Correlations between avalanches in the depinning dynamics of elastic interfaces

Pierre Le Doussal[1] and Thimothée Thiery[2]
[1]*Laboratoire de Physique de l'École Normale Supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université Paris-Diderot, Sorbonne Paris Cité, 24 rue Lhomond, 75005 Paris, France*
[2]*Instituut voor Theoretische Fysica, KU Leuven, Leuven, Belgium*

(Received 19 October 2019; accepted 4 February 2020; published 9 March 2020)

We study the correlations between avalanches in the depinning dynamics of elastic interfaces driven on a random substrate. In the mean-field theory (the Brownian force model), it is known that the avalanches are uncorrelated. Here we obtain a simple field theory which describes the first deviations from this uncorrelated behavior in a $\epsilon = d_c - d$ expansion below the upper critical dimension d_c of the model. We apply it to calculate the correlations between (i) avalanche sizes (ii) avalanche dynamics in two successive avalanches, or more generally, in two avalanches separated by a uniform displacement W of the interface. For (i) we obtain the correlations of the total sizes, of the local sizes, and of the total sizes with given seeds (starting points). For (ii) we obtain the correlations of the velocities, of the durations, and of the avalanche shapes. In general we find that the avalanches are *anticorrelated*, the occurrence of a larger avalanche making more likely the occurrence of a smaller one, and vice versa. Examining the universality of our results leads us to conjecture several exact scaling relations for the critical exponents that characterize the different distributions of correlations. The avalanche size predictions are confronted to numerical simulations for a $d = 1$ interface with short range elasticity. They are also compared to our recent related work on static avalanches (shocks). Finally we show that the naive extrapolation of our result into the thermally activated creep regime at finite temperature predicts strong positive correlations between the forward motion events, as recently observed in numerical simulations.

DOI: 10.1103/PhysRevE.101.032108

I. INTRODUCTION

The motion of elastic interfaces slowly driven in a random medium is not smooth but proceeds via jumps extending over a broad range of space and time scale [1,2]. This avalanche motion is ubiquitous in a number of experimental systems such as magnetic domain walls [3,4], fluid contact lines [5,6], earthquakes [7,8], cracks [9–11], or imbibition fronts [12], often modeled as elastic interfaces. Theoretically, the statics and the dynamics of elastic interfaces has been studied using the functional renormalization group (FRG) [13–24]. The FRG has then been extended to study avalanches, either in the statics (the so-called shocks) [25,26] or near the depinning transition [27–33].

An important question is to quantify the temporal and spatial correlations between successive avalanches. It is well known that in the case of earthquakes strong temporal correlations are observed, called aftershocks [34]. It was believed that in the context of elastic interfaces models, correlations between avalanches arise only if one includes additional mechanisms in the interface dynamics, such as relaxation processes [35,36] or memory effects [37]. In a recent work [38] we have studied correlations between "static avalanches." These are defined by considering the lowest energy configuration (i.e., the ground state) of an elastic interface at equilibrium in presence of both a quenched random potential (which models the random substrate), and an external quadratic potential centered at some position w. One then finds that, as a function of this parameter w, the ground state experiences abrupt reorganizations, called shocks or static avalanches. In Ref. [38] the correlations of the sizes and locations of these shocks were studied. There is no dynamics involved there; one is just probing the structure of the manifold of equilibrium states. In the dynamics by contrast, the interface is driven, and its stationary state is different from the equilibrium one. The avalanches which occur in this stationary state are of a different nature, they exhibit irreversible behavior (and hysteresis) and it remains to study their correlations.

In this paper we study the correlations between avalanches in the depinning dynamics of elastic interfaces driven on a random substrate. The starting point is the mean-field theory, valid in space dimension $d > d_c$, known as the Brownian force model (BFM) [26–28,39,40], a multidimensional generalization of the celebrated Alessandro-Beatrice-Bertotti-Montorsi (ABBM) model [41]. In the BFM, the avalanches are strictly uncorrelated [39]. Here we obtain a simple field theory, based on the FRG, which describes the first deviations from this uncorrelated behavior in a $\epsilon = d_c - d$ expansion below the upper critical dimension of the model [which depends on the range of the elastic interaction, $d_c = 4$ for short-range (SR) elasticity and $d_c = 2$ for usual long-range (LR) elasticity]. The elastic model and the avalanche observables are defined in Sec. II. Section III contains a summary of the main results, which can be read independently of the details of the derivation. The field theory is described in Sec. IV and Appendix C, together with a discussion of the physical origin of the correlations.

We apply our theory to calculate the correlations of two successive avalanches, loosely meaning two avalanches which occur within the same dynamical forward evolution of the interface in a given pinning landscape. It is convenient to study two avalanches separated by a given displacement W of the center of mass of the interface. For $W = 0^+$ this describes immediately successive avalanches. We study two types of information: (i) the correlations of the avalanche sizes, in Sec. V, and (ii) the correlations in the dynamics within each avalanche, in Sec. VII. More precisely for (i) we obtain the correlation between the total sizes, the local sizes, and the total size of avalanches with given seeds (i.e., given the position of their starting points). We show that the first two results are equal to this order in the expansion, i.e., to $O(\epsilon)$, to the ones obtained in the statics [38] for random field disorder (differences are expected to the next order). These results are derived here in a much simpler fashion. The third one, the correlation of the total sizes as the distance between the seeds is varied, was not considered in Ref. [38]. Some of these analytical predictions are confronted, in Sec. VI, to numerical simulations for a $d = 1$ interface with short range elasticity. For the dynamics (ii) we obtain the correlations of the velocities, of the avalanche durations, and of the avalanche shapes, i.e., of the velocity as a function of time at fixed duration. For the latter, a deviation from the famous parabola shape is demonstrated in the correlation. Examining the universal limit of our results, we obtain some nontrivial (and presumably exact, i.e., valid beyond the ϵ expansion) conjectures for a variety of critical exponents that characterize the correlations.

We find that in the depinning dynamics the avalanches are *anticorrelated*, the occurrence of a larger avalanche (in size, in total velocity etc.) making more likely the occurrence of a smaller one, and vice versa. The same was observed in the statics (i.e., for correlations between shocks) for random field disorder, while both positive and negative correlations could occur for random bond disorder depending on W. In the Conclusion we discuss qualitatively some possible extension at finite temperature which indicates instead the occurrence of positive correlations in the creep regime.

II. MODEL AND OBSERVABLES

A. Model

We focus on a d-dimensional elastic interface whose position at point $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}$, is denoted $u(x,t) \in \mathbb{R}$. The origin of time is arbitrary and negative times can be used to specify, e.g., the preparation of the system (see below). The position field satisfies the following overdamped equation of motion with bare friction coefficient η:[1]

$$\eta \partial_t u(x,t) = \nabla_x^2 u(x,t) - m^2[u(x,t) - w(x,t)] + F(u(x,t),x). \tag{1}$$

The random pinning force $F(u,x)$ is chosen Gaussian with correlator

$$\overline{F(u,x)F(u',x')} = \delta^d(x-x')\Delta_b(u-u'), \tag{2}$$

where $\Delta_b(u)$ denotes the bare correlator, assumed to be a symmetric short-range function. Here we have restricted to elastic interfaces with short-range elasticity and the elastic

[1]We use interchangeably $\dot{u}$ or $\partial_t u$ to denote partial derivatives with respect to time.

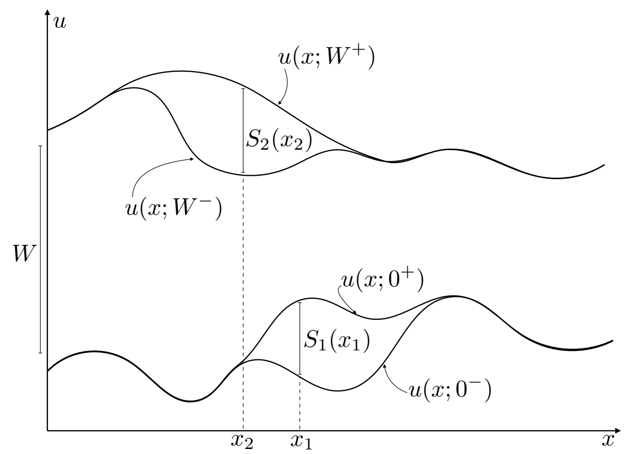


FIG. 1. The quasistatic process $u(x;w)$ for an elastic interface slowly driven in a disordered environment exhibits jumps called avalanches as a function of w (the position of the driving well). Here an avalanche occurred at $w = 0$ and another one at $w = W$ [many others also occurred in between since $u(x;0^+) \neq u(x,W^-)$]. The goal of this article is to study the correlations between these two avalanches, in particular the correlations between the total jumps (the areas of the two avalanches) or between the local jumps $S_1(x_1)$ and $S_2(x_2)$ indicated on the figure.

coefficient [the coefficient in front of the Laplacian in Eq. (1)] has been set to unity by a choice of units. The interface is driven by a parabolic well of stiffness m^2 following some driving protocol $w(x,t)$. We restrict to monotonous driving $\dot{w}(x,t) \geqslant 0$ which leads to only forward motion $\dot{u}(x,t) \geqslant 0$ and to the so-called Middleton attractor. The lateral extension of the interface is noted $L > 0$ and we assume periodic boundary conditions, although this will be unimportant as long as $L \gg 1/m$, the scale over which the interface motion is correlated. Our theory extends to other types of elasticity and more general microscopic disorder as in Ref. [38] but here we focus on this setting for the sake of simplicity.

Our aim is to study the avalanches that occur in the so-called quasistatic limit. There are two main protocols that are largely equivalent.

(1) In the first protocol the interface driving is $w(x,t) = vt$ and we are interested in the stationary state [42] in the limit $v = 0^+$ where the motion of the interface is intermittent. The avalanches are defined by the rapid motion $\dot{u}(x,t) \gg v$ that occur in between quiescence periods [of duration of order $1/(L^d v)$]. The avalanches can be indexed by the time at which they occur and their starting point (x_i, t_i) and we can ask about the correlations of these avalanches. Equivalently the process $u(x;w) = \lim_{v\to 0} u(x, t = w/v)$ exhibits jumps as a function of w that are the avalanches. This process is called the quasistatic process and our goal is to study correlations between different given avalanches as a function of W, the distance along u between two given avalanches, or equivalently $\mathsf{T} = W/v$, the time interval between the two avalanches (see Fig. 1). Note that for $W > 0$ many other avalanches have usually occurred between the two avalanches under study.

(2) In the second protocol we prepare the system in the same stationary state, stop the driving at $w = 0$, and wait for the interface to stop. It is thereby prepared with $\dot{u}(x, t = 0) = 0$ in the so-called Middleton attractor [42]. Then we

apply a kick at $t = 0$, $\dot{w}(x,t) = \delta w(x)\delta(t)$, either (i) local $\delta w(x) = \delta w\, \delta^d(x)$ or (ii) uniform $\delta w(x) = \delta w$. This produces some interface motion, that we call an avalanche. Choosing uniform kicks of vanishing size δw ensures that the interface stops at the position $u(x;\delta w)$ previously defined in the first protocol in the limit $v \to 0^+$. To study the correlation between avalanches, we apply a series of such kicks, waiting each time for the interface to stop before applying the next kick. After $n \gg 1$ kicks the driving is now at $w = n\delta w$ and the position of the interface is $u(x;w)$.

For the sake of simplicity we use in this paper the language of the second protocol and more often consider uniform kicks (although the dependence in the positions of the avalanche starting points, the seeds, will be investigated using a local kick; see below). We derive results using the functional renormalization group and these will be accurate and universal in the limit of small m (with still $L \gg 1/m$), in an expansion in $\epsilon = 4 - d$ around the upper-critical dimension $d_{\rm uc} = 4$ of the model. We thus restrict ourselves to $d < 4$, although equivalent results for $d = 4$ could be obtained.

B. Observables

We focus on the correlations between two avalanches, the first occurring after a uniform kick of size δw_1 at $w = 0$ and the second occurring after a uniform kick of size δw_2 at $w = W$. If $W > 0$ potentially many avalanches have also occurred in between. We start by considering the full velocity field inside an avalanche. It is convenient to adopt the following representation. We define two copies of the velocity field shifted in time $\dot{u}_1(x,\tau_1)$ and $\dot{u}_2(x,\tau_2)$ where τ_1 and τ_2 denote the time since the beginning of respectively the first and second avalanche (i.e., the first and the second kick, recalling that, according to the second protocol described above there has been many kicks in between to move the interface from $w = 0$ to $w = W$).

1. Velocity fields during avalanches and their associated densities

We consider a general observable of the form (i.e., the generating function of the velocity field)

$$G[\lambda] = \overline{e^{\int d^d x dt \lambda(x,t)\dot{u}(x,t)}}. \tag{3}$$

Decomposing the source field $\lambda(x,t)$ as done above for the velocity, this can be rewritten

$$G[\lambda] = G[\lambda_1,\lambda_2] = \overline{e^{\lambda_1\cdot\dot{u}_1 + \lambda_2\cdot\dot{u}_2}}, \tag{4}$$

where here and below we denote for each $i = 1, 2$

$$\lambda_i \cdot \dot{u}_i = \int d^d x \int_0^{+\infty} d\tau_i \lambda_i(x,\tau_i)\dot{u}_i(x,\tau_i), \tag{5}$$

where the source field λ_1 (respectively λ_2) is only nonvanishing in the first (respectively the second) avalanche.

The generating function $G[\lambda]$ depends on the size of the kicks δw_i and can be expanded as follows:

$$G[\lambda_1,\lambda_2] = 1 + \sum_{i=1,2} \delta w_i \int D\dot{u}_i \rho[\dot{u}_i](e^{\lambda_i\cdot\dot{u}_i} - 1)$$
$$+ \delta w_1 \delta w_2 \int D[\dot{u}_1,\dot{u}_2]\rho_W[\dot{u}_1,\dot{u}_2](e^{\lambda_1\cdot\dot{u}_1} - 1)$$
$$\times (e^{\lambda_2\cdot\dot{u}_2} - 1) + O(\delta w_1^2, \delta w_2^2). \tag{6}$$

This is obtained by decomposing the process into events where either an avalanche occurs $\dot{u} > 0$ or does not occur $\dot{u} = 0$.[2] The terms of order δw_i account for the contribution to G coming from events where an avalanche occurred at $w = 0$ or $w = W$, while the term of order $\delta w_1 \delta w_2$ accounts for the contribution to G where avalanches occurred at both positions. Here the factors $\rho[\dot{u}_i]$ denote the (equal by stationarity of the protocol considered here) functional densities of the instantaneous velocity field taking the configuration $\dot{u}_i(x,\tau)$ during the corresponding avalanche (and $\int D\dot{u}_i$ denotes a functional integral). It is normalized as $\int D\dot{u}_i \rho[\dot{u}_i] = \rho_0$ where $\rho_0 \sim L^d$ is the total density of avalanches per unit of driving δw. Similarly $\rho_W[\dot{u}_1,\dot{u}_2]$ is the joint density, i.e., it is proportional to the number of events where two avalanches occurred at $w = 0$ and $w = W$ with velocity fields $\dot{u}_i(x,\tau)$. A more detailed discussion of the formula (6) is given in Appendix B. If avalanches were independent, as is the case for the mean-field BFM (where they form a Levy jump process [26,39]) one would have $\rho_W[\dot{u}_1,\dot{u}_2] = \rho[\dot{u}_1]\rho[\dot{u}_2]$. The present theory goes beyond the independent avalanche process, and allows us to compute the connected joint density,

$$\rho_W^c[\dot{u}_1,\dot{u}_2] = \rho_W[\dot{u}_1,\dot{u}_2] - \rho[\dot{u}_1]\rho[\dot{u}_2], \tag{7}$$

which vanishes in the mean-field theory.

2. Total and local avalanche size

The theory presented in this paper allows us to study any correlation between the two velocity fields in the two avalanches. In this paper, to make calculations and results explicit, we will first focus on the total and local size of the two avalanches (the velocities being studied later in Sec. VII). The local size of avalanche $i = 1, 2$ is defined as the total displacement of the interface at a given point (see Fig. 1),

$$S_i(x) := \int_0^{+\infty} d\tau_i \dot{u}_i(x,\tau_i), \tag{8}$$

and the total size is given by

$$S_i = \int d^d x S_i(x), \tag{9}$$

that is, the area spanned by the avalanche.

The densities for these quantities can be obtained by considering the generating function (4) for source fields chosen as $\lambda_i(x,\tau) = \lambda_i$, for the total size, and as $\lambda_i(x,\tau) = \lambda_i \delta^d(x - x_i)$ for the local sizes $S(x_i)$. Expanding in powers of δw_i gives, for the total size,

$$G[\lambda_1,\lambda_2] = 1 + \sum_{i=1,2} \delta w_i \int_0^{+\infty} dS_i \rho(S_i)(e^{\lambda_i S_i} - 1) \tag{10}$$
$$+ \delta w_1 \delta w_2 \int_0^{+\infty} dS_1 \int_0^{+\infty} dS_2 \rho_W(S_1,S_2)$$
$$\times (e^{\lambda_1 S_1} - 1)(e^{\lambda_2 S_2} - 1) + O(\delta w_1^2, \delta w_2^2), \tag{11}$$

[2]Here by an avalanche we mean a motion $\dot{u} = O(1)$. For $\delta w > 0$ in the BFM there is always an avalanche, however most of them are of vanishing sizes and velocities as $\delta w \to 0^+$.

where $\rho(S)$ is the single avalanche total size density (per unit w) normalized as $\int_0^{+\infty} dS S\rho(S) = 1$ and $\rho_W(S_1, S_2)$ is the joint density of the total sizes S_1, S_2 in the two avalanches. We also define the connected joint density as

$$\rho_W^c(S_1, S_2) = \rho_W(S_1, S_2) - \rho(S_1)\rho(S_2), \tag{12}$$

which vanishes in mean-field theory (the BFM) and, as we show below, is $O(\epsilon)$ where $\epsilon = d_c - d$, near the upper critical dimension $d_c = 4$. Similarly, for the local size we have the same expansion (10) with $S_i \to S_i(x_i)$ and the corresponding densities $\rho(S_i(x_i))$ and $\rho_W(S_1(x_1), S_2(x_2))$. In Sec. V C we will also study the total size of avalanches conditioned on starting at a given point. The associated densities are defined in the same way.

III. SUMMARY OF MAIN RESULTS

We give here a sample of the results obtained below, with a short discussion, and refer to the corresponding sections below for detailed derivations and analysis.

Let us recall that the mass m in the equation of motion (1) provides an upper cutoff S_m for the total sizes of the avalanches, such that the size distribution decays fast with S beyond $S \sim S_m$. This cutoff is visible if the system length $L \gg L_m = 1/m$, where L_m is the cutoff length in the internal direction. The typical displacement in the direction of motion of the largest avalanches is then $u \sim m^{-\zeta}$, where ζ is the roughness exponent at depinning, which leads to $S_m \sim m^{-d-\zeta}$ at small m.

