

Adaptive voter model on simplicial complexesLeonhard Horstmeyer^{1,2} and Christian Kuehn^{3,1}¹*Complexity Science Hub Vienna, Josefstädterstrasse 39, A-1090 Vienna, Austria*²*Basic Research Community for Physics, Mariannenstrasse 89, 04315 Leipzig, Germany*³*Faculty of Mathematics, Technical University of Munich, Boltzmannstrasse 3, 85748 Garching München, Germany*

(Received 14 September 2019; revised manuscript received 30 November 2019; accepted 7 January 2020; published 12 February 2020)

Collective decision making processes lie at the heart of many social, political, and economic challenges. The classical voter model is a well-established conceptual model to study such processes. In this work, we define a form of adaptive (or coevolutionary) voter model posed on a simplicial complex, i.e., on a certain class of hypernetworks or hypergraphs. We use the persuasion rule along edges of the classical voter model and the recently studied rewiring rule of edges towards like-minded nodes, and introduce a peer-pressure rule applied to three nodes connected via a 2-simplex. This simplicial adaptive voter model is studied via numerical simulation. We show that adding the effect of peer pressure to an adaptive voter model leaves its fragmentation transition, i.e., the transition upon varying the rewiring rate from a single majority state into a fragmented state of two different opinion subgraphs, intact. Yet, above and below the fragmentation transition, we observe that the peer pressure has substantial quantitative effects. It accelerates the transition to a single-opinion state below the transition and also speeds up the system dynamics towards fragmentation above the transition. Furthermore, we quantify that there is a multiscale hierarchy in the model leading to the depletion of 2-simplices, before the depletion of active edges. This leads to the conjecture that many other dynamic network models on simplicial complexes may show a similar behavior with respect to the sequential evolution of simplices of different dimensions.

DOI: [10.1103/PhysRevE.101.022305](https://doi.org/10.1103/PhysRevE.101.022305)**I. INTRODUCTION**

Contact and voter processes are a key theme of many disciplines including economics, epidemiology, mathematics, physics, and social science [1]. One natural setting for these processes is an underlying (complex) network, or graph, on which a population of individuals interacts [2]. For example, in the context of opinion formation, which we focus upon here, two individuals, nodes, or vertices may hold different opinions. For simplicity, let us assume that there are only two opinions possible; e.g., we can only vote for two possible parties. The classical voter model [3–5] describes the evolution of opinion dynamics on a fixed network by a Markov chain, either in discrete or continuous time; here we use the discrete-time variant. At each time step, an edge is selected at random. If both vertices hold the same opinion, nothing happens. If they hold opposite opinions, then one adapts the opinion of the other with equal probability. Studying the long-time behavior of such a dynamical system is already highly nontrivial as it does depend crucially on the network structure (see, e.g., [5–11]).

However, an important element of realism is missing in the classical voter model. Social interactions in large populations almost never take place on a fixed network. In fact, with whom we are in contact may also depend upon the difference or similarity of opinions [12]. This viewpoint has led to the development of adaptive, or coevolutionary, network models, in which there is interacting dynamics on and off the network (see, e.g., [13–19]). More precisely, the most common version of the adaptive voter model allows in addition to

persuasion events also for rewiring events, where edges between opposite-minded vertices are rewired to edges between like-minded vertices. This makes the process much more realistic as it allows one to study via a relatively simple model complex self-adapting network structures. There is already a substantial literature on adaptive voter models [20–27].

Yet, changes in communication and social network formation are clear indications that not only binary interactions matter in opinion formation [28,29]. These higher-order interactions have recently started to appear as a new focus in the analysis of complex network data sets [30]. Yet, there are currently no available standard adaptive or coevolutionary dynamics models taking into account higher-order interactions, i.e., allowing for full adaptivity of higher-order structures (but see [31] for an adaptive swarm dynamics model with fixed quadratic interactions on graphs, see [32,33] for work on the majority voter model on fixed graphs, and see [34] for a network generation model creating triangles via triadic closure).

In this work, we propose and study a minimalistic extension of the adaptive voter model to include higher-order interaction between individuals. This model takes into account the well-known effect of peer pressure, which has been studied widely in many scientific fields [35–37]. For example, if three individuals are connected in a friendship, and there is a disagreement in opinions, it is very likely that the majority opinion within the group of three prevails; i.e., a peer-pressure effect has occurred. To model whether a fully connected subgraph of three vertices is in a close enough friendship or not, we need an additional structure beyond vertices and edges.

A very general underlying structure would be to consider a hypergraph [38] instead of a usual graph or network. Although this is possible, we are looking to develop a minimal and mathematically elegant formulation to capture the essential effects of peer-pressure opinion dynamics. One natural choice in this context is to restrict ourselves to simplicial complexes [39], where a triangle of connected nodes is in close friendship interaction if there is a 2-simplex between them in addition to the 1-simplices (the edges) connecting them. Of course, the model we develop here could be generalized very naturally beyond 2-simplices but we postpone this more involved generalization to future work.

In this work, we define the simplicial adaptive voter model and study its dynamics numerically by direct simulation. On the one hand, our model turns out to preserve some key features of the standard adaptive voter model regarding metastability and diffusive absorption [40] into a single opinion for low rewiring as well as a fragmentation transition for high rewiring. The quantitative structure of these transitions turns out to be significantly affected by peer pressure. In particular, absorption occurs faster, and the fragmentation transition occurs earlier with respect to the rewiring frequency. From the viewpoint of opinion dynamics, this can be interpreted in the sense that societies are driven faster into mono-opinion or polarized or fragmented opinion states if peer-pressure effects occur. These effects could evidently be induced from social network interactions; i.e., there is a potential danger that we are going to observe fragmented or polarized societies much earlier and faster than classical models would anticipate. Furthermore, we also find a highly interesting mathematical effect in the simplicial adaptive voter model. It does happen frequently that due to rewiring of edges, even with replacement of lost simplices, there are eventually no active simplices with different opinions left. This effect tends to occur before the final asymptotic dynamics of the voter process has been reached, i.e., when there are no active edges left with different opinions. This leads to the conjecture that dynamical models on simplicial complexes can display a multiscale [41] hierarchy, where higher-order simplices equilibrate before lower-order simplices do [40,42].

II. THE ADAPTIVE SIMPLEX VOTER MODEL

In this section, we introduce the variants of the voter models in more detail. We are going to provide some basic background and references and then define the simplicial adaptive voter model.

