms of $\hat{b}^\dagger_{p_1}$, $\hat{b}_{p_1}$, $\hat{b}^\dagger_{p_2}$, $\hat{b}_{p_2}$, $\hat{b}^\dagger_{p_3}$, $\hat{b}_{p_3}$, $\hat{b}^\dagger_{p_4}$, and $\hat{b}_{p_4}$:

$$\begin{aligned}&\delta(p_1 + p_2 - p_3 - p_4)\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\hat{a}_{p_3}\hat{a}_{p_4}\\ &= \delta(p_1 + p_2 - p_3 - p_4)\big[u_{p_1}\hat{b}^\dagger_{p_1} - v_{p_1}\hat{b}_{-p_1}\big]\big[u_{p_2}\hat{b}^\dagger_{p_2} - v_{p_2}\hat{b}_{-p_2}\big]\big[u_{p_3}\hat{b}_{p_3} - v_{p_3}\hat{b}^\dagger_{-p_3}\big]\big[u_{p_4}\hat{b}_{p_4} - v_{p_4}\hat{b}^\dagger_{-p_4}\big]\\ &= \delta(p_1 + p_2 - p_3 - p_4)\big[u_{p_1}u_{p_2}u_{p_3}u_{p_4}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4} - v_{p_1}u_{p_2}u_{p_3}u_{p_4}\hat{b}_{-p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\end{aligned}$$

$$
\begin{aligned}
&- u_{p_1} v_{p_2} u_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \hat{b}_{p_3} \hat{b}_{p_4} + v_{p_1} v_{p_2} u_{p_3} u_{p_4} \hat{b}_{-p_1} \hat{b}_{-p_2} \hat{b}_{p_3} \hat{b}_{p_4} - u_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} \\
&+ v_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}_{-p_1} \hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} + u_{p_1} v_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} - v_{p_1} v_{p_2} v_{p_3} u_{p_4} \hat{b}_{-p_1} \hat{b}_{-p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} \\
&- u_{p_1} u_{p_2} u_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}^\dagger_{-p_4} + v_{p_1} u_{p_2} u_{p_3} v_{p_4} \hat{b}_{-p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}^\dagger_{-p_4} + u_{p_1} v_{p_2} u_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \hat{b}_{p_3} \hat{b}^\dagger_{-p_4} \\
&- v_{p_1} v_{p_2} u_{p_3} v_{p_4} \hat{b}_{-p_1} \hat{b}_{-p_2} \hat{b}_{p_3} \hat{b}^\dagger_{-p_4} + u_{p_1} u_{p_2} v_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_3} \hat{b}^\dagger_{-p_4} - v_{p_1} u_{p_2} v_{p_3} v_{p_4} \hat{b}_{-p_1} \hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_3} \hat{b}^\dagger_{-p_4} \\
&- u_{p_1} v_{p_2} v_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \hat{b}^\dagger_{-p_3} \hat{b}^\dagger_{-p_4} + v_{p_1} v_{p_2} v_{p_3} v_{p_4} \hat{b}_{-p_1} \hat{b}_{-p_2} \hat{b}^\dagger_{-p_3} \hat{b}^\dagger_{-p_4} \big].
\end{aligned} \tag{B1}
$$

Similarly as for $\hat{H}_2$, we perform several changes of variables, in combination with evaluating the commutators, to obtain

$$
\begin{aligned}
&\delta(p_1 + p_2 - p_3 - p_4) u_{p_1} v_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} \\
&\quad = \delta(p_1 + p_2 - p_3 - p_4) u_{p_1} v_{p_2} v_{p_3} u_{p_4} \big[\hat{b}^\dagger_{p_1} \hat{b}^\dagger_{-p_3} \hat{b}_{-p_2} \hat{b}_{p_4} + \delta(p_2 - p_3) \hat{b}^\dagger_{p_1} \hat{b}_{p_4} \big] \\
&\to \delta(p_1 + p_2 - p_3 - p_4) u_{p_1} v_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + u^2_{p_1} v^2_{p_2} \hat{b}^\dagger_{p_1} \hat{b}_{p_1}, \\
&\delta(p_1 + p_2 - p_3 - p_4) u_{p_1} v_{p_2} u_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \hat{b}_{p_3} \hat{b}^\dagger_{-p_4} \\
&\quad = \delta(p_1 + p_2 - p_3 - p_4) u_{p_1} v_{p_2} u_{p_3} v_{p_4} \big[\hat{b}^\dagger_{p_1} \hat{b}^\dagger_{-p_4} \hat{b}_{-p_2} \hat{b}_{p_3} + \delta(p_2 - p_4) \hat{b}^\dagger_{p_1} \hat{b}_{p_3} + \delta(p_3 + p_4) \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \big] \\
&\to \delta(p_1 + p_2 - p_3 - p_4) u_{p_1} v_{p_2} u_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + \big(u^2_{p_1} v^2_{p_2} + u_{p_1} v_{p_1} u_{p_2} v_{p_2} \big) \hat{b}^\dagger_{p_1} \hat{b}_{p_1}, \\
&\delta(p_1 + p_2 - p_3 - p_4) v_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}_{-p_1} \hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} \\
&\quad = \delta(p_1 + p_2 - p_3 - p_4) v_{p_1} u_{p_2} v_{p_3} u_{p_4} \big[\hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{-p_1} \hat{b}_{p_4} + \delta(p_2 + p_1) \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} + \delta(p_1 - p_3) \hat{b}^\dagger_{p_2} \hat{b}_{p_4} \big] \\
&\to \delta(p_1 + p_2 - p_3 - p_4) v_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + \big(u^2_{p_1} v^2_{p_2} + u_{p_1} v_{p_1} u_{p_2} v_{p_2} \big) \hat{b}^\dagger_{p_1} \hat{b}_{p_1}, \\
&\delta(p_1 + p_2 - p_3 - p_4) v_{p_1} u_{p_2} u_{p_3} v_{p_4} \hat{b}_{-p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}^\dagger_{-p_4} \\
&\quad = \delta(p_1 + p_2 - p_3 - p_4) v_{p_1} u_{p_2} u_{p_3} v_{p_4} \big[\hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_4} \hat{b}_{-p_1} \hat{b}_{p_3} + \delta(p_1 + p_2) \hat{b}_{p_3} \hat{b}^\dagger_{-p_4} + \delta(p_3 + p_4) \hat{b}^\dagger_{p_2} \hat{b}_{-p_1} + \delta(p_1 - p_4) \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \big] \\
&\to \delta(p_1 + p_2 - p_3 - p_4) v_{p_1} u_{p_2} u_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + \big(u^2_{p_1} v^2_{p_2} + u_{p_1} v_{p_1} u_{p_2} v_{p_2} \big) \hat{b}^\dagger_{p_1} \hat{b}_{p_1} + u_{p_1} v_{p_1} u_{p_2} v_{p_2} \hat{b}_{p_1} \hat{b}^\dagger_{p_1}, \\
&\delta(p_1 + p_2 - p_3 - p_4) v_{p_1} v_{p_2} v_{p_3} v_{p_4} \hat{b}_{-p_1} \hat{b}_{-p_2} \hat{b}^\dagger_{-p_3} \hat{b}^\dagger_{-p_4} \to \delta(p_1 + p_2 - p_3 - p_4) v_{p_1} v_{p_2} v_{p_3} v_{p_4} \hat{b}_{p_1} \hat{b}_{p_2} \hat{b}^\dagger_{p_3} \hat{b}^\dagger_{p_4} \\
&\quad = \delta(p_1 + p_2 - p_3 - p_4) v_{p_1} v_{p_2} v_{p_3} v_{p_4} \big[\hat{b}^\dagger_{p_3} \hat{b}^\dagger_{p_4} \hat{b}_{p_1} \hat{b}_{p_2} + \delta(p_2 - p_3) \hat{b}_{p_1} \hat{b}^\dagger_{p_4} + \delta(p_2 - p_4) \hat{b}_{p_1} \hat{b}^\dagger_{p_3} + \delta(p_1 - p_3) \hat{b}^\dagger_{p_4} \hat{b}_{p_2} \\
&\qquad + \delta(p_1 - p_4) \hat{b}^\dagger_{p_3} \hat{b}_{p_2} \big] \\
&\to \delta(p_1 + p_2 - p_3 - p_4) v_{p_1} v_{p_2} v_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + 2 v^2_{p_1} v^2_{p_2} \hat{b}^\dagger_{p_1} \hat{b}_{p_1} + 2 v^2_{p_1} v^2_{p_2} \hat{b}_{p_1} \hat{b}^\dagger_{p_1},
\end{aligned} \tag{B2}
$$