The first set of results concerns the correlations between the total sizes S_1 and S_2 of two avalanches separated by W in the direction of motion (see Fig. 1). We obtain that the dimensionless normalized covariance is given by

$$\frac{\langle S_1 S_2\rangle_W^c}{\langle S\rangle^2} = -\frac{\Delta''(W)}{m^4 L^d} = -A_d \frac{\tilde{\Delta}''(W m^{\zeta})}{(mL)^d}, \tag{13}$$

where $\langle S\rangle$ is the mean size of a single avalanche. This result is exact, where the function $\Delta(u)$ denotes the renormalized disorder correlator defined below in Eq. (25). In the limit $m \to 0$, the dimensionless function $\tilde{\Delta}(u)$ converges to a fixed point, which can be calculated in a $d = 4 - \epsilon$ expansion, see (26), or alternatively, measured in a numerical simulation, as we obtain for $d = 1$ in Fig. 4 below. The comparison is plotted in Fig. 2. One thus finds that the distance W in the direction of motion over which correlations extend scales naturally as $W \sim m^{-\zeta}$. Since $\tilde{\Delta}''(u) > 0$ (see Fig. 5 below), the avalanche sizes are *anticorrelated* as can be seen in Fig. 2. A larger avalanche is more likely to be followed, up to that distance $m^{-\zeta}$, by smaller ones, and vice versa. Equation (13) quantifies that result. Note that the result for successive avalanches (meaning, occurring immediately after) can be obtained setting $W = 0^+$ in that formula. Of course these will usually not occur in the same region of space, hence the factor $1/(mL)^d$ in Eq. (13). More detailed results are given below, such as higher joint size cumulants in Eqs. (44) and (49), and the connected joint density $\rho_W^c(S_1, S_2)$, defined in Eq. (12) above, in Eqs. (45)–(48). In particular we arrive at

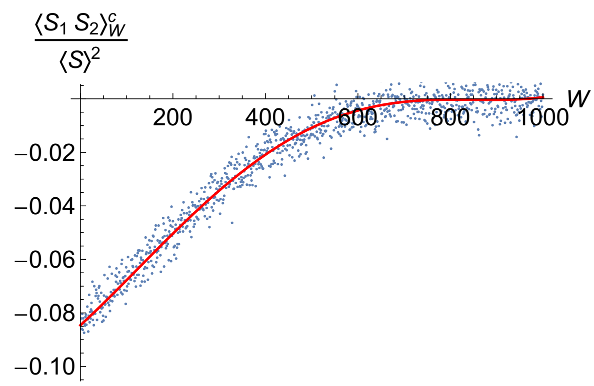


FIG. 2. Comparison between the measurement of $\langle S_1 S_2\rangle_W^c$ (blue dots) and our prediction (13) (exact result). The blue dots corresponds to direct measurements of the correlations between avalanches, each dot corresponding to an average over avalanches for a given W. The dispersion of the cloud of dots gives an estimate of the accuracy of the measurement.

the conjecture, for small avalanches (much smaller than S_m),

$$\rho_W^c(S_1, S_2) \sim \frac{1}{S_1^{\tau_c} S_2^{\tau_c}}, \quad \tau_c = 2 - \frac{2+d}{d+\zeta} \tag{14}$$

with $\tau_c = 1/2$ in mean field.

It is important to obtain also spatial information about avalanche correlations. There are two main types of observables. Either one measures the local jump $S(x)$ at a given point x, for a uniform driving of the interface (as above). Or one performs a kick at some local position x and measures the total size of the resulting avalanche (which thus has a seed in x). One can then correlate this information in two distinct avalanches. The first result obtains the correlations between the local sizes of the same two avalanches, $S_1(x_1)$ and $S_2(x_2)$, at two different points. Our leading order result, extended to $d = 1$, reads

$$\frac{\langle S_1(x_1) S_2(x_2)\rangle_W^c}{\langle S(x)\rangle^2} =_{d=1} -m^{-3}\frac{\Delta''(W)}{4}(1 + m|x_2 - x_1|)e^{-m|x_2-x_1|}. \tag{15}$$

The second observable is the joint connected moment when the two seeds are separated by $x = x_2 - x_1$ (in dimensionless units),

$$\langle S_1 S_2\rangle_{\rho_W^{c,x}} = -\hat{\Delta}''(W) 2^{-1-d/2}\pi^{-d/2}x^{2-d/2}K_{(d-4)/2}(x); \tag{16}$$

see below for precise definitions. More results about the spatial dependence of the size correlations are obtained in Secs. V B and V C.

Some of the above results for the size correlation bear similarities with the one obtained in Ref. [38] for the "static avalanches" (the dependence on the position of the seeds was not discussed there). Let us point out however that the results are *different*. Indeed the renormalized disorder correlator $\Delta(W)$ which enters all formulas flows to *different fixed points* in the statics and in the driven dynamics. In particular, while positive correlations do occur for short range disorder in the statics, they do not occur in the dynamics (i.e., at depinning), and only anticorrelations occur.

The second set of results obtained in this paper concern purely dynamical observables which do not have any analog in the static avalanches. Let us mention some of them. One finds that the connected density correlation between the total velocities $\dot{u}_i = \int d^d x\, \dot{u}(x, \tau_i)$, $i = 1, 2$ in two avalanches, defined in Eq. (7), is given by

$$\rho_W^c(\dot{u}_1, \dot{u}_2) = -\frac{L^d}{m^4 v_m^2 S_m^2} \Delta''(W)\, r_{t_1/\tau_m}\left(\frac{\dot{u}_1}{v_m}\right) r_{t_2/\tau_m}\left(\frac{\dot{u}_2}{v_m}\right), \tag{17}$$

where $v_m = S_m/\tau_m$ is the velocity of large avalanches of sizes S_m, τ_m their typical time scale of duration, and the function $r_t(\dot{u})$ is given by

$$r_t(\dot{u}) = \frac{d}{dt} \frac{(-t + e^t - 1)e^{t - e^t \dot{u}/(e^t - 1)}}{(e^t - 1)^2}. \tag{18}$$

As above, this is the leading order result within the $d = 4 - \epsilon$ expansion, which can be extrapolated to $d < 4$. Similarly one can ask about the correlation of the durations T_1 and T_2 of two avalanches. We find that they are described by

$$\rho_W^c(T_1, T_2) = -\frac{L^d}{m^4 S_m^2 \tau_m^2} \Delta''(W) d(T_1/\tau_m) d(T_2/\tau_m), \tag{19}$$

where the function $d(T)$ is given by

$$d(T) = \frac{e^T [e^T (T - 2) + T + 2]}{(e^T - 1)^3} \tag{20}$$

a result valid for a spatially uniform driving.

Again one may ask about the spatial structure, for a driving with seeds separated by x. The Fourier transform in x of the connected density correlation takes the form for any d,

$$\rho_W^{c,q,f}(T_1, T_2) = \frac{1}{(T_1 T_2)^{\alpha_c^1}} G_d\left(q T_1^{1/z}, q T_2^{1/z}\right),$$
$$\alpha_c^1 = 1 - \frac{1}{2z}(4 - d - 2\zeta), \tag{21}$$

where G_d is a scaling function calculated within the epsilon expansion in Sec. VII A. Finally in Sec. VII B the correlation of the *shapes* at fixed duration of the two avalanches is calculated. For small avalanches, we find that, to leading order in the epsilon expansion, the correlation between the celebrated parabolic shape of small avalanches vanishes, i.e., the $O(T_1 T_2)$ term, however there is a correlation to the order $O(T_1^3 T_2^3)$.

The main message of this section and of our results is that the correlations between observables in the successive avalanches near the depinning of an elastic interface do exist, and are not at all negligible. The calculation shows that they arise mostly from the large avalanches (at the level of the large avalanche cutoff, here S_m, controlled by the mass m), although there is also some amount of correlations for the small avalanches, characterized by some nontrivial power law exponents and universal scaling functions.

We now turn to the derivation of these results.

IV. DYNAMICAL FIELD THEORY FOR VELOCITY FIELD CORRELATIONS IN TWO AVALANCHES

We now present the dynamical field theory which allows us to calculate the densities previously introduced to leading order in the $\epsilon = d_c - d$ expansion. We also comment on the physical origin of the correlations.

A. Field theory

We now go back to consider the generating function $G[\lambda_1, \lambda_2]$ in Eq. (4) for general sources. Our main result, justified in Appendix C, is that, to lowest order in an expansion around mean field (i.e., independent avalanches), the generating function which only measures the dynamics during the two avalanches separated by W (see the Introduction) can be written as a functional average,

$$G[\lambda_1, \lambda_2] = \int D[\tilde{u}_1, \dot{u}_1, \tilde{u}_2, \dot{u}_2] \times e^{\sum_{i=1,2}[\lambda_i \cdot \dot{u}_i + m^2 \delta w_i \int d^d x \tilde{u}_i(x, \tau_i = 0)] - S[\tilde{u}_1, \dot{u}_1, \tilde{u}_2, \dot{u}_2]}, \tag{22}$$

over the following dynamical action:

$$S[\tilde{u}_1, \dot{u}_1, \tilde{u}_2, \dot{u}_2] = S_{\rm BFM}[\tilde{u}_1, \dot{u}_1] + S_{\rm BFM}[\tilde{u}_2, \dot{u}_2] + \Delta''(W) \times \int d^d x \int_{\tau_1, \tau_2 > 0} \tilde{u}_1(x, \tau_1) \tilde{u}_2(x, \tau_2) \times \dot{u}_1(x, \tau_1) \dot{u}_2(x, \tau_1). \tag{23}$$

Here $S_{\rm BFM}$ is the dynamical action associated to the BFM model,

$$S_{\rm BFM}[\dot{u}, \tilde{u}] = \int d^d x \left[\int_{\tau > 0} \tilde{u}(x, \tau)(\eta \partial_\tau - \nabla^2 + m^2)\dot{u}(x, \tau) - \sigma \int_{\tau > 0} \tilde{u}(x, \tau)^2 \dot{u}(x, \tau)\right], \tag{24}$$

following a uniform kick δw at time $\tau = 0$. Here $\sigma = -\Delta'(0^+) > 0$ and here and below $\Delta(w)$ denotes the renormalized disorder correlator defined from the two point correlation function of the position of the center of mass, $u(w) = L^{-d} \int d^d x u(x, t = w/v)$, as

$$\Delta(w - w') = m^4 L^d \overline{[u(w) - w][u(w') - w']}^c, \tag{25}$$

which is implicitly a function of m and reproduces the bare disorder correlator in the limit of large m, i.e., $\Delta_b(w) = \lim_{m \to +\infty} \Delta(w)$. Here we focus on the universal $m \to 0$ limit, where $\Delta(w)$ has been shown [18,23] to take the scaling form

$$\Delta(w) = A_d m^{\epsilon - 2\zeta} \tilde{\Delta}(w m^\zeta), \quad A_{d=4} = 8\pi^2,$$
$$\tilde{\Delta}^{*\prime\prime}(0^+) = \frac{1 - \zeta_1}{3}\epsilon + O(\epsilon^2) \tag{26}$$

with $A_d = 2^{d-1}\pi^{d/2}/\Gamma(3 - \frac{d}{2})$. Here $\tilde{\Delta}(w)$ converges as $m \to 0$ to the FRG fixed point $\tilde{\Delta}^*(w)$, which is uniformly of order $O(\epsilon)$ and solution of the FRG fixed point equation, where $\zeta = \zeta_1 \epsilon + O(\epsilon^2)$ is the roughness exponent, with $\zeta_1 = 1/3$ for interface depinning. Note that the depinning fixed point has one undetermined, nonuniversal constant κ, i.e., one can write $\tilde{\Delta}^*(w) = \kappa^2 \Delta^*(w/\kappa)$, however $\tilde{\Delta}^{*\prime\prime}(0^+)$ is fully universal, given in Eq. (26). The corresponding form for long-range elasticity can be found in Ref. [38]. To lowest order in ϵ the model is thus equivalent to two BFM models with a single inclusion (at some fixed τ_1, τ_2) of the vertex $\Delta''(W)$ in Eq. (23) [see also (29) below, the result being later summed over all τ_1, τ_2]. Diagrammatically it is represented by two trees (the two BFM's) joined (i.e., correlated) by a single vertex $\Delta''(W)$, as in Fig. 106 of [38] (the same diagram holds for the dynamics, with time running upward).

The above theory allows us to calculate exactly the correlations of the velocity field at order $O(\delta w_1 \delta w_2)$ and to order $O(\epsilon)$ in the stationary setting defined in the Introduction. Expanding Eq. (22) in powers of $\delta w_1, \delta w_2$ and comparing with Eq. (6), identifying the terms, we obtain the Laplace transform of the single avalanche velocity field density in terms of the correlation of the response field as

$$\int D\dot{u}_1 \rho[\dot{u}_1][e^{\lambda_1 \cdot \dot{u}_1} - 1] = m^2 \int D[\tilde{u}_1, \dot{u}_1] \int dx \tilde{u}_1(x, 0) e^{\lambda_1 \cdot \dot{u}_1 - S_{\rm BFM}[\tilde{u}_1, \dot{u}_1]}, \quad (27)$$

as well as the joint density of velocity fields in the two avalanches, as

$$\int D[\dot{u}_1, \dot{u}_2] \rho_W[\dot{u}_1, \dot{u}_2](e^{\lambda_1 \cdot \dot{u}_1} - 1)(e^{\lambda_2 \cdot \dot{u}_2} - 1) = m^4 \int D[\tilde{u}_1, \dot{u}_1, \tilde{u}_2, \dot{u}_2] \int dx_1 \tilde{u}_1(x_1, 0) \int dx_2 \tilde{u}_2(x_2, 0) \times e^{\sum_{i=1,2} \lambda_i \cdot \dot{u}_i - S[\tilde{u}_1, \dot{u}_1, \tilde{u}_2, \dot{u}_2]}. \quad (28)$$

We now make these results more explicit by focusing on simpler observables, namely the joint densities and correlations of the total sizes S_1, S_2, as well as the local sizes $S_1(x), S_2(y)$ of the two avalanches.

Remark 1. To be more precise, we stress that the above theory is devised to calculate the $O(\epsilon)$ result for the connected correlations between two avalanches. For the velocity field statistics inside a single avalanche it only leads to the mean-field result, i.e., $O(\epsilon^0)$. To obtain the $O(\epsilon)$ correction to the latter, one needs to add other terms [involving $\Delta''(0)$] in the action as was done in Ref. [28] (see also Appendix C). This is not our purpose here, and one should remember that only the results that we obtain for the *connected* correlations (the sole purpose of this paper) between avalanches are exact up to order $O(\epsilon)$.

Remark 2. The above setup and results can be easily generalized to study correlations between avalanches that are conditioned on starting at any given position (called the "seed") along the interface. Replacing $\int dx_1 \tilde{u}_1(x_1, 0) \int dx_2 \tilde{u}_2(x_2, 0) \to \tilde{u}_1(x_1, 0) \tilde{u}_2(x_2, 0)$ in Eq. (28) for given x_1 and x_2 fixed indeed selects the avalanches that have started at $x = x_1$ at $w = 0$ and at $x = x_2$ at $w = W$. The easiest way to see this is to slightly modify our protocol by triggering the avalanches at $w = 0$ and $w = W$ by local kicks $\delta w_i(x) = \delta^d(x - x_i)\delta w_i$ of vanishing size δw_i. Such kicks can indeed only trigger avalanches with seeds x_1 and x_2. One then easily generalizes (10) to this case; see also Appendix B. This is used in Sec. V C. We refer the reader to [31,33] for more details about this seed centering procedure in the field theory.

Remark 3. One can study truly successive avalanches by simply considering the limit $W = 0^+$. The present theory applies as long as the second avalanche starts well after the first one is finished. It would be interesting to study overlapping avalanches, but this goes beyond this paper (see discussion in Appendix B).

B. Origin of correlations

Before we go further here let us comment here on the physical origin of the correlations. The correlations originate from the fact that the time derivatives of the pinning forces $\partial_{\tau_i} F(u_i(x, \tau_i))$, which enter the equation of motion for the velocity, are correlated, their covariance being

$$\overline{\partial_{\tau_1} F(u_1(x, \tau_1), x) \partial_{\tau_2} F(u_2(x', \tau_2), x')} = \partial_{\tau_1} \partial_{\tau_2} \Delta[u_1(x, \tau_1) - u_2(x, \tau_2) + W] \delta^d(x - x') \simeq -\dot{u}_1(x, \tau_1) \dot{u}_2(x, \tau_2) \Delta''(W) \delta^d(x - x'). \quad (29)$$

This is because two displacement fields in the two avalanches see the same static random pinning force landscape. This landscape is correlated in the direction of the motion, and the correlation, at fixed value of $u_1(x, \tau_1) - u_2(x, \tau_2)$, extends to an arbitrary time difference [we recall that τ_i are counted from the beginning times T_i of each avalanche, e.g., hence $u_i(x, \tau_i) = u(x, t = \mathsf{T}_i + \tau_i)$, where $\mathsf{T}_2 - \mathsf{T}_1 \sim W/v$ is a very large time].

One could object, however, that the SR correlator of the bare (i.e., microscopic) model, $\Delta_b(w)$, has correlations only on short distance $w \sim r_f$, leading to correlations of the displacements only within the (small and fixed) Larkin volume. However, the *renormalized* correlator $\Delta(w)$, which includes the effect of the interplay of elasticity and disorder at all scales, is correlated on the much larger distance $w \sim m^{-\zeta}$. It is this renormalized correlator that must be used here. It is possible to prove this fact [and also justify that it is $\sigma = -\Delta'(0^+)$ which must be used in the BFM] by consideration of the effective action of the theory. Some steps in that direction are provided in Appendix C. Physically, correlations live on the scale $w \sim m^{-\zeta}$ because this is the scale of the displacement of the interface during avalanches. In the dynamics we always find that the correlations are negative. It can be seen already from the negative correlation of the forces in Eq. (29) [since $\Delta''(W) > 0$ near the depinning fixed point and the velocities are positive]. One finds that if one avalanche occurred and the interface moved on a distance $m^{-\zeta}$, the driving has to "catch up" with this scale $w \sim m^{-\zeta}$ in order for the interface to forget that this avalanche has already occurred.

V. ANALYTICAL RESULTS: CORRELATIONS OF THE AVALANCHE SIZES, TOTAL AND LOCAL

A. Joint density of total sizes

We first consider the total sizes of the avalanches defined in Eq. (9).

1. Expressions for Laplace transform

Following the same steps as in the previous section, expanding (22) in powers of $\delta w_1, \delta w_2$ in the special case of constant sources $\lambda_i(x, \tau) = \lambda_i$, and comparing with (10) we obtain the Laplace transform of $\rho(S)$ as

$$\int dS_1 \rho(S_1)[e^{\lambda_1 S_1} - 1] = m^2 \int D\tilde{u}_1 D\dot{u}_1 \int dx \tilde{u}_1(x,0) \times e^{\lambda_1 \int dx \int_0^{+\infty} d\tau_1 \dot{u}_1(x,\tau_1) - S_{\rm BFM}[\tilde{u}_1,\dot{u}_1]}, \quad (30)$$

a formula similar to (27), and the Laplace transform of $\rho_W(S_1,S_2)$ as

$$\int dS_1 dS_2 \rho_W(S_1,S_2)[e^{\lambda_1 S_1} - 1][e^{\lambda_2 S_2} - 1] = m^4 \int D\tilde{u}_1 D\dot{u}_1 \int D\tilde{u}_2 D\dot{u}_2 \int dx_1 \tilde{u}_1(x_1,0) \times \int dx_2 \tilde{u}_2(x_2,0) e^{\sum_{i=1,2} \lambda_i \int d^d x \int_0^{+\infty} d\tau_i \dot{u}_i(x,\tau_i) - S[\tilde{u}_1,\dot{u}_1,\tilde{u}_2,\dot{u}_2]}, \quad (31)$$

a formula similar to (28). We now compute explicitly the right-hand side (rhs) of these equations.

2. Review of the calculation of $\rho(S)$

The rhs of (30) can be obtained from a standard calculation within the BFM. The main observation is that the field $\dot{u}_1$ in the exponential on the rhs of (30) appears only linearly [28]. Hence integrating over it leads to a delta function which constrains $\tilde{u}(x,t)$ to be a solution of the so-called instanton equation equation [28], in the present case given by

$$\tilde{u}(x,t) = \tilde{u}, \quad -m^2\tilde{u} + \sigma\tilde{u}^2 = -\lambda, \quad (32)$$

which is

$$\tilde{u} = \frac{1}{m^2 S_m} Z(\lambda S_m), \quad Z(\lambda) = \frac{1}{2}(1 - \sqrt{1-4\lambda}), \quad (33)$$

where $S_m = \frac{\sigma}{m^4}$ is the typical scale of the largest avalanches in the BFM. This leads to

$$\int dS_1 \rho(S_1)[e^{\lambda_1 S_1} - 1] = \frac{1}{S_m} L^d Z(S_m \lambda), \quad (34)$$

which leads to the total size density

$$\rho(S) = \frac{L^d}{S_m^2} \hat{\rho}(S/S_m), \quad \hat{\rho}(s) = \frac{1}{2\sqrt{\pi} s^{3/2}} e^{-s/4}, \quad (35)$$

which is the classic result for the BFM [exact in our setting at order $O(\epsilon^0)$]. Note that $\int_0^{+\infty} ds s \hat{\rho}(s) = 1$ and $\int_0^{+\infty} ds s^2 \hat{\rho}(s) = 2$.

