Study of the Hopf functional equation for turbulence: Duhamel principle and dynamical scaling

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We consider a formulation for the Hopf functional differential equation which governs statistical solutions of the Navier-Stokes equations. By introducing an exponential operator with a functional derivative, we recast the Hopf equation as an integro-differential functional equation by the Duhamel principle. On this basis we introduce a successive approximation to the Hopf equation. As an illustration we take the Burgers equation and carry out the approximations to the leading order. Scale invariance of the statistical Navier-Stokes equations in d dimensions is formulated and contrasted with that of the deterministic Navier-Stokes equations. For the statistical Navier-Stokes equations, critical scale invariance is achieved for the characteristic functional of the dth derivative of the vector potential in d dimensions. The deterministic equations corresponding to this choice of the dependent variable acquire the linear Fokker-Planck operator under dynamic scaling. In three dimensions it is the vorticity gradient that behaves like a fundamental solution (more precisely, source-type solution) of deterministic Navier-Stokes equations in the long-time limit. Physical applications of these ideas include study of a self-similar decaying profile of fluid flows. Moreover, we reveal typical physical properties in the late-stage evolution by combining statistical scale invariance and the source-type solution. This yields an asymptotic form of the Hopf functional in the long-time limit, improving the well-known Hopf-Titt solution. In particular, we present analyses for the Burgers equations to illustrate the main ideas and indicate a similar analysis for the Navier-Stokes equations.

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I. INTRODUCTION

The problem of Navier-Stokes turbulence remains a major challenge in theoretical physics and mathematics. In particular deriving the statistical properties of solutions to the Navier-Stokes equations (i.e., the governing equations) in a purely deductive manner has been regarded as a difficult task. On the other hand there are attempts to describe statistical properties of turbulence on the basis of approximations relying on physical ideas. In this paper we consider and revisit a formulation from first principles.

Roughly speaking, there are two different ways of writing down equations that govern statistical solutions of the Navier-Stokes equations. One method uses the characteristic functional of the velocity field as the basic variable, which is the Fourier transform of the probability measure of the velocity, and its governing equation is called the Hopf equation. The other method deals directly with the probability measure of the velocity field and the corresponding Liouville equation is called the Hopf-Foias equation (see, e.g., [1–9]). See also [10–20] for related works and [21–38] for mathematical works.

It is fair to say that methods of solving the Hopf equation are at the moment underdeveloped. There are at least two different approaches to determine the Hopf functional. One is to try solving the functional equation as is, taking realizability into account. This approach faces formidable difficulty. The other one starts from deterministic solutions of a nonlinear partial differential equation (PDE), and after taking the average we can try to determine a functional form of the Hopf functional at least asymptotically. While the latter approach is available only when the deterministic PDE is explicitly solvable, it does give a hint as to how the functional actually behaves. As our understanding of the Hopf equation is limited at the moment, it makes sense to combine both approaches.

In view of physical applications, we recall that self-similar solutions often reveal typical properties of nonlinear problems in fluid mechanics [39]. To exemplify our approach, here we first consider the Burgers equations and indicate extensions to handle the Navier-Stokes equations. The Burgers equations on their own footing appear as a physical model for compressible fluid motion (see, e.g., [40]). We show in particular how the source-type solution (to be defined below) determines the Hopf functional for the Burgers equations in its long-time limit through self-similarity, when we choose the dependent variable suitably. This generalizes the well-known expression of the Hopf-Titt solution of the final period decay, which totally neglects the nonlinear terms. It should be noted that an explicit form of the source-type solutions for the multidimensional Burgers equations is obtained as a by-product, which has not been reported before. We will clarify which ideas carry over to the Navier-Stokes equations in two and three dimensions.

The purpose of this paper is twofold. First, we will recast the Hopf functional differential equation (FDE) into an integral equation by introducing a kind of Duhamel principle and thereby yielding successive approximations systematically. In so doing we will make use of symbolic manipulations. Second, we will clarify the concept of scale invariance for the

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statistical Navier-Stokes equations by extending a previous work by Rosen [41] and combine it with the source-type solutions of the deterministic equations.