as well as

$$
\begin{aligned}
&\delta(p_1 + p_2 - p_3 - p_4) v_{p_1} u_{p_2} u_{p_3} u_{p_4} \hat{b}_{-p_1} \hat{b}^\dagger_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} \to \delta(p_1 - p_2 - p_3 - p_4) u_{p_1} v_{p_2} u_{p_3} u_{p_4} \hat{b}_{p_2} \hat{b}^\dagger_{p_1} \hat{b}_{p_3} \hat{b}_{p_4} \\
&= \delta(p_1 - p_2 - p_3 - p_4) u_{p_1} v_{p_2} u_{p_3} u_{p_4} \big[\hat{b}^\dagger_{p_1} \hat{b}_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + \delta(p_1 - p_2) \hat{b}_{p_3} \hat{b}_{p_4} \big] \\
&\to \delta(p_1 - p_2 - p_3 - p_4) u_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + u^2_{p_1} u_{p_2} v_{p_2} \hat{b}_{p_1} \hat{b}_{-p_1}, \\
&\delta(p_1 + p_2 - p_3 - p_4) u_{p_1} v_{p_2} u_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{-p_2} \hat{b}_{p_3} \hat{b}_{p_4} \\
&\to \delta(p_1 - p_2 - p_3 - p_4) u_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{p_2} \hat{b}_{p_3} \hat{b}_{p_4}, \\
&\delta(p_1 + p_2 - p_3 - p_4) u_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_1} \hat{b}^\dagger_{p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} \to \delta(p_1 - p_2 - p_3 - p_4) u_{p_1} u_{p_2} v_{p_3} u_{p_4} \hat{b}^\dagger_{p_4} \hat{b}^\dagger_{p_3} \hat{b}^\dagger_{p_2} \hat{b}_{p_1}, \\
&\delta(p_1 + p_2 - p_3 - p_4) v_{p_1} v_{p_2} v_{p_3} u_{p_4} \hat{b}_{-p_1} \hat{b}_{-p_2} \hat{b}^\dagger_{-p_3} \hat{b}_{p_4} \to \delta(p_1 - p_2 - p_3 - p_4) v_{p_1} v_{p_2} v_{p_3} u_{p_4} \hat{b}_{p_2} \hat{b}_{p_3} \hat{b}^\dagger_{p_1} \hat{b}_{p_4} \\
&= \delta(p_1 - p_2 - p_3 - p_4) v_{p_1} v_{p_2} v_{p_3} u_{p_4} \big[\hat{b}^\dagger_{p_1} \hat{b}_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + \delta(p_1 - p_3) \hat{b}_{p_2} \hat{b}_{p_4} + \delta(p_1 - p_2) \hat{b}_{p_3} \hat{b}_{p_4} \big] \\
&\to \delta(p_1 - p_2 - p_3 - p_4) v_{p_1} v_{p_2} u_{p_3} v_{p_4} \hat{b}^\dagger_{p_1} \hat{b}_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} + 2 \hat{b}_{p_1} \hat{b}_{-p_1} v_{p_1} u_{p_1} v^2_{p_2},
\end{aligned} \tag{B3}
$$