3. Calculation of $\rho_W(S_1,S_2)$

The rhs of (31) can be obtained from a modification of the previous calculation. The difficulty is the term proportional to $\Delta''(W)\dot{u}_1\dot{u}_2$ in the action S in Eq. (23). It can be decoupled by the following calculational trick. We introduce formal centered Gaussian noise fields $\xi_i(x,t)$ as

$$e^{-\Delta''(W) \int d^d x \int_{\tau_1,\tau_2>0} \tilde{u}_1(x,\tau_1)\tilde{u}_2(x,\tau_2)\dot{u}_1(x,\tau_1)\dot{u}_2(x,\tau_1)} = \left\langle e^{\sum_{i=1,2} \int d^d x \int_0^{+\infty} d\tau_i \xi_i(x)\tilde{u}_i(x,\tau_i)\dot{u}_i(x,\tau_i)} \right\rangle_\xi, \quad (36)$$

where by definition

$$\langle \xi_i(x)\xi_j(x')\rangle_\xi = -(1-\delta_{ij})\Delta''(W)\delta^d(x-x'). \quad (37)$$

For a given noise $\xi_i(x)$, the velocity fields now appear linearly and one can integrate over them (as in the BFM). This, for each realization of the noise ξ, constrains the response fields $\tilde{u}_i(x,t)$ to obey two decoupled instanton equations, whose solutions are time independent but space inhomogeneous, $\tilde{u}_i(x,t) = \tilde{u}_i(x)$, where $\tilde{u}_i(x)$ for $i = 1, 2$ are solutions of

$$(\nabla_x^2 - m^2)\tilde{u}_i(x) + \sigma\tilde{u}_i(x)^2 = -\xi_i(x)\tilde{u}_i(x) - \lambda_i. \quad (38)$$

The solutions of these equations are coupled because the noises ξ_1 and ξ_2 are not independent, and correlated as in Eq. (37). From these solutions, and from (31), one obtains the Laplace transform of the joint density as

$$\int dS_1 dS_2 \rho_W(S_1,S_2)[e^{\lambda_1 S_1} - 1][e^{\lambda_2 S_2} - 1] = m^4 \int d^d x_1 \int d^d x_2 \langle \tilde{u}_1(x_1)\tilde{u}_2(x_2)\rangle_\xi. \quad (39)$$

Being interested in computing $\rho_W(S_1,S_2)$ in first order in $\Delta''(W)$ [which is itself $O(\epsilon)$] implies that we only need to solve perturbatively (38) to first order in ξ. To this order the solutions can be written as

$$\tilde{u}_i(x) = \tilde{u}_i^0 + \tilde{u}_i^1(x), \quad \tilde{u}_i^0 = \frac{1}{m^2 S_m} Z(S_m\lambda_i), \quad (40)$$

and in Fourier space

$$\tilde{u}_i^1(q) = \frac{1}{m^2 S_m} \frac{Z(S_m\lambda_i)}{q^2 + m^2 - 2m^2 Z(S_m\lambda_i)} \xi_i(q), \quad (41)$$

leading to our main result for the connected joint density,

$$\int dS_1 dS_2 \rho_W^c(S_1,S_2)[e^{\lambda_1 S_1} - 1][e^{\lambda_2 S_2} - 1] = -\Delta''(W) \frac{L^d}{m^4 S_m^2} \frac{Z(S_m\lambda_1)}{1 - 2Z(S_m\lambda_1)} \frac{Z(S_m\lambda_2)}{1 - 2Z(S_m\lambda_2)}, \quad (42)$$

where the part proportional to $\tilde{u}_1^0\tilde{u}_2^0$ cancels in the connected density.

The formula (42) is formally identical to the result Eq. (71) in Ref. [38] for the statics. This shows that the present theory reproduces the results of (42) in a much simpler fashion. However, one must stress that the renormalized correlator $\Delta(W)$ is different in the dynamics from its value, e.g., for random bond statics, leading to a numerically different result.

By expanding (42) in powers of λ_i one obtains the integer moments over ρ_W^c which we denote $\langle \dots \rangle_W^c$. Similarly the averages over the single avalanche density ρ are denoted as $\langle \dots \rangle$.[3] We give here two explicit formulas, which are tested in the numerics in Sec. VI below. First one finds

$$\frac{\langle S_1 S_2\rangle_W^c}{\langle S\rangle^2} = -\frac{\Delta''(W)}{m^4 L^d}, \quad (43)$$

[3]Since below we only consider the moment ratio the global normalization drops out and we can define $\langle \cdots \rangle_W^c = \int dS_1 dS_2 \dots \rho_W^c(S_1,S_2)$ and $\langle \cdots \rangle = \int dS \dots \rho(S)$.

which is in fact an exact result, and can be seen to follow from the definition (25) of the renormalized disorder correlator. The proof is identical to the statics case, to which we refer [Eq. (8) and Secs. III F. and IV E in Ref. [38]]. Another result is

$$\frac{\langle S_1^2 S_2\rangle_W^c}{\langle S^2\rangle\langle S\rangle} = \frac{\langle S_1 S_2^2\rangle_W^c}{\langle S^2\rangle\langle S\rangle} = -3\frac{\Delta''(W)}{m^4 L^d}, \tag{44}$$

which holds only to order $O(\epsilon)$. Note that to the order that we calculate here $\langle S_1^2 S_2\rangle_W^c = \langle S_1 S_2^2\rangle_W^c$ and more generally, at this order the correlation between the two avalanches is symmetric (which is likely not to hold to higher orders in the ϵ expansion contrary to the statics).

As in Ref. [38], (42) can be inverse Laplace transformed, and leads to the complete $O(\epsilon)$ result for the connected density,

$$\rho_W^c(S_1, S_2) = -\frac{\Delta''(W)}{L^d m^4}\frac{S_1 S_2}{4S_m^2}\rho(S_1)\rho(S_2) + O(\epsilon^2), \tag{45}$$

which again, is formally identical to (10) in Ref. [38]. We recall the definition

$$S_m = \frac{|\Delta'(0^+)|}{m^4} = \frac{\langle S^2\rangle}{2\langle S\rangle} \tag{46}$$

valid beyond mean field. We can rewrite the connected density in the form

$$\rho_W^c(S_1, S_2) = \frac{1}{(Lm)^d}\frac{L^{2d}}{S_m^4}\mathcal{F}_d\left(\frac{W}{W_m}, \frac{S_1}{S_m}, \frac{S_2}{S_m}\right), \tag{47}$$

where $\mathcal{F}_d$ is a universal function, with

$$\mathcal{F}_d(w, s_1, s_2) \simeq -\frac{A_d\tilde{\Delta}^{*\prime\prime}(w)}{16\pi\sqrt{s_1 s_2}}e^{-(s_1+s_2)/4} + O(\epsilon^2), \tag{48}$$

correcting the sign misprint in (13), (89), and (90) of [38], where the prefactor is given by (26), and fully universal at $w=0$, i.e., $\Delta^{*\prime\prime}(0^+) = \frac{2}{9}\epsilon$ and $A_4 = 8\pi^2$. The scale $W_m \simeq \kappa m^{-\zeta}$ contains one nonuniversal amplitude, related however to $S_m = -\Delta'(0^+)/m^4 = -A_d\tilde{\Delta}'(0^+)m^{-d-\zeta} \simeq A_d\kappa\epsilon\sqrt{1-2\zeta_1}m^{-d-\zeta}$ [see formula (90) and below in Ref. [38]] which can be independently measured from (46), allowing us to determine κ.

Finally, note that for any real $p_1, p_2, q_1, q_2 > -1/2$ with $p_1+p_2 = q_1+q_2$ we predict the following dimensionless ratio:

$$\frac{\langle S_1^{p_1} S_2^{p_2}\rangle_W^c}{\langle S_1^{q_1} S_2^{q_2}\rangle_W^c} = \frac{\Gamma\left(p_1+\frac{1}{2}\right)\Gamma\left(p_2+\frac{1}{2}\right)}{\Gamma\left(q_1+\frac{1}{2}\right)\Gamma\left(q_2+\frac{1}{2}\right)}. \tag{49}$$

B. Joint density of local sizes

1. Dimensionless units

In order to lighten notations and calculations, from now on we switch to dimensionless units. We introduce the characteristic scales of avalanches. The lateral extension $1/m$, total size $S_m = \sigma/m^4$, duration $\tau_m = \eta/m^2$, velocity $v_m = m^d S_m/\tau_m$. We rescale space and time as $x \to x/m$, $t \to \tau_m t$. The fields are rescaled as $u \to m^d S_m u$, $\dot{u} \to v_m\dot{u}$, $\tilde{u} \to \frac{1}{m^2 S_m}\tilde{u}$. Avalanche local and total sizes are rescaled accordingly as $S(x) \to m^d S_m S(x)$, $S \to S_m S$. We also use that the renormalized disorder correlator Δ takes a scaling form $\Delta(W) = m^{\epsilon-2\zeta}\hat{\Delta}(m^\zeta W)$ with ζ the roughness exponent of the interface. We rescale the distance between avalanches as $W \to m^{-\zeta}W$. This is equivalent to setting $\eta = m = \sigma = 1$ in the above theory with also the replacement $\Delta''(W) \to \hat{\Delta}''(W)$. We will reintroduce the full dimensions explicitly for some results, which can be done easily using Appendix A.

2. Expressions for Laplace transforms

To study correlations between avalanche local sizes $S_1(x_1)$ and $S_2(x_2)$, see Fig. 1, we now do as in Sec. V A but using source fields $\lambda_1(x, \tau_1) = \lambda_1\delta^d(x - x_1)$ and $\lambda_2(x, \tau_2) = \lambda_2\delta^d(x - x_2)$. Expanding (22) in powers of $\delta w_1, \delta w_2$ in the special case of these sources we obtain the Laplace transform of $\rho(S_1(x_1))$ as [a formula similar to (27)]

$$\int dS_1(x_1)\rho(S_1(x_1))[e^{\lambda_1 S_1(x_1)} - 1] = \int D\tilde{u}_1 D\dot{u}_1\int dx\tilde{u}_1(x,0)e^{\lambda_1\int_0^{+\infty}d\tau_1\dot{u}_1(x_1,\tau_1) - S_{\rm BFM}[\tilde{u}_1,\dot{u}_1]}, \tag{50}$$

and of $\rho_W(S_1(x_1), S_2(x_2))$ as [a formula similar to (28)]

$$\begin{aligned}&\int dS_1(x_1)dS_2(x_2)\rho_W(S_1(x_1), S_2(x_2))[e^{\lambda_1 S_1(x_2)} - 1][e^{\lambda_2 S_2(x_2)} - 1]\\ &= \int D\tilde{u}_1 D\dot{u}_1\int D\tilde{u}_2 D\dot{u}_2\int dy_1\tilde{u}_1(y_1,0)\int dy_2\tilde{u}_2(y_2,0)e^{\sum_{i=1,2}\lambda_i\int_0^{+\infty}d\tau_i\dot{u}_i(x_i,\tau_i) - S[\tilde{u}_1,\dot{u}_1,\tilde{u}_2,\dot{u}_2]}.\end{aligned} \tag{51}$$

We now compute explicitly the rhs of these equations following the same procedure as in the previous section.

3. Review of the calculation of ρ(S(x))

As in Sec. V A, it can be seen that $\dot{u}_i$ only appears linearly in the exponential in the rhs of (50). Integrating over it creates a Dirac delta functional and constrains the response field $\tilde{u}_i$ as $\tilde{u}_i(x, \tau_i) = \tilde{u}_i(x_i)$ with $\tilde{u}_i(x_i)$ the solution of the following space inhomogeneous instanton equation:

$$(\nabla_x^2 - 1)\tilde{u}_i(x) + (\tilde{u}_i(x))^2 = -\lambda_i\delta^d(x - x_i). \tag{52}$$

The Laplace transform of $\rho(S(x_i))$ is then computed as, using (50),

$$\int dS(x_i)\rho(S(x_i))[e^{\lambda_i S(x_i)} - 1] = \int d^d x\, \tilde{u}_i(x). \tag{53}$$

As discussed in Refs. [25,28,40] the solution $\tilde{u}_i(x)$ can be exactly obtained in $d = 1$ as

$$\tilde{u}_i(x) = \tilde{u}_i^0(x) := \frac{6\big(1 - z_i^2\big)e^{-|x-x_i|}}{[1 + z_i + (1 - z_i)e^{-|x-x_i|}]^2}, \tag{54}$$

where $z_i(\lambda_i)$ is one of the solutions of

$$\lambda_i = 3z_i\big(1 - z_i^2\big). \tag{55}$$

The right solution satisfies the following properties: it is defined for $\lambda_i \in]-\infty, 2/\sqrt{3}[$, decreases from $z_i(-\infty) = \infty$ to $z_c = z_i(2/\sqrt{3}) = 1/\sqrt{3}$, and approaches 1 as λ_i approaches 0. It is possible to perform the Laplace inversion leading to [25,28,40]

$$\rho(S(x_i) = S_0) = \frac{2}{\pi S_0} K_{1/3}\left(\frac{2S_0}{\sqrt{3}}\right). \tag{56}$$

4. Calculation of $\rho_W(S_1(x_1), S_2(x_2))$

To compute $\rho_W(S_1(x_1), S_2(x_2))$ we follow the same steps as in Sec. V A 3 and linearize in the $\dot{u}_i$ fields the argument of the exponential in the rhs of (51) by reintroducing the formal Gaussian fields $\xi_i(x)$. This leads to the expression

$$\int dS_1(x_1)dS_2(x_2)\rho_W(S_1(x_1), S_2(x_2))[e^{\lambda_1 S_1(x_1)} - 1][e^{\lambda_2 S_2(x_2)} - 1] = \left\langle \int_{y_1,y_2} d^d y_1 d^d y_2 \tilde{u}_1(y_1)\tilde{u}_2(y_2)\right\rangle_\xi , \tag{57}$$

in terms of the solution of the following space inhomogeneous instanton equation:

$$\big(\nabla_x^2 - 1\big)\tilde{u}_i(x) + \tilde{u}_i(x)^2 = -\xi_i(x)\tilde{u}_i(x) - \lambda_i\delta^d(x - x_i). \tag{58}$$

Again to obtain the result at order $O(\epsilon)$ it is sufficient to solve perturbatively (58) to first order in ξ. This leads to

$$\tilde{u}_i(x) = \tilde{u}_i^0(x) + \int d^d y\, G_i(x, y)\xi_i(y)\tilde{u}_i^0(y) + O(\xi^2), \tag{59}$$

where $\tilde{u}_i^0(x)$ is the solution of (52), which in $d = 1$ is given in (54). We have introduced the propagators $G_i(x, y)$ satisfying the following equations:

$$\big[\nabla_x^2 - 1 + 2\tilde{u}_i^0(x)\big]G_i(x, y) = -\delta^d(x - y). \tag{60}$$

Inserting this formal solution in Eq. (94) and using the noise correlations (37), we finally obtain the Laplace transform of the connected density for the local size as

$$\int dS_1(x_1)dS_2(x_2)\rho_W^c(S_1(x_1), S_2(x_2))[e^{\lambda_1 S_1(x_1)} - 1][e^{\lambda_2 S_2(x_2)} - 1] \tag{61}$$

$$= -\hat{\Delta}''(W)\int d^d z_1 \int d^d z_2 \int d^d y G_1(z_1, y)G_2(z_2, y)\tilde{u}_1^0(y)\tilde{u}_2^0(y). \tag{62}$$

Again, it can be seen that this result reproduces the equivalent results obtained for shocks in the static. This is most easily seen by comparing this expression with the expression (D16) in Appendix D of [38].

From these expressions one easily obtains by expanding in λ_i a few integer moments of the connected distribution (see [38]). Here we only give the explicit result for the first moment in $d = 1$ that will be compared with numerical simulations in Sec. VI. One easily obtains from (55) that $z_i(\lambda_i) = 1 - \frac{1}{6}\lambda_i + O(\lambda_i^2)$. Then from (54) one obtains $\tilde{u}_i(x) = \frac{\lambda_i}{2}e^{-|x-x_i|} + O(\lambda_i^2)$. From (60) one obtains $G_i(x, y) = \frac{1}{2}e^{-|x-y|} + O(\lambda)$ and thus we obtain from (61) that

$$\langle S_1(x_1)S_2(x_2)\rangle_W^c =_{d=1} -\hat{\Delta}''(W)\int dz_1 dz_1 dy \frac{1}{16}e^{-|z_1-y|-|z_2-y|-|y-x_1|-|y-x_2|}$$

$$= -\frac{\hat{\Delta}''(W)}{4}\int dy e^{-|y-x_1|-|y-x_2|} = -\frac{\hat{\Delta}''(W)}{4}(1 + |x_2 - x_1|)e^{-|x_2-x_1|}. \tag{63}$$

Reintroducing the units and normalizing, one gets

$$\frac{\langle S_1(x_1)S_2(x_2)\rangle_W^c}{\langle S(x)\rangle^2} =_{d=1} -m^{-3}\frac{\Delta''(W)}{4}(1 + m|x_2 - x_1|)e^{-m|x_2-x_1|}, \tag{64}$$

a result that reproduces the result (112) of [38].

C. Joint density of total sizes for given positions of the seeds

1. Seed centering and Laplace transform

The explicit results we have obtained up to now are formally equivalent to the results obtained for shocks in the statics [38], rederived in the dynamics in a much simpler fashion using our general result (22)–(24). We now obtain results relevant for the dynamics and not considered in Ref. [38], by looking at the correlations between the total size S_1 and S_2 of avalanches occurring at a distance W and having seeds x_1 and x_2 (see Fig. 3). By "seed" we mean the first point of the interface that moves when the avalanche starts. This concept is much more natural in the dynamics than in the statics. We now introduce $\rho^x(S)$ and $\rho^{x_1,x_2}(S_1,S_2)$, the total size density for an avalanche that starts at x and the total size density for avalanches that start at x_1 at $w=0$ and at x_2 at $w=W$. By translational invariance along the internal direction we have

$$\rho^x(S) = \frac{1}{L^d}\rho(S), \quad \rho_W^{x_1,x_2}(S_1,S_2) \equiv \rho_W^{x_2-x_1}(S_1,S_2), \tag{65}$$

with the normalization

$$\int d^dx \rho_W^x(S_1,S_2) = \frac{1}{L^d}\rho_W(S_1,S_2). \tag{66}$$

Our theory predicts that the Laplace transform of these densities are obtained as in Eqs. (30) and (31) but with the response fields in front of the exponential not integrated over but restricted at x_1 or x_2 (see the remark in Sec. IV for a justification). More precisely we get

$$\int dS_1 \rho^{x_1}(S_1)[e^{\lambda_1 S_1}-1] = \int D\tilde{u}_1 D\dot{u}_1 \tilde{u}_1(x_1,0) e^{\lambda_1\int dx \int_0^{+\infty} d\tau_1 \dot{u}_1(x,\tau_1) - S_{\rm BFM}[\tilde{u}_1,\dot{u}_1]}, \tag{67}$$

and

$$\int dS_1 dS_2 \rho_W^{x_1,x_2}(S_1,S_2)[e^{\lambda_1 S_1}-1][e^{\lambda_2 S_2}-1] = \int D\tilde{u}_1 D\dot{u}_1 \int D\tilde{u}_2 D\dot{u}_2 \tilde{u}_1(x_1,0)\tilde{u}_2(x_2,0) e^{\sum_{i=1,2}\lambda_i\int d^dx \int_0^{+\infty} d\tau_i \dot{u}_i(x,\tau_i) - S[\tilde{u}_1,\dot{u}_1,\tilde{u}_2,\dot{u}_2]}. \tag{68}$$

2. Calculation of the densities

Proceeding as in Sec. V A, it is easily seen that the seed centering procedure does not affect the instanton equations satisfied by the response fields and, as mentioned above, the only modification that is necessary is to keep track of the spatial dependence of the solutions of the instanton equation at $x=x_1$ and $x=x_2$. More precisely, from (67) one easily obtains the trivial result $\rho^x(S) = \frac{1}{L^d}\rho(S)$ and from (68) one obtains

$$\int dS_1 dS_2 \rho_W^{x_1,x_2}(S_1,S_2)[e^{\lambda_1 S_1}-1][e^{\lambda_2 S_2}-1] = \langle \tilde{u}_1(x_1)\tilde{u}_2(x_2)\rangle_\xi,$$

with $\tilde{u}_i(x_i)$ still given by the solutions of the instanton equation (38) of Sec. V A, a result that should be compared to (42). Using the perturbative solution (41) in adimensional units we obtain the Laplace transform of the connected density $\rho_W^{c,x_2-x_1}(S_1,S_2) = \rho_W^{x_2-x_1}(S_1,S_2) - \rho^{x_1}(S_1)\rho^{x_2}(S_2)$ exactly at order $O(\epsilon)$ as

$$\int dS_1 dS_2 \rho_W^{c,x_2-x_1}(S_1,S_2)[e^{\lambda_1 S_1}-1][e^{\lambda_2 S_2}-1] = -\hat{\Delta}''(W)\int \frac{d^dq}{(2\pi)^d} e^{iq(x_1-x_2)} \frac{Z(\lambda_1)}{1-2Z(\lambda_1)+q^2}\frac{Z(\lambda_2)}{1-2Z(\lambda_2)+q^2}.$$