Consider a simplicial 2-complex \mathcal{S} , which consists of zero-dimensional 0-simplices (or vertices) \mathcal{V} , one-dimensional 1-simplices (or edges) \mathcal{E} , and two-dimensional 2-simplices \mathcal{T} . Recall that for a simplicial complex one requires that each face of a simplex is again in the simplicial complex, and that the nonempty intersection of two simplices is a face for each of the two simplices. In our modeling context this makes particular sense since a triadic friendship does generally also contain friendships between the respective three individuals. These friendships are represented by the faces of the 2-simplex, which are the edges. We want to define a minimal adaptive voter model on the space of simplicial complexes with vertices labeled by two possible opinions. For notational

simplicity and convenience we allow for two states -1 and 1 and use the labels

$$\circ = \text{vertex of state } -1, \quad \text{and} \quad \bullet = \text{vertex of state } 1.$$

The possible edges



are either state homogeneous (*inactive*) or state inhomogeneous (*active*). The 2-simplices can occur in any of the following four configurations:



where the first and the last one are state and edge homogeneous, while the second and third are state and edge heterogeneous. Note that we use double edges to indicate a 2-simplex in comparison to a triangle, i.e., a full subgraph on three nodes, which is not part of a 2-simplex. The interior of a 2-simplex is color coded with current majority opinion within the triangle.

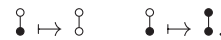
Let us recall the classical adaptive voter model on a graph $(\mathcal{V}, \mathcal{E})$. At each time step an edge $e \in \mathcal{E}$ is chosen at random and one of the following possibilities can then occur:

(C1a) *Social avoidance*: If e is active, then with probability $p \in [0, 1]$ one of the vertices (chosen with equal probability) rewires the edge to a vertex with its own opinion, which is chosen at random from the remaining vertices. We can represent this rule graphically by



Note that the probability p is a very crucial parameter in the adaptive voter model.

(C1b) *Personal discussion*: If e is active, then with probability $1 - p$ one of the two vertices of the edge e is chosen at random with probability $1/2$ and it adapts the opinion of the other vertex:



(C2) *Inert situation*: If e is inactive, then nothing happens.

Of course, the rules imply a conservation law of the number of edges as well as the number of vertices, which tends to be helpful to reduce dimensionality, to simplify the mathematical analysis, and to benchmark computations by checking whether the conservation laws hold. Next, we describe a minimal model extension to include the role of 2-simplices, which is also meaningful for applications. Consider the simplicial complex $(\mathcal{V}, \mathcal{E}, \mathcal{T})$ and another probability $q \in [0, 1]$, that encodes the peer pressure. Again we select an edge e at random at each event time step. Then we use the following rules:

(R0) If e is not part of a 2-simplex, then the classical rules apply.

(R1a) If e is part of at least one 2-simplex and of type $\circ\bullet$, then with probability p one of its vertices (chosen with equal probability) rewires. This is like the classical rule (C1a), except that all simplices attached to that edge break up, e.g.,



and triangles are chosen randomly for conversion into their respective 2-simplices,



to preserve the total number of simplices.

(R1b) If e is part of at least one 2-simplex and of type \triangle , then with probability $1 - p$ persuasion is chosen. With probability q a majority rule is implemented and with probability $1 - q$ the classical rule (C1b) is implemented. When the majority rule applies, one of the simplices attached to e is chosen and the majority persuades the minority with certainty



When the classical rule is chosen one of the two vertices of e is chosen with equal probability to persuade the respective other.

(R2) If e is part of at least one 2-simplex and not of type \triangle it stays inert.

Rules (R0)–(R2) are a very natural extension of the classical adaptive voter model rules (C1) and (C2). Again, conservation of vertices and edges is guaranteed. The new rule (R1) tries to conserve simplices as long as possible. However, due to rewiring, one may eventually not have any triangles left that can be converted into 2-simplices. A subsequent rewiring event within a 2-simplex would therefore reduce the overall content of 2-simplices and thus violate the conservation of simplices. This leaves two natural options:

(1) We stop the simulation precisely at the first time when the triangles have been depleted.

(2) We continue with the simulation despite the violation of the simplex conservation until an absorbing state of the Markov chain is reached.

We always indicate in our simulations which option we have chosen. Our main new results are just focusing on using the first option as it provides us with the regime where the simplex density s is fixed and thus acts as a parameter of the model.

III. RESULTS

Before describing our simulation results for (R0)–(R2), we briefly recall the well-known results for the classical adaptive voter model (C1)–(C2) on a graph.

A. The classical coevolving voter model

The classical coevolving voter model [27,40] corresponds to $q = 0$. There one observes two different phases—the active and the frozen phase—along the parameter p , separated by a fragmentation transition at p_c . In the active phase ($p < p_c$) the dynamics evolves towards a slow manifold of strictly positive active edge densities and then follows a random walk along this manifold towards a state with a giant component, all of whose members are of the same state and all of whose active edges have consequently vanished (cf. Fig. 1). In the frozen phase ($p > p_c$) the dynamics evolves towards a fragmented state in which two disconnected and internally state-uniform components exist. These two phases still exist in the simplicial coevolving voter model. This shows that our model is really a minimal extension as several main effects are preserved. We

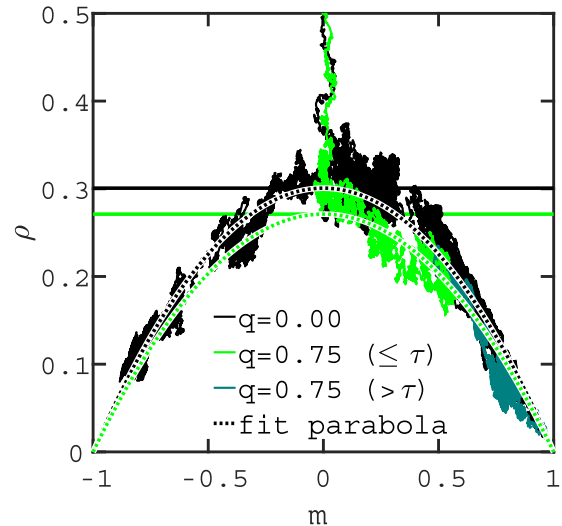


FIG. 1. We show two sample paths of the dynamics in the (m, ρ) space at a rewiring probability of $p = 0.55$: The black path shows the scenario where peer pressure is absent, i.e., $q = 0$. The green and cyan paths show the scenario where the peer pressure is at $q = 3/4$, respectively before the depletion of triangles at time τ and thereafter. The dotted grey lines are best fits of the paths to the parametrized parabola $\rho = \xi_p(1 - m^2)$, respectively for $q = 0$ and $q = 0.75$ before the depletion of triangles. The horizontal lines indicate the respective values of their apexes ξ_p . The size of the network is $N = 500$, its mean degree is $\mu = 8$, and the simplex-per-edge degree is $s = 0.2$. There is an initial population of triangles, such that the triangles-per-edge degree is $t = 0.8$.

are interested in the behavior of the transient and limiting behavior as q is deformed away from zero. When $q > 0$ the simplices start to have an effect, because the majority rule inside a simplex is governed to some extent (i.e., via $q > 0$) by the peer pressure.