and

$$\begin{aligned}
&\delta(p_1+p_2-p_3-p_4)u_{p_1}u_{p_2}u_{p_3}v_{p_4}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}^\dagger_{-p_4}\to\delta(p_1-p_2-p_3-p_4)u_{p_1}u_{p_2}u_{p_3}v_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\hat{b}^\dagger_{p_4}\\
=&\delta(p_1-p_2-p_3-p_4)u_{p_1}u_{p_2}u_{p_3}v_{p_4}\big[\hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}+\delta(p_1-p_4)\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\big]\\
\to&\,\delta(p_1-p_2-p_3-p_4)u_{p_1}u_{p_2}v_{p_3}u_{p_4}\hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}+u^2_{p_1}u_{p_2}v_{p_2}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_1},\\
&\delta(p_1+p_2-p_3-p_4)v_{p_1}u_{p_2}v_{p_3}v_{p_4}\hat{b}_{-p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_3}\hat{b}^\dagger_{-p_4}\to\delta(p_1-p_2-p_3-p_4)v_{p_1}u_{p_2}v_{p_3}v_{p_4}\hat{b}_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_4}\\
=&\delta(p_1-p_2-p_3-p_4)v_{p_1}u_{p_2}v_{p_3}v_{p_4}\big[\hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}+\delta(p_1-p_2)\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_4}+\delta(p_1-p_3)\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_4}+\delta(p_1-p_4)\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_3}\big]\\
\to&\,\delta(p_1-p_2-p_3-p_4)v_{p_1}v_{p_2}u_{p_3}v_{p_4}\hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}+\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_1}\big[v^2_{p_1}u_{p_2}v_{p_2}+2u_{p_1}v_{p_1}v^2_{p_2}\big],\\
&\delta(p_1+p_2-p_3-p_4)v_{p_1}v_{p_2}u_{p_3}v_{p_4}\hat{b}_{-p_1}\hat{b}_{-p_2}\hat{b}_{p_3}\hat{b}^\dagger_{-p_4}\to\delta(p_1-p_2-p_3-p_4)v_{p_1}v_{p_2}u_{p_3}v_{p_4}\hat{b}_{p_4}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}^\dagger_{p_1}\\
=&\,\delta(p_1-p_2-p_3-p_4)v_{p_1}v_{p_2}u_{p_3}v_{p_4}\big[\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}+\delta(p_1-p_2)\hat{b}_{p_3}\hat{b}_{p_4}+\delta(p_1-p_3)\hat{b}_{p_2}\hat{b}_{p_4}+\delta(p_1-p_4)\hat{b}_{p_2}\hat{b}_{p_3}\big]\\
\to&\,\delta(p_1-p_2-p_3-p_4)v_{p_1}v_{p_2}u_{p_3}v_{p_4}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}+\hat{b}_{p_1}\hat{b}_{-p_1}\big[v^2_{p_1}u_{p_2}v_{p_2}+2u_{p_1}v_{p_1}v^2_{p_2}\big]\\
&\delta(p_1+p_2-p_3-p_4)u_{p_1}v_{p_2}v_{p_3}v_{p_4}\hat{b}^\dagger_{p_1}\hat{b}_{-p_2}\hat{b}^\dagger_{-p_3}\hat{b}^\dagger_{-p_4}\to\delta(p_1-p_2-p_3-p_4)v_{p_1}u_{p_2}v_{p_3}v_{p_4}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_4}\\
=&\,\delta(p_1-p_2-p_3-p_4)v_{p_1}u_{p_2}v_{p_3}v_{p_4}\big[\hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}+\delta(p_1-p_3)\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_4}+\delta(p_1-p_4)\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_3}\big]\\
\to&\,\delta(p_1-p_2-p_3-p_4)v_{p_1}v_{p_2}u_{p_3}v_{p_4}\hat{b}^\dagger_{p_4}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}+2\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{-p_1}u_{p_1}v_{p_1}v^2_{p_2}.
\end{aligned}\tag{B4}$$

We finally perform the change of variables

$$\begin{aligned}
&\delta(p_1+p_2-p_3-p_4)\big[u_{p_1}u_{p_2}v_{p_3}v_{p_4}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{-p_3}\hat{b}^\dagger_{-p_4}+v_{p_1}v_{p_2}u_{p_3}u_{p_4}\hat{b}_{-p_1}\hat{b}_{-p_2}\hat{b}_{p_3}\hat{b}_{p_4}\big]\\
\to&\,\delta(p_1+p_2+p_3+p_4)u_{p_1}u_{p_2}v_{p_3}v_{p_4}\big[\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_4}+\hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\big].
\end{aligned}\tag{B5}$$

Combining (B1)–(B5), we find the new form (30) for $\hat{H}_3$.

APPENDIX C

Let us first compute

$$\big[\hat{b}^\dagger_p\hat{b}_p,\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big]=\hat{b}^\dagger_p\hat{b}_p\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}-\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}^\dagger_p\hat{b}_p=(\delta(p'_1-p_1)-\delta(p-p_2)-\delta(p-p_3))\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}.\tag{C1}$$

We now perform the computation $[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1},[\hat{b}^\dagger_p\hat{b}_p,\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}]]$. To this end, we compute

$$\begin{aligned}
\big[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1},\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big]&=\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}-\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1}\\
&=\delta(p'_1-p_1)\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p_2}\hat{b}_{p_3}-\delta(p_3-p'_2)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p'_3}\hat{b}_{p_2}\hat{b}_{p'_1}-\delta(p_3-p'_3)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p_2}\hat{b}_{p'_1}\\
&\quad-\delta(p_2-p'_2)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p'_3}\hat{b}_{p_3}\hat{b}_{p'_1}-\delta(p_2-p'_3)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p_3}\hat{b}_{p'_1}-\delta(p_2-p'_3)\delta(p_3-p'_2)\hat{b}^\dagger_{p_1}\hat{b}_{p'_1}\\
&\quad-\delta(p_2-p'_2)\delta(p_3-p'_3)\hat{b}^\dagger_{p_1}\hat{b}_{p_1}.
\end{aligned}\tag{C2}$$