The rest of this paper is organized as follows. In Sec. II, after reviewing its scaling property, we convert the Hopf equation for the three-dimensional (3D) Navier-Stokes equations into an integral equation with the use of an exponential operator. We introduce a successive approximation on this basis. In Sec. III, the dynamic scaling property for the one-dimensional (1D) statistical Burgers equation is described. In Sec. IV, the dynamic scaling property for d-dimensional statistical Navier-Stokes equations is described. In Sec. V, the implications of the source-type solutions on the late-stage behavior of the Hopf functional are discussed. Section VI is devoted to a summary and outlook. In Appendix A, a formal derivation of the action of the exponential operator is stated. In Appendix B the leading-order approximation is presented for the Burgers equation. In Appendix C, an error estimate of the successive approximation is derived. Finally, Appendix D recalls the self-similar solution of the Burgers equation.

II. HOPF EQUATION FOR THE BURGERS EQUATION

There are many publications on the Hopf equations, for example, [42–49]. We will revisit this equation from a fundamental viewpoint. To illustrate the basic idea, we mainly consider the Burgers equation for simplicity. Statistical solutions of this equation have been studied in many works, such as [50–58].

The 1D Burgers equation written in standard notations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfies invariance under the following set of (static) scaling transformations:

$$x \to \lambda x, \quad t \to \lambda^2 t, \quad u \to \lambda^{-1} u,$$

where $\lambda(>0)$ is an arbitrary parameter. Hence, we have the following property.

Property 1. If $u(x,t)$ is a solution to (1), so is $\lambda u(\lambda x, \lambda^2 t)$.

The characteristic functional for the velocity is defined by

$$\Phi[\theta(x),t] = \left\langle \exp\left(i\int_{-\infty}^{\infty} u(x,t)\theta(x)dx\right)\right\rangle,$$

where $\langle\ \ \rangle$ denotes an ensemble average taken with respect to an initial velocity distribution. It satisfies the FDE

$$\frac{\partial \Phi}{\partial t} = L\Phi, \tag{2}$$

where [59]

$$L\Phi \equiv \frac{i}{2}\int \theta(x)\frac{\partial}{\partial x}\frac{\delta^2\Phi}{\delta\theta(x)^2}dx + \nu\int\theta(x)\frac{\partial^2}{\partial x^2}\frac{\delta\Phi}{\delta\theta(x)}dx.$$

The Hopf functional Φ satisfies some realizability conditions. It is required that

$$\Phi[\theta(x)]|_{\theta(x)\equiv 0} = 1$$

and positive-definiteness

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\Phi[\theta_k(x)-\theta_l(x)]c_k c_l^* = \left\langle\left|\sum_{k=1}^{n} c_k \exp\left(i\int u(x,t)\theta(x)dx\right)\right|^2\right\rangle \geqslant 0$$

hold for $n = 1, 2, 3, \ldots$. We are interested in solutions of (2) that satisfy those conditions at any time $t \geqslant 0$. Even though it is a linear equation, no general method for its solutions is known.

Scale invariance of the Hopf equation for the 3D Navier-Stokes equations has been studied in [41]. For the statistical solutions of the 1D Burgers equations, the corresponding argument goes as follows. Under the following set of transformations,

$$\theta(x) \to \theta(\lambda x), \quad u(x) \to \lambda u(\lambda x),$$

where $\lambda(>0)$ is an arbitrary parameter, the Hopf equation becomes

$$\frac{\partial\Phi}{\partial t} = \lambda^2 L\Phi[\theta(\lambda x), t].$$

This can be made invariant by scaling the time variable as $t \to \lambda^{-2}t$. Hence, we have the following property.

Property 2. If $\Phi[\theta(x),t]$ is a solution to (2), so is $\Phi[\theta(\lambda x), \lambda^{-2}t]$.

