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Comment on "Osmotic pressure of compressed lattice knots"

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In a recent paper, E. J. Janse van Rensburg has presented computational data enumerating the conformations of closed circular self-avoiding lattice polymers with knots confined in a cubic box, and claimed to have observed a negative osmotic pressure in the system. The purpose of this comment is to state that osmotic pressure by a self-avoiding polymer, knotted or otherwise, is positive, which means a polymer pushes confining walls outwards, and the statement of the opposite is a mistake.

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Confinement of a polymer chain is a traditional subject in polymer physics textbooks [1–4], and it is of a considerable practical importance [5,6]. Also, a self-avoiding walk on a cubic lattice is a classical model of a polymer chain [7,8]. Combining the two, maximally compact lattice chains (of some L^3 monomers confined in a $L \times L \times L$ cube, exactly enumerated for up to $L = 4$ [9,10]) has a rich track record of providing useful insights into the protein folding problem [11–13]. Adding a new twist to this story, E. J. Janse van Rensburg in his paper [14] presented large-scale Monte Carlo data on confined closed circular polymers with knots. Specifically, he obtained accurate estimates of $p_{N,L}(K)$: the number of closed self-avoiding paths on a cubic lattice with $N \leqslant L^3$ monomers, fully confined within a cube of size L up to $L \leqslant 15$, and having the topology of the knot K (specifically, knots $K = 0_1, 3_1, 4_1$ in standard knot theory notations were examined). Interpreting $F_K(N, L) = -\ln [p_{N,L}(K)]$ as the free energy, the data should allow for complete thermodynamic characterization of the model. For instance, how do confined polymers press on the confining walls, and how does this pressure depend on the knot type?

Because of the beauty and importance of the question, I am writing this note to make a statement that, contrary to the conclusions of the work [14], polymer osmotic pressure in this system is always positive: a polymer exerts force in the outward direction on the confining walls. It is true for any knot K. The opposite claim of finding a negative polymer osmotic pressure is an error.

The argument rigorously proving my statement is actually very simple: obviously $p_{N,L+1}(K) \geqslant p_{N,L}(K)$, because every state of a polymer realizable in a smaller cube of size L is also realizable in the larger cube $L + 1$, meaning $F_K(N, L + 1) - F_K(N, L) \leqslant 0$. Since osmotic pressure is the partial derivative of free energy with respect to volume $V = L^3$ at constant N,

$$\Pi = -\left(\frac{\partial F}{\partial V}\right)_N , \tag{1}$$

and as long as we are willing to accept "differentiation" with respect to volume changing in discrete steps for a lattice model, we conclude that the osmotic pressure is positive. The argument above and the conclusion of $\Pi > 0$ are valid for any system size, finite or arbitrarily large.

This completes the main point of the present note: osmotic pressure exerted by the self-avoiding polymer chain of any size is positive, meaning that polymer pushes walls outwards, irrespective of the knot type.

In what follows, I will try to provide also a physical explanation of positivity of osmotic pressure, as well as track down the misconception that could lead to the erroneous conclusion of the negative osmotic pressure in the work [14] based on methods of the previous work [15].

First of all, identifying $F_K(N, L) = -\ln [p_{N,L}(K)]$ as the free energy requires that p includes all translated and rotated copies of any particular conformation that fit in the confining volume. Further, the fact that p is the number of (equally weighted) permissible states means that we are talking about athermal polymer model, in which monomer-monomer, monomer-solvent, and solvent-solvent contacts all have the same statistical weights (what is meant by "solvent" here, as usual, is monomer-free lattice sites). In other words, the only forces operating in the system (apart from monomer-monomer bonds of the chain) are the repulsion forces responsible for self-avoidance. The system with only repulsive forces of course produces positive outward pressure on the confining walls.

To compute osmotic pressure, the author rewrites its definition (1) according to

$$\Pi = -\left(\frac{\partial F}{\partial V}\right)_N = \phi^2 \left(\frac{\partial F_V/\phi}{\partial \phi}\right)_N , \tag{2}$$

with $F_V = F/V$ free energy per unit volume, and $\phi = N/V$. Unfortunately, the author neglected to indicate that the derivatives in this last formula must be taken at constant number of monomers. Indeed, this is because osmotic pressure of a polymer is related to the force exerted by the polymer on the walls, which are permeable for the solvent and not permeable for the polymer. This is important, for the following reason. The very common rewriting of the pressure definition (2) is harmless in the usual statistical physics applications because of the thermodynamic limit: for a very large system, free energy per unit volume, F_V, depends only on the intensive properties, such as $\phi = N/V$, but not on extensive variables, such as N and V separately. For a finite system, such as the ones considered here, this is not the case; accordingly, while the standard definition (1), or its rewriting (2) with fixed N, still produces correct result for the pressure, an attempt to perform a derivative while allowing N to change, for instance, keeping constant volume V, leads to erroneous results:

$$\Pi_{\text{wrong}} = \phi^2 \left(\frac{\partial F_V/\phi}{\partial \phi} \right)_V \neq \Pi. \tag{3}$$

