


Universal distributions from non-Hermitian perturbation of zero modesM. Kieburg ^{*}*School of Mathematics and Statistics, University of Melbourne, 813 Swanston Street, Parkville, Melbourne VIC 3010, Australia*A. Mielke [†]*Faculty of Physics, Bielefeld University, P.O. Box 100131, D-33501 Bielefeld, Germany*M. Rud[‡] and K. Splittorff [§]*Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, 2100 Copenhagen, Denmark*

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Hermitian operators with exact zero modes subject to non-Hermitian perturbations are considered. Specific focus is on the distribution of the former zero eigenvalues of the Hermitian operators. The broadening of these zero modes is found to follow an elliptic Gaussian random matrix ensemble of fixed size, where the symmetry class of the perturbation determines the behavior of the modes. This distribution follows from a central limit theorem of matrices and is shown to be robust to deformations.

DOI: [10.1103/PhysRevE.101.032117](https://doi.org/10.1103/PhysRevE.101.032117)**I. INTRODUCTION**

The microscopic spectrum and the zero modes of a Hermitian Hamiltonian matrix are of particular interest since these are usually closely linked to the symmetries and topology of the system. Studies of the smallest eigenvalues and zero modes therefore allow us to extract fundamental information about the system. This fact also extends to the Dirac operator in quantum chromodynamics (QCD), where the average spectral properties of the microscopic eigenvalues are intimately linked to chiral symmetry breaking [1] and the zero modes by the Atiyah-Singer index theorem follow from the topology of the gauge field.

Because of universality [2], the average properties of the microscopic spectrum of the QCD Dirac operator may be studied either by random matrix theory [2–4] or by means of effective field theory [5]. Similarly, the smallest eigenvalues of the Hamiltonian for solid-state systems carrying Majorana modes may be described either by random matrix theory or the associated σ -model [6,7].

In physical realizations, the symmetries and the topology of the ideal system is often perturbed, causing the zero modes to move slightly away from the origin. This poses a question: *Is it possible in the presence of the perturbation to distinguish modes that have their origin in the topological zero modes of the unperturbed system?* This question is highly relevant for the study of Majorana modes [6,8–14] as well as in QCD [15–19]. A first answer was found in Refs. [15–17] in the context of QCD, where it was realized that indeed it is possible

to distinguish would-be zero modes, since their distribution have very specific spectral statistics and scale differently with the volume than the rest of the small eigenvalues. Surprisingly, it was found that the spectral statistics of the would-be zero modes is given by finite-size Gaussian random matrix theory. This at first came as a surprise as random matrix universality is usually only established in the limit of infinite matrix dimension. This new form of universality was shown [20] to have its origin in a kind of matrix-valued central limit theorem that applies to the perturbation matrix for the zero modes. The new finite-size Gaussian random matrix universality for the would-be zero modes is reached in the limit where the size of the remaining system is taken to infinity. In Ref. [20], Hermitian perturbations of the original ideal Hermitian system were considered, and it was possible to show that universal distributions of perturbed zero modes exist for all universality classes.

In the present work, we generalize this statement to weakly non-Hermitian perturbations. Systems with weak non-Hermiticity have been subject to several studies previously, both in the context of solid state physics [21–23] as well as for the strong interactions [24–28]. These studies for instance provided new insight into the nature of Andreev and Majorana modes, [23], and into the sign problem in QCD with chemical potential, see for instance [28]. Two examples of applications are given in Sec. II. We are in particular interested in non-Hermitian perturbations which violate the conditions responsible for the presence of exact zero modes in the original ideal system. The general matrix model we will consider is of the form

$$K = A + P, \quad (1)$$

where A is a Hermitian matrix describing the ideal system with exact symmetries and exact zero modes. The non-Hermitian perturbation is given by P .

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We restrict our study to two classes of perturbations. In the first case, the perturbation P_{unc} takes the form

$$P_{\text{unc}} = \alpha_R U_R S_R U_R^\dagger + i\alpha_I U_I S_I U_I^\dagger, \quad (2)$$

where S_R and iS_I are Hermitian and anti-Hermitian respectively and the real constants, α_R and α_I , are chosen to be small such that the first-order correction dominates. Their ratio will determine the ellipticity of the level density for the broadened zero eigenvalues. We will consider the average spectral properties of the smallest eigenvalues of K where the average is over the unitary matrices U_R and U_I . The average is taken over the respective Haar measure $d\mu(U)d\mu(V)$, where the two matrices U_R and U_I are considered to be statistically independent and there are therefore no correlations between the Hermitian and anti-Hermitian part.

The second class of perturbations P_c that we consider are of the form

$$P_c = \alpha U S V^\dagger. \quad (3)$$

The operator S is fixed and complex-valued. The unitary matrices U and V are drawn from the deformed Haar measure

$$e^{z\text{ReTr}[UV^\dagger]} d\mu(U)d\mu(V), \quad (4)$$

where $z > 0$ is a fixed number setting the eccentricity of the limiting elliptic support of the level density. This freedom of alignment in the complex plane is missing in the first model where the spectrum is either elongated along the real or imaginary axis. Note that the Hermitian and the anti-Hermitian parts of this perturbation are now correlated. For $z \rightarrow \infty$ and fixed matrix size, the two unitary matrices become almost the same $U \approx V$.

We find that the finite-size universality of the would-be zero modes extends to both types of non-Hermitian perturbations. The universal distributions are given by finite-size non-Hermitian Gaussian elliptic random matrix ensembles. We also demonstrate numerically the validity of the universal finite-size distributions for two specific random matrix realizations.

Because of the specific structure of the non-Hermitian perturbation in Eqs. (2) and (3), we may take advantage of the symmetry classification of Hermitian matrices done by Altland and Zirnbauer [29,30]. Those non-Hermitian symmetric matrix spaces are classified in [31–34]. We will consider all those symmetry classes in a unified way for both of our models. Surprisingly, both models yield the same limiting elliptic Gaussian ensemble whose joint matrix distribution have the same formal structure. They only differ in the symmetries of the matrices. Thus, we dub all of these ensembles elliptic Ginibre ensembles.

The presentation of these results is organized as follows. As a start in Section II we introduce two physical realizations of the matrix model, which we will later analyze numerically and compare to our analytical findings. Then follows the main proof, which comes in two steps. The first step is to establish the conditions under which first-order degenerate perturbation theory applies. We establish this in Sec. III, where we derive the conditions under which the characteristic equation of the perturbation factorizes. The second step is to determine the distribution of the perturbation matrices. In Sec. IV we show that they are distributed according to an elliptic Ginibre

ensemble with size determined by the number of zero modes. The conditions under which this holds are derived in Sec. IV. In Sec. V we compare this analytical result with numerical analyses of the two ensembles, which were introduced in Sec. II. In Sec. VI the universality of the distribution of the former zero modes is considered. Finally, in Sec. VII we conclude and discuss our results.

II. SYMMETRY CLASSES—TWO REALIZATIONS

To motivate the study and exemplify the results, we consider two realizations of physical interest. A chiral ensemble, being of relevance to QCD, and an ensemble with particle-hole symmetry relevant for solid-state systems with Majorana zero modes.

A. Chiral ensemble

The QCD Dirac operator for vanishing quark masses exhibits a chiral symmetry and the spontaneous breaking hereof induces a nonzero eigenvalue density of the Dirac operator at the origin [1]. Furthermore, the winding number of the gauge field gives rise to eigenvalues exactly at the origin; see, e.g., Ref. [35]. However, when simulating QCD on the lattice in the Wilson approach, the influence of the nonzero lattice spacing breaks both the chiral symmetry and also perturbs the zero modes. This affects the distribution of the smallest eigenvalues of the Dirac operator in a highly nontrivial manner [15–17]. When a chemical potential is introduced, we break Hermiticity [26,27] as well. It therefore becomes interesting to study the effect of a chiral ensemble perturbed by a non-Hermitian one.

This motivates us to study the following general non-Hermitian perturbation of the chiral random matrix model

$$K^{(N)} = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix} + \alpha_R U_R S_R U_R^\dagger + i\alpha_I U_I S_I U_I^\dagger, \quad (5)$$

where the initial Hamiltonian has dimensions $N \times N$ with $N = 2n + \nu$, and M has dimensions $(n + \nu) \times n$. Thus, for $\alpha_R = \alpha_I = 0$ the model exhibits ν zero modes. This allows us to control the exact number of zero modes, we wish to study. The matrices S_R and S_I have dimensions $N \times N$, are Hermitian and have otherwise no further symmetries. The only average is performed over the unitary matrices U_R and U_I , which are drawn via the Haar measure from the group determined by the symmetries of the ensemble. The real coupling constants α_R and α_I control the magnitude of the Hermitian and the anti-Hermitian part of the perturbation, respectively.