In particular, let us consider the heat diffusion equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}$$

by ignoring the nonlinear term of the Burgers equation. The corresponding FDE reads

$$\frac{\partial\Phi}{\partial t} = \nu\int\theta(x)\frac{\partial^2}{\partial x^2}\frac{\delta\Phi}{\delta\theta(x)}dx,$$

which can be solved explicitly [1,2] as

$$\Phi[\theta(x),t] = \Phi_0\left[\frac{1}{\sqrt{4\pi\nu t}}\int\exp\left(-\frac{(x-y)^2}{4\nu t}\right)\theta(y)dy\right] \equiv \Phi_0[\exp(\nu t\triangle)\theta], \tag{3}$$

where $\triangle = \frac{\partial^2}{\partial x^2}$ denotes the Laplacian. This is called the Hopf-Titt solution and it follows from the self-dual property of g_t:

$$(g_t * u_0, \theta) = (u_0, g_t * \theta), \tag{4}$$

where $g_t(x) = \frac{1}{\sqrt{4\pi\nu t}}\exp\left(-\frac{x^2}{4\nu t}\right)$ denotes the heat kernel and $(f,g) = \int f(x)g(x)dx$ an inner product. In this case, the characteristic functional does not essentially change its form, rather its argument develops following a heat flow. This method of solutions may be regarded as a FDE version of the method of characteristics (see, e.g., [60–62]).

We now consider an operator D, defined formally by

$$D \equiv \int dx\theta(x)\frac{\partial^2}{\partial x^2}\frac{\delta}{\delta\theta(x)} \tag{5}$$

and write the above solution heuristically (symbolically) as follows:

$$\Phi[\theta(x),t] = \exp\left(\nu t \int dx \theta(x)\frac{\partial^2}{\partial x^2}\frac{\delta}{\delta\theta(x)}\right)\Phi_0[\theta] \equiv \exp(\nu t D)\Phi_0[\theta].$$

In other words we define D by its action on $\Phi[\theta]$ as follows:

$$\exp(\nu t D)\Phi[\theta] \equiv \Phi[\exp(\nu t\triangle)\theta]. \tag{6}$$

Note that (5) is a purely symbolic notation the meaning of which is given by (6).

We may regard the operator on the left-hand side as a functional version of the "shift operator" on the basis of the above solution [63]. In other words, we turn a particular solution (3) into a definition of the new operator D. Its meaning is to update all the arguments, $\theta(x)$ in this case, in the operand functional by convoluting the heat kernel.

With this understanding, we can now recast the Hopf equation as follows:

$$\exp(\nu t D)\frac{\partial}{\partial t}\exp(-\nu t D)\Phi[\theta,t] = \frac{i}{2}\int \theta(x)\frac{\partial}{\partial x}\frac{\delta^2\Phi}{\delta\theta(x)^2}dx. \tag{7}$$

This allows us to convert the Hopf equation into an integral equation by a straightforward application of the Duhamel principle:

$$\Phi[\theta(x),t] = \exp(\nu t D)\Phi_0[\theta] + \frac{i}{2}\int_0^t \exp(\nu(t-s)D) \times \int \theta(x)\frac{\partial}{\partial x}\frac{\delta^2\Phi}{\delta\theta(x)^2}[\theta(x),s]dxds. \tag{8}$$

It may be in order to compare (8) with the integral form of deterministic Navier-Stokes equations [64]. Defining yet another operator G by

$$G \equiv \frac{i}{2}\int_0^t ds\exp(\nu(t-s)D)\int dx\theta(x)\frac{\partial}{\partial x}\frac{\delta^2}{\delta\theta(x)^2},$$

the integral form of the Hopf equation can be written

$$\Phi = \tilde{\Phi} + G\Phi,$$

where $\tilde{\Phi} = \Phi_0[\exp(\nu t\triangle)\theta]$ is the Hopf functional for the heat flow. See Appendix A for its formal derivation. It is noted that the operator G depends on time t, while its dependence is suppressed for simplicity. In passing we note that (8) resembles the Lipmann-Schwinger equation in scattering theory in quantum mechanics.

We are in a position to introduce a successive approximation

$$\Phi_{n+1} = \tilde{\Phi} + G\Phi_n \quad (n = 0, 1, 2, \dots)$$

to derive a Neumann series for the Hopf functional

$$\Phi = (I-G)^{-1}\tilde{\Phi} = (I + G + G^2 + G^3 + \dots)\tilde{\Phi}.$$

The zeroth-order approximation is given by the above (3). The first-order approximation is given by

$$\Phi \approx (I+G)\tilde{\Phi},$$

which is reminiscent of the Born approximation in scattering theory.