This can be explicitly computed and we obtain the following exact result $[O(\epsilon)]$ for this Laplace transform:

$$\int dS_1 dS_2 \rho_W^{c,x_2-x_1}(S_1,S_2)[e^{\lambda_1 S_1}-1][e^{\lambda_2 S_2}-1] = -\hat{\Delta}''(W)Z(\lambda_1)Z(\lambda_2)\mathcal{I}(1-2Z(\lambda_1),1-2Z(\lambda_2),|x_1-x_2|),$$

$$\mathcal{I}(a,b,x) = \int \frac{d^dq}{(2\pi)^d}\frac{e^{iqx}}{(a+q^2)(b+q^2)} = \frac{1}{b-a}(\tilde{\mathcal{I}}(a,x)-\tilde{\mathcal{I}}(b,x)),$$

$$\tilde{\mathcal{I}}(a,x) = \frac{1}{(2\pi)^{d/2}} a^{(d-2)/4} x^{(2-d)/2} K_{(d-2)/2}(\sqrt{a}x). \tag{69}$$

This is an exact expression but not easy to inverse Laplace transform to obtain the density. For that purpose it is more convenient to work in Fourier space, defining $\rho_W^q(S_1,S_2) = \int d^dx e^{iqx}\rho_W^x(S_1,S_2)$. We obtain for the connected part

$$\rho_W^{c,q}(S_1,S_2) = -\hat{\Delta}''(W)\psi_q(S_1)\psi_q(S_2), \tag{70}$$

where we have introduced the function $\psi_q(S)$ defined as

$$\psi_q(S) = \hat{\rho}(S)(1+q^2)\frac{S}{2}\left(1-\frac{\sqrt{\pi S}}{2}q^2 e^{q^4 S/4}\text{Erfc}\left(\frac{q^2\sqrt{S}}{2}\right)\right), \tag{71}$$

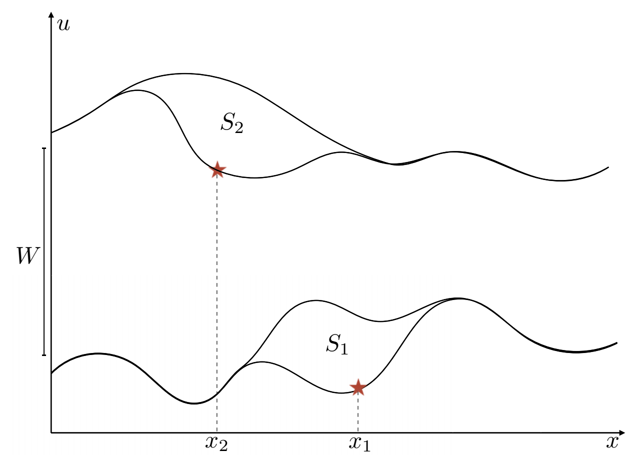


FIG. 3. In Sec. V C we calculate the correlations between the total sizes S_1 and S_2 of two avalanches separated by a distance W along u, with seeds (i.e., starting points) at x_1 and x_2.

with $\hat{\rho}(S)$ still given by (35), and whose Laplace transform is

$$\int dS(e^{\lambda S}-1)\psi_q(S)=\frac{Z(\lambda)}{1-2Z(\lambda)+q^2}. \tag{72}$$

We give here a few of the lowest order moments of the function $\psi_q(S)$:

$$\begin{aligned}\int \psi_q(S)dS&=\frac{1}{2},\\ \int \psi_q(S)SdS&=\frac{1}{1+q^2},\\ \int \psi_q(S)S^2dS&=\frac{2(3+q^2)}{(1+q^2)^2}.\end{aligned} \tag{73}$$

The zeroth order moment allows us, for example, to calculate from (70) the connected total density of avalanches with seeds x_1 and x_2,

$$\begin{aligned}\rho_2^c(W,x_2-x_1)&:=\int dS_1dS_2\rho_W^{c,x_2-x_1}(S_1,S_2)\\ &=-\frac{\hat{\Delta}''(W)}{4}\delta^d(x_2-x_1).\end{aligned} \tag{74}$$

The form of the result shows that the drop in density of an avalanche at $w=W$ caused by the occurrence of an avalanche at $w=0$ with seed x_1 is only felt at microscopic distances. This is not so surprising since the vast majority of avalanches are of microscopic sizes [the density $\rho^x(S)$ is not normalizable at small S]. By contrast, the first joint moment [obtained from the second identity in Eq. (73)] is dominated by large avalanches and reads

$$\begin{aligned}\langle S_1S_2\rangle_{\rho_W^{c,x}}&:=\int dS_1dS_2S_1S_2\rho_W^{c,x}(S_1,S_2)\\ &=-\hat{\Delta}''(W)2^{-1-d/2}\pi^{-d/2}x^{2-d/2}K_{(d-4)/2}(x).\end{aligned} \tag{75}$$

Integrating this result over x leads to the exact relation $L^{-2d}\langle S_1S_2\rangle_W^c=-\frac{\hat{\Delta}''(W)}{L^d}$. This suggests that the above result may be quite accurate since it integrates to an exact result. Surprisingly, the full x dependence of this first joint moment is *identical* to the one of the correlation of the local sizes when the driving is uniform; see Eq. (112) in Ref. [38]. This surprising result does not hold for higher order moments as can easily be seen by comparing the formulas.

3. Universality and the massless limit

We now address in more depth the issue of universality and of the massless limit. In Sec. V A 3 we obtained a universal scaling form (47) for $\rho_W^c(S_1,S_2)$. This form however involves two features which makes it not fully universal: (i) First it depends explicitly on m and S_m, i.e., it is universal (i.e., independent of small scale details) within the model of the driving parabolic well, which provides a large scale cutoff. One can ask if it is possible to obtain results for avalanche correlations which would be independent of the details of large scale cutoff (which we call full universality). (ii) The second feature is that $\rho_W^c(S_1,S_2)$ is proportional to $(mL)^d$, i.e., the number of independent regions along the interface. This makes sense since avalanches are expected to be correlated only if they are separated within a distance $x\sim 1/m$ along the interface.

In the above calculation of $\rho_W^{c,x}(S_1,S_2)$ the factor $(mL)^d$ is absent, since the separation of the seeds, x, is fixed. In Fourier space we now expect, restoring units,

$$\rho_W^{c,q}(S_1,S_2)=\frac{1}{m^dS_m^4}\mathcal{F}_d\left(\frac{q}{m},\frac{W}{W_m},\frac{S_1}{S_m},\frac{S_2}{S_m}\right), \tag{76}$$

where $\mathcal{F}_d$ is a universal function. From our result (70) we see that, restoring units, with $w=W/W_m$ one has the expansion around the upper critical dimension

$$\begin{aligned}\mathcal{F}_d(q,w,s_1,s_2)&=-m^{d-4}\Delta''(W)\psi_q(s_1)\psi_q(s_2)+O(\epsilon^2)\\ &=-A_d\tilde{\Delta}^{*\prime\prime}(w)\psi_q(s_1)\psi_q(s_2)+O(\epsilon^2),\end{aligned} \tag{77}$$

where all factors are universal using the fixed point (26) [apart from a single nonuniversal scale; see discussion below (48)].

Let us now discuss the behavior of (76) in the region $q\gg m$, $S_i\ll S_m$, $W\ll W_m$ where we hope to find a fully universal behavior. It is equivalent to take $m\to 0$ and consider *the massless limit*. It turns out that consideration of this limit, upon some hypothesis, leads to a host of information, i.e., determination of some critical exponents.

Let us first recall the analysis of the massless limit for the single avalanche size density, $\rho(S)$, and its connection to the Narayan-Fisher conjecture [15,16]. The massless limit of the starting equation of motion for the interface, Eq. (1), is obtained by defining $f(x,t)=m^2w(x,t)$, the applied force. One can then take $m\to 0$ at fixed $f(x,t)$ and the equation remains well defined. Then one must define densities per unit force, denoted everywhere with a subscript f, rather than per unit w as we did until now. They simply differ by the factor m^2, e.g.,

$$\rho^f(S)=m^{-2}\rho(S). \tag{78}$$

It is easy to see in the result for the BFM (35) that the massless limit of $\rho^f(S)$ is well defined (i.e., all factors of m cancel), leading to

$$\rho^f(S)=\lim_{m\to 0}m^{-2}\rho(S)=\frac{L^d}{\sigma^{1/2}}\frac{1}{2\sqrt{\pi}S^{3/2}}, \tag{79}$$

which is the fully universal part of the density corresponding to small avalanches $S \ll S_m$. It turns out that this extends beyond mean field. Indeed, for any d one has, via simple dimensional analysis,

$$\rho(S) = \frac{L^d}{S_m^2} r(S/S_m), \quad r(s) \sim_{s\to 0} s^{-\tau}, \tag{80}$$

where τ is the avalanche size exponent. For $\rho^f(S) = \lim_{m\to 0} m^{-2}\rho(S)$ to be finite, we see that we need $m^{-2}S_m^{\tau-2}$ to be finite in the limit $m \to 0$, and using $S_m \sim m^{-(d+\zeta)}$ this is equivalent to

$$\tau = \tau_{\rm NF} = 2 - \frac{2}{d+\zeta}. \tag{81}$$

This relation was tested to one loop in Ref. [28], and is rather natural since we do expect that a universal massless limit exists (a massless field theory). We will thus generally assume that densities per unit force do exist in the limit $m \to 0$. This led in Ref. [29] to predictions for a number of other avalanche exponents, and we call it the *generalized NF conjecture*.

Consider now the massless limit of the joint size density to which we apply similar arguments. We note from (71) that

$$\lim_{m\to 0} m^4 \psi_{q/m}\left(\frac{S}{S_m}\right) = \frac{\sigma}{S}\psi(q(S/\sigma)^{1/4}) \tag{82}$$

$$\psi(q) = \frac{q^2}{4\sqrt{\pi}} - \frac{1}{8}e^{q^4/4}q^4 \text{erfc}\left(\frac{q^2}{2}\right)$$
$$= \frac{q^2}{4\sqrt{\pi}} + O(q^4) = \frac{1}{2\sqrt{\pi}q^2} + O\left(\frac{1}{q^4}\right), \tag{83}$$

where $\psi(q)$ is the massless scaling form of $\psi_q(S)$. Thus our result to $O(\epsilon)$ for the joint density per unit force has a well defined $m \to 0$ limit

$$\begin{aligned}\rho_W^{c,q,f}&(S_1, S_2)\\ &= \lim_{m\to 0} m^{-4}\rho_W^{c,q}(S_1, S_2)\\ &= -\frac{\Delta''(W)}{\sigma^2}\frac{1}{S_1S_2}\psi(q(S_1/\sigma)^{1/4})\psi(q(S_2/\sigma)^{1/4}) + O(\epsilon^2).\end{aligned} \tag{84}$$

Note that as $m \to 0$, $\Delta''(W) = A_d m^\epsilon \tilde{\Delta}^{*\prime\prime}(Wm^\zeta) \simeq A_d\tilde{\Delta}^{*\prime\prime}(0^+) + O(\epsilon^2)$ if W is kept fixed as $m \to 0$. Hence the dependence in W disappears in that limit since there can be avalanches of arbitrary sizes; all avalanches can be considered as successive (i.e., $W = 0^+$).

More generally, we surmise that this massless limit exist in any dimension. Scaling arguments and dimensional analysis then lead to the scaling form[4]

$$\rho_W^{c,q,f}(S_1, S_2) = \frac{\ell_\sigma^{d+4}}{(S_1S_2)^{\tau_c^1}} f_d\left(\ell_\sigma\, qS_1^{1/(d+\zeta)}, \ell_\sigma\, qS_2^{1/(d+\zeta)}\right),$$
$$\tau_c^1 = \frac{1}{2}\left(2 - \frac{4-d-2\zeta}{d+\zeta}\right), \tag{85}$$

where τ_c^1 is a correlation exponent, f_d is a fully universal function, and the scale ℓ_σ is nonfully universal.[5] For the parabolic well model it equals $\ell_\sigma = \lim_{m\to 0} m^{-1}S_m^{-1/(d+\zeta)}$ [hence it can be measured independently using (46)], as easily seen by studying the possible $m \to 0$ limit of the scaling function $\mathcal{F}_d$ in Eq. (76).

We see that for $d = 4 - \epsilon$ and $\zeta = O(\epsilon)$ the form (85) reproduces (84) with $\ell_\sigma = \sigma^{-1/4}$ and

$$\begin{aligned}f_d(q_1, q_2) &= -A_d\tilde{\Delta}^{*\prime\prime}(0^+)\psi(q_1)\psi(q_2) + O(\epsilon^2)\\ &= -\frac{16\pi^2\epsilon}{9}\psi(q_1)\psi(q_2) + O(\epsilon^2),\end{aligned} \tag{86}$$

where $\psi(q)$ is given in Eq. (83).

Let us now study the $q = 0$ limit, i.e., the uniform driving studied in Sec. V A 3. For a fixed $m > 0$ one has

$$\begin{aligned}\rho_W^c(S_1, S_2) &= L^d\rho_W^{c,q=0}(S_1, S_2)\\ &= -\frac{L^d}{m^dS_m^4}m^{d-4}\Delta''(W)\frac{S_1S_2}{4S_m^2}\hat{\rho}(S_1/S_m)\hat{\rho}(S_2/S_m)\end{aligned} \tag{87}$$

using (76), (77), and, from (71), that $\psi_{q=0}(s) = \hat{\rho}(s)\frac{s}{2}$. It coincides with (45) upon using (35). Its $m \to 0$ limit reads

$$\begin{aligned}\rho_W^{c,f}(S_1, S_2) &= m^{-4}\rho_W^c(S_1, S_2)\\ &= -(mL)^d\frac{A_4\tilde{\Delta}^{*\prime\prime}(0^+)}{\sigma^3}\frac{1}{16\pi S_1^{1/2}S_2^{1/2}} + O(\epsilon^2),\end{aligned} \tag{88}$$

Note that this result cannot be obtained from taking the $q \to 0$ limit of (84) since $\psi(q) \sim q^2$ as $q \to 0$. Hence there is a noncommutation of limits $m \to 0$ and $q \to 0$.

It is reasonable to surmise that in any dimension, as $m \to 0$[6]

$$\rho_W^{c,f}(S_1, S_2) \sim (mL)^d\frac{1}{S_1^{\tau_c}S_2^{\tau_c}} \tag{89}$$

with $\tau_c = 1/2$ in mean field, i.e., for $d = d_c = 4$ here, the factor $(mL)^d$ being the number of independent regions. One can obtain this factor by considering the $q \to 0$ limit of the massless result on one hand, and the $q = 0$ massive result at small m on the other, and requiring matching upon setting $q = m$. This determines the $q \to 0$ behavior of the scaling function f_d as

$$f_d\left(qS_1^{1/(d+\zeta)}, qS_2^{1/(d+\zeta)}\right) \simeq_{q\to 0} q^d(S_1S_2)^{(1/2)[d/(d+\zeta)]}, \tag{90}$$

so that, substituting $q = m$ in (85), we indeed obtain

$$\rho_W^{c,f}(S_1, S_2) = L^d\rho_W^{c,q=m,f}(S_1, S_2) \sim (Lm)^d\frac{1}{S_1^{\tau_c}S_2^{\tau_c}},$$
$$\tau_c = 2 - \frac{2+d}{d+\zeta}, \tag{91}$$

[4]Note that an additional dependence in $qW^{1/\zeta}$ cannot be ruled out, although it does not seem to appear to $O(\epsilon)$ (see remark above). Thus, to be fully consistent, in Eq. (85) we have in mind here $W = 0^+$, i.e., successive avalanches.

[5]ℓ_σ has dimension $(x^\zeta/u)^{1/(d+\zeta)}$ where x and u are lengths in internal and displacement directions respectively.

[6]One cannot exclude an additional factor $g_d(S_1/S_2)$ (not present to this order) which we ignore here for simplicity (it does not affect the discussion of the critical exponent τ_c defined here for $S_1 \sim S_2$).

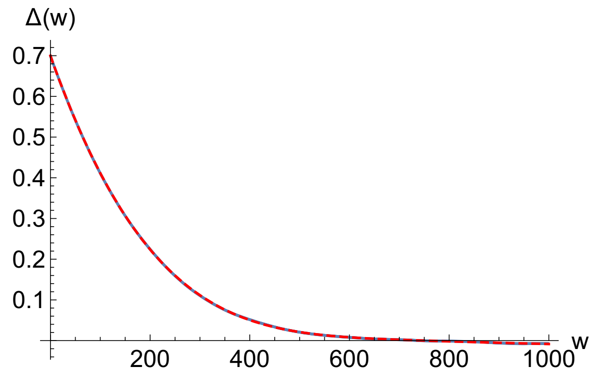


FIG. 4. The renormalized disorder second cumulant measured in the simulations (blue dots) and its polynomial fit (red line).

where the exponent τ_c is thus fully determined, via this generalized NF argument (recovering $\tau_c = 1/2$ in the mean field for $d = 4$, $\zeta = 0$).

VI. NUMERICS

In this section we compare some of our results with the simulation of a $d = 1$ elastic interface with short-ranged elasticity in a short-ranged correlated disordered landscape.

A. Protocol

To perform numerical simulations we choose a Gaussian disorder $F(u, x)$ with a correlator $\overline{F(u,x)F(u',x')} = \delta(x - x')\Delta_0(u - u')$ with $\Delta_0(u) = \sigma\delta u e^{-|u|/\delta u}$ with δu the microscopic correlation length of the disorder. As explained in [33], this can be realized by taking $F(u, x)$ as a collection (indexed by x) of independent Ornstein-Uhlenbeck processes (in the direction u). More precisely the model we study can be defined as, for any driving protocol $w(x, t)$,

$$\eta\partial_t u(x,t) = \nabla_x^2 u(x,t) - m^2[u(x,t) - w(x,t)] + F(u(x,t), x),$$
$$\partial_u F(u,x) = \sqrt{2\sigma}\xi(u,x) - \frac{1}{\delta u}F(u,x) \quad (92)$$

with $\xi(u, x)$ a unit centered two dimensional Gaussian white noise, $\overline{\xi(u,x)\xi(u',x')} = \delta(u - u')\delta(x - x')$. The advantage of this setup is that one can directly obtain an autonomous equation for the velocity field $\dot{u}(x, t)$: the above model is equivalent to

$$\eta\dot{u}(x,t) = \nabla^2\dot{u}(x,t) + m^2[\dot{w}(t) - \dot{u}(x,t)] + \partial_t \mathsf{F}(x,t),$$
$$\partial_t \mathsf{F}(x,t) = \sqrt{2\sigma\dot{u}(x,t)}\chi(x,t) - \frac{\dot{u}(x,t)}{\delta u}\mathsf{F}(x,t), \quad (93)$$

with $\chi(x, t)$ a centered Gaussian white noise $\overline{\chi(x,t)\chi(x',t')} = \delta^d(x - x')\delta(t - t')$ and we have the equality in law $\mathsf{F}(x, t) \sim F(x, u(x, t))$.

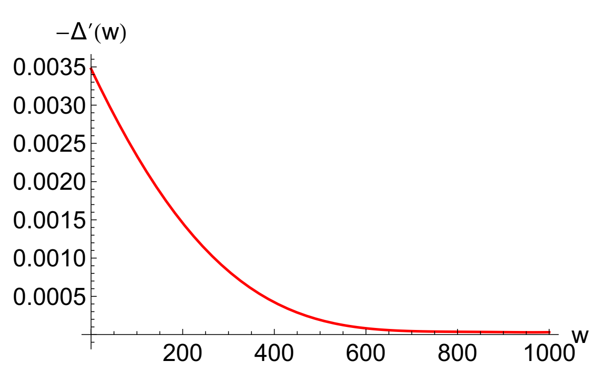
We take as initial conditions $\dot{u}(x, t = 0) = \mathsf{F}(x, t = 0) = 0$ and then apply a sequence of kicks of size $\delta w = 1$, which amounts at setting $\dot{u}(x,t) = \frac{m^2}{\eta}\delta w$ at the beginning of each kick and wait for the interface to stop before applying the next kick. The motion of the interface between each kick is measured by integrating the velocity field in between two kicks and this defines for each kick an avalanche S_x. The avalanche at the nth kick is said to have been triggered at $w = n\delta w$. We wait for the system to reach a stationary state before measuring anything. Averages are obtained using 50 independent "experiences," each experience consisting in 2×10^6 kicks, and we have thus simulated 10^8 avalanches. Correlations between avalanches are measured for avalanches inside a window of 1000 successive kicks.

