

Effective ergodicity breaking phase transition in a driven-dissipative system

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We show that the Olami-Feder-Christensen model exhibits an effective ergodicity breaking transition as the noise is varied. Above the critical noise, the system is effectively ergodic because the time-averaged stress on each site converges to the global spatial average. In contrast, below the critical noise, the stress on individual sites becomes trapped in different limit cycles, and the system is not ergodic. To characterize this transition, we use ideas from the study of dynamical systems and compute recurrence plots and the recurrence rate. The order parameter is identified as the recurrence rate averaged over all sites and exhibits a jump at the critical noise. We also use ideas from percolation theory and analyze the clusters of failed sites to find numerical evidence that the transition, when approached from above, can be characterized by exponents that are consistent with hyperscaling.

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I. INTRODUCTION

The Fermi-Pasta-Ulam-Tsingou model [1] is a well-known example of a system that exhibits broken ergodicity. The model exhibits quasiregular dynamics below the critical energy threshold and reaches equipartition above the threshold. Understanding the nature of this type of transition may help extend the tools of equilibrium statistical mechanics to driven-dissipative systems [2], active matter [3], and other nonequilibrium phenomena.

In this paper we consider the nearest-neighbor Olami-Feder-Christensen (OFC) model [2]. This model is a driven-dissipative system that has been of particular interest in the context of the study of earthquakes. We simulate the OFC model on a square lattice of length L with $N = L^2$ sites and periodic boundary conditions. Each site i is initially assigned a stress $\sigma_i = \sigma_R \pm 0.25r$, where r is a uniform random number between ± 1 . At each time step or plate update is found the initiating site, which is the site with the maximum stress. The stress on all sites is increased by the same amount such that the stress on the initiating site equals σ_F . This procedure corresponds to the zero-velocity limit of the loading plate in the Rundle-Jackson model [4]. When site i fails, its stress is reset to the residual stress $\sigma_{R,i} = \sigma_R + r\eta$ and the stress $(1 - \lambda)[(\sigma_i - \sigma_{R,i})/4]$ is transferred to its four nearest neighbors. The magnitude of the noise is η . The value of the dissipation parameter λ is $0 \leq \lambda < 1$. Sites fail when the stress is greater than or equal to σ_F . The failure of a site may cause neighboring sites to fail, triggering an avalanche. The stress is redistributed until the stress on all sites is less than σ_F . This process concludes one plate update.

Grassberger [5] showed that the OFC model with periodic boundary conditions is deterministic for zero noise and that the dynamics appears to be stochastic for sufficiently high values of the noise. However, the transition between the two types of behavior was not explored.

In this paper, we show that there is a effective ergodicity breaking transition in the nearest-neighbor OFC model as the noise is varied. In Sec. II we show that the OFC model is effectively ergodic [6–8] for $\eta > \eta_c$, but is not effectively ergodic which implies not ergodic for $\eta < \eta_c$, where η_c is the critical noise. In Sec. III, we characterize the high and low noise phases using recurrence plots, which are commonly used in nonlinear dynamics [9]. From the recurrence plots of the stress on given sites, we calculate the recurrence rate [10] and define the recurrence fraction f_R as the recurrence rate averaged over all sites. The recurrence fraction acts as a order parameter and differentiates the high- and low-noise phases.

We also use ideas from percolation theory to examine the critical behavior as the critical noise is approached from above. In Sec. IV we treat all sites that fail in a plate update as part of the same cluster and determine the mean cluster size χ and the mean radius of gyration, R_G [11]. We determine the exponents γ and ν associated with the divergence of χ and R_G , respectively, as $\eta \rightarrow \eta_c$ [11]. We also measure the Fisher exponents τ and σ for the cluster distribution near the critical noise [11,12]. Our measured exponents are consistent with hyperscaling.

The numerical results reported in the following are for $\sigma_F = 2.0$ and $\sigma_R = 1.0$, $\lambda = 0.01$, and $L = 500$. Our results for $\lambda \in [0.005, 0.1]$ are qualitatively similar. For all our runs, we discarded at least the first 5×10^6 plate updates before recording data for 3×10^6 plate updates.