In order to apply this approximation in practice, numerical evaluation of the above integral would be needed. Here we restrict attention to leading-order analysis to see how the successive approximations look and what conditions are required to assure convergence of the successive approximations. Taking the initial Hopf functional of the following Gaussian form,

$$\Phi_0[\theta] = \exp\left(-\frac{1}{2}\iint Q(x',x'')\theta(x')\theta(x'')dx'dx''\right),$$

where $Q(x',x'') = \langle u(x',0)u(x'',0)\rangle$ denotes the initial velocity correlation function, we find

$$\begin{aligned} G\tilde{\Phi} = -\frac{i}{2}\int_0^t ds\Bigg\{&\frac{-1}{4\pi\nu s}\iiint dxdx'dx'' e^{\nu(t-s)\triangle}\theta(x)Q(x',x'')\frac{x'+x''-2x}{2\nu s}e^{-\frac{(x'-x)^2}{4\nu s}-\frac{(x''-x)^2}{4\nu s}} \\ &+\frac{1}{2\pi\nu s}\int dx e^{\nu(t-s)\triangle}\theta(x)\iint dx'dx''Q(x',x'')e^{-\frac{(x'-x)^2}{4\nu s}}e^{\nu t\triangle}\theta(x'') \\ &\times\iint dx'dx''Q(x',x'')\frac{x'-x}{2\nu s}e^{-\frac{(x'-x)^2}{4\nu s}}e^{\nu t\triangle}\theta(x'')\Bigg\}\Phi_0[e^{\nu t\triangle}\theta], \end{aligned} \tag{9}$$

where we have denoted a function of $x'' e^{\nu t\triangle}\theta(x'') = \frac{1}{\sqrt{4\pi\nu t}}\int\exp(-\frac{(x''-y)^2}{4\nu t})\theta(y)dy$. See Appendix B for the derivations.

It is in order to derive a sufficient condition for the convergence of the above successive approximation. Define the operator norm of G by

$$\|G\| \equiv \sup_{\tilde{\Phi}}\frac{\|G\tilde{\Phi}\|}{\|\tilde{\Phi}\|}.$$

By a standard argument [65,66], provided that $\|G\| < 1$ we have

$$\|(I-G)^{-1} - (I+G+\dots+G^{n-1})\| \leqslant \sum_{m=n}^{\infty}\|G\|^m = \|G\|^n(1-\|G\|)^{-1} \to 0, \quad \text{as } n\to\infty.$$

Because

$$\|G\tilde{\Phi}\| \leqslant \frac{\|\tilde{\Phi}\|}{2}\left\|\int_0^t ds \int e^{\nu(t-s)\triangle}\theta(x)\frac{\partial}{\partial x}\left\{e^{\nu s(\triangle'+\triangle'')}Q(x',x'') - \left(\int dx'' e^{\nu s\triangle'}Q(x',x'')e^{\nu t\triangle}\theta(x'')\right)^2\right\}dx\right\|,$$

we find that

$$\frac{1}{2}\left\|\int_0^t ds \int e^{\nu(t-s)\triangle}\theta(x)\frac{\partial}{\partial x}\left\{e^{\nu s(\triangle'+\triangle'')}Q(x',x'') - \left(\int dx'' e^{\nu s\triangle'}Q(x',x'')e^{\nu t\triangle}\theta(x'')\right)^2\right\}dx\right\| < 1$$

is sufficient for the convergence. It is shown in Appendix C that the condition is satisfied, for example, if

$$\frac{C}{\nu}\left\|\frac{\partial\theta}{\partial x}\right\|_{L^1}\|Q(x,y)\|_{L^1(\mathbb{R}^2)}\left(1+\|Q(x,y)\|_{L^1(\mathbb{R}^2)}\|\theta\|_{L^\infty}^2\right) < 1, \tag{10}$$

where C is a nondimensional constant. In particular, it is satisfied when ν is large.

III. DYNAMIC SCALING FOR THE HOPF EQUATION FOR THE BURGERS EQUATION

In [41], scale invariance of the 3D Navier-Stokes equations has been discussed and used to study a self-similar decaying process of turbulence. We present a variant of its argument adapted to one spatial dimension.

A. Dynamic scaling: Deterministic version

We will be interested in a decaying process of the Burgers "turbulence." We consider solutions with forward self-similarity, where the relevant scaling parameter is $\sqrt{2a(t+t_*)}$ with a constant $a(>0)$.