In the results reported here we have taken an interface of lateral extension $L = 1024$ with periodic boundary conditions, discretized with 1024 points. The parameters are chosen as $m = 20/L \simeq 0.02$. The kicks are of size $\delta w = 1$ and the microscopic disorder correlation length is taken as $\delta u = 5\delta w = 5$. The discretization in time is handled using an algorithm similar to the one introduced in Ref. [43] and we take a time step $\delta t = 0.025$.

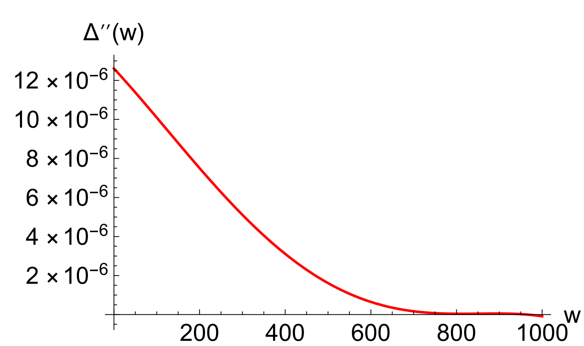
B. Results

1. Renormalized disorder correlator

Central to our results is the measurement of the renormalized disorder second cumulant $\Delta(w - w') = L^d m^4\overline{[u(w) - w][u(w') - w']}^c$ where $u(w)$ is the position of the interface in the end of the $w/\delta w$-th kick. The plot of



(a)



(b)

FIG. 5. First (a) and second (b) derivative of the renormalized disorder second cumulant as obtained using the polynomial fit of the measured renormalized disorder second cumulant.

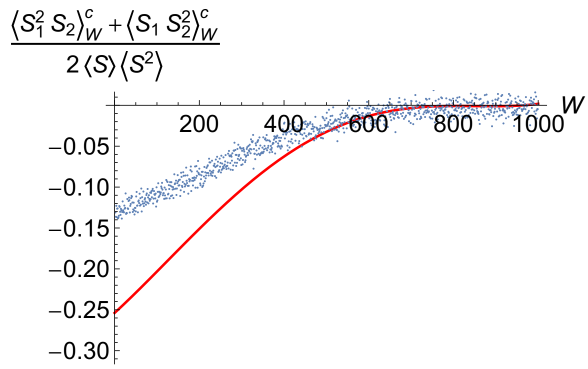


FIG. 6. Comparison between the measurement of $\frac{1}{2}(\langle S_1 S_2^2\rangle_W^c + \langle S_1^2 S_2\rangle_W^c)$ (blue dots) and our prediction (44) [order $O(\epsilon)$ result]. The blue dots correspond to direct measurements of the correlations between avalanches, each dot corresponding to an average over avalanches for a given W. The dispersion of the cloud of dots gives an estimate of the accuracy of the measurement.

$\Delta(w)$ is presented in Fig. 4. To obtain a good measurement of the derivative $\Delta'(w)$ and $\Delta''(w)$ we fitted $\Delta(w)$ with a polynomial of order 7 and differentiated directly the fitted polynomial. A plot of $\Delta'(w)$ and $\Delta''(w)$ is also given in Fig. 5.

2. Total avalanche sizes correlations

The measurement of $\langle S_1 S_2\rangle_W^c$ was shown in Sec. III, in Fig. 2. We show in Fig. 6 the comparison between the measurements of $\frac{1}{2}(\langle S_1 S_2^2\rangle_W^c + \langle S_1^2 S_2\rangle_W^c)$. Both are compared with our predictions (43) and (44). The analytical result for $\langle S_1 S_2\rangle_W^c$ is exact and the agreement is, as expected, perfect. The analytical result for $\frac{1}{2}(\langle S_1 S_2^2\rangle_W^c + \langle S_1^2 S_2\rangle_W^c)$ is only a $O(\epsilon)$ approximation and appears to overestimate the correlations.

3. Local avalanche sizes correlations

We show in Fig. 7 the comparison between the measurements of $\langle S_{10} S_{2x}\rangle_W^c$ and our prediction (64). Despite the fact that (64) is only valid up to order $O(\epsilon)$ (here $\epsilon = 3$), it is clear that it is a very good approximation. Also it seems that our result tends to slightly underestimate the correlations between avalanches at short distance and overestimate the correlations at large distance.

VII. ANALYTICAL RESULTS: DYNAMICAL CORRELATIONS

A. Correlations of the total velocities in an avalanche, and of the avalanche durations

In this section we study the correlation of the global velocities (center of mass velocities) in the two avalanches. As a by-product we also obtain the correlation between the avalanche durations. Let us define the total (areal) velocities in the two avalanches, which we denote $\dot u_1 \equiv \dot u_1(t_1) = \int d^d x \dot u_1(x, t_1)$ and $\dot u_1 \equiv \dot u_2(t_2) = \int d^d x \dot u_1(x, t_2)$. The time is counted from the kick in each avalanche, i.e., each avalanche starts at $t_i = 0$.

Similar methods as in Sec. V B give, for the joint density $\rho_W^{x_1,x_2}(\dot u_1, \dot u_2)$ at fixed positions of the seeds x_1, x_2,

$$\begin{aligned}\int d\dot u_1 d\dot u_2 \rho_W^{x_1,x_2}(\dot u_1, \dot u_2)[e^{\lambda_1 \dot u_1} - 1][e^{\lambda_2 \dot u_2} - 1]\\ = \langle \tilde u_1(x_1, 0)\tilde u_2(x_2, 0)\rangle_\xi,\end{aligned} \tag{94}$$

where $\tilde u_i(x, t)$ are the solutions of the time-dependent space inhomogeneous instanton equation

$$\begin{aligned}\left(\partial_t + \nabla_x^2 - 1\right)\tilde u_i(x, t) + \tilde u_i(x, t)^2\\ = -\xi_i(x)\tilde u_i(x, t) - \lambda_i \delta(t - t_i)\end{aligned} \tag{95}$$

with $\tilde u_i(x, t > t_i) = 0$, which are needed only to first order in each ξ_i. As in Sec. V B we introduce

$$\tilde u_i(x, t) = \tilde u_i^0(t) + \tilde u_i^1(x, t), \tag{96}$$

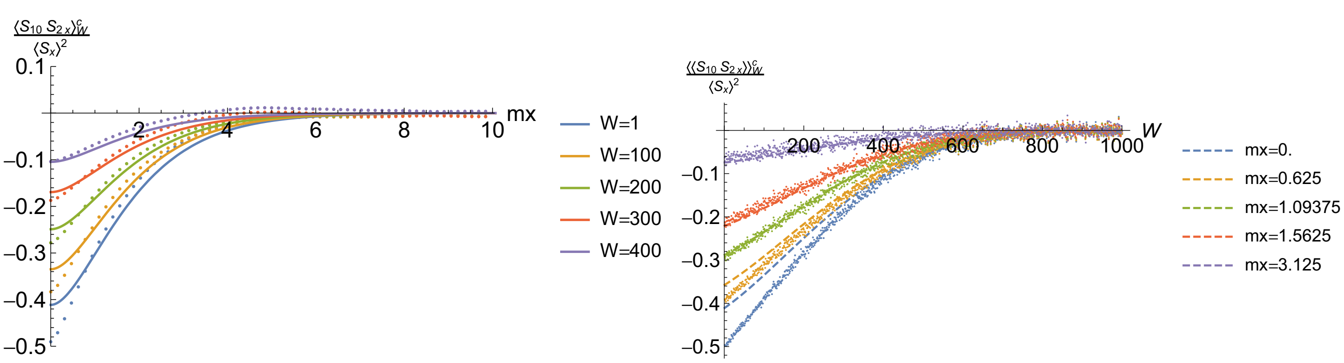
where $\tilde u_i^0(t)$ is the solution for $\xi_i(x) = 0$, which satisfies

$$(\partial_t - 1)\tilde u_i^0(t) + \tilde u_i^0(t)^2 = -\lambda_i \delta(t - t_i) \tag{97}$$

with $\tilde u_i^0(t > t_i) = 0$. The solution is well known to be [27]

$$\tilde u_i^0(t) = \frac{\lambda_i}{\lambda_i + (1 - \lambda_i)e^{t_i - t}}\theta(t < t_i). \tag{98}$$

Let us recall that from this solution one obtains the single avalanche (time-dependent) density in the BFM, denoted $\rho(\dot u) \equiv \rho_t(\dot u)$, of the total velocity $\dot u(t) = \int d^d x \dot u(x, t)$, by



(a) (b)

FIG. 7. Comparison between the measurement of $\langle S_{10} S_{2x}\rangle_W^c$ (dots) and our prediction (64) [plain and dashed lines, order $O(\epsilon)$ approximation] as a function of mx for a few values of W (a) or as a function of W for a few values of mx (b).

Laplace inversion of

$$\int_0^{+\infty} d\dot{u}\rho_t(\dot{u})(e^{\lambda\dot{u}}-1) = L^d\tilde{u}^0(0) = L^d\frac{\lambda}{\lambda+(1-\lambda)e^t} \Rightarrow \rho_t(\dot{u}) = L^d\frac{e^{t-e^t\dot{u}/(e^t-1)}}{(e^t-1)^2}, \quad \dot{u}>0. \tag{99}$$

By integration over time one recovers the well known result for the mean density of total velocity $\rho_{v=0^+}(\dot{u})$ for a uniform driving in the limit $v=0^+$, $\rho_{v=0^+}(\dot{u}) = \int_0^{+\infty} dt\,\rho_t(\dot{u}) = \frac{L^d}{\dot{u}}e^{-\dot{u}}$ (see [27,28,44]). The same calculation also gives the density of the *avalanche duration* T in the BFM,[7]

$$\rho(T) = -\partial_t\int_0^{+\infty} d\dot{u}\rho_t(\dot{u})|_{t=T} = \frac{L^d}{4\sinh^2(T/2)}, \tag{100}$$

which reads, in dimensionful units, $\rho(T) = \frac{L^d}{S_m\tau_m 4\sinh^2(T/2\tau_m)}$.

To obtain the connected joint density, we need to calculate $\tilde{u}_i^1(x,t)$ to first order in ξ_i. For that purpose we introduce the dressed response kernel

$$[\partial_t+\nabla_x^2-1+2\tilde{u}_i^0(t)]G_i(x,t;y,t') = -\delta^d(x-y)\delta(t-t') \tag{101}$$

with $G_i(x,t;y,t')=0$ for $t>t'$ (i.e., time is in effect reversed as compared to a standard response function). It reads in Fourier, for $t,t'<t_i$,

$$G_i(q,t,t') = e^{-(q^2+1)(t'-t)+2\int_t^{t'}\tilde{u}_i^0(s)ds}\theta(t'-t) = e^{-(q^2+1)(t'-t)}\frac{(1-\lambda_i+\lambda_i e^{t'-t_i})^2}{(1-\lambda_i+\lambda_i e^{t-t_i})^2}\theta(t'-t). \tag{102}$$

We obtain

$$\tilde{u}_i^1(x,t) = \int d^dy\,G_i(x,t;y,t')\xi_i(y)\tilde{u}_i^0(t') + O(\xi^2), \tag{103}$$

which leads to the Laplace transform of the connected density as

$$\int d\dot{u}_1 d\dot{u}_2\rho_W^{c,x_1,x_2}(\dot{u}_1,\dot{u}_2)[e^{\lambda_1\dot{u}_1}-1][e^{\lambda_2\dot{u}_2}-1] = -\hat{\Delta}''(W)\int d^dy\int_0^{t_1}dt'\int_0^{t_2}dt''G_1(x_1,0;y,t')G_2(x_2,0;y,t'')\tilde{u}_1^0(t')\tilde{u}_2^0(t''). \tag{104}$$

It is more convenient to work in Fourier space and define

$$\rho_W^{c,x_1,x_2}(\dot{u}_1,\dot{u}_2) = \int\frac{d^dq}{(2\pi)^d}e^{-iq(x_1-x_2)}\rho_W^{c,q}(\dot{u}_1,\dot{u}_2). \tag{105}$$

We finally obtain the Laplace transform of the connected part of the joint density of total velocities in the two avalanches, for a fixed driving wave vector q, as

$$\int d\dot{u}_1 d\dot{u}_2\rho_W^{c,q}(\dot{u}_1,\dot{u}_2)[e^{\lambda_1\dot{u}_1}-1][e^{\lambda_2\dot{u}_2}-1] = -\hat{\Delta}''(W)F_{\lambda_1}(q,t_1)F_{\lambda_2}(q,t_2), \tag{106}$$

where we have defined

$$F_\lambda(q,t) = \lambda\frac{(1-q^2-\lambda)e^{-(1+q^2)t}+(1-\lambda)(q^2-1)e^{-t}+\lambda q^2e^{-2t}}{q^2(q^2-1)(1-\lambda+\lambda e^{-t})^2}. \tag{107}$$

We now analyze this formula in various cases: (i) homogeneous driving (ii) fixed distance between the seeds (iii) massless limit (leading to conjectures for the correlation exponents in any dimension).

1. Homogeneous driving

Velocities. For the homogeneous driving $\delta w(x)=\delta w$, the Laplace transform of the connected joint density simplifies into

$$\int d\dot{u}_1 d\dot{u}_2\rho_W^{c}(\dot{u}_1,\dot{u}_2)[e^{\lambda_1\dot{u}_1}-1][e^{\lambda_2\dot{u}_2}-1] = -L^d\hat{\Delta}''(W)F_{\lambda_1}(t_1)F_{\lambda_2}(t_2), \tag{108}$$

where we denote

$$F_\lambda(t) = F_\lambda(q=0,t) = -\frac{\lambda e^{-t}(\lambda(t-1+e^{-t})-t)}{(1-\lambda+\lambda e^{-t})^2}. \tag{109}$$

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[7]Indeed one also has that $\int_0^{+\infty}d\dot{u}\rho_t(\dot{u}) = \frac{1}{e^t-1} = \partial_{\delta w=0^+}[1-p_{\delta w}(t)]$, where $1-p_{\delta w}(t)$ is the probability that the velocity is nonzero at time t. This comes from the definition of the density $\rho(\dot{u}) = \partial_{\delta w=0^+}P_{\delta w}(\dot{u})$ from the PDF of the total velocity, which reads $P_{\delta w}(\dot{u}) = p_{\delta w}(t)\delta(\dot{u}) + [1-p_{\delta w}(t)]\tilde{P}_{\delta w}(\dot{u})$ where $\tilde{P}_{\delta w}$ is the smooth normalized PDF for $\dot{u}>0$ [see (26) and (28) in Ref. [44] for exact expressions]. Note the extra delta function piece which is usually not considered in the expression for the density.

It is possible to perform the inverse Laplace transform explicitly and obtain

$$\rho_W^c(\dot u_1,\dot u_2) = -L^d\,\hat\Delta''(W)\,r_{t_1}(\dot u_1)\,r_{t_2}(\dot u_2) \tag{110}$$

$$r_t(\dot u) = \frac{e^{t-e^t u/(e^t-1)}\{e^t[-(t+1)\dot u + e^t(t+\dot u-2)+4]-t-2\}}{(e^t-1)^4} = \frac{d}{dt}\frac{(-t+e^t-1)e^{t-e^t\dot u/(e^t-1)}}{(e^t-1)^2}. \tag{111}$$

Restoring the units it reads

$$\rho_W^c(\dot u_1,\dot u_2) = -\frac{L^d}{m^4 v_m^2 S_m^2}\Delta''(W)\,r_{t_1/\tau_m}\left(\frac{\dot u_1}{v_m}\right) r_{t_2/\tau_m}\left(\frac{\dot u_2}{v_m}\right), \tag{112}$$

where $v_m = S_m/\tau_m$. One can check that $\int_0^{+\infty} dt_1 \int_0^{+\infty} dt_2 \langle \dot u_1 \dot u_2\rangle_W^c$ calculated with this formula coincides with the result for $\langle S_1 S_2\rangle_W^c$ obtained above in Eq. (43) (which, we recall, was an exact result, i.e., valid beyond the ϵ expansion).

We can calculate the joint density of the mean total velocity, averaged over all the avalanche. In dimensionless units, using that $\int_0^{+\infty} dt\, r_t(\dot u) = e^{-\dot u}$, it reads simply

$$\int_0^{+\infty} dt_1 \int_0^{+\infty} dt_2 \rho_W^c(\dot u_1,\dot u_2) = -L^d\,\hat\Delta''(W) e^{-\dot u_1 - \dot u_2}. \tag{113}$$

Note that it is regular at small $\dot u$, unlike the single avalanche density (see above) $\rho(\dot u) = \frac{1}{\dot u} e^{-\dot u}$.

Let us obtain some cumulants. One has (in dimensionful units)

$$\frac{\langle \dot u_1 \dot u_2\rangle_W^c}{\langle \dot u_1\rangle\langle \dot u_2\rangle} = -\frac{\Delta''(W)}{m^4 L^d}\frac{t_1 t_2}{\tau_m^2} \tag{114}$$

and (in dimensionless units)

$$\frac{\langle \dot u_1 \dot u_2^3\rangle_W^c}{\langle \dot u_1^2 \dot u_2^2\rangle_W^c} = \frac{3e^{t_1-t_2}(e^{t_2}-1)[-3t_2+e^{t_2}(t_2+2)-2]}{2[-2t_1+e^{t_1}(t_1+1)-1][-2t_2+e^{t_2}(t_2+1)-1]}. \tag{115}$$

Durations. Finally we can obtain the correlation between the durations T_1 and T_2 of the two avalanches. Integrating over $\dot u$,

$$\int_0^{+\infty} d\dot u\, r_t(\dot u) = \frac{e^t(t-1)+1}{(e^t-1)^2}, \tag{116}$$

hence we obtain[8]

$$\rho_W^c(T_1,T_2) = -L^d\,\hat\Delta''(W) d(T_1) d(T_2), \tag{117}$$

$$d(T) = -\partial_T \int_0^{+\infty} d\dot u\, r_T(\dot u) = \frac{e^T[e^T(T-2)+T+2]}{(e^T-1)^3} \simeq_{T\ll 1} \frac{1}{6} - \frac{T^2}{60} + O(T^3) \tag{118}$$

$$\simeq_{T\gg 1} \simeq Te^{-T}, \tag{119}$$

where $d(T)$ is a decreasing function of T. Note that the dimensionful version is

$$\rho_W^c(T_1,T_2) = -\frac{L^d}{m^4 S_m^2 \tau_m^2}\Delta''(W) d(T_1/\tau_m) d(T_2/\tau_m). \tag{120}$$

2. Fixed distance between the seeds

Velocities. We now calculate the connected joint density $\rho_W^{c,x}(\dot u_1,\dot u_2)$, for the total velocities (at times t_1 and t_2 respectively) of two avalanches starting a distance x apart (i.e., in x_1 and x_2 with $x_2 - x_1 = x$). The definitions are similar to those in Sec. V C. One finds

$$\rho_W^{c,x}(\dot u_1,\dot u_2) = \int \frac{d^d q}{(2\pi)^d} e^{iqx}\,\rho_W^{c,q}(\dot u_1,\dot u_2), \tag{121}$$

$$\rho_W^{c,q}(\dot u_1,\dot u_2) = -\hat\Delta''(W)\,r_{q,t_1}(\dot u_1) r_{q,t_2}(\dot u_2), \tag{122}$$

[8]Again, one shows that $\int_0^{+\infty}\int_0^{+\infty} d\dot u_1 d\dot u_2 \rho_W(\dot u_1,\dot u_2) = \partial_{\delta w_1}|_{\delta w_1=0}\partial_{\delta w_2}|_{\delta w_2=0}\text{Prob}(\dot u_1>0,\dot u_2>0) = \rho_W(T_1,T_2)$ since $\text{Prob}(\dot u_1>0,\dot u_2>0) = \text{Prob}(T_1>t_1, T_2>t_2)$.

where

$$r_{q,t}(\dot u) = \frac{e^{-q^2 t - e^t \dot u/(e^t-1)+t}}{q^2(q^2-1)(e^t-1)^4}[A(q,t) + \dot u B(q,t)], \tag{123}$$

$$B(q,t) = e^t(-q^2 e^t + e^{q^2 t} + q^2 - 1), \tag{124}$$

$$A(q,t) = -2q^2 e^t - 2q^2 e^{q^2 t + t} + (q^2-1)(1 + e^{(q^2+2)t}) + (q^2+1)(e^{2t} + e^{q^2 t}). \tag{125}$$

Restoring the units it reads

$$\rho_W^{c,q}(\dot u_1, \dot u_2) = -\frac{1}{m^4 v_m^2 S_m^2}\Delta''(W)\, r_{q/m,t_1/\tau_m}\left(\frac{\dot u_1}{v_m}\right) r_{q/m,t_2/\tau_m}\left(\frac{\dot u_1}{v_m}\right), \tag{126}$$

where $v_m = S_m/\tau_m$. Again one checks that $\int_0^{+\infty} dt_1 \int_0^{+\infty} dt_2 \langle \dot u_1 \dot u_2\rangle_W^{c,q}$ calculated with this formula coincides with the result for $\langle S_1 S_2\rangle_W^{c,q}$ obtained in Sec. V C 2.