It was observed in the classical case that it is helpful to view the dynamics within a compact region under a suitable projection [27]. To this end let σ_+ and σ_- denote the relative densities of the two opinions ± 1 across all nodes. Note that $0 \leq \sigma_{\pm} \leq 1$ and $\sigma_+ + \sigma_- = 1$. Then we denote the difference in opinion, i.e., the majority disparity, by $m = \sigma_+ - \sigma_-$. In a statistical physics context we can also draw the analogy of m to the magnetization, e.g., when thinking of the classical Ising model. Furthermore, we denote the active link density by ρ . It is very helpful to use the coarse-grained (m, ρ) coordinates to understand the dynamics. The network initially loses active links. Either all active links are depleted directly without any of the opinions becoming dominant in the process or the active links reach a quasistationary density at a positive value. There it enters into a random walk on a neighborhood of a parabolic-shaped region defined via the relation $\rho = \xi_p(1 - m^2)$, for some constant ξ_p indexed by p . In that case one opinion may gain the majority, as m deviates from zero along that region. Which of these two scenarios happens depends on whether $p < p_c$ or $p > p_c$ (see Fig. 1). On the parabola the active edge density evolves much slower than initially, which is why one may refer to this region as the slow or inertial manifold. Eventually the random walk hits either of the end points $(m, \rho) = (\pm 1, 0)$, corresponding

to a giant component of a single opinion. The value ξ_p is a characteristic p -dependent value of the slow manifold and corresponds to the quasistationary density of active links at vanishing majority disparity. In [27] they consider the time average of all the quasistationary densities of surviving runs ρ_{surv} , which is then taken as the order parameter of the model; however, ξ_p is another legitimate choice.

In this work, we perform a numerical study of the q -deformed simplicial coevolving voter model. In the following we describe the implementation and the results.

B. Initialization

First we initialize a random simplicial complex $(\mathcal{V}, \mathcal{E}, \mathcal{S})$ and assign an equal amount of $+1$ and -1 states to the vertices at random. We want to construct this complex from the data consisting of the number of vertices N , of edges E , and of 2-simplices S , or alternatively the mean degree $\mu = 2E/N$ and the 2-simplices-per-edge degree $s = 3S/E$. An important aspect of the dynamics of (R0)–(R2) is the simplex-preserving transformation of a triangle into a 2-simplex whenever a 2-simplex is destroyed by a rewiring action. Due to these transformations one also requires an initial population of triangles in the network, an amount T with triangles-per-edge degree $t = 3T/E$, so that a significant fraction of edges should be part of triangles. There are at least two ways to create a random simplicial complex from the data (N, E, S, T) . One option is to pick E edges uniformly at random from the list of $\binom{N}{2}$ possible vertex pairs. However, there is a chance that not enough triangles are created to declare S of them as 2-simplices and T as triangles. Another method is to pick $S + T$ triangles from the $\binom{N}{3}$ combinations of unordered nonrepeating three-tuples. An amount S are declared 2-simplices and the rest, i.e., T , remain triangles. All edges that were thus created and are part of triangles or 2-simplices become part of the edge set \mathcal{E} . If there are not yet E edges, ones picks uniformly at random the remaining edges from vertex pairs that are not yet part of the edge set and that will not produce further triangles. If more than E edges were introduced already by forming triangles and simplices, the method has failed. The first method tries to reduce degree correlations and works well for large μ and very low s . The second method aims to reduce correlations between the number of simplices per edge at the cost of degree correlations and is guaranteed to work when $3(S + T) \leq E$. We choose the second method to allow for larger values of s . In Appendix A we show the details of this method.

C. Fragmentation transition

For $q > 0$ the network dynamics also falls onto a parabola-shaped region in (m, ρ) space, on which the active edges evolve much slower. This is exemplified in Fig. 1, where we compare two sample paths with peer pressures $q = 0$ and $q = 0.75$ at a rewiring rate $p = 0.55$. However, the presence of the peer pressure lowers the quasistationary densities. This effect can be explained by the enhanced force to eradicate active edges: Consider a heterogeneous 2-simplex in which one node is of opinion $+1$ and two are of opinion -1 . It has two active edges and we suppose one is chosen for an update. If the node with the minority opinion convinces the

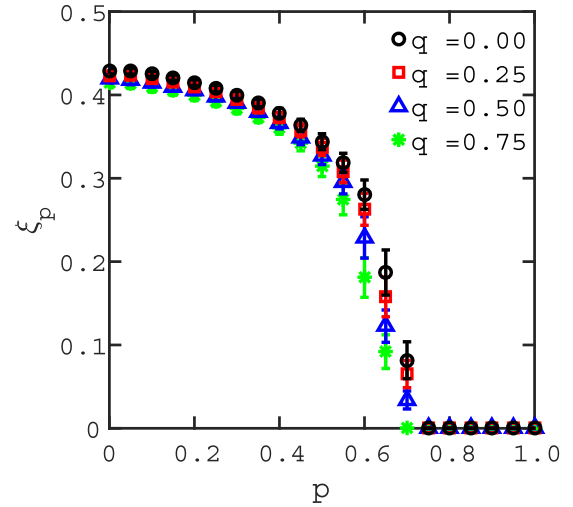


FIG. 2. We compare the curves of the order parameter for various peer pressures. The estimates of the order parameters ξ_p , i.e., the apexes of the parabolic regions (cf. Fig. 1), are plotted for a range of rewiring probabilities between zero and 1 and peer pressures $q \in \{0, 0.25, 0.5, 0.75\}$, which are respectively color coded with black, red, blue, and green. Each data point is generated from 200 runs and shows the mean and the standard deviations of the fitted values for ξ_p . Here, the size of the network is $N = 500$, its mean degree is $\mu = 8$, and the simplices-per-edge degree is $s = 0.2$. We also have an initial population of triangles such that the triangles-per-edge degree is $t = 0.8$. However, the parameter t does not directly affect the curves for ξ_p , but only the triangle depletion time.

neighbor with the majority opinion, then there are still two active edges in the simplex. If, on the other hand, the majority convinces the minority node, there are none. Thus, the higher the probability of a majority rule, the higher the tendency to reduce active edges. The same argument holds of course for a 2-simplex with the opposite majority. This effect happens irrespective of system size, edge, or 2-simplex densities, as long as they are positive.

In Fig. 2 we show the order parameter curves for various values of q with $N = 500$ at an edge-per-vertex degree $\mu = 8$ and a simplices-per-edge degree $s = 0.2$. We observe that the peer pressure shifts the critical threshold to lower rewiring probabilities. This also follows from the previous observation. If the active edge densities are reduced by the majority rule, then the active edges will vanish at lower rewiring probabilities. This means that the slow manifold is never reached and consequently a fragmentation takes place rather than a random walk towards any of the single-opinioned final states. The variance of the order parameter increases towards the fragmentation threshold, as is expected. We also note that a higher initial simplices-per-edge degree s makes the effect more pronounced.