We generalize static scaling transformations using a dynamically rescaled time variable. Applying a set of dynamic scaling transformations

$$\xi = \frac{x}{\sqrt{2a(t+t_*)}}, \quad \tau = \frac{1}{2a}\log\frac{t+t_*}{t_*},$$
$$u(x,t) = \frac{1}{\sqrt{2a(t+t_*)}}U(\xi,\tau)$$

to the Burgers equation with $2at_* = 1$, we obtain

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial \xi} = a\frac{\partial}{\partial \xi}(\xi U) + \nu\frac{\partial^2 U}{\partial \xi^2}. \tag{11}$$

Note that the dynamically scaled Burgers equation, when linearized, coincides with a linear Fokker-Planck equation, that is, a Fokker-Planck equation with a linear drift term, associated with the Ornstein-Uhlenbeck process. It should be noted that the associated stochastic process reaches stationarity while Brownian motion (i.e., the Wiener process) does not. Note the distinction in behaviors of D and D^* below. It has a conservative term in the form $a\partial_\xi(\xi U)$, rather than the drift term $a\xi\partial_\xi U$ only. It is known that the solution to (11) converges to a steady solution as $\tau \to \infty$. See Sec. V and Appendix D for the steady solution.

B. Dynamic scaling: Statistical version

In principle there are two ways to derive the Hopf equation for the dynamically scaled Burgers equation. We will show that they actually lead to the same result. To avoid proliferation of notations, we use the same symbol Φ for the Hopf functional for the dynamically scaled equations, which can be distinguished by the argument.

Method 1. We first apply dynamic scaling and then move onto a statistical description.

By (11) it is straightforward to check that the characteristic functional of the velocity

$$\Phi[\theta(\xi),\tau] = \left\langle \exp\left(i\int_{-\infty}^{\infty} U(\xi,\tau)\theta(\xi)d\xi\right)\right\rangle$$

satisfies

$$\begin{aligned}\frac{\partial\Phi}{\partial\tau} &= \frac{i}{2}\int\theta(\xi)\frac{\partial}{\partial\xi}\frac{\delta^2\Phi}{\delta\theta(\xi)^2}d\xi + \nu\int\theta(\xi)\frac{\partial^2}{\partial\xi^2}\frac{\delta\Phi}{\delta\theta(\xi)}d\xi \\ &\quad + a\int\theta(\xi)\frac{\partial}{\partial\xi}\left(\xi\frac{\delta\Phi}{\delta\theta(\xi)}\right)d\xi \\ &= \frac{i}{2}\int\theta(\xi)\frac{\partial}{\partial\xi}\frac{\delta^2\Phi}{\delta\theta(\xi)^2}d\xi + \nu\int\frac{\partial^2\theta}{\partial\xi^2}\frac{\delta\Phi}{\delta\theta(\xi)}d\xi \\ &\quad - a\int\frac{\delta\Phi}{\delta\theta(\xi)}\xi\frac{\partial\theta}{\partial\xi}d\xi,\end{aligned} \tag{12}$$

that is,

$$\begin{aligned}\frac{\partial\Phi}{\partial\tau} &= \frac{i}{2}\int\theta(\xi)\frac{\partial}{\partial\xi}\frac{\delta^2\Phi}{\delta\theta(\xi)^2}d\xi \\ &\quad + \int\theta(\xi)\left(\nu\frac{\partial^2}{\partial\xi^2} + a\frac{\partial}{\partial\xi}\xi\right)\frac{\delta\Phi}{\delta\theta(\xi)}d\xi.\end{aligned} \tag{13}$$

Method 2. We first consider a statistical description and then apply dynamic scaling.