Taking into account the fact $p_1=p_2+p_3$ and $p'_1=p'_2+p'_3$, it now follows straightforwardly from Wick's theorem that

$$\begin{aligned}
&\big\langle\big[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1},\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\big]\big\rangle\delta(p_1-p_2-p_3)\delta(p'_1-p'_2-p'_3)\\
&\quad=\delta(p_1-p'_1)\delta(p_2-p'_2)\delta(p_3-p'_3)\delta(p_1-p_2-p_3)[f(p_2)f(p_3)-f(p_1)f(p_2)-f(p_1)f(p_3)-f(p_1)]\\
&\qquad+\delta(p_1-p'_1)\delta(p_2-p'_3)\delta(p_3-p'_2)\delta(p_1-p_2-p_3)[f(p_2)f(p_3)-f(p_1)f(p_2)-f(p_1)f(p_3)-f(p_1)],
\end{aligned}\tag{C3}$$

which implies

$$\begin{aligned}
&\big\langle\big[\hat{b}^\dagger_{p'_3}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_1},[\hat{b}^\dagger_p\hat{b}_p,\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}]\big]\big\rangle\delta(p_1-p_2-p_3)\delta(p'_1-p'_2-p'_3)\\
&\quad=[\delta(p_1-p'_1)\delta(p_2-p'_2)\delta(p_3-p'_3)+\delta(p_1-p'_1)\delta(p_2-p'_3)\delta(p_3-p'_2)]\\
&\qquad\times[\delta(p-p_1)-\delta(p-p_2)-\delta(p-p_3)]\delta(p_1-p_2-p_3)\\
&\qquad\times[f(p_2)f(p_3)-f(p_1)f(p_2)-f(p_1)f(p_3)-f(p_1)].
\end{aligned}\tag{C4}$$

In the above computation, for $p'_1=p_1$, there are two choices of p'_2 and p'_3, $p'_2=p_2$, $p'_3=p_3$ and $p'_2=p_3$, $p'_3=p_2$.

A similar procedure also gives

$$\begin{aligned}\langle[\hat{b}^\dagger_{p'_1}\hat{b}_{p'_2}\hat{b}_{p'_3},[\hat{b}^\dagger_p\hat{b}_p,\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}]]\rangle\delta(p_1-p_2-p_3)\delta(p'_1-p'_2-p'_3)\\ =[\delta(p_1-p'_1)\delta(p_2-p'_2)\delta(p_3-p'_3)+\delta(p_1-p'_1)\delta(p_2-p'_3)\delta(p_3-p'_2)]\\ \times[\delta(p-p_1)-\delta(p-p_2)-\delta(p-p_3)]\delta(p_1-p_2-p_3)\\ \times[f(p_2)f(p_3)-f(p_1)f(p_2)-f(p_1)f(p_3)-f(p_1)].\end{aligned} \tag{C5}$$

Since in the above procedure, the nonlinearity $f(p_2)f(p_3)[f(p_1)+1]-f(p_1)[f(p_2)+1][f(p_3)+1]$ appears four times, we multiply the factor $\pi\frac{g^2n}{V}$ by 4 and obtain the first collision operator C_{12}.

APPENDIX D

We first compute

$$\begin{aligned}[\hat{b}^\dagger_p\hat{b}_p,\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}]&=\hat{b}^\dagger_p\hat{b}_p\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}-\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\hat{b}^\dagger_p\hat{b}_p\\ &=[\delta(p-p_1)+\delta(p-p_2)-\delta(p-p_3)-\delta(p-p_4)]\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}.\end{aligned} \tag{D1}$$

We now analyze the commutator $[\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4},[\hat{b}^\dagger_p\hat{b}_p,\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}]]$. To this end, we compute

$$\begin{aligned}[\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4},\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}]&=\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4}\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}-\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\hat{b}_{p'_4}\\ &=\delta(p'_3-p_2)\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}^\dagger_{p_1}\hat{b}_{p'_4}\hat{b}_{p_3}\hat{b}_{p_4}+\delta(p'_4-p_2)\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}^\dagger_{p_1}\hat{b}_{p'_3}\hat{b}_{p_3}\hat{b}_{p_4}\\ &\quad+\delta(p'_3-p_1)\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}^\dagger_{p_2}\hat{b}_{p'_4}\hat{b}_{p_3}\hat{b}_{p_4}+\delta(p'_4-p_1)\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}^\dagger_{p_2}\hat{b}_{p'_3}\hat{b}_{p_3}\hat{b}_{p_4}\\ &\quad+\delta(p'_3-p_1)\delta(p'_4-p_2)\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p_3}\hat{b}_{p_4}+\delta(p'_4-p_1)\delta(p'_3-p_2)\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}_{p_3}\hat{b}_{p_4}\\ &\quad-\delta(p_3-p'_2)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p'_1}\hat{b}_{p_4}\hat{b}_{p'_3}\hat{b}_{p'_4}-\delta(p_4-p'_2)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p'_1}\hat{b}_{p_3}\hat{b}_{p'_3}\hat{b}_{p'_4}\\ &\quad-\delta(p_3-p'_1)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p'_2}\hat{b}_{p_4}\hat{b}_{p'_3}\hat{b}_{p'_4}-\delta(p_4-p'_1)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p'_2}\hat{b}_{p_3}\hat{b}_{p'_3}\hat{b}_{p'_4}\\ &\quad-\delta(p_3-p'_1)\delta(p_4-p'_2)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p'_3}\hat{b}_{p'_4}-\delta(p_4-p'_1)\delta(p_3-p'_2)\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}_{p'_3}\hat{b}_{p'_4}.\end{aligned} \tag{D2}$$