Once the quasistationary states are reached, they can be sampled and the quasistationary density ξ_p can be inferred. This quantity does not depend on the initial triangles-per-edge degree t directly. It may, however, happen that the quasistationary parabola is not reached before all triangles are depleted (in Fig. 1 it depletes after reaching the parabola). In that case simplices also deplete and the simplex density

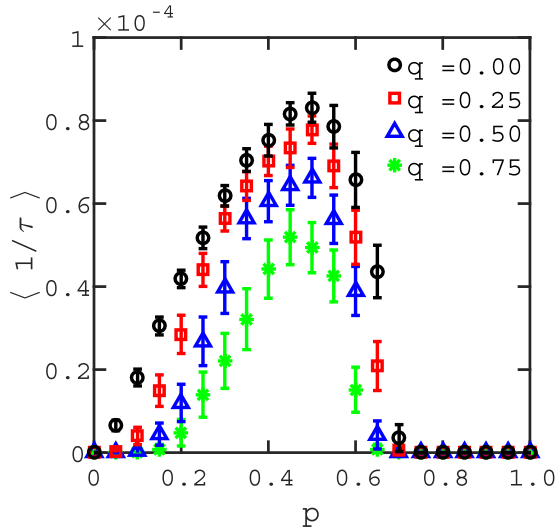


FIG. 3. We show the average inverse depletion time of triangles for rewiring probabilities in the entire range between zero and 1 and for peer pressures $q \in \{0, 0.25, 0.5, 0.75\}$. The initial triangles-per-edge degree t is 0.8 and all other parameters N , μ , and s are as above.

s cannot act as a constant of the motion. The depletion of triangles prior to reaching the parabola happens when there are not enough triangles initially, i.e., t is very low, or when the system becomes very large. The latter reason is due to a classical result about Erdős-Rényi graphs [43] with a given mean degree μ , namely, that the number of triangles is independent of the size N . Thus the triangles per edge decay as N^{-1} , rewirings will not create new triangles, and they deplete potentially before reaching the parabola. In the next section we therefore cover the triangle depletion in more detail.

D. Depletion of triangles

In the classical coevolving voter model there is a depletion of active edges. One crucial feature of the coevolving voter model on a simplicial complex is the depletion of the higher-order structures, which in our case are the 2-simplices and the triangles. Heterogeneous simplices are either homogenized by the peer pressure or destroyed via rewiring. As simplices are destroyed new ones are created by converting a randomly chosen triangle into a 2-simplex. In some parameter regimes this process depletes triangles at a higher rate than their production via rewiring. Thus, in these regimes there is a finite first triangle-depletion time τ , at which no triangles are left for conversion into 2-simplices.

In Fig. 3 we show the average inverse triangle-depletion time $\langle \tau^{-1} \rangle$ for a range of rewiring probabilities and peer pressures at an initial triangles-per-edge degree of $t = 0.8$. A value of zero implies an infinite depletion time, or one that is as long as the duration of a simulation. Of course for $p = 0$, in the absence of rewiring events, the depletion time is infinite. As the rewiring probability is increased, we observe a rise of $\langle \tau^{-1} \rangle$ up to a point where a maximum is reached, succeeded by a drop back to zero around the fragmentation threshold. This qualitative behavior can be observed for all values of q , with the additional effect that $\langle \tau^{-1} \rangle$ is lowered as q becomes larger. As the initial triangles-per-edge degree t is increased,

the depletion times naturally become longer. In the following we explain the behavior of the $\langle \tau^{-1} \rangle$ curves.

Our argument is based on the net decay of triangles, i.e., the expected difference between their production and their destruction. The only way that a destruction or production of triangles can occur is by way of rewiring events. We also remark that the chance for a rewiring to turn a triplet of vertices with two edges into a triangle with three edges rises as the mean degree of the network increases. If at the same time the rewiring link, prior to its rewiring, has a lower probability of being part of a triangle, then there is a net production of triangles.

With this in mind we can explain both the rise and the fall of the curves as p is increased. First the decline: When p is getting close to the fragmentation transition, the density of active edges ρ decreases. Consequently, the density of $++$ links ($--$ links), say, ρ_+ (ρ_-), is higher than in uncorrelated networks and so is the mean degree $\mu_{\pm} = 2\rho_{\pm}E/N_{\pm}$ of the subgraphs with $+1$ states (-1 states), where N_{\pm} are their respective sizes. An approximation of the respective mean degrees μ_{\pm} via the simplifying assumption $\rho_+/\rho_- \approx N_+^2/N_-^2$, corresponding to an uncorrelated scattering of edges on the two components, and the identity $Nm = N_+ - N_-$ yields $\mu_{\pm} \approx \mu(1 - \rho)(1 \mp m)$ to first order in m . See Appendix B for a derivation. This relation supports the intuition that the minority component is becoming more densely connected as active edges are depleting and $|m|$ is increasing. Therefore, as the active link density decreases, a rewiring event has an increasing probability to produce a triangle rather than to destroy one, given the sparseness of active edges. This effect is enhanced as p is approaching the fragmentation transition with low quasistationary values of ρ . There, the production may compete with the loss leading to a positive stationary abundance of triangles and a diverging triangle depletion time τ . This effect is of course more pronounced for higher peer pressures as they tend to reduce ρ even further.

We now argue for the initial rise of $\langle \tau^{-1} \rangle$. As long as triangles are not produced to a level that compensates their loss, as it happens close to the transition, there is an exponential depletion of them after a finite time. The rate of this depletion depends on the rewiring probability. The higher p , the faster they deplete. This effect is opposing the one mentioned before. One may think of it as follows: The values of ρ and m determine how many triangles plus simplex-declared triangles, i.e., $T + S$, the network can maintain on average. The arrival at that quasistationary state happens at an exponential rate that is of course proportional to p . If, however, that state can only support less than an amount S of triangles and 2-simplices, then the triangles will die very likely and will do so much quicker when p is higher. The effect of peer pressure is that ρ is decreased and the majority disparity in the network is enhanced. Both of these imply an increased production of triangles and thus a longer triangle depletion time.

Finally we also note that beyond the fragmentation threshold the dynamics hits the absorbing fragmented $\rho = 0$ state directly without entering the parabolic quasistationary region. Therefore, it hits an absorbing state ever quicker as the rewiring probability p takes ever larger values. In that regime one cannot make very meaningful statements about the

triangle depletion time τ , since it is bounded by the extinction time of the active edges.

We now discuss some direct consequences of the triangle depletion. The conservation of simplices is lost once there are no convertible triangles left. This implies also that the simplices-per-edge degree s decreases and in turn results in a weakening net effect of the peer pressure up to a point, where possibly all 2-simplices are gone. Thus, s ceases to be a fixed parameter of the model. In order to examine the pure effect of the peer pressure q at a given parameter set (N, μ, s) we therefore measure quantities only as long as the number of simplices is conserved, i.e., until the triangle depletion time τ .