We will show that the perturbation matrix for the former zero modes is distributed according to a Gaussian in the complex plane with standard deviations $\sigma_l = \alpha_l \sqrt{\text{Tr}(S_l^{(j)})^2 / (\tilde{\gamma}_l^{(j)}(N_l^{(j)})^2)}$ where $l = R, I$ is the real and imaginary direction respectively. The parameters $\tilde{\gamma}_l^{(j)}$ depend on the symmetry class [see Eq. (28)]. It is basically the exponent of the determinant when performing a multivariate Gaussian integral, see Table II in Ref. [20]. Because the elements are Gaussian, the statistics of the broadening of the former zero eigenvalues are those of the elliptic Ginibre

ensemble, the one-point function of which is known [36],

$$R_1(\lambda) = \sqrt{\omega_\tau(\lambda)\omega_\tau(\lambda^*)} \sum_{n=0}^{\nu-1} \frac{\tau^n}{2^n n!} H_n\left(\frac{\lambda}{\sqrt{2\tau}}\right) H_n\left(\frac{\lambda^*}{\sqrt{2\tau}}\right), \quad (6)$$

where

$$\omega_\tau(\lambda) = \frac{1}{\pi\sqrt{1-\tau^2}} \exp\left[-\frac{|\lambda|^2}{1-\tau^2} + \frac{\tau(\lambda^2 + \lambda^{*2})}{2(1-\tau^2)}\right], \quad (7)$$

and the factor τ is defined as

$$\frac{\sqrt{1+\tau}}{\sqrt{1-\tau}} = \frac{\alpha_R \sqrt{\text{Tr}(S_R)^2}}{\alpha_I \sqrt{\text{Tr}(S_I)^2}}. \quad (8)$$

B. Majorana ensemble

In the study of superconductors carrying topological zero modes, see for instance Refs. [6,9,12,14,37–40], it becomes relevant to study the spectral properties of a Hamiltonian consisting of the direct sum of two antisymmetric Hermitian ensembles (Class D in the Cartan classification) that are the same up to a sign. This corresponds to particle-hole-symmetry [7,12,40], and the antisymmetric matrices are of odd size, which gives the system two identical zero modes. It has been pointed out that it is difficult to experimentally distinguish such an ensemble where the zero modes have been perturbed from an ensemble with an accumulation of eigenvalues around the origin [10]. It is therefore advantageous to study perturbations arising from thermal fluctuations that couple the two sectors. We here consider a generalization where the perturbation is non-Hermitian. Such a non-Hermiticity may arise from a coupling to an environment; see Ref. [23] for a relevant example.

In terms of a random matrix model this ensemble is

$$K^{(N)} = \begin{pmatrix} iM & 0 \\ 0 & -iM \end{pmatrix} + \alpha_R O_R \begin{pmatrix} 0 & iW_R \\ -iW_R^T & 0 \end{pmatrix} O_R^T + i\alpha_I O_I \begin{pmatrix} 0 & iW_I \\ -iW_I^T & 0 \end{pmatrix} O_I^T, \quad M = -M^T, \quad (9)$$

where M is real antisymmetric and has dimensions $2n + \nu$. W_R and W_I also have dimensions $2n + \nu$, are real but have no further symmetries. This ensemble is defined for all ν , but we need only $\nu = 0, 1$. The average is done over the orthogonal matrices O_R and O_I drawn from the Haar measure of the orthogonal group. As above, α_R and α_I are real and control the magnitude of the Hermitian and anti-Hermitian part of the perturbation, respectively.

As mentioned above, we will show that the perturbation matrix of the zero modes has a Gaussian distribution in the complex plane with standard deviations $\sigma_l = \alpha_l \sqrt{\text{Tr}(S_l^{(j)})^2 / [\tilde{\gamma}_l^{(j)} (N_l^{(j)})^2]}$, where $l = R, I$. This ensemble exhibits two exact zero modes when $\alpha_R = \alpha_I = 0$, hence, for ν odd the perturbation matrix is of dimension $(4n + 2\nu) \times (4n + 2\nu)$ and therefore $N = 4n + 2\nu$. The full ensemble matrix $K^{(N)}$ is antisymmetric, $K^T = -K$, which makes the perturbation matrix antisymmetric as well. If the perturbation matrix is distributed according to a Gaussian, then the broadened zero modes will be as well; see the ensuing example with

a 2×2 antisymmetric matrix:

$$0 = \det \left[i \begin{pmatrix} -\lambda & x \\ -x & -\lambda \end{pmatrix} \right] \Rightarrow \lambda^2 - x^2 = 0. \quad (10)$$

The probability density for elements with Gaussian weights is defined as

$$P(\lambda) = \int_{-\infty}^{\infty} dx \delta(\lambda - x) \frac{1}{\sqrt{\pi\sigma^2}} e^{-x^2/\sigma^2} = \frac{1}{\sqrt{\pi\sigma^2}} e^{-\lambda^2/\sigma^2}. \quad (11)$$

Therefore, the broadened zero modes are compared to Gaussian distributions in both the real and imaginary direction with the shown standard deviations,

$$p(\lambda) = \frac{\exp[-(\text{Re}[\lambda])^2/\sigma_R^2 - (\text{Im}[\lambda])^2/\sigma_I^2]}{\pi\sigma_R\sigma_I}. \quad (12)$$

Note that our result for the model (2) holds for all non-Hermitian classes. This is clear because we will show that our results hold for S_R and S_I in any Hermitian symmetry class. This will certainly give an overcounting, but we are guaranteed to cover all classes with noncorrelated Hermitian and anti-Hermitian part.

For the model (3), this is less clear, primarily because the non-Hermitian matrices have not yet been unambiguously classified [31–34]. We still conjecture that they hold in full generality, because symmetry constraints will merely give conditions on U and V . It is likely that the integrals with substructure may be solved the same way as for the different Hermitian classes.

III. FACTORIZATION OF THE CHARACTERISTIC EQUATION

First we show that, for sufficiently weak perturbation, the spectrum of the former zero modes decouples from the bulk. This corresponds to focusing on the part of the perturbation which corresponds to first-order degenerate perturbation theory. The approach here follows closely the one in Ref. [20]. We may deal with both forms, Eqs. (2) and (3), in a unified fashion. Our starting point is the following:

$$T = \alpha U S V^\dagger, \quad (13)$$

with

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad (14)$$

unitary and divide it into the sectors

$$T = \begin{pmatrix} T_1 & T_2 \\ T_4 & T_3 \end{pmatrix}. \quad (15)$$

We have organized the basis such that T_3 will correspond to the zero modes of A that is given in this basis as follows:

$$A = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}. \quad (16)$$

The notation is chosen in such a way to line up with S_3 , S_{R3} , and S_{I3} , as in Ref. [20].

Although the form (13) resembles the second model (3) instead of the first one (2), both can be dealt here in the same way. One needs to keep in mind that the result in

Ref. [20] gives a bound on $\alpha\|S\|_{\text{op}}$, where $\|\cdot\|_{\text{op}}$ is the operator norm (the largest singular value). As we shall see, the corresponding bound here will depend on $\|\alpha_R U_R S_R U_R^\dagger + i\alpha_I U_I S_I U_I^\dagger\|_{\text{op}}$ or $\|\alpha U S V^\dagger\|_{\text{op}} = \alpha\|S\|_{\text{op}}$. We underline that the operator norm is invariant under left and right multiplication of unitary matrices. Thus, when defining $\tilde{U} = U_R^\dagger U_I$ we can identify $S = \alpha_R S_R + i\alpha_I \tilde{U} S_I \tilde{U}^\dagger$. Since \tilde{U} is Haar distributed, we can have any combination of the eigenvalues of the real eigenvalues of $\alpha_R S_R$ and the imaginary eigenvalues of $i\alpha_I S_I$.

Therefore, we have the inequality $\|\alpha_R S_R + i\alpha_I \tilde{U} S_I \tilde{U}^\dagger\|_{\text{op}} \leq \sqrt{\alpha_R^2 \|S_R\|_{\text{op}}^2 + \alpha_I^2 \|S_I\|_{\text{op}}^2}$. The equality is given by the worst case when S_R , S_I , and \tilde{U} are diagonal and the eigenspaces of the largest eigenvalues of S_R and S_I are the same. In practice, this will not be the case, and the inequality will hold instead. Hence, we need to replace the operator norm by the norm $\sqrt{\alpha_R^2 \|S_R\|_{\text{op}}^2 + \alpha_I^2 \|S_I\|_{\text{op}}^2}$ for the estimates in the first model.