II. BREAKDOWN OF EFFECTIVE ERGODICITY

For systems with many degrees of freedom, it is difficult to verify if a system is ergodic and ergodicity can be checked rigorously for only a few simple systems [13]. Instead, we use the stress fluctuation metric [6–8] to determine if the system is effectively ergodic and to study the transition between phases that are not necessarily in equilibrium. The stress fluctuation

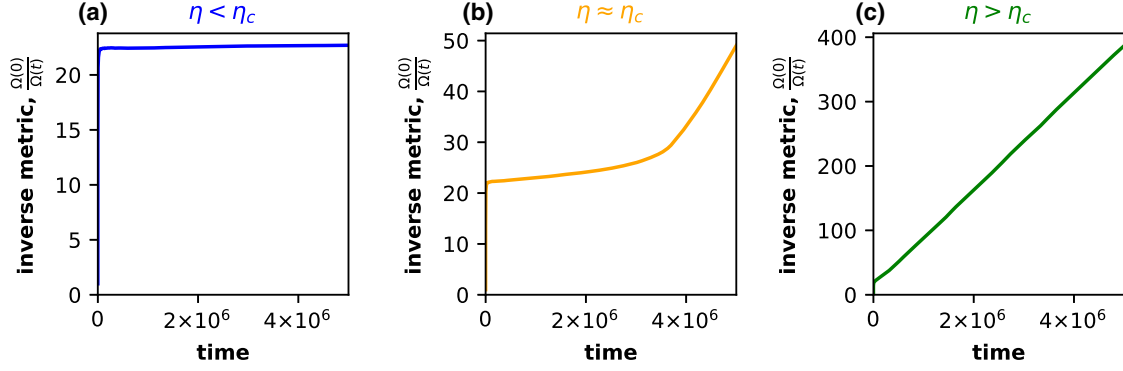


FIG. 1. The normalized inverse stress fluctuation metric as a function of time shows the breakdown of effective ergodicity at the critical noise. (a) For $\eta = 0.06 < \eta_c$ the inverse metric is flat, indicating that the system is not ergodic. (b) For $\eta = \eta_c \approx 0.071$, $\Omega(0)/\Omega(t)$ is initially flat but becomes approximately linear during the observation time. (c) For $\eta = 0.08 > \eta_c$, $\Omega(0)/\Omega(t)$ is linear, and the system is effectively ergodic.

metric describes how the spatial variance of the time-averaged stress on each site behaves for very long times. A spatially homogeneous system is effectively ergodic if the time average of the stress on each site approaches the same value. An analysis of the temporal properties of the metric can be found in Ref. [7].

We define the time average of the stress at site i , up to time t , as

$$\bar{\sigma}_i(t) = \frac{1}{t} \sum_{t'=1}^t \sigma_i(t'), \quad (1)$$

where t represents the number of plate updates. The spatial average of the time-averaged stress on all sites is

$$\langle \sigma(t) \rangle = \frac{1}{N} \sum_{i=1}^N \bar{\sigma}_i(t). \quad (2)$$

The stress fluctuation metric is defined as

$$\Omega(t) = \frac{1}{N} \sum_{i=1}^N [\bar{\sigma}_i(t) - \langle \sigma(t) \rangle]^2. \quad (3)$$

If the system is effectively ergodic, $\Omega(t)$ approaches zero as $1/t$ [6–8]. The system is not effectively ergodic during the observation time if the metric reaches a finite value, or does not increase linearly. The fluctuation metric provides a necessary but not sufficient condition for ergodicity.

As shown in Fig. 1, the inverse metric increases linearly and the system is effectively ergodic for $\eta > \eta_c \approx 0.071$. For $\eta \approx \eta_c$, the inverse metric is initially flat for some time before showing a slow linear increase. For $\eta < \eta_c$, the inverse metric reaches a plateau, implying that the system is no longer effectively ergodic during our observation time. As the noise is increased past the critical noise, the system transitions from a phase that is not effectively ergodic during the observation time to one that is effectively ergodic.

We can define the mixing time τ_M from the linear behavior of the inverse of the metric for $\eta > \eta_c$ as $\Omega(0)/\Omega(t) = t/\tau_M$. The mixing time is a measure of the how quickly the differences between the time-averaged stress on each site vanishes. Systems for which the differences never approach zero are not ergodic. The mixing time as a function of the noise η is shown

in Fig. 2. We see that there is an apparent divergence in τ_M as η approaches η_c from above, which implies the existence of critical slowing down.