Durations. We can obtain the joint duration density. In dimensionless units, using

$$\int_0^{+\infty} d\dot u\, r_{q,t}(\dot u) = \frac{(q^2-1)e^t - q^2 + e^{(1-q^2)t}}{q^2(q^2-1)(e^t-1)^2}, \tag{127}$$

we find

$$\rho_W^{c,q}(T_1, T_2) = -\hat\Delta''(W)\, d_q(T_1)\, d_q(T_2), \tag{128}$$

$$d_q(t) = -\partial_t \int_0^{+\infty} d\dot u\, r_{q,t}(\dot u) = \frac{e^t((q^2-1)e^t + e^{-q^2 t}[(q^2+1)e^t - q^2 + 1] - q^2 - 1)}{q^2(q^2-1)(e^t-1)^3}. \tag{129}$$

3. Massless limit

We now study these formulas in the massless limit, to extract the fully universal limit. We follow the same strategy as explained in Sec. V C 3.

Velocities. In the limit $m \to 0$, we obtain from (126)

$$\rho_W^{c,q,f}(\dot u_1, \dot u_2) = \lim_{m\to 0} m^{-4}\rho_W^{c,q}(\dot u_1, \dot u_2) = -\Delta''(W)\frac{1}{t_1}\tilde r_{q\sqrt{t_1}}\left(\frac{\dot u_1}{t_1}\right)\frac{1}{t_2}\tilde r_{q\sqrt{t_2}}\left(\frac{\dot u_2}{t_2}\right) + O(\epsilon^2) \tag{130}$$

in the massless units (i.e., such that $\sigma = \eta = 1$), where

$$\tilde r_q(\dot u) = \lim_{m\to 0} m^2 r_{q/m,m^2}(m^2 \dot u) = \frac{e^{-q^2-\dot u}(q^2 + 2 - (q^2+1)\dot u + e^{q^2}(q^2 + \dot u - 2))}{q^4} = \frac{1}{2}\dot u e^{-\dot u} + O(q^2) \tag{131}$$

$$= \frac{e^{-\dot u}}{q^2} + O(q^{-4}). \tag{132}$$

We recall that $\Delta''(W) \simeq A_d \tilde\Delta^{*\prime\prime}(0^+) + O(\epsilon^2)$ if W is kept fixed as $m \to 0$.

We can thus surmise, more generally in the limit $m \to 0$ (and fixed W), from scaling and dimensional analysis, the fully universal scaling form[9]

$$\rho_W^{c,q,f}(\dot u_1, \dot u_2) = \frac{1}{(t_1 t_2)^{\mathsf{a}_c^1}} F_d\left(\frac{\dot u_1}{t_1^{(d+\zeta)/z-1}}, \frac{\dot u_2}{t_2^{(d+\zeta)/z-1}}, q t_1^{1/z}, q t_2\right)^{1/z}, \quad \mathsf{a}_c^1 = \frac{1}{z}\left(\frac{3}{2}d - 2 + 2\zeta\right) - 1 \tag{133}$$

with, in the $d = 4 - \epsilon$ expansion,

$$F_d(\dot u_1, \dot u_2, q_1, q_2) = -A_4\tilde\Delta^{*\prime\prime}(0^+)\tilde r_{q_1}(\dot u_1)\tilde r_{q_2}(\dot u_2) + O(\epsilon^2) \tag{134}$$

with $A_4\tilde\Delta^{*\prime\prime}(0^+) = \frac{16\pi^2}{9}\epsilon$.

Let us study $q = 0$, i.e., the homogeneous driving. From (112) using that $r_{m^2 t}(m^2\dot u) \simeq_{m\to 0} \frac{\dot u}{2t^2 m^2} e^{-\dot u/t}$ we obtain the simple and finite expression in the massless limit,

$$\rho_W^{c,f}(\dot u_1, \dot u_2) = \lim_{m\to 0} m^{-4}\rho_W^c(\dot u_1, \dot u_2) = -L^d \Delta''(W)\frac{\dot u_1}{2t_1^2}e^{-\dot u_1/t_1}\frac{\dot u_2}{2t_2^2}e^{-\dot u_2/t_2} \tag{135}$$

[9] Up to two nonuniversal scales $\ell_\sigma = \lim_{m\to 0} m^{-1} S_m^{-1/(d+\zeta)}$ and $\ell_\eta = \lim_{m\to 0} m^{-1}\tau_m^{-1/z}$.

in the massless units. At variance with the joint size densities, there is no factor $(mL)^d$ (although of course there is a factor L^d). Hence this expression already has a fully universal limit. The origin of this surprising fact is that now there is a commutation of limits $q \to 0$ and $m \to 0$, as can be seen from (131). Presumably it occurs because the times t_1, t_2 provide some natural cutoff.[10] If we surmise that for this observable this property holds more generally we obtain

$$\rho_W^{c,f}(\dot{u}_1, \dot{u}_2) = \frac{L^d}{(t_1 t_2)^{(1/z)[(3/2)d-2+2\zeta]-1}} F_d\left(\frac{\dot{u}_1}{t_1^{(d+\zeta)/z-1}}, \frac{\dot{u}_2}{t_2^{(d+\zeta)/z-1}}, 0, 0\right). \tag{136}$$

Durations. Let us now discuss the joint density of the avalanche durations. Restoring the units in Eq. (128) we have

$$\rho_W^{c,q}(T_1, T_2) = -\frac{1}{m^4 S_m^2 \tau_m^2} \Delta''(W)\, d_{q/m}(T_1/\tau_m)\, d_{q/m}(T_2/\tau_m). \tag{137}$$

In the limit $m \to 0$ we obtain

$$\rho_W^{c,q,f}(T_1, T_2) = \lim_{m\to 0} m^{-4} \rho_W^{c,q}(T_1, T_2) = -\Delta''(W) \frac{1}{T_1 T_2}\, \tilde{d}\big(q T_1^{1/2}\big)\, \tilde{d}\big(q T_2^{1/2}\big) \tag{138}$$

in massless units with

$$\tilde{d}(q) = \frac{q^2 - 2 + e^{-q^2}(q^2+2)}{q^4} = \frac{q^2}{6} + O(q^4) = \frac{1}{q^2} + O\left(\frac{1}{q^4}\right). \tag{139}$$

We can compare with the massless limit of (120),

$$\rho_W^{c,f}(T_1, T_2) \simeq_{m\to 0} -(mL)^d m^{4-d} \Delta''(W) \tfrac{1}{36} + O(\epsilon^2) \tag{140}$$

in massless units. As was the case for the size joint density there is a factor $(mL)^d$ and noncommutation of limits $q \to 0$ and $m \to 0$. The matching, i.e., setting $q = m$ and $m \to 0$ into (138) and recovering (140) [to $O(\epsilon)$] also works, as was the case for the size density. Another sign of the noncommuting limits is that if one integrates the $q = 0$ result (135) over $\dot{u}_1, \dot{u}_2$ and takes $\partial_{t_1} \partial_{t_2}$ one obtains zero, while the correct subleading term in m is (140).

More generally we can thus surmise, from scaling and dimensional analysis, the fully universal scaling form (up to two nonuniversal scales) in the massless limit in general dimension d,

$$\rho_W^{c,q,f}(T_1, T_2) = \frac{1}{(T_1 T_2)^{\alpha_c^1}} G_d\big(q T_1^{1/z}, q T_2^{1/z}\big), \alpha_c^1 = 1 - \frac{1}{2z}(4 - d - 2\zeta), \tag{141}$$

with, in the $d = 4 - \epsilon$ expansion, $G_d(q_1, q_2) = -A_4 \tilde{\Delta}^{*\prime\prime}(0^+) \tilde{d}(q_1) \tilde{d}(q_2) + O(\epsilon^2)$, with $A_4 \tilde{\Delta}^{*\prime\prime}(0^+) = \frac{16\pi^2}{9}\epsilon$. By the same reasoning which led to (91) we can also surmise that the rhs of (141) must behave as q^d at small q leading to

$$\rho_W^{c,f}(T_1, T_2) \sim (mL)^d \frac{1}{T_1^{\alpha_c} T_2^{\alpha_c}}, \quad \alpha_c = 1 - \frac{1}{z}(2 - \zeta) \tag{142}$$

with $\alpha_c = 0$ in mean field, i.e., for $d = d_c = 4$, consistent with (140).

B. Correlation of the shapes of two avalanches

Consider now the joint density of the total velocities $\dot{u}_i = \dot{u}_i(t_i) = \int d^d x\, \dot{u}_i(x, t_i)$, $i = 1, 2$, and $\dot{u}_i' = \dot{u}_i(t_i') = \int d^d x\, \dot{u}_i(x, t_i')$, $i = 1, 2$ in two avalanches, the times $0 < t_1 < t_1'$ and $0 < t_2 < t_2'$ being counted from the beginning of each avalanche. Its Laplace transform satisfies

$$\int d\dot{u}_1 d\dot{u}_1' d\dot{u}_2 d\dot{u}_2' \rho_W^{x_1,x_2}(\dot{u}_1, \dot{u}_1', \dot{u}_2, \dot{u}_2')[e^{\lambda_1 \dot{u}_1 + \lambda_1' \dot{u}_1'} - 1][e^{\lambda_2 \dot{u}_2 + \lambda_2' \dot{u}_2'} - 1] = \langle \tilde{u}_1(x_1, 0) \tilde{u}_2(x_2, 0)\rangle_\xi, \tag{143}$$

where now $\tilde{u}_i(x, t)$ are solutions of (95) with the source $-\lambda_i \delta(t - t_i) - \lambda_i' \delta(t - t_i')$ with $t_i < t_i'$.

Here we are only interested in the correlation of the shape of each avalanche. Let us first recall the definition of the mean shape for a single avalanche, at fixed avalanche duration T: it is the mean velocity as a function of time, conditioned to the avalanche duration

$$\langle \dot{u}_i(t_i)\rangle_{T_i} = \frac{\int d\dot{u}_i \dot{u}_i \rho(\dot{u}_i, T_i)}{\rho(T_i)}, \tag{144}$$

[10] Note however, interestingly, the noncommutation of limits $m \to 0$ and integration over time, as integrating the massless result (135) over time gives an extra factor $1/4$ as compared to taking the small mass (or velocity) limit of the formula (113) which was integrated over time at finite m. This is because the time scale τ_m diverges in that limit, while (135) is dominated by time scales t_1, t_2.

where $\rho(\dot{u}_i, T_i)$ is the joint density of the velocity and duration in an avalanche. Here we are interested in the correlation of the shapes,

$$\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle^{x_1,x_2}_{T_1,T_2} = \frac{\int d\dot{u}_1 d\dot{u}_2 \dot{u}_1\dot{u}_2 \rho_W^{x_1x_2}(\dot{u}_1, T_1, \dot{u}_2, T_2)}{\rho_W^{x_1x_2}(T_1, T_2)}, \tag{145}$$

with fixed positions of the seeds at x_1, x_2, and in the correlation of the shapes for a homogeneous driving,

$$\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2} = \frac{\int d\dot{u}_1 d\dot{u}_2 \dot{u}_1\dot{u}_2 \rho_W(\dot{u}_1, T_1, \dot{u}_2, T_2)}{\rho_W(T_1, T_2)}. \tag{146}$$

The denominator, i.e., the joint density of durations $\rho_W^{x_1x_2}(T_1, T_2)$ was studied in the previous section. The numerator can be obtained by taking a derivative of (143) with respect to λ_1 and λ_2 at $\lambda_1 = \lambda_2 = 0$, and setting $\lambda_i' = -\infty$ [which implies $\dot{u}_i(t_i') = 0$ hence $T_i < t_i'$] and, to obtain the joint density with durations T_i, taking the derivative with respect to t_i',

$$\int d\dot{u}_1 d\dot{u}_2 \dot{u}_1\dot{u}_2 \rho_W^{x_1x_2}(\dot{u}_1, T_1, \dot{u}_2, T_2) = \partial_{t_1'}\big|_{t_1'=T_1}\partial_{t_2'}\big|_{t_2'=T_2} \lim_{\lambda_1',\lambda_2'\to-\infty} \partial_{\lambda_1}\big|_{\lambda_1=0}\partial_{\lambda_2}\big|_{\lambda_2=0}\langle \tilde{u}_1(x_1,0)\tilde{u}_2(x_2,0)\rangle_\xi. \tag{147}$$

Let us first describe the solution of (95) to order 0 in ξ and recall the calculation of the shape in the BFM. The solution of

$$(\partial_t - 1)\tilde{u}_i^0(t) + \tilde{u}_i^0(t)^2 = -\lambda_i\delta(t - t_i) - \lambda_i'\delta(t - t_i') \tag{148}$$

with $\tilde{u}_i^0(t > t_i') = 0$ is [28,44]

$$\tilde{u}_i^0(t) = \frac{\lambda_i'}{\lambda_i' + (1 - \lambda_i')e^{t_i'-t}}, \quad t_i < t < t_i', \tag{149}$$

$$\tilde{u}_i^0(t) = \left(1 - \frac{\lambda_i\lambda_i' e^{t_i} - (1 - \lambda_i)(1 - \lambda_i')e^{t_i'}}{(1 + \lambda_i)\lambda_i' e^{t_i} + (1 - \lambda_i')\lambda_i e^{t_i'}} e^{t_i - t}\right)^{-1}, \quad t < t_i. \tag{150}$$

It allows us to obtain the mean shape of a single avalanche (144) within the BFM as

$$\langle \dot{u}_i(t_i)\rangle_{T_i} = \frac{1}{\rho^{x_i}(T_i)}\partial_{t_i'}\big|_{t_i'=T_i}\partial_{\lambda_i}\big|_{\lambda_i=0}\,\tilde{u}_i^0(0)\big|_{\lambda_i'=-\infty}. \tag{151}$$

Thus, for the remainder of the calculation we only need $\tilde{u}_i^0(0)$ for $\lambda_i' = -\infty$ and to first order in λ_i, which reads

$$\tilde{u}_i^0(t) = \frac{\theta(t_i' - t)}{1 - e^{t_i'-t}}\left(1 + \lambda_i \frac{e^{-t_i}(e^{t_i} - e^{t_i'})^2}{e^t - e^{t_i'}}\theta(t_i - t)\right), \tag{152}$$

where we discard higher order terms in λ_i. This leads to the classical BFM result for the shape [28,44],

$$\begin{aligned}\langle \dot{u}_i(t_i)\rangle_{T_i} &= \frac{1}{\rho^{x_i}(T_i)}\partial_{t_i'}\big|_{t_i'=T_i}\frac{e^{-t_i}(e^{t_i} - e^{t_i'})^2}{(1 - e^{t_i'})^2} = \frac{2(1 - e^{-t_i})(e^{T_i} - e^{t_i})}{e^{T_i} - 1} \\ &= s_0(t_i, T_i) = \frac{4\sinh\frac{t_i}{2}\sinh\frac{T_i - t_i}{2}}{\sinh\frac{T_i}{2}} = 2T_i z(1 - z) + O(T^3), \quad z = \frac{t_i}{T_i},\end{aligned} \tag{153}$$

using (100). Restoring the units the BFM result reads

$$\langle \dot{u}(t)\rangle_T = v_m\, s_0\left(\frac{t}{\tau_m}, \frac{T}{\tau_m}\right) \simeq \frac{v_m}{\tau_m}\, T\, 2z(1 - z) + O(T^3) \tag{154}$$

with $v_m = S_m/\tau_m$, which has a well defined massless limit since in the BFM $\lim_{m\to0}\frac{v_m}{\tau_m} = \sigma/\eta^2 = 1$ in dimensionless units. Note that the next order term $O(T^3)$ is $\sim 1/\tau_m^2 \sim m^4$ higher order in that limit.

Now we study (95) to the desired order $O(\xi)$. As in Sec. VII A we obtain

$$\tilde{u}_i(x, t) = \tilde{u}_i^0(t) + \int d^d y\, G_i(x, t; y, t')\xi_i(y)\tilde{u}_i^0(t') + O(\xi^2). \tag{155}$$

Inserting in Eq. (147) and going to Fourier space, the connected correlation

$$\rho_W^{c,x_1,x_2}(\dot{u}_1, T_1, \dot{u}_2, T_2) = \int \frac{d^d q}{(2\pi)^d} e^{-iq(x_1-x_2)}\rho_W^{c,q}(\dot{u}_1, T_1, \dot{u}_2, T_2) \tag{156}$$

becomes (taking into account that the disconnected piece has been substracted)

$$\int d\dot{u}_1 d\dot{u}_2 \dot{u}_1 \dot{u}_2 \rho_W^{c,q}(\dot{u}_1, T_1, \dot{u}_2, T_2) = -\hat{\Delta}''(W)\, \hat{r}_{q,t_1}(T_1)\hat{r}_{q,t_2}(T_2),$$
$$\hat{r}_{q,t_i}(T_i) = \partial_{t_i'}\big|_{t_i'=T_i} \partial_{\lambda_i}\big|_{\lambda_i=0} \int_0^{t_i'} dt' G_i(q,0,t')\tilde{u}_i^0(t'), \tag{157}$$

where we must insert (152) and the propagator to the needed order,

$$\begin{aligned} G_i(q,t,t') &= e^{-(q^2+1)(t'-t)+2\int_t^{t'} \tilde{u}_i^0(s)ds}\theta(t'-t) \\ &= e^{-(q^2+1)(t'-t)} \frac{(e^{t_i'}-e^{t'})^2}{(e^{t_i'}-e^{t})^2}\theta(t<t'<t_i')\bigg(1+\lambda_1 \frac{2e^{-t_i}(e^{t_i}-e^{t_i'})^2(e^{t}-e^{t'})}{(e^{t}-e^{t_i'})(e^{t_i'}-e^{t'})}\theta(t<t'<t_i) \\ &\quad +\lambda_1 \frac{2e^{-t_i}(e^{t_i}-e^{t})(e^{t_i}-e^{t_i'})}{e^{t}-e^{t_i'}}\theta(t<t_i<t')\bigg). \end{aligned} \tag{158}$$

From this one obtains the connected shape correlation. Since the connected parts of the densities are $O(\epsilon)$ one can expand (145) as

$$\begin{aligned} \langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2}^{c,x_1,x_2} &= \langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2}^{x_1,x_2} - \langle \dot{u}_1(t_1)\rangle_{T_1}\langle \dot{u}_2(t_2)\rangle_{T_2} \\ &= \frac{\int d\dot{u}_1 d\dot{u}_2 \dot{u}_1\dot{u}_2 \rho_W^{c,x_1x_2}(\dot{u}_1,T_1,\dot{u}_2,T_2)}{L^{-2d}\rho(T_1)\rho(T_2)} - \langle \dot{u}_1(t_1)\rangle_{T_1}\langle \dot{u}_2(t_2)\rangle_{T_2}\frac{\rho_W^{c,x_1x_2}(T_1,T_2)}{L^{-2d}\rho(T_1)\rho(T_2)} + O(\epsilon^2). \end{aligned} \tag{159}$$

We recall that the nonconnected parts are x_1, x_2 independent and $\rho^x(T) = L^{-d}\rho(T)$. Thus, to this order, one can easily Fourier transform and write the shape correlation at fixed seed positions as

$$\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2}^{c,x_1,x_2} = \int \frac{d^dq}{(2\pi)^d} e^{-iq(x_1-x_2)}\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2}^{c,q}, \tag{160}$$

$$\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2}^{c,q} = -\hat{\Delta}''(W)\left[\frac{\hat{r}_{q,t_1}(T_1)\hat{r}_{q,t_2}(T_2)}{L^{-2d}\rho(T_1)\rho(T_2)} - \frac{d_q(T_1)d_q(T_2)}{L^{-2d}\rho(T_1)\rho(T_2)}\langle \dot{u}_1(t_1)\rangle_{T_1}\langle \dot{u}_1(t_2)\rangle_{T_2}\right] + O(\epsilon^2), \tag{161}$$

where we have used (157). We recall that $\rho(T)$ is given in (100) and $d_q(T)$ in Eq. (128). For the shape correlation at uniform driving we obtain

$$\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2}^{c} = -L^d\hat{\Delta}''(W)\left[\frac{\hat{r}_{t_1}(T_1)\hat{r}_{t_2}(T_2)}{\rho(T_1)\rho(T_2)} - \frac{d(T_1)d(T_2)}{\rho(T_1)\rho(T_2)}\langle \dot{u}_1(t_1)\rangle_{T_1}\langle \dot{u}_1(t_2)\rangle_{T_2}\right] + O(\epsilon^2), \tag{162}$$

where $\hat{r}_t(T) = \hat{r}_{q=0,t}(T)$ and $d(T) = d_{q=0}(T)$ is given in Eq. (117). In both formulas, for $\langle \dot{u}(t)\rangle_T$ one can insert into this order the BFM shape given in Eq. (153).