We consider the characteristic polynomial

$$\begin{aligned} \det(K^{(N)} - \lambda \mathbb{1}_N) &= \det \begin{pmatrix} A' + T_1 - \lambda \mathbb{1}_{N-\nu} & T_2 \\ T_4 & T_3 - \lambda \mathbb{1}_\nu \end{pmatrix} \\ &= \det(A' - \lambda \mathbb{1}_{N-\nu}) \det(\mathbb{1}_{N-\nu} + (A' - \lambda \mathbb{1}_{N-\nu})^{-1} T_1) \\ &\quad \times \det[T_3 - \lambda \mathbb{1}_\nu - T_4(\mathbb{1}_{N-\nu} + (A' - \lambda \mathbb{1}_{N-\nu})^{-1} T_1)^{-1} (A' - \lambda \mathbb{1}_{N-\nu})^{-1} T_2], \end{aligned} \quad (17)$$

where we first pulled out the factor $A' - \lambda \mathbb{1}_{N-\nu}$ and then the factor $\mathbb{1}_{N-\nu} + (A' - \lambda \mathbb{1}_{N-\nu})^{-1} T_1$. Next, we use $T_1 = U_1 \alpha S V_1^\dagger$ and express the inverse as a Neumann series

$$(\mathbb{1}_{N-\nu} + (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1 \alpha S V_1^\dagger)^{-1} = \sum_{j=0}^{\infty} [-(A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1 \alpha S V_1^\dagger]^j,$$

which can be exploited as follows:

$$\begin{aligned} &U_2 \alpha S (\mathbb{1}_{N-\nu} - V_1^\dagger (\mathbb{1}_{N-\nu} + (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1 \alpha S V_1^\dagger)^{-1} (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1 \alpha S V_2^\dagger) \\ &= U_2 \alpha S \left(\mathbb{1}_{N-\nu} + \sum_{j=1}^{\infty} [-V_1^\dagger (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1 \alpha S]^j \right) V_2^\dagger = U_2 \alpha S [\mathbb{1}_{N-\nu} + V_1^\dagger (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1 \alpha S]^{-1} V_2^\dagger. \end{aligned}$$

Here, we have made use of $T_2 = U_1 \alpha S V_2^\dagger$ and $T_4 = U_2 \alpha S V_1^\dagger$. Inserting this result into Eq. (17), the characteristic polynomial becomes

$$\det(K^{(N)} - \lambda \mathbb{1}_N) = \det(A' - \lambda \mathbb{1}_{N-\nu}) \det(\mathbb{1}_N + \alpha S V_1^\dagger (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1) \det(U_2 [\mathbb{1}_N + \alpha S V_1^\dagger (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1]^{-1} \alpha S U_2^\dagger - \lambda \mathbb{1}_\nu), \quad (18)$$

which is in full analogy to the Hermitian case in Ref. [20].

The same analogy carries even further. To make the first-order perturbation theory exact, we need the smallest eigenvalues of A' , given by $\|(A')^{-1}\|_{\text{op}}^{-1}$, to not interact with the broadened spectrum of the zero modes. The latter is represented by the operator in the third determinant of Eq. (18). Hence, we need

$$\|(A')^{-1}\|_{\text{op}}^{-1} \gg \|U_2 [\mathbb{1}_N + \alpha S V_1^\dagger (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1]^{-1} \alpha S U_2^\dagger\|_{\text{op}} = \alpha \|[\mathbb{1}_N + \alpha S V_1^\dagger (A' - \lambda \mathbb{1}_{N-\nu})^{-1} U_1]^{-1} S\|_{\text{op}}. \quad (19)$$

We underline that λ is of the order of the operator on the right hand side so that we can drop it in every combination of the form $A' - \lambda \mathbb{1}_{N-\nu}$. This simplifies the characteristic polynomial to

$$\det(K^{(N)} - \lambda \mathbb{1}_N) \approx \det(A') \det(\mathbb{1}_N + \alpha S V_1^\dagger (A')^{-1} U_1) \det(U_2 [\mathbb{1}_N + \alpha S V_1^\dagger (A')^{-1} U_1]^{-1} \alpha S U_2^\dagger - \lambda \mathbb{1}_\nu). \quad (20)$$

Furthermore, we would like to suppress the operator $\alpha S V_1^\dagger (A')^{-1} U_1$ compared to the identity matrix $\mathbb{1}_N$. In particular, we need $\|\alpha S V_1^\dagger (A')^{-1}\|_{\text{op}} \ll 1$. To understand of what order the generic value of $\|\alpha S V_1^\dagger (A')^{-1}\|_{\text{op}}$ is, we choose an arbitrary vector $|\chi\rangle \in \mathbb{C}^N$ and consider the m 'th moments of the squared norm of $V_1^\dagger (A')^{-1} U_1 \alpha S |\chi\rangle$, which gives the conditions

$$\int_{\mathcal{K}} d\mu(U) (\langle \chi | \alpha S^\dagger U_1^\dagger (A')^{-2} U_1 \alpha S | \chi \rangle)^m \leq c_m \left(\frac{\alpha^2 \text{Tr}(A')^{-2} \langle \chi | S^\dagger S | \chi \rangle}{N} \right)^m \ll 1. \quad (21)$$

Here c_m is some constant of order 1 and \mathcal{K} is the Haar measure of the appropriate unitary group. Since the vector $|\chi\rangle$ is arbitrary, we can also choose the eigenvector to the largest singular value $\|S\|_{\text{op}}$ of S . Combining Eqs. (19) and (21), we need to assume

$$\alpha \frac{\sqrt{\text{Tr}(A')^{-2}} \|S\|_{\text{op}}}{\sqrt{N}} \ll 1 \quad (22)$$

for the second model, which indeed also implies $\|A^{-1}\|_{\text{op}}^{-1} \gg \alpha \|S\|_{\text{op}}$, and

$$\frac{\sqrt{\text{Tr}(A')^{-2}} \|\alpha_R U_R S_R U_R^\dagger + i\alpha_I U_I S_I U_I^\dagger\|_{\text{op}}}{\sqrt{N}} \leq \sqrt{\frac{\text{Tr}(A')^{-2} (\alpha_R^2 \|S_R\|_{\text{op}}^2 + \alpha_I^2 \|S_I\|_{\text{op}}^2)}{N}} \ll 1 \quad (23)$$

for the first model. These two conditions allow us to restrict the discussion to first order perturbation theory. In practice this means that we only need to study the spectrum of

$$\alpha_R S_{R3} + i\alpha_I S_{I3} = \alpha_R U_{R2} S_R U_{R2}^\dagger + i\alpha_I U_{I2} S_I U_{I2}^\dagger \quad \text{or} \quad \alpha S_3 = \alpha U_2 S V_2^\dagger, \quad (24)$$

as these matrices determine the distribution of the former zero modes. We will derive the distribution of these matrices in the next section.

Before coming to calculating this distribution, we would like to emphasize the following. Although first order perturbation theory of the first model may dictate that the spectrum of a purely real or imaginary perturbation only spreads in the according direction, eventually the spectrum will start to invade the whole complex plane.

IV. DISTRIBUTION OF S_3 —A MATRIX-VALUED CENTRAL LIMIT THEOREM

We move on with a derivation of the distribution of the perturbation matrix and how this broadens the zero modes of our Hamiltonian. We consider the two classes, Eqs. (2) and (3), in turn.

A. Perturbation with uncorrelated Hermitian and anti-Hermitian parts

We start by considering perturbations of the form P_{unc} of Eq. (2). Since the Hermitian and anti-Hermitian parts are uncorrelated the result from the Hermitian case [20] can be exploited for the two parts independently. To apply this result we need the additional conditions

$$\lim_{N \rightarrow \infty} \frac{\sqrt{\text{Tr}(S_{R/I})^2}}{\|S_{R/I}\|_{\text{op}}} = \infty \quad (25)$$

and

$$\text{Tr}(S_{R/I}) = 0. \quad (26)$$

These essentially state that a big portion of the singular values have to be of the same order as the largest one. Looking into the proof of the Hermitian case in Ref. [20], this condition guarantees that the distributions of S_{R3} and S_{I3} converge to Gaussians. Without it, deviations from the Gaussian behavior are quite likely.

$$\tilde{\gamma}_l^{(j)} = \begin{cases} \frac{1}{4}, & \text{nonchiral and an antiunitary symmetry,} \\ 1, & \text{nonchiral and no antiunitary symmetry or one of the two Bogoliubov–de Gennes operators,} \\ \frac{p_l^{(j)} n_l^{(j)}}{(N_l^{(j)})^2}, & \text{chiral and no antiunitary symmetry,} \\ \frac{p_l^{(j)} n_l^{(j)}}{2(N_l^{(j)})^2}, & \text{the remaining two chiral operator classes.} \end{cases} \quad (28)$$

For the standard chiral ensembles we have the operator dimensions $p_l^{(j)}, n_l^{(j)} \propto N_l^{(j)} \gg 1$ with $p_l^{(j)} + n_l^{(j)} = N_l^{(j)}$. To be precise, we need to assume $\lim_{N \rightarrow \infty} N_l^{(j)} = \infty$, $\text{Tr} S_l^{(j)} = 0$ as

For the sake of generality, we choose S_R and S_I being independently a direct sum of operators in one of the ten symmetry classes rather than just a single one. The direct sum is highly important because some ensembles in the Magnea–Bernard–LeClair classification [31–33] have real and imaginary parts that decompose in direct sums, e.g., the Wilson–Dirac operator [[15–17], [19]]. This happens exactly then when S has a pseudo-Hermiticity property, meaning $S^\dagger = \gamma_5 S \gamma_5$ with $\gamma_5 = \gamma_5^\dagger = \gamma_5^{-1}$. Then there is a basis where the Hermitian part is block diagonal and the anti-Hermitian part is chiral, meaning the direct sum has maximal two components in the standard symmetry classification by Magnea [32]. The role of Hermitian and anti-Hermitian part may be reversed when having $S^\dagger = -\gamma_5 S \gamma_5$. The notation of γ_5 is reminiscent of the γ_5 matrix in the four dimensional Dirac theory, like the QCD Dirac operator.