III. RECURRENCE PLOTS

We can analyze the dynamical transition by using recurrence plots [9,10] to explore the short-term dynamics of the system. The recurrence plots are generated as follows: If the stress on a given site at times t and $t' > t$ differs by less than ϵ , then the corresponding point on the two-dimensional recurrence plot assumes the value one. Otherwise, the value is zero. From the time series of the stress on a given site, the recurrence matrix, $R(t, t')$, is defined as

$$R_{t,t'}(\epsilon) = \begin{cases} 1 & \text{for } |\sigma(t') - \sigma(t)| < \epsilon \\ 0 & \text{for } |\sigma(t') - \sigma(t)| \geq \epsilon. \end{cases} \quad (4)$$

A common choice for the threshold ϵ is 10% of the range of values that the states can take [10], which in our case is one, leading to the choice $\epsilon = 0.1$. We will show results only for $\epsilon = 0.1$, but our results for other values of ϵ are consistent.

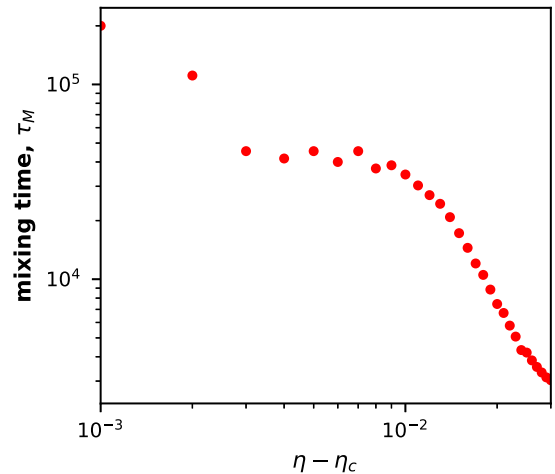


FIG. 2. As the noise η approaches the critical noise η_c there is a rapid increase in the mixing time τ_M .

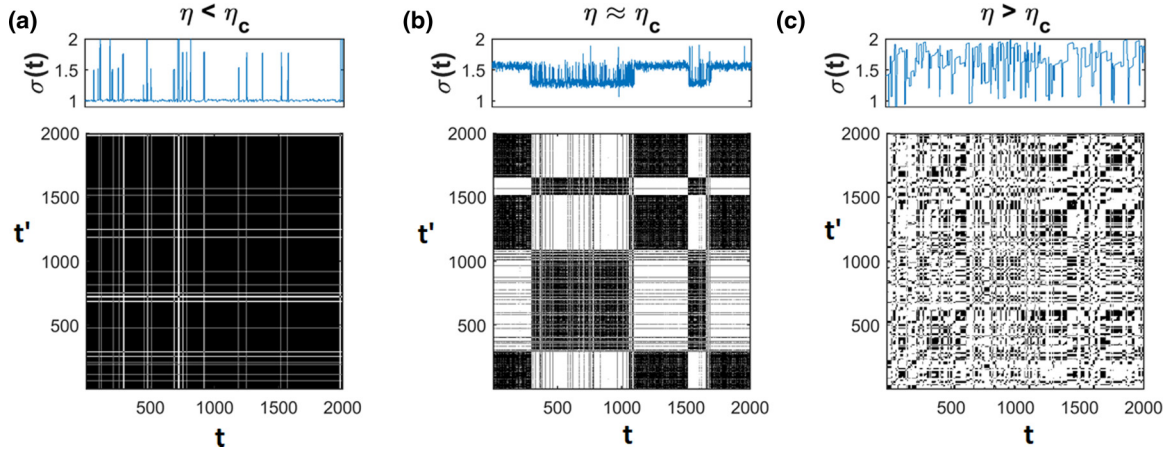


FIG. 3. The dynamics of the stress on a site transitions from recurrent to stochastic as the noise η is increased past the critical noise η_c . The time series of the stress $\sigma(t)$ on a given site and the corresponding recurrence map is shown in the top and bottom rows, respectively. The dark points in the recurrence map corresponds to the recurrences. (a) For $\eta < \eta_c$ the dynamics is strongly recurrent, and the values of the stress on the site are confined to a small subset of the possible values. (b) For $\eta \approx \eta_c$, shown in the middle column, the dynamics is quasistationary. (c) If $\eta > \eta_c$, the dynamics is stochastic.