Consider a rewiring rate below the fragmentation transition, i.e., $p < p_c$. If τ is less than the time at which the slow manifold is reached, then one cannot take unbiased data from the slow manifold, as 2-simplices have already diminished. When triangles are depleted after reaching the slow manifold, but before reaching an absorbing state, then one may take unbiased data from the slow manifold for times less than τ . These data are suitable for computing the apex ξ_p of the parabola that is fitted through the sample path in the (m, ρ) plane (cf. Fig. 1). It is not suitable for extracting the average quasistationary active link density of surviving runs, ρ_{surv} , because systematic biases towards higher values would be introduced when data are only taken until some time τ . Consequently we plot ξ_p rather than ρ_{surv} .

E. Depletion of active edges

There is a fast evolution before the dynamics reaches either the slow parabolic manifold or a fragmented state. During this fast evolution active edges are converted into inactive edges by persuasion until the quasistationary density is reached. It is therefore expected that a higher peer pressure leads to a faster initial depletion of active edges. The expected initial depletion rate of active edges is given by

$$\left\langle \frac{d}{dt} \Big|_{\rho_t} \rho_t \right\rangle.$$

In our simulations we estimate the derivative $d\rho/dt|_0$ discretely via $(\rho_s - \rho_0)/s$ for low s . For a given initialization of the simplicial complex we perform 250 runs and compute the average slope at zero for that run. This is repeated for various initializations with the same parameter settings and then averaged to estimate the expected value.

In Fig. 4 we plot the average initial depletion rate of active edges for various values of p and q . Higher peer pressures enhance the depletion rate, as expected. This effect is diminished for higher rewiring rates up to the point $p = 1$ where persuasion does not exist anymore and the peer pressure has no effect.

F. Diffusion and drift velocity

We have seen that the peer pressure enhances the depletion rate of active edges during the fast dynamics either towards fragmentation, in the regime $p > p_c$, or before reaching the slow manifold, for $p < p_c$. Here, we study the transient behavior towards the absorbing states. As can be seen in Fig. 1 the parabolic region is not as well explored for $q = 0.75$ as

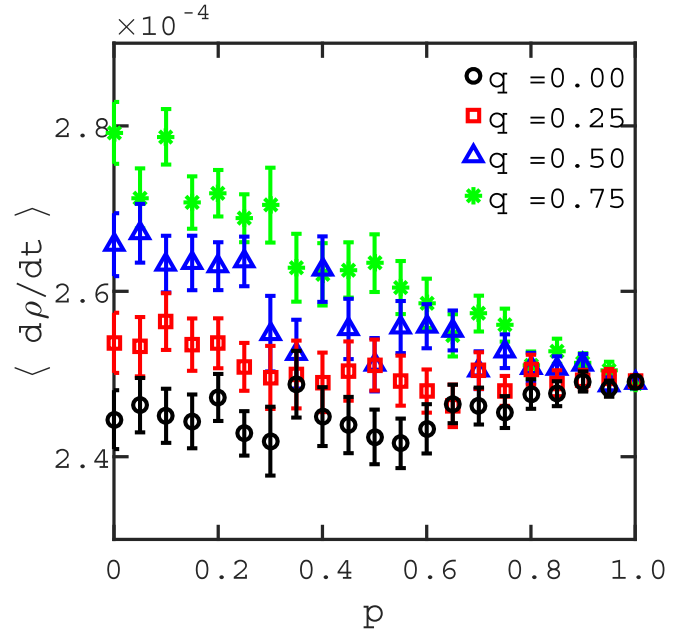


FIG. 4. We show the average initial depletion rate of active edges for rewiring probabilities in the entire range between zero and 1 and for peer pressures $q \in \{0, 0.25, 0.5, 0.75\}$. All other parameters, i.e., N, μ , and s , are as above.

it is for $q = 0$. For $q = 0.75$ it can be seen often, as in the figure, that only one side of the parabola is visited, indicating that a majority grows once a bias towards one opinion exists. We investigate this behavior by looking at the average rate of change of the majority disparity and its squared value, which are proxies for drift and diffusion respectively. A pure Brownian motion has a vanishing drift. Therefore, we are interested in the extent to which the process is not like a pure Brownian motion. A process with a higher radial drift is expected to hit the absorbing states at $m = \pm 1$ earlier. We do not measure the hitting times directly, however, because they may be larger than τ cf. Sec. III D).

Consider a sample path (m_t, ρ_t) [44]. The mean rate of change of the squared distance from $m = 0$ at time s is given by

$$\left\langle \frac{d}{dt} \Big|_s m_t^2 \right\rangle. \quad (1)$$

We are interested in this quantity not at any arbitrary point, but precisely as the dynamics is on the slow manifold and has vanishing m . Hence, we consider this average, conditional on the event that $m_s = 0$ for some time s at which the slow manifold has been reached. For a one-dimensional Brownian motion along the m direction this quantity would be twice the diffusion constant D . In our simulations we measure it as follows. First we evolve the dynamics until the quasistationary parabolic region is reached. After that, whenever the dynamics passes through the $m = 0$ line we approximate the average rate of change (1) via finite differences at that time instance. This is then repeated for many runs and initializations, keeping the parameters fixed.

In Fig. 5 we show how the diffusion (1) changes as the rewiring and peer pressure are varied. The first observation

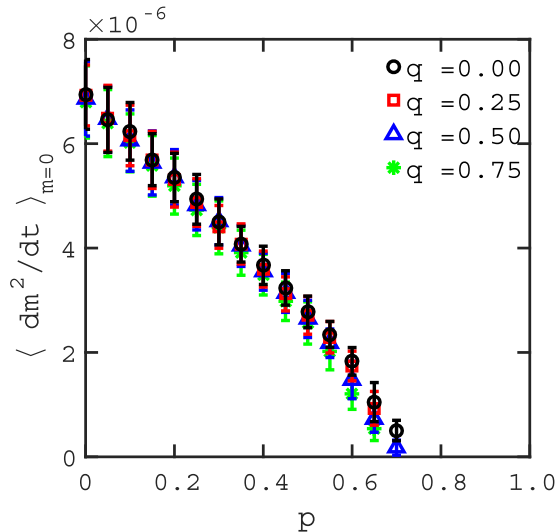


FIG. 5. This plot shows the diffusivity of the dynamics along the direction of m at $m = 0$. It is measured in terms of the mean rate of change of m^2 for rewiring probabilities in the range between zero and 1 and for peer pressures $q \in \{0, 0.25, 0.5, 0.75\}$. All other parameters, i.e., N , μ , s , and t , are as above.

is that the peer pressure does not influence the diffusion very much. There is even a slight tendency towards lower diffusions for higher peer pressures. As the rewiring probability increases, the diffusion decreases linearly for all peer pressures. There are two reasons that account for this effect, both of which are based on the fact that only persuasion can change m . First of all, given a certain amount of active edges one expects fewer changes in m for lower persuasion probabilities, i.e., higher rewiring probabilities. Second, higher rewiring probabilities decrease the quasistationary level of active edges, so that persuasions cannot yield as much change in majority disparity.