The dimension of the subspaces for the decompositions $S_R = \bigoplus_{j_R} S_R^{(j_R)}$ and $S_I = \bigoplus_{j_I} S_I^{(j_I)}$ are $N_R^{(j_R)}$ and $N_I^{(j_I)}$, respectively. The corresponding unitary matrices U_R and U_I have an according decomposition $U_R = \bigoplus_{j_R} U_R^{(j_R)}$ and $U_I = \bigoplus_{j_I} U_I^{(j_I)}$, where $S_R^{(j_R)} \rightarrow U_R^{(j_R)} S_R^{(j_R)} (U_R^{(j_R)})^\dagger$ and $S_I^{(j_I)} \rightarrow U_I^{(j_I)} S_I^{(j_I)} (U_I^{(j_I)})^\dagger$ keeps the respective global symmetries invariant and generates the largest compact groups distributed along the corresponding Haar measure. Then, the distribution of each matrix $S_{l,j}^{(j)} = U_l^{(j)} S_l^{(j)} (U_l^{(j)})^\dagger$ with $l = R, I$ is for large N given by [20]

$$p(S_{l,j}^{(j)}) = \frac{\exp[-\tilde{\gamma}_l^{(j)} (N_l^{(j)})^2 \text{Tr}(S_{l,j}^{(j)})^2 / \text{Tr}(S_l^{(j)})^2]}{\int d\tilde{S} \exp[-\tilde{\gamma}_l^{(j)} (N_l^{(j)})^2 \text{Tr}\tilde{S}^2 / \text{Tr}(S_l^{(j)})^2]}, \quad (27)$$

with the denominator properly normalizing the distribution and $\tilde{\gamma}_l^{(j)}$ a parameter of order one that depends on the chosen symmetry class of the Altland–Zirnbauer classification [29,30], i.e.,

well as $\lim_{N \rightarrow \infty} \sqrt{\text{Tr}(S_l^{(j)})^2} / \|S_l^{(j)}\|_{\text{op}} = \infty$ for each $l = R, I$, and j to find this result. The traceless condition is a nontrivial restriction for a decomposition into a true direct sum. This

has to be seen in contrast to when no direct sum is present. Here the whole operator is only shifted by a scalar times the identity matrix. This is not true for a direct sum, where we get a shift by a diagonal matrix consisting of blocks of different scalars in front of the identity matrix when we do not have the trace condition. This has indeed physical effects as known from the Wilson–Dirac random matrix model [15–17], [19] and its corresponding lattice QCD-operator [41].

In total, the distribution of the full perturbation $S'_3 = \alpha_R S_{R3} + i\alpha_I S_{I3} = \alpha_R \bigoplus_{j_R} S_R^{(j_R)} + i\alpha_I \bigoplus_{j_I} S_I^{(j_I)}$ converges in the large N -limit to the joint Gaussian probability density

$$p(S'_3) = \prod_{l=R,I} \prod_{j_l} p(S'_{l,j_l}). \quad (29)$$

This constitutes the matrix central limit theorem for the perturbation matrices S'_3 mentioned in the introduction. When assuming that for both $l=R, I$ there is a $\sigma_l = \alpha_l \sqrt{\text{Tr}(S_l^{(j_l)})^2 / [\tilde{\gamma}_l^{(j_l)}(N_l^{(j_l)})^2]}$ for all j , then we have the simplification of the distribution to the elliptic Ginibre ensemble for one of the non-Hermitian ensembles [31,31],

$$p(S'_3) = \frac{\exp[-\text{Tr}(S_{R3})^2/\sigma_R^2 - \text{Tr}(S_{I3})^2/\sigma_I^2]}{\int d\tilde{S}_{R3} d\tilde{S}_{I3} \exp[-\text{Tr}(\tilde{S}_{R3})^2/\sigma_R^2 - \text{Tr}(\tilde{S}_{I3})^2/\sigma_I^2]}. \quad (30)$$

Let us summarize and underline once again that the homogeneity $\sigma_l = \alpha_l \sqrt{\frac{\text{Tr}(S_l^{(j_l)})^2}{\tilde{\gamma}_l^{(j_l)}(N_l^{(j_l)})^2}}$ and being traceless $\text{Tr}S_l^{(j)} = 0$ are for the single components of a direct sum of operators not always guaranteed, e.g., for the Wilson–Dirac operator [[15–17], [19]]. Thus, the Gaussian distribution becomes noncentered and has a nontrivial covariance matrix. Nevertheless, the joint distribution of $S'_3 = \alpha_R U_{R2} S_R U_{R2}^\dagger + i\alpha_I U_{I2} S_I U_{I2}^\dagger$ remains Gaussian as long as the conditions $\lim_{N \rightarrow \infty} \sqrt{\text{Tr}(S_l^{(j)})^2} / \|S_l^{(j)}\|_{\text{op}} = \infty$ for all $l=R, I$, and j as well as Eq. (23) are satisfied.

B. Perturbation with correlated Hermitian and anti-Hermitian parts and deformed Haar measure

We now turn to the second form where the perturbation is given by Eq. (3). The condition that corresponds to Eq. (25) reads

$$\lim_{N \rightarrow \infty} q^{(N)} = \infty, \quad (31)$$

where

$$q^{(N)} = \frac{\sqrt{\text{Tr}SS^\dagger}}{\|S\|_{\text{op}}}. \quad (32)$$

We again also assume $\text{Tr}S = 0$ to simplify the problem. As introduced in Eqs. (3) and (4), we consider the perturbation

$$K = A + \alpha USV^\dagger. \quad (33)$$

This time the perturbation does not need to decompose into a direct sum since we always consider the sum of the Hermitian and anti-Hermitian part as a whole. This detail can be implemented in $S = e^{i\varphi}[\cos(\vartheta)S_R + i\sin(\vartheta)S_I]$, where S_R and S_I are the two Hermitian components in one of the non-Hermitian symmetry classes [32] and $\vartheta, \varphi \in [-\pi, \pi]$ embed

the perturbation somehow in the complex matrices. Since the phase $e^{i\varphi}$ is only a scalar rotation, we can set it to 1 for the computation and later multiply it to the matrix again.

To understand the nature of the unitary matrices U and V , we need to consider different cases that concerns the global symmetries. One symmetry is a possible pseudo-Hermiticity like $S^\dagger = \gamma_5 S \gamma_5$. Here we have the relation $V = \gamma_5 U \gamma_5$. In the case of pseudosymmetry like $S^T = \pm \gamma_5 S \gamma_5$, this relation reads $V = \tilde{\gamma}_5 U^* \tilde{\gamma}_5$ with a different $\tilde{\gamma}_5 = \tilde{\gamma}_5^\dagger = \tilde{\gamma}_5^{-1}$. Certainly, S may fulfill none or both. In the latter situation S as well as U satisfy reality conditions, $S^* = \pm \gamma_5 \tilde{\gamma}_5 S \tilde{\gamma}_5 \gamma_5$ and $U^* = \gamma_5 \tilde{\gamma}_5 U \tilde{\gamma}_5 \gamma_5$.

To simplify the situation and not to discuss all four cases, separately, we exploit the fact that operators which do not satisfy one or both of these symmetries can be embedded always in those where this is indeed the case. To this aim we consider instead of $K = A + \alpha USV^\dagger$ the enlarged matrices,

$$\begin{aligned} & \text{diag}(K, K^*, K^\dagger, K^T) \\ &= \text{diag}(A, A^*, A^\dagger, A^T) + \alpha \text{diag}(U, U^*, V, V^*) \\ & \quad \times \text{diag}(S, S^*, S^\dagger, S^T) \tilde{\gamma}_5^{(N)} \text{diag}(U^\dagger, U^T, V^\dagger, V^T) \tilde{\gamma}_5, \end{aligned} \quad (34)$$

with

$$\tilde{\gamma}_5^{(N)} = \begin{pmatrix} 0 & \mathbb{1}_{2N} \\ \mathbb{1}_{2N} & 0 \end{pmatrix}. \quad (35)$$

We relabel the dimension $4N = \hat{N}$, $\text{diag}(K, K^*, K^\dagger, K^T) = \hat{K}$, $\text{diag}(S, S^*, S^\dagger, S^T) = \hat{S}$, $\text{diag}(U, U^*, V, V^*) = \hat{U}$, and so forth. We can therefore, without restriction of generality, assume that S or better $\hat{U} \hat{S} \tilde{\gamma}_5^{(N)} \hat{U}^\dagger \tilde{\gamma}_5^{(N)}$ satisfies a pseudo-Hermiticity condition with the matrix $\tilde{\gamma}_5$ and a pseudosymmetry condition with the matrix

$$\tilde{\gamma}_5^{(N)} = \begin{pmatrix} 0 & 0 & \mathbb{1}_N \\ 0 & \mathbb{1}_N & 0 \\ \mathbb{1}_N & 0 & 0 \end{pmatrix}, \quad (36)$$

and, therefore, also a reality condition which may lead to one of the three number fields. Indeed, S may additionally satisfy commutation relations with fixed matrices. This is usually the case when it has a block-diagonal or chiral structure or when it has had already a pseudo-Hermiticity and/or pseudosymmetry condition before the embedding (34). Moreover, the unitary matrix \hat{U} might be real, complex or quaternion keeping the symmetry under the complex conjugation as well as block-diagonal or a full matrix that keeps the possible chiral or block structures invariant. Those choices still reflect the Altland–Zirnbauer classification [29,30].