The top row in Fig. 3 shows the time series of the stress on a given site and the bottom row shows the corresponding recurrence plot. For $\eta < \eta_c$ in Fig. 3(a), the time series of the stress is confined to a narrow set of values. The recurrence plot confirms that the trajectory is strongly recurrent. For $\eta \approx \eta_c$ in Fig. 3(b), the time series is quasistationary, and the recurrence map shows alternating black and white bands. The dynamics is stochastic for $\eta > \eta_c$, as shown by the time series and the corresponding recurrence map in Fig. 3(c). We conclude from the recurrent plots that the dynamics of the stress on a given site undergoes a significant change as the noise is varied from $\eta < \eta_c$ to $\eta > \eta_c$.

We use the recurrence plots to introduce a scalar order parameter to differentiate between the phases above and below the critical noise. The recurrence rate RR [10] measures the fraction of times the stress on a site returns to the neighborhood of its original value, averaged over all initial conditions. The recurrence rate RR for trajectories of duration T is given by

$$\text{RR}^{(i)} = \frac{1}{T^2} \sum_{t,t'=1}^T R_{t,t'}^{(i)}(\epsilon). \quad (5)$$

We define the recurrence fraction f_R as the recurrence rate averaged over all sites. We find that f_R averaged over 50 sites, chosen at random, gives a good measure of the average recurrence rate. If $f_R \approx 1$, the system is in the ordered state for which the trajectories of the stress on individual sites appear regular and are restricted to a subset of the allowed values of stress as shown in Fig. 3(a). If $f_R \approx 2\epsilon$, the system is in the disordered phase, and the local trajectories fluctuate almost randomly with no well-defined order [see Fig. 3(c)]. In Fig. 3(b), we see quasiperiodic changes in the stress trajectory, and f_R for $\eta \approx \eta_c$ is larger than for $\eta > \eta_c$. Note that f_R in Fig. 4 appears to show a discontinuous jump at η_c . However, we are limited by the rapid increase of the mixing time near $\eta \approx \eta_c$ in determining if there is an actual jump in the order parameter. The rapid change in f_R as the noise is varied

implies a divergence in the fluctuations, which we will explore by using the cluster analysis of the failed sites.

IV. CLUSTER ANALYSIS

In equilibrium statistical mechanics, we can learn about the nature of a transition from the geometric properties of the fluctuations [14,15] for certain systems. We map the failed sites onto a percolation problem by assuming that an event of s sites corresponds to a percolation cluster [12,14,16,17]. When the noise is greater than η_c , the system is effectively ergodic and the distribution of the clusters can be fit to a power law with an exponential cutoff as

$$n_s \sim s^{-\tau} \exp[-(\eta - \eta_c)s^\sigma], \quad (6)$$

where n_s is the number of clusters with s failed sites and τ and σ are the Fisher exponents. From Fig. 5 these exponents are estimated to be $\tau \approx 1.04 \pm 0.14$, and $\sigma = 0.43 \pm 0.03$.

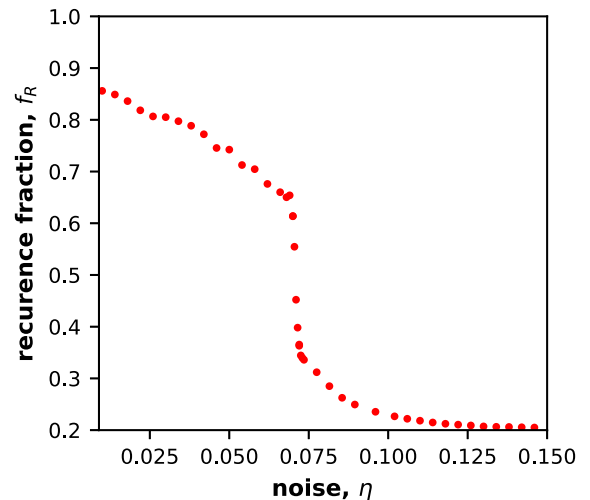


FIG. 4. The recurrence fraction f_R as a function of η shows a transition from recurrent to stochastic dynamics for η near η_c .