We write

$$\Phi[\theta(x),t] = \Psi[\zeta(y),\tau],$$

where $\zeta(y) \equiv \theta(\sqrt{2a(t+t_*)}x)$, $y = \sqrt{2a(t+t_*)}x$. By $\frac{\partial\zeta}{\partial t} = \frac{1}{2(t+t_*)}y\frac{\partial\theta}{\partial y}$ we find

$$\begin{aligned}\frac{\partial\Phi}{\partial t} &= \int\frac{\delta\Psi}{\delta\zeta}\frac{\partial\theta}{\partial t}dy + \frac{\partial\Psi}{\partial\tau}\frac{\partial\tau}{\partial t} \\ &= \int\frac{\delta\Psi}{\delta\zeta}\frac{1}{2(t+t_*)}y\frac{\partial\theta}{\partial y}dy + \frac{\partial\Psi}{\partial\tau}\frac{1}{2a(t+t_*)}.\end{aligned}$$

By the scaling of Φ, the right-hand side equals $\frac{1}{\lambda^2}L$, that is,

$$\frac{1}{2a(t+t_*)}\left(\frac{i}{2}\int\zeta(y)\frac{\partial}{\partial y}\frac{\delta^2\Psi}{\delta\zeta(y)^2}dy + \nu\int\zeta(y)\frac{\partial^2}{\partial y^2}\frac{\delta\Psi}{\delta\zeta(y)}dy\right).$$

Hence we obtain

$$\frac{\partial\Psi}{\partial\tau}+a\int\frac{\delta\Psi}{\delta\zeta}y\frac{\partial\zeta}{\partial y}dy$$
$$=\frac{i}{2}\int\zeta(y)\frac{\partial}{\partial y}\frac{\delta^2\Psi}{\delta\zeta(y)^2}dy+\nu\int\zeta(y)\frac{\partial^2}{\partial y^2}\frac{\delta\Psi}{\delta\zeta(y)}dy,$$

which matches the previous expression (13) by rewriting $\zeta(y)\to\theta(\xi)$.

C. Operator D^* for the modified heat kernel

Consider a linearization of (11), the linear Fokker-Planck equation:

$$\frac{\partial V}{\partial\tau}=a\frac{\partial}{\partial\xi}(\xi V)+\nu\frac{\partial^2 V}{\partial\xi^2}.$$

Its solution is given by

$$V(\xi,\tau)=\tilde{g}_\tau * V_0\equiv\left(\frac{a}{2\pi\nu(1-e^{-2a\tau})}\right)^{1/2}$$
$$\times\int_{\mathbb{R}^1}e^{a\tau}V_0(e^{a\tau}y)\exp\left(-\frac{a}{2\nu}\frac{(\xi-y)^2}{1-e^{-2a\tau}}\right)dy,$$

where $\tilde{g}_\tau$ denotes a modified heat kernel and $*$ denotes convolution as defined above. It is convenient to define an operator D^* and write symbolically the solution to the above Fokker-Planck solution as

$$V(\xi,\tau)=\exp(\nu\tau D^*)V_0=\exp[\tau\{\nu\triangle+a\partial_\xi(\xi\cdot)\}]V_0.$$

In the limit of $\tau\to\infty$ we have

$$V(\xi,\tau)\to\exp\left(\frac{\nu}{2a}D\right)V_0(\xi)\Big|_{V_0=M\delta}$$
$$=M\sqrt{\frac{a}{2\pi\nu}}\exp\left(-\frac{a}{2\nu}\xi^2\right),$$

where $M=\int V_0(\eta)d\eta$. Here we have made use of the formula

$$\frac{1}{\epsilon}f\left(\frac{x}{\epsilon}\right)\to M\delta(x),\quad\text{as }\epsilon\to 0,$$

which holds for any localized function f with $M=\int f(x)dx$.

D. Duhamel principle for the scaled Hopf equation

Replacing $\nu\triangle$ with $\nu\triangle+a\partial_\xi(\xi\cdot)$ in (7) we can write the dynamically scaled version of the Hopf equation as

$$\exp(\nu\tau D^*)\frac{\partial}{\partial\tau}\exp(-\nu\tau D^*)\Phi[\theta(\xi),\tau]$$
$$=\frac{i}{2}\int\theta(\xi)\frac{\partial}{\partial\xi}\frac{\delta^2\Phi}{\delta\theta(\xi)^2}d\xi,$$

from which we find

$$\Phi[\theta(\xi),\tau]=\exp(\nu\tau D^*)\Phi_0[\theta(\xi)]+\frac{i}{2}\int_0^\tau\exp[\nu(\tau-s)D^*]$$
$$\times\int\theta(\xi)\frac{\partial}{\partial\xi}\frac{\delta^2\Phi}{\delta\theta(\xi)^2}[\theta(\xi),s]d\xi ds. \quad (14)$$

We are interested in studying the long-time limit $\Phi[\theta(\xi)]=\lim_{\tau\to\infty}\Phi[\theta(\xi),\tau]$, but the evaluation of the second term on the right-hand side faces difficulty. In Sec. V we will show how we may obtain an asymptotic expression $\Phi[\theta(\xi)]$, working directly from the definition of the Hopf functional.