Once this is settled, we can go over to the distribution of the unitary matrix U , which is

$$d\tilde{\mu}(\hat{U}) = \frac{\exp(z \text{Tr}[\hat{U} \tilde{\gamma}_5^{(N)} \hat{U}^\dagger \tilde{\gamma}_5^{(N)}]/4) d\mu(\hat{U})}{\int \exp(z \text{Tr}[\hat{U} \tilde{\gamma}_5^{(N)} \hat{U}^\dagger \tilde{\gamma}_5^{(N)}]/4) d\mu(\hat{U})} \quad \text{with } z > 0. \quad (37)$$

In the case when neither a pseudo-Hermiticity or pseudosymmetry originally existed, this measure agrees with Eq. (4) after employing the embedding (34). In any case, the deformation of the Haar measure in Eq. (37) has the effect that for $z \rightarrow \infty$, the unitary matrix starts to align to the

relation $\hat{U} = \hat{\gamma}_5^{(N)} \hat{U} \hat{\gamma}_5^{(N)}$. Employing the original model with the embedding (34), this would translate to $U = V$. When z is correlated with the angle ϑ such that $\vartheta \rightarrow 0$ or $\vartheta \rightarrow \pi/2$ for $z \rightarrow \infty$, the perturbation becomes essentially a Hermitian matrix. In comparison, the limit $z \rightarrow 0$ yields the unmodified Haar measure which always corresponds to the full non-Hermitian case. Since we have now two parameters, z and τ , which have the same impact, we can eliminate one of those. Therefore, we choose S to be in one of the ten symmetry classes of the Altland–Zirnbauer scheme [29,30], meaning $\vartheta = 0, \pi/2$. Then, we select one non-Hermitian symmetric space in the Magnea scheme [32], where S is either the Hermitian or anti-Hermitian part. Afterwards, we employ the embedding (34). The unitary matrix space for \hat{U} is then the maximally compact subgroup of the invariance group of these symmetric spaces.

Let us give three examples to illustrate this construction:

(1) The first one is the choice of S being a real symmetric matrix and embedded in the complex symmetric matrices. The invariance group of the complex symmetric matrices is given by the map $P \rightarrow GPG^T$ with $G \in \text{Gl}(N, \mathbb{C})$ and, therefore, the general linear complex group. The corresponding maximally compact subgroup are the unitary matrices with the map $P \rightarrow UPU^T$. Thus, $V = U^*$ and U is a unitary matrix. The enlarged unitary matrix is $\hat{U} = \text{diag}(U, U^*, U^*, U)$. In the limit for $z \rightarrow \infty$, U becomes a real orthogonal matrix.

(2) For the second example we again choose S to be real symmetric but now embedded in the real matrices. The invariance group of the real matrices is given by the map $P \rightarrow G_1PG_2^T$ with $(G_1, G_2) \in \text{Gl}(N, \mathbb{R}) \times \text{Gl}(N, \mathbb{R})$. This time the maximally compact subgroup is the product of two real

orthogonal groups $O(N) \times O(N)$ with the map $P \rightarrow UPV^T$ with the enlarged unitary matrix $\hat{U} = \text{diag}(U, U, V, V)$. When we take $z \rightarrow \infty$, the two matrices reduce to a single one via $U = V$.

(3) The last example is given by the choice of S being the direct sum of two Hermitian matrices of dimensions $n \times n$ and $(n + \nu) \times (n + \nu)$ embedded in the $\gamma_5 = \text{diag}(\mathbb{1}_n, -\mathbb{1}_{n+\nu})$ -Hermitian matrices, i.e., $P^\dagger = \gamma_5 P \gamma_5$. The invariance group of these pseudo-Hermitian matrices is determined by the map $P \rightarrow GP\gamma_5G^\dagger\gamma_5$ with $G \in \text{Gl}(N, \mathbb{C})$ and its maximally compact subgroup is the unitary group $U(N)$. Thence, the matrix U is unitary and $V = \gamma_5 U \gamma_5$ and $\hat{U} = \text{diag}(U, U^*, \gamma_5 U \gamma_5, \gamma_5 U^* \gamma_5)$. In this case, the limit $z \rightarrow \infty$ leads to a block-diagonal form of $U = \gamma_5 U \gamma_5$.

The derivation of the large N limit follows closely the one for the Hermitian case, see Ref. [20]. The normalized Haar measure $d\mu(\hat{U})$ for \hat{U} over the compact group \mathcal{K} can be expressed in terms of a Dirac δ function in the matrix group $\mathcal{G} \supset \mathcal{K}$ which keeps the non-Hermitian matrices invariant, i.e.,

$$\int_{\mathcal{K}} f(\hat{U}) d\mu(\hat{U}) = \frac{\int_{\mathcal{G}} f(\hat{U}) \delta(\mathbb{1}_{\hat{N}} - \hat{U} \hat{U}^\dagger) d\hat{U}}{\int_{\mathcal{G}} \delta(\mathbb{1}_{\hat{N}} - \hat{U} \hat{U}^\dagger) d\hat{U}}, \quad (38)$$

for an arbitrary suitably integrable function f . As we have seen in the three examples and can be readily checked in general, the closure of these matrix groups are always vector spaces, in particular some kind of embeddings of Cartesian products of one of the three general linear groups. The Dirac δ function may, however, be expressed as a Fourier–Laplace transform on the Hermitian matrix space $\mathcal{H} = \text{span}(\mathcal{G}\mathcal{G}^\dagger) = \text{span}\{GG^\dagger | G \in \mathcal{G}\}$, where span is the linear span, i.e.,

$$\int_{\mathcal{K}} f(\hat{U}) d\mu(\hat{U}) = \lim_{\epsilon \rightarrow 0} \frac{\int_{\mathcal{G}} d\hat{U} \int_{\mathcal{H}} d\hat{Q} f(\hat{U}) \exp[\epsilon \text{Tr}(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 + \xi \text{Tr}(\mathbb{1}_{\hat{N}} - \hat{U} \hat{U}^\dagger)(\mathbb{1}_{\hat{N}} - i\hat{Q})]}{\int_{\mathcal{G}} d\hat{U} \int_{\mathcal{H}} d\hat{Q} \exp[\epsilon \text{Tr}(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 + \xi \text{Tr}(\mathbb{1}_{\hat{N}} - \hat{U} \hat{U}^\dagger)(\mathbb{1}_{\hat{N}} - i\hat{Q})]}. \quad (39)$$

The matrix \hat{Q} has the form $\text{diag}(Q_1, Q_1^*, Q_2, Q_2^*)$ with Q_1 and Q_2 Hermitian matrices which may satisfy some relations if U and V do. The auxiliary variable $\epsilon > 0$ renders the integrals absolutely integrable and the parameter $\xi > 0$ will be chosen appropriately when performing the saddle point analysis. The variable ξ is independent of ϵ but depends on N, z as well as the chosen symmetry class as we will see later on.

The function f in the denominator is the deformation of the Haar measure $e^{z \text{Tr}(\hat{U} \hat{\gamma}_5^{(N)} \hat{U}^\dagger \hat{\gamma}_5^{(N)})/4}$ and in the numerator we have additionally the Dirac δ function,

$$\delta(\hat{S}'_3 - \kappa \hat{U}_2 \hat{S} \hat{\gamma}_5^{(N)} \hat{U}_2^\dagger \hat{\gamma}_5^{(v)}) = \lim_{\epsilon \rightarrow 0} \frac{\int_{\mathcal{M}} dH \exp[-\epsilon \text{Tr}(\hat{H} \hat{\gamma}_5^{(v)})^2 + i \text{Tr} \hat{H} (\hat{S}'_3 - \kappa \hat{U}_2 \hat{S} \hat{\gamma}_5^{(N)} \hat{U}_2^\dagger \hat{\gamma}_5^{(v)})]}{\int_{\mathcal{M}} dS'_3 \int_{\mathcal{M}} dH \exp[-\text{Tr}(\hat{H} \hat{\gamma}_5^{(v)})^2 + i \text{Tr} \hat{H} \hat{S}'_3]}, \quad (40)$$

with $\kappa > 0$ again a parameter that has to be appropriately chosen. The matrix \hat{H} depends on H like $\hat{H} = \text{diag}(H, H^*, H^\dagger, H^T)$ and the same holds for \hat{S}'_3 and S'_3 . Already in the first model we have seen that it is proportional to $N/\sqrt{\text{Tr} S S^\dagger}$ which is also true here. The matrix space \mathcal{M} is one of the symmetric non-Hermitian matrix spaces given in Magnea's work [32] (note that with the embedding all classes have pseudo-Hermiticity). We would like to point out that we need the matrix $\hat{\gamma}_5^{(j)}$ in two different dimensions, see Eq. (35), namely, $j = N$ for the symmetry of S and $j = \nu$ for the projected subspace of the broadened zero modes.