We also look at the average drift velocity of the majority disparity. Since the definition of the dynamics and hence its probability laws are invariant with respect to the discrete symmetry of interchanging the opinions, we expect on average no drift velocity at $m = 0$. Therefore, we study the average drift velocity at some nonzero majority disparity $m^+ \neq 0$:

$$\left\langle \frac{d}{dt} \Big|_{m_t} \right\rangle_s,$$

where s is such that $m_s = m^+$. We expect that the drift velocities at $-m^+$ should be minus the drift at m^+ , in the sense of their probability laws. This is due to the discrete reflection symmetry. Hence the quantity of interested is the radial drift velocity, pointing away from the origin.

In Fig. 6 we plot the average radial drift velocity at $m^+ = 0.2$. We find that there is no drift velocity at $q = 0$. The majority disparity in the classical coevolving voter model is therefore more like a pure diffusion process. When $q > 0$ we do, however, observe a strong deviation. The higher the peer pressure q the stronger is the average drift velocity towards the extreme points $m = \pm 1$. This effect can be explained by the ratio of heterogeneous 2-simplices. When $m_t = m^+$ there are more \circ -simplices than \bullet -simplices. Thus the majority rule

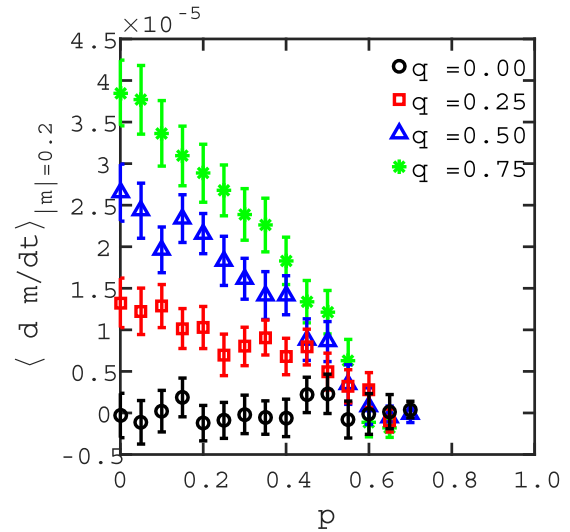


FIG. 6. We show the average radial drift velocity of m at $m = \pm 0.2$ for rewiring probabilities in the range between zero and 1 and for peer pressures $q \in \{0, 0.25, 0.5, 0.75\}$. There are no data points for rewiring rates above the fragmentation transition because the dynamics does not enter the slow manifold on which it can explore the regions of higher majority disparity m . All other parameters, i.e., N , μ , s , and t , are as above.

tends to further increase the amount of \circ nodes. The argument holds *mutatis mutandis* for $m_t = -m^+$. These drifts also explain why it becomes more unlikely to change majorities once there is a bias.

In summary, we also see in the slow regime on the parabolic-shaped manifold a similar effect as in the fast regime. The peer pressure enhances the drift towards a single majority opinion; i.e., polarization is enhanced.

IV. CONCLUSIONS AND OUTLOOK

We have shown how to naturally (from the viewpoint of applications) and minimally (from the mathematical perspective) extend the adaptive voter model to a model on simplicial complexes. It seems now plausible as further steps to also extend other adaptive contact processes to simplicial complexes, e.g., epidemic spreading models. Then we demonstrated that the main structural features of the adaptive voter model remain in the simplicial version, which still yields a fragmentation transition upon varying the rewiring rate. Yet, the quantitative properties are changed and we observe faster transitions to a single-opinion absorbing state or towards a fragmented two-opinion state. This is in line with heuristic arguments that peer-pressure effects may lead to polarization; in fact, our model seems to be one of the simplest mathematical manifestations of this effect. We expect that similar effects do occur in far more complicated models involving very large-scale higher-order structures such as interactions on social networks. Furthermore, we found that the simplicial adaptive voter model often displays multiple time scales, where the higher-order 2-simplices die out before the active edges decay to zero. This multiscale effect leads one to the conjecture that simplicial dynamics models could be analyzed order-by-order with respect to the dimension of the simplices.

ACKNOWLEDGMENTS

L.H. thanks the Austrian Science (FWF) for support via the grant for Project No. P29252. C.K. thanks the Volkswagen-Stiftung for partial support via a Lichtenberg Professorship and the EU for partial support within the TiPES project funded the European Union's Horizon 2020 research and innovation programme under Grant Agreement No. 820970.

APPENDIX A: ALGORITHM FOR THE INITIALIZATION OF A RANDOM SIMPLICIAL COMPLEX

Let there be N vertices. We would like to draw uniformly at random T distinct elements from the set of all unordered K -tuples without repetition. For the purpose of this paper we are randomly sampling triangles from the vertex list, so that $K = 3$, but we expose the method more generally. This set has cardinality $\binom{N}{K}$. Storing such a list is prohibitively large when N is large and K larger than 2 and smaller than $N - 1$. Thus it is advantageous to find an algorithm that needs less storage complexity. The idea of the algorithm is to take a natural number n and find the corresponding K -tuple in *colex*, which is one of the two canonical orderings defined on unordered tuples without repetitions. We recall that $u = (u_1, u_2, \dots, u_K)$ is less than $v = (v_1, v_2, \dots, v_K)$ in *colex* [45] if and only if $u_k < v_k$ for the last k where u_k and v_k are different. This mapping from $n \mapsto (u_1, u_2, \dots, u_K)$ is done by iteratively determining the entries, starting from the last one. Suppose the $(k+1)$ th entry u_{k+1} was found to be $i+1$, where $0 \leq i \leq (N-1)$. Then we want to determine the k th entry. To this end we define $F_{i,k}$ as the number of tuples whose k th entry, u_k , equals i , with all entries above the k th kept fixed, since they are already determined. It is given by $F_{i,k} = \binom{i-1}{k-1}$, because we have to fill $k-1$ slots from a set of $j-1$ numbers. The total number of K -tuples whose $(k+1)$ th entry is known to be $u_{k+1} = i+1$ is denoted by $G_{i,k}$. It is simply given by $G_{i,k} = \sum_{\ell=k}^i F_{\ell,k}$, which sums up all the tuples whose k th entry is less than or equal to i , again keeping higher entries fixed. In other words, $G_{i,k} = F_{k,k} + F_{k+1,k} + \dots + F_{i,k}$ and clearly there cannot be any summands where $\ell < k$ in *colex*. One may find F and G iteratively,

$$F_{i+1,k} = F_{i,k} \frac{i}{i-k+1},$$

$$G_{i+1,k} = G_{i,k} + F_{i,k},$$

where we use the hockey-stick identity for binomial coefficients.