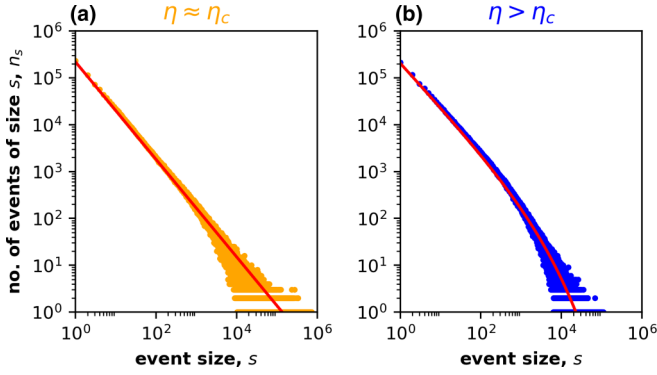


FIG. 5. The measured value of Fisher exponents τ and σ is consistent with the hyperscaling laws. The number n_s of clusters of size s can be fit to Eq. (6) to estimate the exponents τ and σ [18]. (a) We find $\tau \approx 1.04 \pm 0.14$ for $\eta = 0.071$. (b) For $\eta = 0.08$ we estimate $\sigma = 0.43 \pm 0.03$.

We measure the mean event size (analogous to the susceptibility in thermal systems) χ and the connectedness length (analogous to the correlation length) ξ , which are defined as [11]

$$\chi = \frac{\sum_s s^2 n_s}{\sum_s s n_s}, \quad (7)$$

$$\xi = \frac{\sum_s s R_G^2(s) n_s}{\sum_s s n_s}, \quad (8)$$

where $R_G(s)$ is the mean radius of gyration of clusters with s sites [11]. From Fig. 6 we see that χ diverges as $(\eta - \eta_c)^{-\gamma}$ with the exponent $\gamma \approx 2.01 \pm 0.14$. Similarly, from Fig. 7 we find that $\xi \sim (\eta - \eta_c)^{-\nu}$ with $\nu \approx 1.25 \pm 0.13$. The divergence of the mean cluster size is used to estimate that the critical noise is $\eta_c \approx 0.071$. This value of the critical noise is consistent the value of η at which the jump in the recurrence fraction occurs, as seen in Fig. 4. We are unable to determine the heat capacity and the exponent α directly because we have not identified an energy in the nearest-neighbor OFC

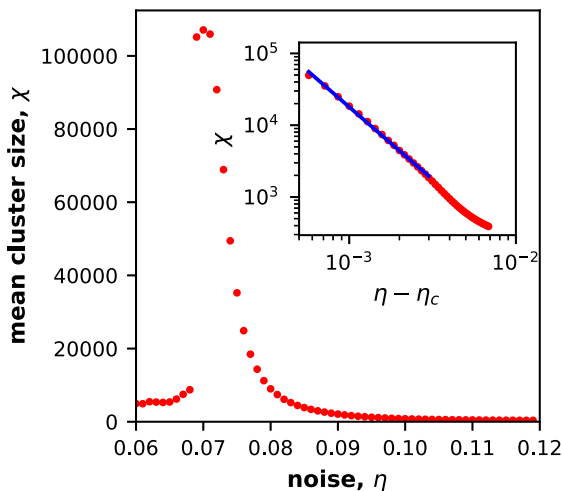


FIG. 6. The critical exponent γ determined from the power-law behavior of the mean cluster size χ is estimated to be $\approx 2.01 \pm 0.14$.

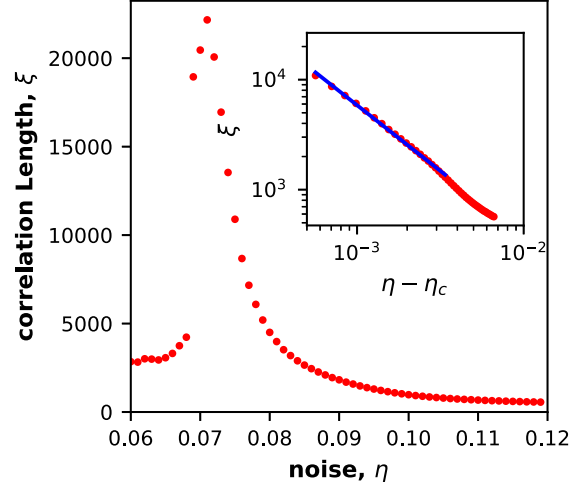


FIG. 7. The critical exponent ν characterizes the divergence of the connectedness length ξ and is estimated to be $\approx 1.20 \pm 0.13$.

model. We have shown that the nearest-neighbor OFC model is not ergodic for $\eta < \eta_c$. In the nonergodic regime, tools of statistical mechanics may not apply, and we find that we are unable to properly fit critical exponents for $\eta < \eta_c$.