Here we will only discuss the final formula for the shape at homogeneous driving, i.e., for $q = 0$. The formula at finite q are given in Appendix D.

Denoting $z = t/T$, we obtain the building blocks of (162) as

$$\begin{aligned} A_T(t) = \frac{\hat{r}_t(T)}{L^{-d}\rho(T)} &= \frac{\sinh(T-t) + (t-2T)\cosh(t) - (t+T)\cosh(t-T) + \sinh(t) + 2T - \sinh(T) + T\cosh(T)}{\sinh^2\frac{T}{2}} \\ &= \frac{1}{3}T^3 z(1-z)[1+z(1-z)] + O(T^5) \end{aligned} \tag{163}$$

and

$$B_T(t) = \frac{d(T)}{L^{-d}\rho(T)}\langle \dot{u}(t)\rangle_T = \frac{4\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{T-t}{2}\right)}{\sinh\left(\frac{T}{2}\right)}\left[T\coth\left(\frac{T}{2}\right) - 2\right] = \frac{1}{3}T^3 z(1-z) + O(T^5). \tag{164}$$

From them one obtains the explicit expression for (162) in the form, restoring units

$$\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle_{T_1,T_2}^{c} = -(mL)^{-d}v_m^2 m^{d-4}\Delta''(W)\left[A_{T_1/\tau_m}\left(\frac{t_1}{\tau_m}\right)A_{T_2/\tau_m}\left(\frac{t_2}{\tau_m}\right) - B_{T_1/\tau_m}\left(\frac{t_1}{\tau_m}\right)B_{T_2/\tau_m}\left(\frac{t_2}{\tau_m}\right)\right] + O(\epsilon^2) \tag{165}$$

with $v_m = S_m/\tau_m$. Note that since $A_T(T-t) = A_T(t)$ and $B_T(T-t) = B_T(t)$ we find that to this order the shape correlations are symmetric in independently changing each $t_i \to T_i - t_i$. As is well known this property of the mean shape for a single avalanche does not hold to the next order in ϵ, the corrections having been obtained in Ref. [29].

We now display the shape correlation explicitly for small avalanches. This is equivalent to considering the small mass limit. Putting together the above results we find in the small T_1, T_2 limit, with $z_1 = t_1/T_1$ and $z_2 = t_2/T_2$,

$$\langle \dot{u}_1(t_1)\dot{u}_2(t_2)\rangle^c_{T_1,T_2} = -(mL)^{-d}\frac{S_m^2}{\tau_m^8}m^{d-4}\Delta''(W) \times \frac{1}{9}T_1^3T_2^3 z_1(1-z_1)z_2(1-z_2)\{[1+z_1(1-z_1)][1+z_2(1-z_2)]-1\}. \tag{166}$$

The factor $(mL)^{-d}$ is expected since, in order to be correlated, the avalanches should take place in the same region of size $1/m$ along the interface. We see that there is indeed a correlation between the shapes of the avalanches. However it is of order $O(T_1^3T_2^3)$, i.e., loosely speaking it arises as a correlation between the subleading $O(T^3)$ terms in the avalanche shape in Eq. (154). As a consequence it is $O(m^4)$ in the limit of small m. Hence the correlation of the fully universal part, which corresponds to the parabolic form for the mean shape, vanishes, but there is a nontrivial correlation at the next leading order.

Finally, performing the double integral $\int_0^{T_1} dt_1 \int_0^{T_2} dt_2$ on (167) we obtain an interesting observable, the correlation between the total sizes of two avalanches, at fixed durations, $\langle S_1S_2\rangle^c_{T_1,T_2}$, which reads

$$\langle S_1S_2\rangle^c_{T_1,T_2} = -(mL)^{-d}S_m^2m^{d-4}\Delta''(W)[A(T_1/\tau_m)A(T_2/\tau_m) - B(T_1/\tau_m)B(T_2/\tau_m)] + O(\epsilon^2) \tag{167}$$

$$= -(mL)^{-d}\frac{S_m^2}{\tau_m^8}m^{d-4}\Delta''(W)\frac{11}{8100}T_1^4T_2^4 + O\big(T_1^6, T_2^6\big) \tag{168}$$

with

$$A(T) = T\frac{T(\cosh(T)+2) - 3\sinh(T)}{\sinh^2\left(\frac{T}{2}\right)} = \frac{T^4}{15} + O(T^6),$$

$$B(T) = 2\left[T\coth\left(\frac{T}{2}\right) - 2\right]^2 = \frac{T^4}{18} + O(T^6). \tag{169}$$

We note the sum rule (in dimensionless units)

$$\int_0^{+\infty} dT A(T)\rho(T) = \int_0^{+\infty} dT A(T)\frac{1}{4\sinh^2(T/2)} = 1. \tag{170}$$

Hence multiplying (167) by $\rho(T_1)\rho(T_2)$, the first term leads exactly to (43) (with $\langle S\rangle = L^d$). The second term comes from the $O(\epsilon)$ correlation between T_1 and T_2 together with the precise definition of the connected shape in Eq. (159). Similarly there is a sum rule when performing the double integral $\int_0^{T_1} dt_1 \int_0^{T_2} dt_2$ on Eq. (157). Indeed, upon further integration $\int_0^{+\infty} dT_1 dT_2$, it should give back (70). Using (73) we see that it implies the sum rule (in dimensionless units)

$$\int_0^{+\infty} dT \int_0^T dt\, \hat{r}_{q,t}(T) = \frac{1}{1+q^2}, \tag{171}$$

which we have checked is indeed obeyed by the result in Appendix D.

VIII. CONCLUSION AND DISCUSSION

In conclusion we have obtained a method to calculate the correlations between successive avalanches in the dynamics of an elastic interface near the depinning transition to leading order in the $\epsilon = d_c - d$ expansion. This approach is technically simpler than the corresponding one developed to study shocks in the statics. We have first calculated correlations of the global and local sizes, which, to the accuracy of $O(\epsilon)$ leads to results formally similar to the one for the shocks in the statics, apart from the fact that the renormalized disorder correlator is different in each case. Next we have calculated an observable which is more natural in the dynamics, the correlation between avalanche sizes with prescribed positions of the seeds (the starting points). The massless limit was studied, leading to fully universal results, and conjectures for the correlation exponents. Some of these results were confronted to numerical simulations of an interface in $d = 1$. In a second part we studied truly dynamical correlations, between the velocities in the two avalanches. We obtained the correlations of the total velocities and of the avalanche durations both for homogeneous driving and for prescribed positions of the seeds. These correlations admit a fully universal massless limit which we studied, leading to further conjectures for correlation exponents. Finally, we calculated the correlation between the shapes of two avalanches. These were found to be subdominant for small avalanches, but nonzero for larger ones. It would be quite useful to probe these correlations further in numerical simulations and in experiments to test the theoretical predictions. These tests should allow us to distinguish the various universality classes for avalanches.

Let us close by indicating an interesting direction for further work. Here we have shown that the correlation between avalanches separated by W in the direction of motion is proportional (to leading order) to $\Delta''(W)$, where $\Delta(w)$ is the renormalized correlator of the pinning force. At the depinning fixed point this quantity is *negative*, leading to anticorrelations. This result is valid at strictly zero temperature. On the other hand, avalanches at very low but finite temperature were studied recently in numerical simulations [45]. There "events" where the interface moves forward without returning, similar to avalanches, were observed. These occur at scales below and around the so-called thermal activation nucleus scale (also called $L_{\rm opt}$). These successive events tend to cluster in the same spatial region and are observed to be very strongly positively correlated (reminiscent of the propagation of a forest fire). At larger scale, they appear to organize into clusters, which behave more like conventional depinning avalanches. On the other hand, the FRG theory of creep, as obtained in [46], predicts a similar crossover in scales from the creep to depinning regimes. It is well known that in the creep regime $\Delta''(W)$ is *very large and positive* within a "thermal boundary layer" for small W, corresponding to the thermal nucleus scale. We claim that this is quite consistent with the observations in Ref. [45] of a strong positive correlation between the events. Obtaining a detailed theory is more challenging, since it requires a precise and operational definition of these events (as we did here for the zero temperature avalanches). However we believe that our result should provide the main guiding idea in that direction.

Note added. A recent preprint by Le Priol *et al.* [47] studies dynamical correlations in crack front propagation near depinning, in and outside of the avalanche regime, and observes some regime of negative velocity velocity correlations, globally consistent with the prediction of dynamical avalanche anticorrelations obtained in this work.

ACKNOWLEDGMENTS

We thank C. Le Priol, L. Ponson, and A. Rosso for interesting discussions. We acknowledge support from ANR Grant No. ANR-17-CE30-0027-01 RaMaTraF. T.T.'s research was supported by a postdoctoral grant from the Research Foundation, Flanders (FWO).

APPENDIX A: RESTORING UNITS

In this Appendix we give useful information on how to restore the dimensionful units in the formula for the problem with a mass $m > 0$. To check units (and restore them) one must convert all quantities in units of m, S_m, τ_m which are the natural units. The conversion goes as follows:

$$\begin{aligned}
&[x], [L] = m^{-1}, \quad [w(x,t)], [u(x,t)] = S_m m^d, \quad [t] = \tau_m,\\
&[\dot u(x,t)] = S_m m^d/\tau_m, \quad [\dot u^{\rm tot}] = v_m := S_m/\tau_m,\\
&[\Delta(w)] = m^{4-d}[w^2] = S_m^2 m^{4+d}, \quad [\Delta'(w)] = S_m m^4,\\
&[\Delta''(w)] = m^{4-d}. \qquad \text{(A1)}
\end{aligned}$$

As dimensional relations these are exact (i.e., up to dimensionless prefactors) in any dimension. Note that the relation $-\Delta'(0) = S_m m^4$ is exact. Let us give more details.

For kicks, source, and response field, one has $\dot w(x,t) = \delta w(x)\delta(t)$, with for a uniform kick $\delta w(x) = \delta w$, then $[\delta w(x)] = [\delta w] = S_m m^d$. For the source, $[\lambda(x,t)] = 1/S_m$, with the same unit for $\lambda(x,t) = \lambda$, conjugated to S. For the response field $[\tilde u(x,t)] = 1/(S_m m^2)$.

For local sizes, one has $[S(x)] = S_m m^d$ and $\lambda(x,t) = \lambda\delta^d(x)$ implies $[\lambda] = 1/(m^d S_m)$.

For densities, one must distinguish densities for different driving, and for different observables, which have all different dimensions. The density with respect to a uniform driving is $[\rho(S)] = L^d/S_m^2$, $[\rho(\dot u^{\rm tot})] = L^d/(S_m v_m)$. The joint densities are $[\rho_W^c(S_1,S_2)] = [\rho_W(S_1,S_2)] = [\rho(S_1)\rho(S_2)]$. The density with respect to uniform driving of local size is $[\rho(S(x))] = 1/(S_m^2 m^{2d})$ and $[\rho_W(S_1(x_1),S_2(x_2))] = 1/(S_m^4 m^{4d})$. The densities with fixed kick positions have dimension $[\rho^x(S)] = 1/S_m^2$, $[\rho_W^{c,x_1,x_2}(S_1,S_2)] = [\rho_W^{c,x}(S_1,S_2)] = 1/S_m^4$, and in Fourier space $[\rho_W^{c,q}(S_1,S_2)] = 1/(m^d S_m^4)$. Similarly one has $[\rho^x(\dot u)] = 1/(S_m v_m)$ $[\rho_W^{c,x_1,x_2}(\dot u_1,\dot u_2)] = 1/(S_m^2 v_m^2)$ $[\rho_W^{c,q}(\dot u_1,\dot u_2)] = 1/(m^d S_m^2 v_m^2)$.

All the above assumes that one converts $L = 1/m$. There are also some rules for how L appears in the formula. For instance densities with uniform driving are $\rho(S) \sim L^d$, $\rho_W(S_1,S_2) \sim L^{2d}$, while connected densities are $\rho_W^c(S_1,S_2) \sim L^d$. The densities with fixed seed positions are all $O(1)$.

In a second stage one can make explicit the dependence in m, and introduce the roughness and dynamical exponents, i.e., write $S_m = A_S m^{-(d+\zeta)}$ and $\tau_m = A_\tau m^{-z}$, and write $\Delta(w) = m^{\epsilon-2\zeta}\hat\Delta(m^\zeta w)$.

Finally, what we call the "massless dimensionless units" are such that $\sigma = \eta = 1$.

APPENDIX B: AVALANCHE DECOMPOSITION

Let us justify further the formula (6). The avalanche picture is the following. In the limit of very slow driving, one can assume that the part of the velocity field $\dot u \equiv \dot u(x,t)$ which is $O(1)$ can be decomposed in a sum over discrete events called avalanches, schematically

$$\dot u = \sum_\alpha \dot u^{(\alpha)}. \qquad \text{(B1)}$$

Each $\dot u^{(\alpha)}$ is a random velocity field (inside one avalanche). It is either nonzero (the avalanche has occurred) or zero $\dot u^{(\alpha)} = 0$ with finite probability (the avalanche has not occurred). In practice it means that the velocity is not $O(1)$; it can be nonzero but vanishes as the driving vanishes.

Now we can use the identity

$$e^{\lambda\cdot\dot u} = \prod_\alpha e^{\lambda\cdot\dot u^{(\alpha)}} = \sum_{n=0}^{+\infty}\sum_{1<\alpha_1<\alpha_2<\ldots\alpha_n}\prod_{j=1}^n \left(e^{\lambda\cdot\dot u^{(\alpha_j)}} - 1\right). \qquad \text{(B2)}$$

If we want to think of each avalanche α to be triggered by a small kick δw_α with probability proportional to δw_α, we can average (B2) and obtain

$$\begin{aligned}
G[\lambda] &= \overline{\prod_\alpha e^{\lambda\cdot\dot u^{(\alpha)}}}\\
&= \sum_{n=0}^{+\infty}\sum_{1<\alpha_1<\alpha_2<\ldots\alpha_n}\prod_{j=1}^n \delta w_{\alpha_j}\int\prod_{j=1}^n D[\dot u^{(\alpha_j)}]\\
&\quad\times \rho_{\alpha_1,\ldots\alpha_n}(\dot u^{(\alpha_1)},\ldots,\dot u^{(\alpha_n)})\prod_{j=1}^n\left(e^{\lambda\cdot\dot u^{(\alpha_j)}} - 1\right), \qquad \text{(B3)}
\end{aligned}$$

where

$$\left(\prod_{j=1}^n \delta w_{\alpha_j}\right)\rho_{\alpha_1,\ldots\alpha_n}(\dot u^{(\alpha_1)},\ldots,\dot u^{(\alpha_n)}) \qquad \text{(B4)}$$

is the probability that the avalanches $\alpha_1, \dots \alpha_n$ have occurred, and $\rho_{\alpha_1,\dots\alpha_n}(\dot u^{(\alpha_1)}, \dots, \dot u^{(\alpha_n)})$ is the associated joint density for the avalanche velocities to take values $\dot u^{(\alpha_1)}, \dots, \dot u^{(\alpha_n)}$. In the BFM the avalanches are independent, and these densities are just products $\rho_{\alpha_1,\dots\alpha_n}(\dot u^{(\alpha_1)}, \dots, \dot u^{(\alpha_n)}) = \prod_{j=1}^n \rho_{\alpha_j}(\dot u^{(\alpha_j)})$ and one obtains $e^{\lambda\cdot\dot u} = e^{\sum_\alpha \delta w_\alpha \int D[\dot u^{(\alpha)}]\rho_\alpha(\dot u^{(\alpha)})(e^{\lambda\cdot\dot u^{(\alpha)}}-1)}$.

If we consider a source λ which is nonzero only for two specific avalanches, we see that we obtain the formula (6) since all other terms vanish. There is a small subtlety however concerning, e.g., the terms $\delta w_1^2, \delta w_2^2$ in Eq. (6) (not of interest there). The above picture is correct for distinct kicks δw_α. There are additional terms in $G[\lambda]$ containing powers of δw_α^p with $p > 1$. Those are obtained by a small modification, namely there can be in the sum (B1) p_α replica of the same avalanche (i.e., the kick δw_α can trigger p avalanches). In the BFM the p_α are distributed according to the Poisson distribution. Here we are not interested in these terms and we can use $p_\alpha = 1$. Hence the above picture is sufficient.

One can now compare with the expansion of (C2) in powers of $\dot w_{xt}$, namely

$$G[\lambda] = 1 + m^2 \int_{xt} \dot w_{xt}\langle \tilde u_{xt}\rangle_{S_\lambda} + \frac{1}{2}m^4 \int_{xt,y't'} \dot w_{xt}\dot w_{yt'}\langle \tilde u_{xt}\tilde u_{yt'}\rangle_{S_\lambda} + \cdots, \quad \text{(B5)}$$

where the brackets denote expectations of the response fields in the theory $S_\lambda = S - \int_{xt}\lambda_{xt}\dot u_{xt}$ (normalized since $G[\lambda] = 1$ when $\dot w_{xt} = 0$). Choosing $\dot w_{xt}$ to be a series of kicks at well separated times (much larger than the typical duration of an avalanche) $\dot w_{xt} = \sum_\alpha \delta w_\alpha(x)\delta(t - t_\alpha)$, and inserting in Eq. (B5) we obtain an expansion similar to (B3), and we can identify, e.g.,

$$\int D[\dot u^{(\alpha)}]\rho_\alpha^x(\dot u^{(\alpha)})(e^{\lambda\cdot\dot u^{(\alpha)}} - 1) = m^2\langle \tilde u_{x,t_\alpha}\rangle_{S_\lambda}, \quad \text{(B6)}$$

$$\int D[\dot u^{(\alpha)}]D[\dot u^{(\beta)}]\rho_{\alpha,\beta}^{x,y}(\dot u^{(\alpha)}, \dot u^{(\beta)})\left(e^{\lambda\cdot\dot u^{(\alpha)}} - 1\right)\left(e^{\lambda\cdot\dot u^{(\beta)}} - 1\right) = m^4\langle \tilde u_{x,t_\alpha}\tilde u_{y,t_\beta}\rangle_{S_\lambda}, \quad \text{(B7)}$$

and so on ($\alpha \neq \beta$ in the second relation). In the BFM $\langle \tilde u_{x,t}\rangle_{S_\lambda} = \tilde u^\lambda_{x,t}$, $\langle \tilde u_{x,t}\tilde u_{y,t'}\rangle_{S_\lambda} = \tilde u^\lambda_{x,t}\tilde u^\lambda_{y,t'}$ and so on in terms of the solution of the instanton equation, and the densities factorize. The calculations performed in the main text amount to calculating these expectation values beyond the BFM. As said above the two times t, t' in Eq. (B7) are very far away and chosen to belong to different avalanches. The response field correlation in Eq. (B7) when t, t' are distant of order τ_m instead allows us to study overlapping avalanches, which goes beyond the present study.

APPENDIX C: DERIVATION OF THE ACTION

In this Appendix we justify our main result (22)–(24) about the simplified field theory which allows us to compute correlations between several avalanches at order $O(\epsilon)$ in the depinning dynamics of elastic interfaces. To this aim, it is easier to consider the first protocol (see Sec. II A) where the interface is driven at a constant velocity $v \to 0^+$. For compactness we denote the space and time dependence in subscript, i.e., $u(x,t) \equiv u_{xt}$, $\lambda(x,t) \equiv \lambda_{xt}$, and so on.