Although the above statement is obvious mathematically, it might be useful to appreciate its physical meaning by a simple illustrative calculation (even if for a system that does not pretend for being literally an analog of a lattice knot). Suppose there are some N noninteracting particles (ideal gas) confined in a box of volume V, and in addition there is also one special particle that is attached by some spring to an immobile point in the middle of the box. Assuming the spring is tight, this special particle never touches container walls and thus does not contribute anything to the pressure. If the free energy of the special particle on the spring is g, then (Helmholtz) free energy of the whole system is $F = F_{\text{id}} + g$, where F_{id} is the ideal gas free energy. Clearly, then $F_V/\phi = k_BT \ln \phi + \frac{g}{V\phi}$. As expected, the first (ideal gas) term is a function of ϕ only, while the second term is not. Although this second term is irrelevant in thermodynamic limit, it might not be negligible for a finite system. Nevertheless, the correct formula (1) or (2) produces the right result for the pressure, $\Pi = k_BT\phi$, while differentiation with respect to ϕ at constant V, as in (3), gives $\Pi_{\text{wrong}} = k_BT\phi - g/V$, which may indeed be negative—but it is not a pressure.

Trying to explain his claim of negative polymer osmotic pressure, the author says, for instance [14]: "At negative osmotic pressure the lattice unknot will add length. Similarly, at positive osmotic pressure the lattice unknot will shed length and become smaller." Leaving for now aside the (very valid!) question of how to realize physically a polymer which adds and sheds monomers while maintaining its ring structure and even its topology, I note here that exchange of particles is controlled by chemical potential, not by osmotic pressure. To this end, since a lattice model is "incompressible," in the sense that $N + N_s = V$, where N_s is the number of "solvent molecules," or empty lattice sites, inside the confinement volume, polymer osmotic pressure (1) is equal to the chemical potential of a solvent (assuming that lattice volume per site is set to unity): $\Pi = -(\partial F/\partial V)_N = (\partial F/\partial N_s)_N = \mu_s$. However, if adding or losing monomers is an issue, then we should introduce the chemical potential of monomers. That can be defined in two ways, as $\mu = (\partial F/\partial N)_{N_s}$ or $\mu_{\text{ex}} = (\partial F/\partial N)_V$; the former corresponds to adding or removing monomers by simultaneously changing volume, while the latter describes exchanging one monomer for one solvent molecule (based on the fact that every monomer occupies just one lattice site; see Ref. [8] for a more general lattice model). Formally, one can say that similarly to $\Pi = \mu_s$, also $\mu = \Pi_s$: the chemical potential of monomers μ is equal to osmotic pressure of solvent. Is osmotic pressure of a solvent a meaningful quantity? Although mathematically possible to define, physically osmotic pressure of a solvent would require a membrane that is permeable for monomers and impermeable for a solvent—something quite difficult to imagine in the real physical world. And in any case none of the chemical potentials are equal to Π_{wrong} (3).

Thinking beyond the paper [14], it would be interesting to check the conjecture that in the thermodynamic limit osmotic pressure is insensitive to one single knot buried somewhere in the middle (somewhat like free energy F_V was getting insensitive to the added special particle in my illustrative example above). If this conjecture is correct, then the osmotic pressure dependence on knots is intrinsically a finite-size effect. At the same time, one could also consider a thermodynamic limit of a different kind, in which a composite knot of, say, M trefoils is taken, with $M \to \infty$ together with $N \to \infty$ and $V \to \infty$ while $N/V = \text{const}$ and $M/N = \text{const}$. This and many other open questions emphasize the interest of the problem.

To conclude, it would be very interesting to find out how the polymer osmotic pressure of compressed knots depends on the knot type. The lattice model is a sufficiently good tool for that purpose. However, we know in advance that this pressure is positive for any knot.

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