The last entry, u_K , is easy to determine as follows: We note that the first $G_{K,K}$ tuples end with K . There is literally only one, namely, $(1, 2, 3, \dots, K)$. The first $G_{K+1,K}$ tuples end with K or $K+1$ and there are $1+K$ of them. The first $G_{j,K}$ tuples end with any number smaller than or equal to j . Thus we find u_K by searching for j such that $G_{j,K} \geq n > G_{j-1,K}$, or equivalently $\min\{j : G_{j,K} \geq n\}$. Now that we know u_K , our problem reduces to finding the right K -tuple in the range between $G_{u_K-1,K}$ and $G_{u_K,K}$. This is equivalent to finding the $(K-1)$ -tuple at position $n' = n - G_{u_K-1,K}$, with the slight modification that these tuples must be taken from the smaller set $\{1, \dots, u_K - 1\}$. So now there are $G_{j,K-1}$ such tuples whose $(K-1)$ th entry is j or less. We determine the $(K-1)$ th entry again by finding the minimal j such that $G_{j,K-1} \geq n'$. We repeat this procedure until all entries have been determined. So in general we have the following iterative algorithm that uses an auxiliary set of variables $\mathbf{n} = (n_1, n_2, \dots, n_K)$, initialized as $n_K = n$:

$$u_k = \min\{j : n_k \leq G_{j,k}\},$$

$$n_{k-1} = n_k - G_{u_k-1,k}.$$

We also define $G_{k-1,k} := 0$ to resolve the problem that arises when $u_k = k$.

APPENDIX B: APPROXIMATION OF THE SUBGRAPH-DEGREES μ_{\pm} IN THE (m, ρ) SPACE

Let \mathcal{G} be a graph with N vertices, E edges, and mean degree $\mu = 2E/N$. Its nodes are either in the $+1$ or -1 state. Let $\mathcal{G}_{\pm} \subseteq \mathcal{G}$ be the subgraph consisting of all the nodes in the ± 1 states and their links. We denote the mean degrees of \mathcal{G}_{\pm} by μ_{\pm} . We also define three types of links $(++)$, $(--)$, and $(+-)$ and their respective densities ρ_+ , ρ_- , and ρ as a fraction of E . Thus we have

$$\rho_+ + \rho_- + \rho = 1. \quad (\text{B1})$$

The numbers of nodes in state $+1$ or -1 are N_+ or N_- and their respective fractions of the entire set of vertices is denoted by $\sigma_+ = N_+/N$ or $\sigma_- = N_-/N$. Their difference is denoted by $m = \sigma_+ - \sigma_-$. Therefore, $N_{\pm} = \frac{N}{2}(1 \pm m)$, which can be used to obtain the following expression for the mean degree in the subgraphs \mathcal{G}_{\pm} with $E_{\pm} = \rho_{\pm}E$ edges:

$$\mu_{\pm} := 2\rho_{\pm}E/N_{\pm} = \frac{4E}{N(1 \pm m)}\rho_{\pm} = \frac{2\mu}{1 \pm m}\rho_{\pm}. \quad (\text{B2})$$

So far no assumptions were made about the graph. We would like to approximate the link densities in terms of the effective coordinates m and ρ . When $m = 0$ a reasonable assumption is $\rho_+ = \rho_-$ due to symmetry, which implies by (B1) that $\rho_{\pm} = \frac{1}{2}(1 - \rho)$ and consequently that

$$\mu_{\pm}|_{m=0} = \mu(1 - \rho) \quad (\text{B3})$$

at $m = 0$. One may obtain an approximation for the case when m deviates slightly from zero by requiring that it satisfies (B3) for $m \rightarrow 0$. A first approximation is obtained by making a decorrelation assumption. Suppose we throw L links onto the vertex clouds of \mathcal{G}_{\pm} , whose total amount of vertex pairs are $P_{\pm} = \frac{1}{2}N_{\pm}(N_{\pm} - 1)$, respectively. Thus, on average there will be a total of $L\rho_{\pm} = LP_{\pm}/(P_- + P_+)$ links in the respective subgraphs and their expected ratio becomes

$$\frac{\rho_+}{\rho_-} = \frac{N_+(N_+ - 1)}{N_-(N_- - 1)} \approx \frac{N_+^2}{N_-^2}. \quad (\text{B4})$$

Consequently we can plug this approximation into (B1) and then (B2) and expand the resulting expression to first order in m ,

$$\mu_{\pm} \approx \frac{2\mu}{1 \pm m} \frac{1 - \rho}{1 + \frac{(1 \mp m)^2}{(1 \pm m)^2}} = \mu(1 - \rho)(1 \mp m) + O(m^2),$$

which also satisfies $\lim_{m \rightarrow 0} \mu_{\pm} = \mu(1 - \rho)$ and resembles (B3). Thus for small $m \ll 1$ we can say that the minority component is getting more densely connected, the higher m

deviates from zero and the closer ρ gets to zero. When one of the few active edges rewires in this regime, it will be rewired to one of the two subgraphs \mathcal{G}_{\pm} whose higher link densities

enhance the chance for triangle production and reduce the chance of triangle destruction due to the sparseness of active links.