Our starting point is that the MSR action for the velocity theory is exactly given by $S[\tilde u, \dot u] = S_0[\tilde u, \dot u] + S_{\rm dis}[\tilde u, \dot u] - m^2\int_{xt}\dot w_{xt}\tilde u_{xt}$ with [see Eqs. (301)–(303) in Ref. [28]],

$$S_0[\tilde u, \dot u] = \int_{xt}\tilde u_{xt}\left(\eta\partial_t - \nabla_x^2 + m^2\right)\dot u_{xt},$$
$$S_{\rm dis}[\tilde u, \dot u] = -\sigma\int_{xt}\tilde u_{xt}^2\dot u_{xt} + \frac{1}{2}\int_{xtt'}\tilde u_{xt}\tilde u_{xt'}\dot u_{xt}\dot u_{xt'}\Delta''_{\rm reg} \times (u_{xt} - u_{xt'}), \quad \text{(C1)}$$

where $\sigma = -\Delta'(0^+)$ and $\Delta''_{\rm reg}(u)$ is the regularized version of the renormalized disorder correlator $\Delta(u)$ that is smooth at 0 and defined by $\Delta_{\rm reg}(u) = \Delta(u) + \sigma|u|$.

This action allows us to calculate observables of the velocity field as, for any source λ_{xt},

$$G[\lambda] = \overline{e^{\int_{xt}\lambda_{xt}\dot u_{xt}}} = \int \mathcal{D}[\tilde u, \dot u]e^{\int_{xt}\lambda_{xt}\dot u_{xt} - S_0[\tilde u,\dot u] - S_{\rm dis}[\tilde u,\dot u] + m^2\int_{xt}\dot w_{xt}\tilde u_{xt}}. \quad \text{(C2)}$$

Let us consider the slow uniform driving $\dot w_{xt} = v \to 0^+$. Being interested in correlations between different avalanches, we can consider a source of the form

$$\lambda_{xt} = \lambda_1(x,t)\theta(t)\theta(\mathsf{T}_1 - t) + \lambda_2(x,t)\theta(t - W/v)\theta(W/v + \mathsf{T}_2 - t), \quad \text{(C3)}$$

that is, as a source that is active in two different time windows $[0, \mathsf{T}_1]$ and $[W/v, W/v + \mathsf{T}_2]$ and probes the result of two avalanches eventually occurring at times $t \in [0, \mathsf{T}_1]$ and $t \in [W/v, W/v + \mathsf{T}_2]$. Taking first the limit $v \to 0$ with T_1 and T_2 fixed, it is clear that if the interface moves during both time windows, this is due to *different avalanches* since the duration of an avalanche is $O(v^0)$.

It is a crucial point that near the critical dimension one has $u_{xt} = vt + O(\epsilon)$, while $\Delta''_{\rm reg}$ is also uniformly $O(\epsilon)$. That means that, at order $O(\epsilon)$, we can replace in the above action $\Delta''_{\rm reg}(u_{xt} - u_{xt'}) \to \Delta''_{\rm reg}[v(t - t')] + O(\epsilon^2)$. We now rescale the fields $\tilde u$ and $\dot u$ by introducing the characteristic scales of the avalanche motion: we rescale $t \to \tau_m t$, $x \to m^{-1}x$, and $u \to m^d S_m u$ with $\tau_m = \eta/m^2$ and $S_m = \sigma/m^4$. That leads to a rescaling of the fields as $\dot u \to v_m\dot u$ and $\tilde u \to \frac{1}{m^2 S_m}\tilde u$ with $v_m = m^d S_m/\tau_m$. We also use that for m close to 0 the renormalized disorder correlator Δ takes a scaling form $\Delta(u) = m^{\epsilon - 2\zeta}\hat\Delta(m^\zeta u)$ with $\hat\Delta$ a function that converges to a FRG fixed point uniformly of order $O(\epsilon)$ in the $m \to 0$ limit and ζ the roughness exponent of the interface. Rescaling finally the source field as $\lambda_{xt} \to \frac{1}{S_m}\lambda_{xt}$ and the driving velocity as $v \to v_m v$ we can decompose the action for these rescaled variables between the tree action $[O(\epsilon^0)]$ and one-loop corrections $[O(\epsilon)]$

$$G[\lambda] = \overline{e^{\int_{xt}\lambda_{xt}\dot u_{xt}}} = \int \mathcal{D}[\tilde u, \dot u]e^{\int_{xt}\lambda_{xt}\dot u_{xt} - S_{\rm tree}[\tilde u,\dot u] - \delta S_{\rm loop}[\tilde u,\dot u] + v\int_{xt}\tilde u_{xt}}, \quad \text{(C4)}$$

with the tree action corresponding to the rescaled version of the BFM action,

$$S_{\rm tree}[\tilde u, \dot u] = \int_{xt}\tilde u_{xt}\left(\partial_t - \nabla_x^2 + 1\right)\dot u_{xt} - \int_{xt}\tilde u_{xt}^2\dot u_{xt} \sim O(1), \quad \text{(C5)}$$

and the $O(\epsilon)$ corrections,

$$\delta S_{\rm loop}[\tilde{u}, \dot{u}] = \frac{1}{2}\int_{xtt'} \tilde{u}_{xt}\tilde{u}_{xt'}\dot{u}_{xt}\dot{u}_{xt'}\hat{\Delta}''_{\rm reg}[m^\zeta v_m v \tau_m (t-t')] \sim O(\epsilon). \tag{C6}$$

On the other hand, expanding G at order v^2 we obtain

$$G[\lambda] = 1 + v\int \mathcal{D}[\tilde{u},\dot{u}]\int_{xt}\tilde{u}_{xt} e^{\int_{xt}\lambda_{xt}\dot{u}_{xt} - S_{\rm tree}[\tilde{u},\dot{u}] - \delta S_{\rm loop}[\tilde{u},\dot{u}]} + \frac{v^2}{2}\int \mathcal{D}[\tilde{u},\dot{u}]\int_{xt}\tilde{u}_{xt}\int_{x't}\tilde{u}_{x't'} \times e^{\int_{xt}\lambda_{xt}\dot{u}_{xt} - S_{\rm tree}[\tilde{u},\dot{u}] - \delta S_{\rm loop}[\tilde{u},\dot{u}]} + O(v^3). \tag{C7}$$

As it has been already discussed numerous times in that context, e.g., [28,33], each response field $\tilde{u}_{xt}$ present in front of the dynamical path weight in Eq. (C7) generates the contribution to the observable $G[\lambda]$ that comes from an avalanche starting at time t at position x. The first line in Eq. (C7) thus corresponds to the contribution from single avalanches (a single avalanche occurred with probability of order v), while the second line corresponds to the contribution from two avalanches (two avalanches occurred with probability of order v^2), and contains the correlations between the two avalanches. Let us now think a bit diagrammatically about the calculation of an observable like (C3) at order $O(\epsilon)$. This can be performed by an expansion in λ_{xt} and $\delta S_{\rm loop}$ of the $O(v^2)$ term in Eq. (C7) and computing the resulting correlation functions involving the fields $\tilde{u}_{xt}$ and $\dot{u}_{xt}$ within the $S_{\rm tree}$ action. Since the tree action can only connect fields at times differing by a time scale at most of order $O(1) \ll 1/v$ and that the $O(\epsilon)$ interaction vertex $\delta S_{\rm loop}[\tilde{u},\dot{u}]$ can only be used once at this order, it is clear that we can only get diagrams of two types at order $O(\epsilon)$. The first type are diagrams where the $\delta S_{\rm loop}$ term was used to contract fields in the same time window. This leads to diagrams where fields on the two time windows are disconnected. These do not participate to the correlations between the two avalanches at $w = 0$ and $w = W$. For these diagrams we can replace in the limit $v \to 0$, $\hat{\Delta}''_{\rm reg}[m^\zeta v_m v\tau_m(t-t')] \to \hat{\Delta}''_{\rm reg}(0)$ in $\delta S_{\rm loop}$. In the second type of diagrams the $\delta S_{\rm loop}$ term is used to contract fields in different time windows. These are the only diagrams that contribute to correlations between avalanches at $w = 0$ and $w = W$. For these diagrams we can replace in the limit $v \to 0$, $\hat{\Delta}''_{\rm reg}[m^\zeta v_m v\tau_m(t-t')] \to \hat{\Delta}''_{\rm reg}(m^\zeta W)$ in $\delta S_{\rm loop}$.

In the above discussion, the first type of diagrams contains the diagrams that lead to the $O(\epsilon)$ corrections to the single avalanche statistics as studied in Refs. [28,33]. The second type of diagrams on the other hand generates the correlations between avalanches as studied in this paper. Since fields living inside the two different time windows (avalanches) can only be connected once by the $\hat{\Delta}''(W)$ interaction vertex we can formally introduce two different copies of the fields, one for each time window, the two copies being only connected by the $\hat{\Delta}''(W)$ interaction vertex. The contribution to $G[\lambda]$ coming from avalanches starting at position x_1 at time $t = 0$ and at position x_2 at time $t = W/v$ can then be targeted by restricting the response fields outside the exponential in the second line of (C7) to (x_1, t_1) and (x_2, t_2). Once this has been done one can send the time window T_1 and T_2 to infinity to ensure that the avalanche terminates inside the time window with probability 1 (the order $\mathsf{T}_i \ll 1/v$ holds since the limit $v \to 0$ has already been taken). Going back to the original units of the problem, one then sees that (C7) leads to (28) and more generally we obtain the formulation of the theory presented in the text in Eqs. (22)–(24).

APPENDIX D: FORMULA FOR THE SHAPE CORRELATION AT FIXED SEED POSITIONS

We give here the formula for arbitrary q in Eq. (163) We have

$$\partial_{\lambda_i}\big|_{\lambda_i=0}\int_0^{t'_i} dt' G_i(q,0,t')\tilde{u}^0_i(t') = \int_0^{t_i} dt' e^{-(q^2+1)t'}\frac{e^{t'-t_i}(e^{t'_i}-e^{t_i})^2}{(e^{t'_i}-1)^2} \tag{D1}$$

$$-2\int_0^{t_i} dt' e^{-(q^2+1)t'}\frac{e^{t'-t_i}(e^{t'_i}-e^{t_i})^2(e^{t'}-1)}{(e^{t'_i}-1)^3} - 2\int_{t_i}^{t'_i} dt' e^{-(q^2+1)t'}\frac{e^{t'-t_i}(e^{t'_i}-e^{t'})}{(e^{t'_i}-1)^3}(e^{t_i}-1)(e^{t'_i}-e^{t_i}) \tag{D2}$$

$$= \frac{e^{-t_i}(e^{t'_i}-e^{t_i})}{(e^{t'_i}-1)^2}\left((e^{t'_i}-e^{t_i})\int_0^{t_i} dt' e^{-q^2t'} - 2\frac{(e^{t'_i}-e^{t_i})}{(e^{t'_i}-1)}\int_0^{t_i} dt' e^{-q^2t'}(e^{t'}-1) - 2\frac{(e^{t_i}-1)}{(e^{t'_i}-1)}\int_{t_i}^{t'_i} dt' e^{-q^2t'}(e^{t'_i}-e^{t'})\right)$$

$$= \frac{e^{-t_i}(e^{t'_i}-e^{t_i})\left(\left(1-e^{-q^2t_i}\right)(e^{t'_i}-e^{t_i}) + \frac{2\left(e^{q^2(-t_i)+t'_i+t_i}+e^{t'_i-q^2t'_i}-e^{q^2(-t'_i)+t'_i+t_i}-e^{t_i-q^2t_i}-e^{t'_i}+e^{t_i}\right)}{(q^2-1)(e^{t'_i}-1)}\right)}{q^2(e^{t'_i}-1)^2} \tag{D3}$$

from which we obtain the result by a simple derivative,

$$r_{q,t_i}(T_i) = \partial_{t'_i}\big|_{t'_i=T_i}\partial_{\lambda_i}\big|_{\lambda_i=0}\int_0^{t'_i} dt' G_i(q,0,t')\tilde{u}^0_i(t'). \tag{D4}$$

[1] D. Fisher, Collective transport in random media: From superconductors to earthquakes, Phys. Rep. **301**, 113 (1998).

[2] J. Sethna, K. Dahmen, and C. Myers, Crackling noise, Nature (London) **410**, 242 (2001).

[3] S. Zapperi, P. Cizeau, G. Durin, and H. E. Stanley, Dynamics of a ferromagnetic domain wall: Avalanches, depinning transition, and the Barkhausen effect, Phys. Rev. B **58**, 6353 (1998).
[4] G. Durin and S. Zapperi, Scaling Exponents for Barkhausen Avalanches in Polycrystalline and Amorphous Ferromagnets, Phys. Rev. Lett. **84**, 4705 (2000).
[5] S. Moulinet, C. Guthmann, and E. Rolley, Roughness and dynamics of a contact line of a viscous fluid on a disordered substrate, Eur. Phys. J. E **8**, 437 (2002).
[6] P. Le Doussal, K. J. Wiese, S. Moulinet, and E. Rolley, Height fluctuations of a contact line: A direct measurement of the renormalized disorder correlator, Europhys. Lett. **87**, 56001 (2009).
[7] Y. Ben-Zion and J. Rice, Earthquake failure sequences along a cellular fault zone in a three-dimensional elastic solid containing asperity and nonasperity regions, J. Geophys. Res. **98**, 14109 (1993).
[8] D. S. Fisher, K. Dahmen, S. Ramanathan, and Y. Ben-Zion, Statistics of Earthquakes in Simple Models of Heterogeneous Faults, Phys. Rev. Lett. **78**, 4885 (1997).
[9] L. Ponson, Depinning Transition in Failure of Inhomogeneous Brittle Materials, Phys. Rev. Lett. **103**, 055501 (2009).
[10] S. Santucci, M. Grob, R. Toussaint, J. Schmittbuhl, A. Hansen, and K. J. Maloy, Fracture roughness scaling: A case study on planar cracks, Europhys. Lett. **92**, 44001 (2010).
[11] D. Bonamy, S. Santucci, and L. Ponson, Crackling Dynamics in Material Failure as the Signature of a Self-Organized Dynamic Phase Transition, Phys. Rev. Lett. **101**, 045501 (2008).
[12] R. Planet, S. Santucci, and J. Ortín, Avalanches and Non-Gaussian Fluctuations of the Global Velocity of Imbibition Fronts, Phys. Rev. Lett. **102**, 094502 (2009).
[13] D. S. Fisher, Interface Fluctuations in Disordered Systems: 5-ε Expansion, Phys. Rev. Lett. **56**, 1964 (1986).
[14] T. Nattermann, S. Stepanow, L.-H. Tang, and H. Leschhorn, Dynamics of interface depinning in a disordered medium, J. Phys. II (France) **2**, 1483 (1992).
[15] O. Narayan and D. S. Fisher, Critical behavior of sliding charge-density waves in 4-ε dimensions, Phys. Rev. B **46**, 11520 (1992).
[16] O. Narayan and D. S. Fisher, Threshold critical dynamics of driven interfaces in random media, Phys. Rev. B **48**, 7030 (1993).
[17] P. Chauve, P. Le Doussal, and K. J. Wiese, Renormalization of Pinned Elastic Systems: How does it Work Beyond One Loop? Phys. Rev. Lett. **86**, 1785 (2001).
[18] P. Le Doussal, K. J. Wiese, and P. Chauve, 2-loop functional renormalization group analysis of the depinning transition, Phys. Rev. B **66**, 174201 (2002).
[19] P. Le Doussal, K. J. Wiese, and P. Chauve, Functional renormalization group and the field theory of disordered elastic systems, Phys. Rev. E **69**, 026112 (2004).
[20] P. Le Doussal, Finite temperature Functional RG, droplets and decaying Burgers turbulence, Europhys. Lett. **76**, 457 (2006).
[21] P. Le Doussal, Exact results and open questions in first principle functional RG, Ann. Phys. **325**, 49 (2009).
[22] A. A. Middleton, P. Le Doussal, and K. J. Wiese, Measuring Functional Renormalization Group Fixed-Point Functions for Pinned Manifolds, Phys. Rev. Lett. **98**, 155701 (2007).
[23] A. Rosso, P. Le Doussal, and K. Wiese, Numerical calculation of the functional renormalization group fixed-point functions at the depinning transition, Phys. Rev. B **75**, 220201 (2007).
[24] A. Rosso, P. Le Doussal, and K. Wiese, Avalanche-size distribution at the depinning transition: A numerical test of the theory, Phys. Rev. B **80**, 144204 (2009).
[25] P. Le Doussal and K. J. Wiese, Size distributions of shocks and static avalanches from the functional renormalization group, Phys. Rev. E **79**, 051106 (2009).
[26] P. Le Doussal and K. J. Wiese, First-principle derivation of static avalanche-size distribution, Phys. Rev. E **85**, 061102 (2012).
[27] P. Le Doussal and K. J. Wiese, Distribution of velocities in an avalanche, Europhys. Lett. **97**, 46004 (2012).
[28] P. Le Doussal and K. J. Wiese, Avalanche dynamics of elastic interfaces, Phys. Rev. E **88**, 022106 (2013).
[29] A. Dobrinevski, P. Le Doussal, and K. J. Wiese, Avalanche shape and exponents beyond mean-field theory, Europhys. Lett. **108**, 66002 (2014).
[30] A. Dobrinevski, Field theory of disordered systems – avalanches of an elastic interface in a random medium, arXiv:1312.7156.
[31] T. Thiery, Analytical methods and field theory for disordered systems, arXiv:1705.07457.
[32] L. E. Aragon, A. B. Kolton, P. Le Doussal, K. J. Wiese, and E. A. Jagla, Avalanches in tip-driven interfaces in random media, Europhys. Lett. **113**, 10002 (2016).
[33] T. Thiery and P. Le Doussal, Universality in the mean spatial shape of avalanches, Europhys. Lett. **114**, 36003 (2016).
[34] F. Omori, On the aftershocks of earthquakes, J. Coll. Sci. Imper. Univ. Tokyo **7**, 111 (1895).
[35] E. A. Jagla, F. P. Landes, and A. Rosso, Viscoelastic Effects in Avalanche Dynamics: A Key to Earthquake Statistics, Phys. Rev. Lett. **112**, 174301 (2014).
[36] E. A. Jagla, Aftershock production rate of driven viscoelastic interfaces, Phys. Rev. E **90**, 042129 (2014).
[37] A. Dobrinevski, P. Le Doussal, and K. J. Wiese, Statistics of avalanches with relaxation and Barkhausen noise: A solvable model, Phys. Rev. E **88**, 032106 (2013).
[38] T. Thiery, P. Le Doussal, and K. J. Wiese, Universal correlations between shocks in the ground state of elastic interfaces in disordered media, Phys. Rev. E **94**, 012110 (2016).
[39] T. Thiery, P. Le Doussal, and K. J. Wiese, Spatial shape of avalanches in the brownian force model, J. Stat. Mech.: Theory Exp. (2015) P08019.
[40] M. Delorme, P. Le Doussal, and K. J. Wiese, Distribution of joint local and total size and of extension for avalanches in the Brownian force model, Phys. Rev. E **93**, 052142 (2016).
[41] B. Alessandro, C. Beatrice, G. Bertotti, and A. Montorsi, Domain-wall dynamics and Barkhausen effect in metallic ferromagnetic materials. I. Theory, J. Appl. Phys. **68**, 2901 (1990).
[42] A. A. Middleton, Asymptotic Uniqueness of the Sliding State for Charge-Density Waves, Phys. Rev. Lett. **68**, 670 (1992).
[43] I. Dornic, H. Chaté, and M. A. Munoz, Integration of Langevin Equations with Multiplicative Noise and the Viability of Field Theories for Absorbing Phase Transitions, Phys. Rev. Lett. **94**, 100601 (2005).

[44] A. Dobrinevski, P. Le Doussal, and K. J. Wiese, Nonstationary dynamics of the Alessandro-Beatrice-Bertotti-Montorsi model, Phys. Rev. E **85**, 031105 (2012).
[45] E. Ferrero, L. Foini, T. Giamarchi, A. B. Kolton, and A. Rosso, Spatio-Temporal Patterns in Ultra-Slow Domain Wall Creep Dynamics, Phys. Rev. Lett. **118**, 147208 (2017).
[46] P. Chauve, T. Giamarchi, and P. Le Doussal, Europhys. Lett. **44**, 110 (1998); Phys. Rev. B **62**, 6241 (2000).
[47] C. Le Priol, J. Chopin, P. Le Doussal, L. Ponson, and A. Rosso, Universal Scaling of the Velocity Field in Crack Front Propagation, Phys. Rev. Lett. **124**, 065501 (2020).