Although the exponents γ , ν , τ , and σ were estimated independently, they are related by hyperscaling [19]. To test the consistency of the measured exponents with the scaling relations, we need to add one to the measured value of τ from our simulations, because the clusters in the OFC model are grown from a seed site [11,17]. We define $\tilde{\tau} = \tau + 1$ as the corrected exponent and the hyperscaling relations as $\gamma = (3 - \tilde{\tau})/\sigma$ and $\nu = (\tilde{\tau} - 1)/d\sigma$, where d is the spatial dimension. If we substitute $\gamma = 2$, $\nu = 5/4$, and $d = 2$, we find the Fisher exponents to be $\tau = 10/9$ and $\sigma = 4/9$. The predictions of hyperscaling are consistent with our numerical estimates of $\tau \approx 1.04 \pm 0.14$ and $\sigma = 0.43 \pm 0.03$. If we use the additional scaling relations, $\alpha = 2 - (\tilde{\tau} - 1)/\sigma$ and $\beta = (\tilde{\tau} - 2)/\sigma$, we find $\beta = 1/4$ and $\alpha = -1/2$, corresponding to the critical behavior of the order parameter and the heat capacity. We are unable to estimate β directly using f_R because β is close to zero and the range of η over which power-law behavior is observed is too small. As we have discussed, we have not been able to identify an energy in the OFC model, so we cannot measure the “specific heat” exponent α directly and can only determine its value indirectly using hyperscaling.

Another exponent that we can measure independently is the fractal dimension d_f of the clusters (see Fig. 8). By using the scaling relations we can express the fractal dimension in terms of the measured exponents and the spatial dimension: $d_f = d - \beta/\nu = d - (\tilde{\tau} - 2)/(\sigma\nu)$. Our measured value of $d_f = 1.79 \pm 0.03$ is consistent with the prediction from hyperscaling.

V. LONG-RANGE STRESS TRANSFER

Our results for the nearest-neighbor OFC model also apply to long-range stress transfer. In this case, a failing site distributes its stress equally to all sites within a circle of radius R , which is the stress transfer range. The nearest-neighbor case corresponds to $R = 1$. Figure 9 shows that the value

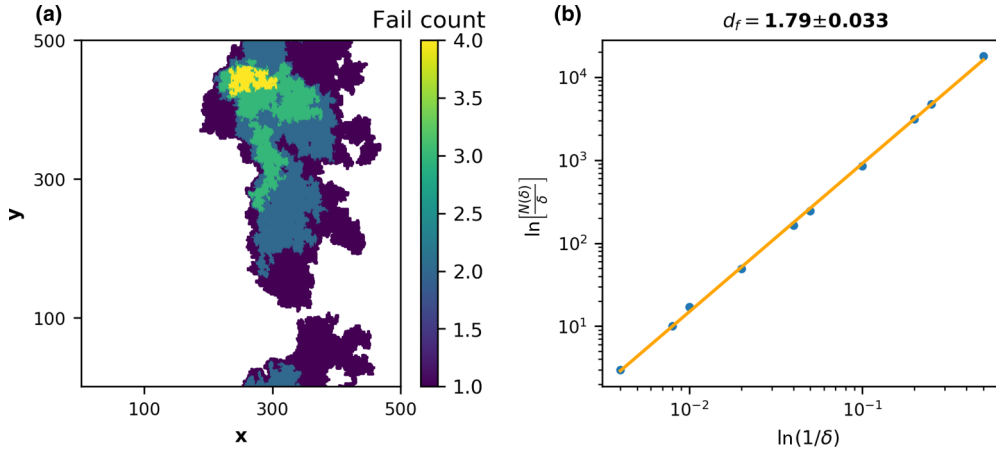


FIG. 8. The measured value of the fractal dimension d_f is consistent with predictions of hyperscaling. (a) The cluster of failed sites appears to be a fractal. (b) The fractal dimension was computed by using a box-counting method. The number of boxes needed to cover the cluster is $N(\delta)$, where δ is the box length. In the limit, that the box length $\delta \rightarrow 0$, and the slope of $\ln[N(\delta)/\delta]$ corresponds to the fractal dimension. The estimated fractal dimension is $d_f = 1.79 \pm 0.03$.

of the critical noise decreases as the range of stress transfer increases. For each value of the range R , the jump in the recurrence fraction f_R , the divergence in the mixing time τ_M and the mean cluster size χ occur at the same value of the noise.