We write $U_2 = \Pi_2 U$ with Π_2 the projection onto the last ν rows and denote the embedded matrices by $\hat{U}_2, \hat{\Pi}_2$, and \hat{U} . Plugging Eq. (40) into Eq. (39), the integral over \hat{U} becomes Gaussian. As already mentioned, \hat{U} might be in one of the three number fields as well as have a block-diagonal structure that certainly contains U and U^* but never U^T or U^\dagger (this follows from the group multiplication property) as we know from the embedding. Thus, $\hat{U} \in \mathcal{G}$ can be understood as a very large real vector whose real entries may appear multiple times. The multiplicity can only be 1,2,4,8 when restricting $P = \alpha U S V^\dagger$ to one of the standard non-Hermitian symmetry classes [31,32].

Let d_G be the dimension of \mathcal{G} . Then, we have the following Gaussian integral

$$\begin{aligned} & \frac{\int_{\mathcal{G}} d\hat{U} \exp\left[\frac{z}{4}\text{Tr}[\hat{U}\hat{\gamma}_5^{(N)}\hat{U}^\dagger\hat{\gamma}_5^{(N)}] - i\kappa\text{Tr}\hat{H}\hat{\Pi}_2\hat{U}\hat{S}\hat{\gamma}_5^{(N)}\hat{U}^\dagger\hat{\Pi}_2^\dagger\hat{\gamma}_5^{(v)} - \xi\text{Tr}\hat{U}\hat{U}^\dagger(\mathbb{1}_{\hat{N}} - i\hat{Q})\right]}{\int_{\mathcal{G}} d\hat{U} \int_{\mathcal{H}} d\hat{Q} \exp\left[\frac{z}{4}\text{Tr}[\hat{U}\hat{\gamma}_5^{(N)}\hat{U}^\dagger\hat{\gamma}_5^{(N)}] + \epsilon\text{Tr}(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 + \xi\text{Tr}(\mathbb{1}_{\hat{N}} - \hat{U}\hat{U}^\dagger)(\mathbb{1}_{\hat{N}} - i\hat{Q})\right]} \\ &= \frac{\det^{-d_G/(2\hat{N}^2)}[\mathbb{1}_{\hat{N}} \otimes (\mathbb{1}_{\hat{N}} - i\hat{Q}) + i\frac{\kappa}{\xi}\hat{\Pi}_2^\dagger\hat{\gamma}_5^{(v)}\hat{H}\hat{\Pi}_2 \otimes \hat{S}\hat{\gamma}_5^{(N)} - \frac{z}{4\xi}\hat{\gamma}_5^{(N)} \otimes \hat{\gamma}_5^{(N)}]}{\int_{\mathcal{H}} d\hat{Q} \det^{-d_G/(4\hat{N})}[(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 - \frac{z^2}{16\xi^2}\mathbb{1}_{\hat{N}}] \exp[\epsilon\text{Tr}(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 + \xi\text{Tr}(\mathbb{1}_{\hat{N}} - i\hat{Q})]}. \end{aligned} \quad (41)$$

We exploited here that half of the eigenvalues of $\hat{\gamma}_5^{(N)}$ are +1 and the other half is -1.

The saddle point analysis of the \hat{Q} integral for large N is dominated by the Lagrangian

$$L(\hat{Q}) = \xi\text{Tr}(\mathbb{1}_{\hat{N}} - i\hat{Q}) - \frac{d_G}{4\hat{N}}\text{Tr}\ln\left[(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 - \frac{z^2}{16\xi^2}\mathbb{1}_{\hat{N}}\right]. \quad (42)$$

The ϵ dependent term is tiny because ϵ will be sent to 0. Moreover, we can read off the scaling of ξ which has to be linear in N . We note that $d_G = \mathcal{O}(N^2)$. Due to the singularity of the integrand at $(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 = z^2/(16\xi^2)$ preventing us from shifting the contour to other saddle points, we have a unique solution at

$$\mathbb{1}_{\hat{N}} - i\hat{Q}_0 = \frac{d_G + \sqrt{d_G^2 + z^2\hat{N}^2}}{4\hat{N}\xi}\mathbb{1}_{\hat{N}} = q\mathbb{1}_{\hat{N}}. \quad (43)$$

When expanding the integrand in $\hat{Q} = \hat{Q}_0 + \delta\hat{Q}/\sqrt{N}$, the integral over the massive modes $\delta\hat{Q}$ becomes a Gaussian and can be carried out. When choosing κ appropriately as we will do soon, the remaining determinant in the numerator will not change this saddle point approximation. We are left with

$$p(S'_3) = \lim_{\epsilon \rightarrow 0} \frac{\int_{\mathcal{M}} dH \exp[-\epsilon\text{Tr}(\hat{H}\hat{\gamma}_5^{(v)})^2 + i\text{Tr}\hat{H}\hat{S}'_3 + \tilde{L}(H)]}{\int_{\mathcal{M}} dS'_3 \int_{\mathcal{M}} dH \exp[-\text{Tr}(\hat{H}\hat{\gamma}_5^{(v)})^2 + i\text{Tr}\hat{H}\hat{S}'_3]}, \quad (44)$$

with

$$\begin{aligned} \tilde{L}(H) &= -\frac{d_G}{2\hat{N}^2}\text{Tr}\ln\left[q\mathbb{1}_{\hat{N}^2} + i\frac{\kappa}{\xi}\hat{\Pi}_2^\dagger\hat{\gamma}_5^{(v)}\hat{H}\hat{\Pi}_2 \otimes \hat{S}\hat{\gamma}_5^{(N)} - \frac{z}{4\xi}\hat{\gamma}_5^{(N)} \otimes \hat{\gamma}_5^{(N)}\right] + \frac{d_G}{4}\ln\left(q^2 - \frac{z^2}{16\xi^2}\right) \\ &= -\frac{d_G}{2\hat{N}^2}\text{Tr}\ln\left[\mathbb{1}_{\hat{N}^2} + i\frac{4\kappa}{16\xi^2q^2 - z^2}\hat{\Pi}_2^\dagger\hat{\gamma}_5^{(v)}\hat{H}\hat{\Pi}_2 \otimes \hat{S}\hat{\gamma}_5^{(N)}(4\xi q\mathbb{1}_{\hat{N}^2} - z\hat{\gamma}_5^{(N)} \otimes \hat{\gamma}_5^{(N)})\right]. \end{aligned} \quad (45)$$

When choosing κ of the order $\mathcal{O}(1/\sqrt{\text{Tr}SS^\dagger})$, we can expand the logarithm in \hat{S} . The first term vanishes exactly because $\text{Tr}\hat{S} = 4\text{ReTr}S = 0$ and $\text{Tr}\hat{S}\hat{\gamma}_5^{(N)} = 0$. The higher-order terms asymptotically vanish in the large N limit because of the estimate

$$\left|\frac{\text{Tr}[\hat{S}(c_1\mathbb{1}_{\hat{N}} + c_2\hat{\gamma}_5^{(N)})^j]}{(\text{Tr}SS^\dagger)^{j/2}}\right| \leq 4(|c_1| + |c_2|)^j \left(\frac{\|\hat{S}\|_{\text{op}}}{\sqrt{\text{Tr}SS^\dagger}}\right)^{j-2} = 4(|c_1| + |c_2|)^j (q^{(N)})^{2-j} \xrightarrow{N \rightarrow \infty} 0, \quad (46)$$

with two constants c_1 and c_2 of order one or smaller, as it is the case for us. For the limit we exploited the assumption (31).

Summarizing the Lagrangian $\tilde{L}(H)$ asymptotes to the quadratic term

$$\tilde{L}(H) \stackrel{N \gg 1}{\approx} -\frac{4d_G\kappa^2}{\hat{N}^2(16\xi^2q^2 - z^2)^2} [16\xi^2q^2\text{Tr}(\hat{\gamma}_5^{(v)}\hat{H})^2 \text{Tr}(\hat{S}\hat{\gamma}_5^{(N)})^2 + z^2\text{Tr}\hat{H}^2 \text{Tr}\hat{S}^2]. \quad (47)$$

The mixed term vanishes because $\text{Tr}\hat{S}^2\hat{\gamma}_5^{(N)} = 0$, and we have employed $\hat{\Pi}_2\hat{\gamma}_5^{(N)}\hat{\Pi}_2^\dagger = \hat{\gamma}_5^{(v)}$. Since we now have a integration-guaranteeing term we can perform the limit $\epsilon \rightarrow 0$, exactly. Hence, we end up with the distribution

$$p(S'_3) = \frac{\int_{\mathcal{M}} dH \exp[i\text{Tr}\hat{H}\hat{S}'_3 + \tilde{L}(H)]}{\int_{\mathcal{M}} dS'_3 \int_{\mathcal{M}} dH \exp[-\text{Tr}(\hat{H}\hat{\gamma}_5^{(v)})^2 + i\text{Tr}\hat{H}\hat{S}'_3]} = \frac{\exp[-\tilde{a}\text{Tr}(\hat{\gamma}_5^{(v)}\hat{S}'_3)^2 + \tilde{b}\text{Tr}(\hat{S}'_3)^2]}{\int_{\mathcal{M}} dS'_3 \exp[-\tilde{a}\text{Tr}(\hat{\gamma}_5^{(v)}\hat{S}'_3)^2 + \tilde{b}\text{Tr}(\hat{S}'_3)^2]}, \quad (48)$$

with

$$\tilde{a} = \frac{\hat{N}^2((4\xi q)^2 - z^2)^2}{16d_G\kappa^2} \frac{(4\xi q)^2\text{Tr}(\hat{S}\hat{\gamma}_5^{(N)})^2}{(4\xi q)^4(\text{Tr}(\hat{S}\hat{\gamma}_5^{(N)})^2)^2 - z^4(\text{Tr}\hat{S}^2)^2}, \quad \tilde{b} = \frac{\hat{N}^2((4\xi q)^2 - z^2)^2}{16d_G\kappa^2} \frac{z^2\text{Tr}\hat{S}^2}{(4\xi q)^4(\text{Tr}(\hat{S}\hat{\gamma}_5^{(N)})^2)^2 - z^4(\text{Tr}\hat{S}^2)^2}. \quad (49)$$