-
- [1] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* (Springer, Berlin, 2013).
- [2] A. Barrat, M. Barthélemy, and A. Vespignani, *Dynamical Processes on Complex Networks* (Cambridge University Press, Cambridge, U.K., 2008).
- [3] I. Dornic, H. Chaté, J. Chave, and H. Hinrichsen, Critical Coarsening without Surface Tension: The Universality Class of the Voter Model, *Phys. Rev. Lett.* **87**, 045701 (2001).
- [4] R. A. Holley and T. M. Liggett, Ergodic theorems for weakly interacting infinite systems and the voter model, *Ann. Probab.* **3**, 643 (1975).
- [5] V. Sood and S. Redner, Voter Model on Heterogeneous Graphs, *Phys. Rev. Lett.* **94**, 178701 (2005).
- [6] C. Castellano, D. Vilone, and A. Vespignani, Incomplete ordering of the voter model on small-world networks, *Europhys. Lett.* **61**, 153 (2003).
- [7] J. P. Gleeson, Binary-State Dynamics on Complex Networks: Pair Approximation and Beyond, *Phys. Rev. X* **3**, 021004 (2013).
- [8] C. Nardini, B. Kozma, and A. Barrat, Who's Talking First? Consensus or Lack Thereof in Coevolving Opinion Formation Models, *Phys. Rev. Lett.* **100**, 158701 (2008).
- [9] V. Sood, T. Antal, and S. Redner, Voter models on heterogeneous networks, *Phys. Rev. E* **77**, 041121 (2008).
- [10] K. Suchecki, V. M. Eguíluz, and M. San Miguel, Voter model dynamics in complex networks: Role of dimensionality, disorder, and degree distribution, *Phys. Rev. E* **72**, 036132 (2005).
- [11] F. Vazquez and V. M. Eguíluz, Analytical solution of the voter model on uncorrelated networks, *New J. Phys.* **10**, 063011 (2008).
- [12] M. McPherson, L. Smith-Lovin, and J. M. Cook, Birds of a feather: Homophily in social networks, *Ann. Rev. Sociol.* **27**, 415 (2001).
- [13] S. Bornholdt and T. Rohlf, Topological Evolution of Dynamical Networks: Global Criticality from Local Dynamics, *Phys. Rev. Lett.* **84**, 6114 (2000).
- [14] T. Gross, C. J. Dommar D'Lima, and B. Blasius, Epidemic Dynamics on an Adaptive Network, *Phys. Rev. Lett.* **96**, 208701 (2006).
- [15] *Adaptive Networks: Theory, Models and Applications*, edited by T. Gross and H. Sayama (Springer, Berlin, 2009).
- [16] L. Horstmeyer, C. Kuehn, and S. Thurner, Network topology near criticality in adaptive epidemics, *Phys. Rev. E* **98**, 042313 (2018).
- [17] S. Jain and S. Krishna, Autocatalytic Sets and the Growth of Complexity in an Evolutionary Model, *Phys. Rev. Lett.* **81**, 5684 (1998).
- [18] C. Kuehn, G. Zschaler, and T. Gross, Early warning signs for saddle-escape transitions in complex networks, *Sci. Rep.* **5**, 13190 (2015).
- [19] J. M. Pacheco, A. Traulsen, and M. A. Nowak, Coevolution of Strategy and Structure in Complex Networks with Dynamical Linking, *Phys. Rev. Lett.* **97**, 258103 (2006).
- [20] I. J. Benczik, S. Z. Benczik, B. Schmittmann, and R. K. P. Zia, Opinion dynamics on an adaptive random network, *Phys. Rev. E* **79**, 046104 (2009).
- [21] G. A. Böhme and T. Gross, Analytical calculation of fragmentation transitions in adaptive networks, *Phys. Rev. E* **83**, 035101(R) (2011).
- [22] R. Durrett, J. P. Gleeson, A. L. Lloyd, P. J. Mucha, F. Shi, D. Sivakoff, J. E. S. Socolar, and C. Varghese, Graph fission in an evolving voter model, *Proc. Natl. Acad. Sci. USA* **109**, 3682 (2012).
- [23] P. P. Klamsner, M. Wiedermann, J. F. Donges, and R. V. Donner, Zealotry effects on opinion dynamics in the adaptive voter model, *Phys. Rev. E* **96**, 052315 (2017).
- [24] B. Kozma and A. Barrat, Consensus formation on adaptive networks, *Phys. Rev. E* **77**, 016102 (2008).
- [25] T. Rogers and T. Gross, Consensus time and conformity in the adaptive voter model, *Phys. Rev. E* **88**, 030102(R) (2013).
- [26] G. Zschaler, G. Böhme, M. Seissinger, C. Huepe, and T. Gross, Early fragmentation in the adaptive voter model on directed networks, *Phys. Rev. E* **85**, 046107 (2012).
- [27] F. Vazquez, V. M. Eguíluz, and M. San Miguel, Generic Absorbing Transition in Coevolution Dynamics, *Phys. Rev. Lett.* **100**, 108702 (2008).
- [28] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo, Opinion dynamics and the evolution of social power in influence networks, *SIAM Rev.* **57**, 367 (2015).
- [29] D. J. Watts and P. S. Dodds, Influentials, networks, and public opinion formation, *J. Consum. Res.* **34**, 441 (2007).
- [30] A. R. Benson, R. Abebe, M. T. Schaub, A. Jadbabaie, and J. Kleinberg, Simplicial closure and higher-order link prediction, *Proc. Natl. Acad. Sci. USA* **115**, E11221 (2018).
- [31] C. Huepe, G. Zschaler, A.-L. Do, and T. Gross, Adaptive-network models of swarm dynamics, *New J. Phys.* **13**, 073022 (2011).
- [32] S. Galam, Minority opinion spreading in random geometry, *Eur. Phys. J. B* **25**, 403 (2002).
- [33] P. L. Krapivsky and S. Redner, Dynamics of Majority Rule in Two-State Interacting Spin Systems, *Phys. Rev. Lett.* **90**, 238701 (2003).
- [34] P. Klimek and S. Thurner, Triadic closure dynamics drives scaling laws in social multiplex networks, *New J. Phys.* **15**, 063008 (2013).
- [35] B. B. Brown, D. R. Clasen, and S. A. Eicher, Perceptions of peer pressure, peer conformity dispositions, and self-reported behavior among adolescents, *Dev. Psychol.* **22**, 521 (1986).
- [36] W. B. Hansen and J. W. Graham, Preventing alcohol, marijuana, and cigarette use among adolescents: Peer pressure resistance training versus establishing conservative norms, *Prev. Med.* **20**, 414 (1991).
- [37] E. Kandel and E. P. Lazear, Peer pressure and partnerships, *J. Political Econ.* **100**, 801 (1992).

- [38] A. Bretto, *Hypergraph Theory: An Introduction* (Springer, Berlin, 2013).
- [39] A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, U.K., 2002).
- [40] G. Demirel, F. Vazquez, G. A. Böhme, and T. Gross, Moment-closure approximations for discrete adaptive networks, *Physica D* **267**, 68 (2014).
- [41] C. Kuehn, Multiscale dynamics of an adaptive catalytic network model, *Math. Model. Nat. Phenom.* **14**, 402 (2019).
- [42] C. Kuehn, Moment closure—a brief review, in *Control of Self-Organizing Nonlinear Systems*, edited by E. Schöll, S. Klapp, and P. Hövel (Springer, Berlin, 2016), pp. 253–271.
- [43] P. Erdős and A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci* **5**, 17 (1960).
- [44] Formally a sample path ω_t is a discrete time-indexed path in the state space Ω , which in this case is the space of $\{-1, 1\}$ -labelled simplicial complexes $(\mathcal{V}, \mathcal{E}, \mathcal{S})$ and m and ρ are random variables on Ω .
- [45] Unordered K -tuples without repetitions are simply sets of the form $u = \{u_1, u_2, \dots, u_K\}$ of cardinality K , for which the order of their elements doesn't matter. When $u_i \in \mathbb{N}$, one may however associate to it uniquely a particular choice of an ordered K -tuple $\hat{u} = (u_1, u_2, \dots, u_K)$ where $u_k < u_\ell$ if and only if $k < \ell$. This is then a unique representation of the unordered tuple u in the realm of ordered tuples, which is of course a much larger space. In abuse of notation we denote the representation \hat{u} of u in the set of ordered tuples again by u . The *colex* order is strictly speaking defined via the representation \hat{u} , rather than u .