Our next task is to perform Wick's theorem to the 12 terms. We only analyze one of them below. The other terms can be done in exactly the same way. We compute

$$\begin{aligned}\delta(p'_4-p_2)\langle\hat{b}^\dagger_{p'_1}\hat{b}^\dagger_{p'_2}\hat{b}^\dagger_{p_1}\hat{b}_{p'_3}\hat{b}_{p_3}\hat{b}_{p_4}\rangle&=\delta(p'_4-p_2)\langle\hat{b}^\dagger_{p'_1}\hat{b}_{p'_3}\rangle\langle\hat{b}^\dagger_{p'_2}\hat{b}_{p_3}\rangle\langle\hat{b}^\dagger_{p_1}\hat{b}_{p_4}\rangle+\delta(p'_4-p_2)\langle\hat{b}^\dagger_{p'_1}\hat{b}_{p'_3}\rangle\langle\hat{b}^\dagger_{p'_2}\hat{b}_{p_4}\rangle\langle\hat{b}^\dagger_{p_1}\hat{b}_{p_3}\rangle\\ &\quad+\delta(p'_4-p_2)\langle\hat{b}^\dagger_{p'_1}\hat{b}_{p_3}\rangle\langle\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\rangle\langle\hat{b}^\dagger_{p_1}\hat{b}_{p_4}\rangle+\delta(p'_4-p_2)\langle\hat{b}^\dagger_{p'_1}\hat{b}_{p_3}\rangle\langle\hat{b}^\dagger_{p'_2}\hat{b}_{p_4}\rangle\langle\hat{b}^\dagger_{p_1}\hat{b}_{p'_3}\rangle\\ &\quad+\delta(p'_4-p_2)\langle\hat{b}^\dagger_{p'_1}\hat{b}_{p_4}\rangle\langle\hat{b}^\dagger_{p'_2}\hat{b}_{p'_3}\rangle\langle\hat{b}^\dagger_{p_1}\hat{b}_{p_3}\rangle+\delta(p'_4-p_2)\langle\hat{b}^\dagger_{p'_1}\hat{b}_{p_4}\rangle\langle\hat{b}^\dagger_{p'_2}\hat{b}_{p_3}\rangle\langle\hat{b}^\dagger_{p_1}\hat{b}_{p'_3}\rangle\\ &=\delta(p'_4-p_2)\delta(p'_1-p'_3)\delta(p'_2-p_3)\delta(p_1-p_4)f(p'_1)f(p'_2)f(p_1)\\ &\quad+\delta(p'_4-p_2)\delta(p'_1-p'_3)\delta(p'_2-p_4)\delta(p_1-p_3)f(p'_1)f(p'_2)f(p_1)\\ &\quad+\delta(p'_4-p_2)\delta(p'_1-p_3)\delta(p'_2-p'_3)\delta(p_1-p_4)f(p'_1)f(p'_2)f(p_1)\\ &\quad+\delta(p'_4-p_2)\delta(p'_1-p_3)\delta(p'_2-p_4)\delta(p_1-p'_3)f(p'_1)f(p'_2)f(p_1)\\ &\quad+\delta(p'_4-p_2)\delta(p'_1-p_4)\delta(p'_2-p'_3)\delta(p_1-p_3)f(p'_1)f(p'_2)f(p_1)\\ &\quad+\delta(p'_4-p_2)\delta(p'_1-p_4)\delta(p'_2-p_3)\delta(p_1-p'_3)f(p'_1)f(p'_2)f(p_1).\end{aligned} \tag{D3}$$

The six terms will be analyzed in detail below, with the notice that $p_1+p_2=p_3+p_4$ and $p'_1+p'_2=p'_3+p'_4$.

(i) The first term $\delta(p'_4-p_2)\delta(p'_1-p'_3)\delta(p'_2-p_3)\delta(p_1-p_4)f(p'_1)f(p'_2)f(p_1)$ appears when $p'_4=p'_2=p_2=p_3$, $p'_1=p'_3$, and $p_1=p_4$. This term will cancel with a similar term coming from $\delta(p_4-p'_2)\langle\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p'_1}\hat{b}_{p_3}\hat{b}_{p'_3}\hat{b}_{p'_4}\rangle$.

(ii) The second term $\delta(p'_4-p_2)\delta(p'_1-p'_3)\delta(p'_2-p_4)\delta(p_1-p_3)f(p'_1)f(p'_2)f(p_1)$ appears when $p'_4=p'_2=p_2=p_4$, $p'_1=p'_3$, and $p_1=p_3$. This term will cancel with a similar term coming from $\delta(p_4-p'_2)\langle\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p'_1}\hat{b}_{p_3}\hat{b}_{p'_3}\hat{b}_{p'_4}\rangle$.

(iii) The fourth term $\delta(p'_4-p_2)\delta(p'_1-p_3)\delta(p'_2-p'_3)\delta(p_1-p_4)f(p'_1)f(p'_2)f(p_1)$ appears when $p'_4=p'_1=p_2=p_3$, $p_4=p_1$, and $p'_2=p'_3$. This term will cancel with a similar term coming from $\delta(p_4-p'_2)\langle\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p'_1}\hat{b}_{p_3}\hat{b}_{p'_3}\hat{b}_{p'_4}\rangle$.

(iv) The fourth term $\delta(p'_4-p_2)\delta(p'_1-p_3)\delta(p'_2-p_4)\delta(p_1-p'_3)f(p'_1)f(p'_2)f(p_1)$ appears when $p'_4=p_2$, $p_4=p'_2$, $p'_1=p_3$, and $p_1=p'_3$. This produces the nonlinearity $\delta(p_1+p_2-p_3-p_4)f(p_1)f(p_3)f(p_4)$.

(v) The fourth term $\delta(p_4' - p_2)\delta(p_1' - p_4)\delta(p_2' - p_3')\delta(p_1 - p_3)f(p_1')f(p_2')f(p_1)$ appears when $p_4' = p_1' = p_2 = p_4$, $p_3 = p_1$, and $p_2' = p_3'$. This term will cancel with a similar term coming from $\delta(p_4 - p_2')\langle \hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_1'}\hat{b}_{p_3}\hat{b}_{p_3'}\hat{b}_{p_4'}\rangle$.

(vi) The fourth term $\delta(p_4' - p_2)\delta(p_1' - p_4)\delta(p_2' - p_3)\delta(p_1 - p_3')f(p_1')f(p_2')f(p_1)$ appears when $p_4' = p_2$, $p_3 = p_2'$, $p_1' = p_4$, and $p_1 = p_3'$. This produces the nonlinearity $\delta(p_1 + p_2 - p_3 - p_4)f(p_1)f(p_3)f(p_4)$.