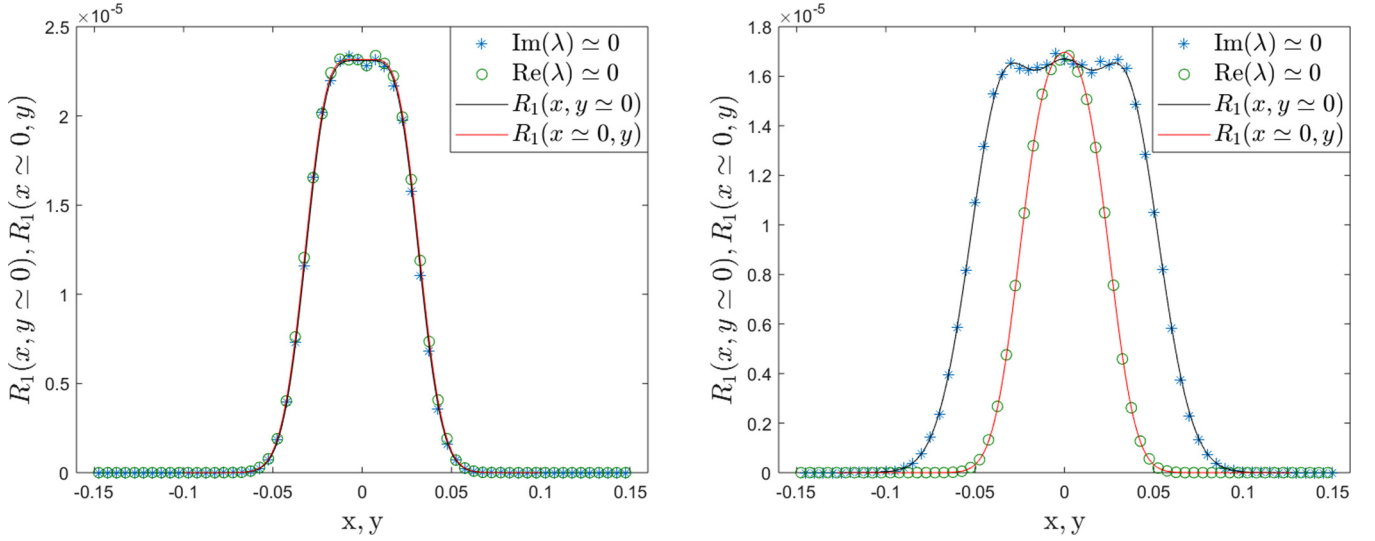


FIG. 1. The eigenvalue distribution of the chiral ensemble. Shown are the distributions of the smallest eigenvalues in a narrow bin either along the real or the imaginary axis. In the first case (left) the perturbation is of equal magnitude in the real and imaginary direction, $\alpha_R = \alpha_I = 0.01$, while in the second case (right) the perturbations are of different magnitude, $\alpha_R = 0.015$ and $\alpha_I = 0.01$. In both figures the matrix dimension is $N = 2n + \nu$ with $n = 40$ and the number of zero modes is $\nu = 3$. The numerical points are obtained from an ensemble of 10^6 matrices. The eigenvalue density in the neighbourhood of the real and the imaginary axis follow the parameter free theoretical prediction given by the circular and the elliptic Ginibre ensemble, respectively.

To find the second line, we have shifted as follows:

$$\hat{H} \rightarrow \hat{H} + 2i\tilde{a}\hat{\gamma}_5^{(\nu)}\hat{S}'_3\hat{\gamma}_5^{(\nu)} - 2i\tilde{b}\hat{S}'_3, \quad (50)$$

in the integrand in the denominator, which canceled the coupling term between \hat{H} and \hat{S}'_3 .

To bring the result (48) into the well-known form of an elliptic Ginibre ensemble [36], we define the ellipticity

$$\begin{aligned} \tau &= \frac{z^2 \text{Tr} \hat{S}^2}{(4\xi q)^2 \text{Tr} (\hat{S}'_5^{(N)})^2} \\ &= \frac{16N^2 z^2}{(d_G + \sqrt{d_G^2 + 16N^2 z^2})^2} \frac{\text{Re}[\text{Tr} S^2]}{\text{Tr} S S^\dagger} \in]-1, 1[\end{aligned} \quad (51)$$

and fix the width of the ensemble

$$\kappa^2 = \frac{\hat{N}^2 [(4\xi q)^2 - z^2]^2}{4d_G (4\xi q)^2 \text{Tr} (\hat{S}'_5^{(N)})^2} = \frac{d_G}{\text{Tr} S S^\dagger}. \quad (52)$$

Then, the distribution simplifies to

$$p(S'_3) = \frac{\exp\left[-\frac{1}{1-\tau^2} \text{Tr} S'_3 S'_3{}^\dagger + \frac{\tau}{1-\tau^2} \text{Re}[\text{Tr} S'_3{}^2]\right]}{\int_{\mathcal{M}} dS'_3 \exp\left[-\frac{1}{1-\tau^2} \text{Tr} S'_3 S'_3{}^\dagger + \frac{\tau}{1-\tau^2} \text{Re}[\text{Tr} S'_3{}^2]\right]}. \quad (53)$$

We would again like to emphasize that this formula holds for all non-Hermitian symmetry classes. The subtle differences are encoded in the specific structure of S'_3 . In all cases the distribution again has the form of a matrix-valued central limit theorem.

In the perturbation P_c we can reintroduce the width κ and the complex phase $e^{i\varphi}$ as well as the coupling constant α so that we finally arrive at the limiting perturbation $P_c = \alpha e^{i\varphi} \kappa S'_3$ where S'_3 is distributed along Eq. (53). The ellipticity vanishes when z or $\text{Re}[\text{Tr} S^2]$ does, while the spectrum becomes quasi-

one-dimensional when $\tau \rightarrow \pm 1$, this means we need both $|z| \rightarrow \infty$ and that $\text{Re}[\text{Tr} S^2] = \pm \text{Tr} S S^\dagger$. The latter is only achieved when S is almost Hermitian or anti-Hermitian in the large N limit otherwise the ensemble only tends to an elliptic Ginibre ensemble with a fixed eccentricity.

The spectral density of Eq. (53) for the case of $\mathcal{M} = \text{Gl}(N, \mathbb{C})$ can be found in Eq. (6).

V. NUMERICAL ANALYSIS

Below we verify numerically that in the neighborhood of the real and imaginary axis of the eigenvalue density of the broadened zero modes follow the form given by the elliptic Ginibre ensemble. We begin with the chiral ensemble,

$$K^{(N)} = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix} + \alpha_R U_R S_R U_R^\dagger + i\alpha_I U_I S_I U_I^\dagger. \quad (54)$$

In this case, the distribution of the smallest eigenvalues is compared to the one-point correlation function (6) with the factor τ determined by the standard deviations found in Sec. IV.

The average is performed over the unitary matrices U_R and U_I . The Hermitian matrices, S_R and S_I , are kept fixed (the i.i.d. elements drawn from a normal distribution and the matrices are then diagonalized). The matrix M is generically complex and of dimension $(n + \nu) \times n$, such that the full initial matrix is of dimension $N = 2n + \nu$. The perturbation matrices are complex too, and have no further symmetries.