VI. DISCUSSION

We have found a novel phase transition in the OFC model as the noise is varied for all stress transfer ranges studied. Below the critical noise η_c , the system is not ergodic and the stress on the sites appears to evolve via limit cycles. In contrast, for $\eta > \eta_c$ the system is effectively ergodic and the dynamics is stochastic. For $\eta < \eta_c$ recurrence plots show that individual sites are trapped in limit cycles with long lifetimes. As η approaches η_c^- , the limit cycles become unstable, and the trajectories of individual sites show deviations from limit cycle behavior. Larger values of η disrupt the limit cycles and, for $\eta > \eta_c$, the trajectories appear random. We find that

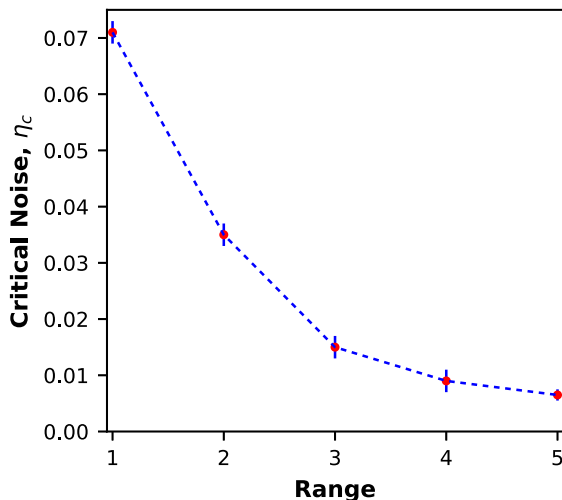


FIG. 9. The critical noise, η_c , in the OFC model decreases as the range of stress transfer is increased.

the recurrence fraction f_R is a convenient choice of order parameter and describes the transition from limit cycle to stochastic behavior. f_R appears to exhibit a discontinuous jump at $\eta = \eta_c$.

We also investigated the transition for $\eta \rightarrow \eta_c^+$ by using a cluster analysis to study properties such as the mean cluster size, the connectedness length, and the Fisher exponents. Our measured numerical values of the exponents are consistent with hyperscaling within statistical error. An unusual feature of this “percolation” description of the transition is that there is no “infinite cluster” or avalanche for η above or below η_c . Hence the percolation exponent β associated with how the probability that a site selected at random belongs to the infinite cluster goes to zero as the transition is approached cannot be measured directly. However, a measurement of the fractal dimension $d_f = d - \beta/\nu$ yields a value of β consistent with the relation $\gamma + 2\beta = d\nu$. Our measured critical exponents do not belong to the two-dimensional random percolation universality class. In the OFC model, the sites that fail are correlated. In contrast, the probability of a site being occupied in random site percolation is independent. The critical exponents of correlated models are often different from uncorrelated models such as random percolation [11].

Our results indicate that the effect of noise in nonequilibrium phase transitions is more subtle than was previously suspected and that noise may play a significant role in transitions in systems such as the Kuramoto model [20] and neural systems modeled by integrate-and-fire neurons [21]. Of particular interest is the role of noise in the behavior of earthquake faults. The OFC model (and the virtually identical Rundle-Jackson model) with long-range stress transfer has been used to study earthquake fault systems [16,17], where the stress transfer occurs over large distances due to elastic forces. Figure 9 shows that the critical noise η_c decreases as the stress transfer range is increased. The noise in real earthquake fault systems is believed to be small [22], so it is possible that earthquake faults operate near the critical noise. Therefore, small changes in the noise due to variations in the water content of rocks or changes in the density of

microcracks could drive the fault system into a different phase where the dynamics are considerably different. Our results suggest one possible mechanism by which a fault may change its behavior from quasiperiodic to a scale-free distribution of events consistent with Gutenberg-Richter scaling [23].

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