As a result, the quantity $\delta(p_4' - p_2)\langle \hat{b}^\dagger_{p_1'}\hat{b}^\dagger_{p_2'}\hat{b}^\dagger_{p_1}\hat{b}_{p_3'}\hat{b}_{p_3}\hat{b}_{p_4}\rangle$ produces two times the nonlinearity $\delta(p_1 + p_2 - p_3 - p_4)f(p_1)f(p_3)f(p_4)$.

Finally, in the end, the nonlinearity $f(p_1)f(p_2)(f(p_3)+1)[f(p_4)+1] - f(p_3)f(p_4)[f(p_2)+1][f(p_1)+1]$ appears four times in the forms

$$\begin{aligned}
&\delta(p_1 + p_2 - p_3 - p_4)\delta(p_1 - p_1')\delta(p_2 - p_2')\delta(p_3 - p_3')\delta(p_4 - p_4')\{f(p_1)f(p_2)[f(p_3)+1][f(p_4)+1]\\
&\quad - f(p_3)f(p_4)[f(p_2)+1][f(p_1)+1]\},\\
&\delta(p_1 + p_2 - p_3 - p_4)\delta(p_1 - p_2')\delta(p_1 - p_2')\delta(p_3 - p_4')\delta(p_4 - p_3')\{f(p_1)f(p_2)[f(p_3)+1][f(p_4)+1]\\
&\quad - f(p_3)f(p_4)[f(p_2)+1][f(p_1)+1]\},\\
&\delta(p_1 + p_2 - p_3 - p_4)\delta(p_1 - p_1')\delta(p_2 - p_2')\delta(p_3 - p_4')\delta(p_4 - p_3')\{f(p_1)f(p_2)[f(p_3)+1][f(p_4)+1]\\
&\quad - f(p_3)f(p_4)[f(p_2)+1][f(p_1)+1]\},
\end{aligned} \tag{D4}$$

and

$$\begin{aligned}
&\delta(p_1 + p_2 - p_3 - p_4)\delta(p_1 - p_2')\delta(p_2 - p_1')\delta(p_3 - p_3')\delta(p_4 - p_4')\{f(p_1)f(p_2)[f(p_3)+1][f(p_4)+1]\\
&\quad - f(p_3)f(p_4)[f(p_2)+1][f(p_1)+1]\}.
\end{aligned}$$

We then multiply the factor $\frac{g^2\pi}{4V^2}$ by 4 and obtain the collision operator C_{22}.

APPENDIX E

Let us first compute

$$\begin{aligned}
\left[\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\right] &= \hat{b}^\dagger_p\hat{b}_p\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4} - \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\hat{b}^\dagger_p\hat{b}_p\\
&= [\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3) - \delta(p - p_4)]\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}.
\end{aligned} \tag{E1}$$

We now analyze the commutator $[\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_1'}, [\hat{b}^\dagger_p\hat{b}_p, \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}]]$. To this end, we compute

$$\begin{aligned}
&\left[\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_1'}, \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\right] = \hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_1'}\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4} - \hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_1'}\\
&= -\big[\delta(p_4 - p_4')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_1'} + \delta(p_4 - p_4')\delta(p_3 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_1'} + \delta(p_4 - p_4')\delta(p_2 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}_{p_3}\hat{b}_{p_1'}\\
&\quad + \delta(p_4 - p_4')\delta(p_2 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2'}\hat{b}_{p_3}\hat{b}_{p_1'} + \delta(p_4 - p_4')\delta(p_3 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}_{p_2}\hat{b}_{p_1'} + \delta(p_4 - p_4')\delta(p_2 - p_3')\delta(p_3 - p_2')\hat{b}^\dagger_{p_1}\hat{b}_{p_1}\\
&\quad + \delta(p_4 - p_4')\delta(p_3 - p_3')\delta(p_2 - p_2')\hat{b}^\dagger_{p_1}\hat{b}_{p_1} + \delta(p_3 - p_4')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_3 - p_4')\delta(p_4 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_1'}\\
&\quad + \delta(p_3 - p_4')\delta(p_2 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_3 - p_4')\delta(p_2 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2'}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_3 - p_4')\delta(p_4 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}_{p_2}\hat{b}_{p_1'}\\
&\quad + \delta(p_3 - p_4')\delta(p_2 - p_3')\delta(p_4 - p_2')\hat{b}^\dagger_{p_1}\hat{b}_{p_1} + \delta(p_3 - p_4')\delta(p_4 - p_3')\delta(p_2 - p_2')\hat{b}^\dagger_{p_1}\hat{b}_{p_1}\\
&\quad + \delta(p_2 - p_4')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_4}\hat{b}_{p_3}\hat{b}_{p_1'} + \delta(p_2 - p_4')\delta(p_3 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2'}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_2 - p_4')\delta(p_4 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}_{p_3}\hat{b}_{p_1'}\\
&\quad + \delta(p_2 - p_4')\delta(p_4 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_2'}\hat{b}_{p_3}\hat{b}_{p_1'} + \delta(p_2 - p_4')\delta(p_3 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_2 - p_4')\delta(p_4 - p_3')\delta(p_3 - p_2')\hat{b}^\dagger_{p_1}\hat{b}_{p_1}\\
&\quad + \delta(p_2 - p_4')\delta(p_3 - p_3')\delta(p_4 - p_2')\hat{b}^\dagger_{p_1}\hat{b}_{p_1} + \delta(p_2 - p_3')\delta(p_3 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_2 - p_3')\delta(p_4 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}_{p_3}\hat{b}_{p_1'}\\
&\quad + \delta(p_3 - p_3')\delta(p_2 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_3 - p_3')\delta(p_4 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}_{p_2}\hat{b}_{p_1'} + \delta(p_4 - p_3')\delta(p_3 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}_{p_2}\hat{b}_{p_1'}\\
&\quad + \delta(p_4 - p_3')\delta(p_2 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}_{p_3}\hat{b}_{p_1'} + \delta(p_2 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_3}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_3 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_4}\hat{b}_{p_1'}\\
&\quad + \delta(p_4 - p_3')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_1'} + \delta(p_2 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}_{p_3}\hat{b}_{p_4}\hat{b}_{p_1'} + \delta(p_3 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}_{p_2}\hat{b}_{p_4}\hat{b}_{p_1'}\\
&\quad + \delta(p_4 - p_2')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_1'}\big] + \delta(p_1 - p_1')\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}.
\end{aligned} \tag{E2}$$

In the above structure, we can see that there are totally 34 terms, classified into three categories:
(i) 10 terms of the type $\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}\hat{b}$.
(ii) 18 terms of the type $\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}$.
(iii) 6 terms of the type $\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}$.