Figure 1 shows the results for $n = 40$, $\nu = 3$ and an ensemble of size 10^6 . Shown is the smallest eigenvalues in a narrow bin along the real and imaginary axis. Left panel shows the symmetric case $\alpha_R = \alpha_I = 0.01$ while the right panel shows $\alpha_R = 0.015$ and $\alpha_I = 0.01$. Also shown along with the numerical points are the parameter free analytic predictions

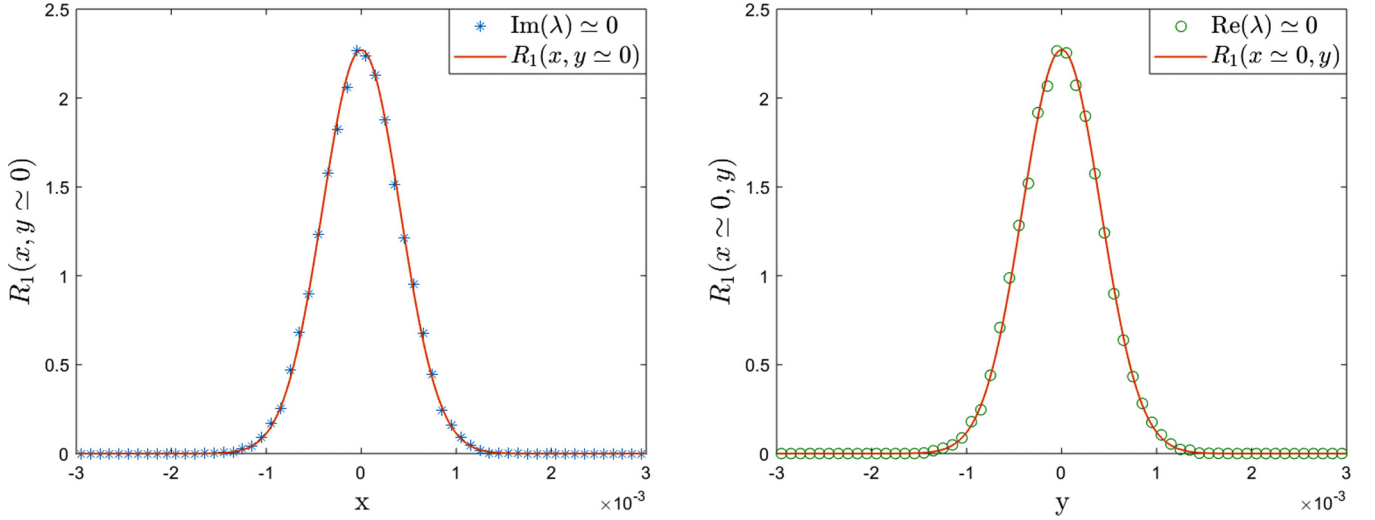


FIG. 2. The eigenvalue distribution of the Majorana ensemble. As for the chiral ensemble we plot the smallest eigenvalues in a narrow bin close to respectively the real (left) or the imaginary (right) axis. In both figures the numerical results are for matrix dimension $N = 4n + 2\nu$ with $n = 20$ and $\nu = 1$ and ensemble size 10^5 . The perturbation is of equal magnitude in the real and imaginary direction, $\alpha_R = \alpha_I = 0.001$.

and we observe complete agreement. Note that in this case the factorization of the characteristic equation, which ensures the application of leading-order perturbation theory, is realized even though the smallest bulk eigenvalues of the unperturbed operator are on the order of $1/n$.

The second ensemble considered is the Majorana ensemble,

$$K^{(N)} = \begin{pmatrix} iM & 0 \\ 0 & -iM \end{pmatrix} + \alpha_R O_R \begin{pmatrix} 0 & iW_R \\ -iW_R^T & 0 \end{pmatrix} O_R^T + i\alpha_I O_I \begin{pmatrix} 0 & iW_I \\ -iW_I^T & 0 \end{pmatrix} O_I^T, \quad (55)$$

$$M = -M^T,$$

where, again, the only average performed is over the orthogonal transformations O_R and O_I . The elements of the matrices M and W are drawn only once as i.i.d. from a

uniform distribution on the interval $[-1, 1]$ and are then kept fixed. M and W are real and of dimension $2n + \nu$, and M is antisymmetric. In Fig. 2 the broadening of the zero modes are compared to Gaussian distributions, see Eq. (12), in both the real and imaginary direction. Again complete agreement is found with the parameter free prediction.

VI. BOUNDS AND GENERALIZATIONS

It is natural to ask the question how universal the result (29) is. The main possible generalization is either a deformation of the Haar measure of \hat{U} or restricting to a subset of the original group. Both kinds of deformations can be implemented by a distribution $g(\hat{U})$, in particular we consider the measure $g(\hat{U})d\tilde{\mu}(\hat{U})$ instead of the measure (37). The impact of this deformation in the calculation appears when we want to integrate over the Gaussian; see Eq. (41). This integral has now $g(\hat{U})$ as a prefactor,

$$\hat{g} \left(\mathbb{1}_{\hat{N}} \otimes (\mathbb{1}_{\hat{N}} - i\hat{Q}) + i\frac{\kappa}{\xi} \hat{\Pi}_2^\dagger \hat{\gamma}_5^{(\nu)} H \hat{\Pi}_2 \otimes \hat{S} \hat{\gamma}_5^{(N)} - \frac{z}{4\xi} \hat{\gamma}_5^{(N)} \otimes \hat{\gamma}_5^{(N)} \right) = \frac{\int_G d\hat{U} g(\hat{U}) \exp \left[\frac{z}{4} \text{Tr} [\hat{U} \hat{\gamma}_5^{(N)} \hat{U}^\dagger \hat{\gamma}_5^{(N)}] - i\kappa \text{Tr} H \hat{\Pi}_2 \hat{U} \hat{S} \hat{\gamma}_5^{(N)} \hat{U}^\dagger \hat{\Pi}_2^\dagger \hat{\gamma}_5^{(\nu)} - \xi \text{Tr} \hat{U} \hat{U}^\dagger (\mathbb{1}_{\hat{N}} - i\hat{Q}) \right]}{\int_G d\hat{U} \exp[-\xi \text{Tr} \hat{U} \hat{U}^\dagger]}. \quad (56)$$

We can still exploit the factorization

$$\hat{g} \left(\mathbb{1}_{\hat{N}} \otimes (\mathbb{1}_{\hat{N}} - i\hat{Q}) + i\frac{\kappa}{\xi} \hat{\Pi}_2^\dagger \hat{\gamma}_5^{(\nu)} H \hat{\Pi}_2 \otimes \hat{S} \hat{\gamma}_5^{(N)} - \frac{z}{4\xi} \hat{\gamma}_5^{(N)} \otimes \hat{\gamma}_5^{(N)} \right) = \det^{-d_G/(4\hat{N})} \left[(\mathbb{1}_{\hat{N}} - i\hat{Q})^2 - \frac{z^2}{16\xi^2} \mathbb{1}_{\hat{N}} \right] \hat{g} \left(\mathbb{1}_{\hat{N}^2} + i\frac{\kappa}{\xi} \hat{\Pi}_2^\dagger \hat{\gamma}_5^{(\nu)} H \hat{\Pi}_2 \otimes \hat{S} \hat{\gamma}_5^{(N)} \left[\mathbb{1}_{\hat{N}} \otimes (\mathbb{1}_{\hat{N}} - i\hat{Q}) - \frac{z}{4\xi} \hat{\gamma}_5^{(N)} \otimes \hat{\gamma}_5^{(N)} \right]^{-1} \right). \quad (57)$$

This result can be achieved by rescaling \hat{U} as follows:

$$\hat{U}_{kl} \rightarrow \sum_{k',l'=1}^{\hat{N}} \hat{U}_{k'l'} \left\{ \left(\mathbb{1}_{\hat{N}} \otimes (\mathbb{1}_{\hat{N}} - i\hat{Q}) - \frac{z}{4\xi} \hat{\gamma}_5^{(N)} \otimes \hat{\gamma}_5^{(N)} \right)^{-1/2} \right\}_{k'k, l'l}. \quad (58)$$

In Eq. (57), we see a sufficient condition for not influencing the result apart from a rescaling of the width κ . We need to assume that the transform \hat{g} of the function $g(\hat{U})$ remains finite and differentiable in the vicinity of the identity matrix $\mathbb{1}_{\hat{N}^2}$. The dependence on N can be weak as it is in the case for the determinant in our computation above. Then, the Taylor expansion of $\ln(\hat{g})$ about $\mathbb{1}_{\hat{N}^2}$ can be carried out up to the second order. The N -dependence of higher order coefficients should not interfere with the behavior of the ratio $1/q^{(N)} \rightarrow 0$, see Eq. (31), so that those terms still vanish in this limit. As a by-product of this calculation, we can read off the rescaling of the width κ which is the factor $1/\sqrt{\hat{g}(\mathbb{1}_{\hat{N}^2})}$.

Though the discussion above has been only for the second model (3), a similar analysis can be expected for the first model (2), as well. As already mentioned, both computations follow along the same steps.

VII. CONCLUSION

The remarkable fact that perturbed zero modes distribute themselves according to finite-size Gaussian random matrix theory has been shown to apply also for non-Hermitian perturbations. In the non-Hermitian case, the distribution of the perturbed zero modes follows the one point function of the elliptic Ginibre ensemble. At first, it may appear highly surprising that finite-size Gaussian random matrix theory can provide universal distributions for would-be zero modes. Usually random matrix theory only yields universal results for infinite-size matrices and the Gaussian in the weight can be replaced by some other weight without altering the universal distribution as long as the support of the eigenvalue density is still a single cut. For the perturbed zero modes, however,

the Gaussian weight in the universal distribution can not be changed. As shown, the Gaussian form results from a matrix version of the central limit theorem which applies to the finite-size perturbation matrix as the size of the full Hamilton goes to infinity. We have derived explicit bounds where the results are applicable. In physical terms, the bounds correspond to the regime where first-order degenerate perturbation theory dominates.

As physical examples, we have shown that the broadened zero modes of both a chiral matrix, with the same symmetry properties as the Dirac operator of QCD, and of a matrix modeling topological superconductors with Majorana particles, follow an elliptic Ginibre ensemble, when perturbed by a non-Hermitian perturbation.

We have demonstrated the universality of the results by considering two different forms of the non-Hermitian perturbation, both leading to the same universal form for the perturbed zero modes. Furthermore we have discussed the stability of the results when restricting the manifold over which the average is performed. It would be most interesting to generalize these universality considerations.

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