We provide below the detailed analysis for all of the 10 terms in the first category. The treatment of the other terms can be done in a similar manner. By Wick's theorem applied to $\delta(p_1 - p_1')\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}$, we have

$$\begin{aligned}
\delta(p_1' - p_1)\langle \hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\rangle &= \delta(p_1' - p_1)\langle \hat{b}^\dagger_{p_4'}\hat{b}_{p_4}\rangle\langle \hat{b}^\dagger_{p_3'}\hat{b}_{p_3}\rangle\langle \hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\rangle + \delta(p_1' - p_1)\langle \hat{b}^\dagger_{p_4'}\hat{b}_{p_4}\rangle\langle \hat{b}^\dagger_{p_3'}\hat{b}_{p_2}\rangle\langle \hat{b}^\dagger_{p_2'}\hat{b}_{p_3}\rangle \\
&\quad + \delta(p_1' - p_1)\langle \hat{b}^\dagger_{p_4'}\hat{b}_{p_2}\rangle\langle \hat{b}^\dagger_{p_3'}\hat{b}_{p_3}\rangle\langle \hat{b}^\dagger_{p_2'}\hat{b}_{p_4}\rangle + \delta(p_1' - p_1)\langle \hat{b}^\dagger_{p_4'}\hat{b}_{p_2}\rangle\langle \hat{b}^\dagger_{p_3'}\hat{b}_{p_4}\rangle\langle \hat{b}^\dagger_{p_2'}\hat{b}_{p_3}\rangle \\
&\quad + \delta(p_1' - p_1)\langle \hat{b}^\dagger_{p_4'}\hat{b}_{p_3}\rangle\langle \hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\rangle\langle \hat{b}^\dagger_{p_3'}\hat{b}_{p_4}\rangle + \delta(p_1' - p_1)\langle \hat{b}^\dagger_{p_4'}\hat{b}_{p_3}\rangle\langle \hat{b}^\dagger_{p_3'}\hat{b}_{p_2}\rangle\langle \hat{b}^\dagger_{p_2'}\hat{b}_{p_4}\rangle \\
&= \delta(p_1' - p_1)\delta(p_4' - p_4)\delta(p_3' - p_3)\delta(p_2' - p_2)f(p_2)f(p_3)f(p_4) \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_4)\delta(p_3' - p_2)\delta(p_2' - p_3)f(p_2)f(p_3)f(p_4) \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_2)\delta(p_3' - p_3)\delta(p_2' - p_4)f(p_2)f(p_3)f(p_4) \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_2)\delta(p_3' - p_4)\delta(p_2' - p_3)f(p_2)f(p_3)f(p_4) \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_3)\delta(p_2' - p_2)\delta(p_3' - p_4)f(p_2)f(p_3)f(p_4) \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_3)\delta(p_3' - p_2)\delta(p_2' - p_4)f(p_2)f(p_3)f(p_4).
\end{aligned} \tag{E3}$$

Now, similar Wick's theorem arguments can be used for $\delta(p_4 - p_4')\hat{b}^\dagger_{p_1}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_1'}$. In this case, we get the sum of two terms:

$$\begin{aligned}
&\delta(p_1 - p_1')\delta(p_2 - p_3')\delta(p_3 - p_2')\delta(p_4 - p_4')f(p_1)f(p_2)f(p_3) \\
&\quad + \delta(p_1 - p_1')\delta(p_2 - p_2')\delta(p_3 - p_3')\delta(p_4 - p_4')f(p_1)f(p_2)f(p_3)
\end{aligned} \tag{E4}$$

and

$$\begin{aligned}
&\delta(p_1 - p_2)\delta(p_3 + p_4)\delta(p_4 - p_4')\delta(p_3 - p_3')f(p_1)f(p_3)f(p_1') \\
&\quad + \delta(p_1 - p_3)\delta(p_2 + p_4)\delta(p_4 - p_4')\delta(p_2 - p_3')f(p_1)f(p_2)f(p_1'),
\end{aligned} \tag{E5}$$

where we have used the fact that $p_1 = p_2 + p_3 + p_4$ and $p_1' = p_2' + p_3' + p_4'$.

Taking the sum with respect to p_1', p_2', p_3', p_4' and p_1, p_2, p_3, p_4 the second term (E5), with the kernel $K^{13}_{1,2,3,4}$ we find

$$\begin{aligned}
&2 \sum_{p_1,p_2,p_3,p_4,p_1',p_2',p_3',p_4',\neq 0} K^{13}_{1,2,3,4}K^{13}_{1',2',3',4'}\delta(p_1 - p_2 - p_3 - p_4)\delta(p_1' - p_2' - p_3' - p_4')\delta(p_1 - p_3) \\
&\quad \times \delta(p_2 + p_4)\delta(p_4 - p_4')\delta(p_2 - p_3')f(p_1)f(p_2)f(p_1'),
\end{aligned} \tag{E6}$$

where $K^{13}_{1',2',3',4'}$ is $K^{13}_{1,2,3,4}$, in which p_1, p_2, p_3, p_4 are replaced by p_1', p_2', p_3', p_4' and we have used the symmetry of p_2 and p_3, to get the factor 2 outside.

The other terms have exactly the same structure and by taking the sum of all of terms like (E4) and (E6), we arrive at two big terms:

$$\begin{aligned}
\mathfrak{A}_1 &:= \delta(p_1' - p_1)\delta(p_4' - p_4)\delta(p_3' - p_3)\delta(p_2' - p_2)[f(p_1)f(p_3)f(p_4) + f(p_1)f(p_2)f(p_4) + f(p_1)f(p_2)f(p_3)] \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_4)\delta(p_3' - p_2)\delta(p_2' - p_3)[f(p_1)f(p_3)f(p_4) + f(p_1)f(p_2)f(p_4) + f(p_1)f(p_2)f(p_3)] \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_2)\delta(p_3' - p_3)\delta(p_2' - p_4)[f(p_1)f(p_3)f(p_4) + f(p_1)f(p_2)f(p_4) + f(p_1)f(p_2)f(p_3)] \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_2)\delta(p_3' - p_4)\delta(p_2' - p_3)[f(p_1)f(p_3)f(p_4) + f(p_1)f(p_2)f(p_4) + f(p_1)f(p_2)f(p_3)] \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_3)\delta(p_2' - p_2)\delta(p_3' - p_4)[f(p_1)f(p_3)f(p_4) + f(p_1)f(p_2)f(p_4) + f(p_1)f(p_2)f(p_3)] \\
&\quad + \delta(p_1' - p_1)\delta(p_4' - p_3)\delta(p_3' - p_2)\delta(p_2' - p_4)[f(p_1)f(p_3)f(p_4) + f(p_1)f(p_2)f(p_4) + f(p_1)f(p_2)f(p_3)],
\end{aligned} \tag{E7}$$

$$\begin{aligned}
\mathfrak{A}_2 &:= 12 \sum_{p_1,p_2,p_3,p_4,p_1',p_2',p_3',p_4',\neq 0} K^{13}_{1,2,3,4}K^{13}_{1',2',3',4'}\delta(p_1 - p_2 - p_3 - p_4)\delta(p_1' - p_2' - p_3' - p_4')\delta(p_1 - p_3) \\
&\quad \times \delta(p_2 + p_4)\delta(p_4 - p_4')\delta(p_2 - p_3')f(p_1)f(p_2)f(p_1'),
\end{aligned} \tag{E8}$$

in which we have taken into account the symmetry of p_2, p_3, p_4 to rearrange the terms and get the factor 12 in front of the sum in $\mathfrak{A}_2$. This term is indeed negligible due to the delta function $\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3) - \delta(p - p_4)$ in (E1). To see this, we apply this delta function to the left-hand side of (E8) and get

$$\begin{aligned}
&\big[\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3) - \delta(p - p_4)\big]\mathfrak{A}_2 \\
&= \big[\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3) - \delta(p - p_4)\big]12 \sum_{p_1,p_2,p_3,p_4,p_1',p_2',p_3',p_4',\neq 0} K^{13}_{1,2,3,4}K^{13}_{1',2',3',4'} \\
&\quad \times \delta(p_1 - p_2 - p_3 - p_4)\delta(p_1' - p_2' - p_3' - p_4')\delta(p_1 - p_3)\delta(p_2 + p_4)\delta(p_4 - p_4')\delta(p_2 - p_3')f(p_1)f(p_2)f(p_1') = 0.
\end{aligned} \tag{E9}$$

The first quantity $\mathfrak{A}_1$ can be combined with (E3), yielding

$$\begin{aligned}
&\delta(p_1'-p_1)\delta(p_4'-p_4)\delta(p_3'-p_3)\delta(p_2'-p_2)[f(p_2)f(p_3)f(p_4)-f(p_1)f(p_3)f(p_4)-f(p_1)f(p_2)f(p_4)-f(p_1)f(p_2)f(p_3)]\\
&+\delta(p_1'-p_1)\delta(p_4'-p_4)\delta(p_3'-p_2)\delta(p_2'-p_3)[f(p_2)f(p_3)f(p_4)-f(p_1)f(p_3)f(p_4)-f(p_1)f(p_2)f(p_4)-f(p_1)f(p_2)f(p_3)]\\
&+\delta(p_1'-p_1)\delta(p_4'-p_2)\delta(p_3'-p_3)\delta(p_2'-p_4)[f(p_2)f(p_3)f(p_4)-f(p_1)f(p_3)f(p_4)-f(p_1)f(p_2)f(p_4)-f(p_1)f(p_2)f(p_3)]\\
&+\delta(p_1'-p_1)\delta(p_4'-p_2)\delta(p_3'-p_4)\delta(p_2'-p_3)[f(p_2)f(p_3)f(p_4)-f(p_1)f(p_3)f(p_4)-f(p_1)f(p_2)f(p_4)-f(p_1)f(p_2)f(p_3)]\\
&+\delta(p_1'-p_1)\delta(p_4'-p_3)\delta(p_2'-p_2)\delta(p_3'-p_4)[f(p_2)f(p_3)f(p_4)-f(p_1)f(p_3)f(p_4)-f(p_1)f(p_2)f(p_4)-f(p_1)f(p_2)f(p_3)]\\
&+\delta(p_1'-p_1)\delta(p_4'-p_3)\delta(p_3'-p_2)\delta(p_2'-p_4)[f(p_2)f(p_3)f(p_4)-f(p_1)f(p_3)f(p_4)-f(p_1)f(p_2)f(p_4)-f(p_1)f(p_2)f(p_3)].
\end{aligned} \tag{E10}$$

Notice that in the above procedure, the nonlinearity $[f(p_2)f(p_3)f(p_4)-f(p_1)f(p_3)f(p_4)-f(p_1)f(p_2)f(p_4)-f(p_1)f(p_2)f(p_3)]$ appears six times.

By similar arguments, applied to terms of the other two categories, we find the full nonlinearity $\{f(p_3)f(p_4)f(p_2)[f(p_1)+1]-f(p_1)[f(p_2)+1][f(p_3)+1][f(p_4)+1]\}$, which also appears six times. Now, due to the commutator $[\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4},[\hat{b}^\dagger_p\hat{b}_p,\hat{b}^\dagger_{p_4'}\hat{b}^\dagger_{p_3'}\hat{b}^\dagger_{p_2'}\hat{b}_{p_1'}]]$, the nonlinearity $\{f(p_3)f(p_4)f(p_2)[f(p_1)+1]-f(p_1)[f(p_2)+1](f(p_3)+1)[f(p_4)+1]\}$ appears 12 times in total. We multiply the factor $\frac{g^2\pi}{4V}$ by 12 and obtain the third collision operator C_{31}.

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