For the scaled-version of the Hopf equation for the linearized equation, that is, the first term on the right-hand side of (14), we have

$$\exp(\nu\tau D^*)\Phi_0[\theta(\xi)]=\langle\exp(i(\tilde{g}_\tau * V_0,\theta)\rangle.$$

Unlike the original heat kernel, self-duality $(\tilde{g}_\tau * V_0,\theta)=(V_0,\tilde{g}_\tau*\theta)$ does *not* hold in general for the modified kernel $\tilde{g}_\tau$. It holds when V_0 is a homogeneous function of degree -1, that is, $\lambda V_0(\lambda x)=V_0(x)$ for all $\lambda>0$ (just like the Dirac delta function).

IV. DYNAMIC SCALING FOR HOPF EQUATION FOR NAVIER-STOKES EQUATIONS

Dynamic scaling can be applied to the Hopf equation for the Navier-Stokes equations without any problem. Scaling properties of the Hopf equation have been discussed in [41], which we generalize here into d spatial dimensions. We will show that if a critical condition in the statistical solution is achieved the Hopf functional takes the simplest form of self-similarity. This is crucial in improving the Hopf-Titt solution with a "near-Gaussian" solution associated with the heat kernel. The final form would be more complicated with other choices of dependent variables.

A. Dynamic scaling (deterministic version)

For the incompressible Navier-Stokes equations written in standard notations

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=\nu\triangle\boldsymbol{u},\quad\nabla\cdot\boldsymbol{u}=0, \quad (15)$$

static scale invariance implies the following.

Property 1'. If $\boldsymbol{u}(\boldsymbol{x},t)$ is a solution to (15), so is $\lambda\boldsymbol{u}(\lambda\boldsymbol{x},\lambda^2 t)$.

By applying the dynamic scaling transforms

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{1}{\sqrt{2a(t+t_*)}}\boldsymbol{U}(\boldsymbol{\xi},\tau),$$
$$\boldsymbol{\xi}=\frac{\boldsymbol{x}}{\sqrt{2a(t+t_*)}},\quad\tau=\int_0^t\frac{ds}{\lambda(s)^2}=\frac{1}{2a}\log\frac{t+t_*}{t_*},$$

we obtain the scaled version of the Navier-Stokes equations, also known as the Leray equations:

$$\frac{\partial\boldsymbol{U}}{\partial\tau}+\boldsymbol{U}\cdot\nabla_\xi\boldsymbol{U}=-\nabla_\xi P+\nu\triangle_\xi\boldsymbol{U}+a(\boldsymbol{\xi}\cdot\nabla_\xi\boldsymbol{U}+\boldsymbol{U}),$$
$$\nabla_\xi\cdot\boldsymbol{U}=0.$$

B. Dynamic scaling (statistical version)

The characteristic functional of the velocity

$$\Phi[\boldsymbol{\theta},t]=\left\langle\exp\left(i\int\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\theta}(\boldsymbol{x})d\boldsymbol{x}\right)\right\rangle$$

satisfies

$$\Phi[\boldsymbol{\theta},t]=\Phi[\boldsymbol{\theta}^\perp,t],\ \Phi[\boldsymbol{\theta},t]^*=\Phi[-\boldsymbol{\theta},t],\ |\Phi[\boldsymbol{\theta},t]|\leqslant 1,$$

where $\boldsymbol{\theta}^\perp$ denotes a solenoidal projection of $\boldsymbol{\theta}$.

The Hopf equation can be written

$$\frac{\partial \Phi}{\partial t} = L\Phi, \tag{16}$$

where

$$L \equiv i \int d\boldsymbol{x}\, \theta_j^{\perp}(\boldsymbol{x}) \frac{\partial}{\partial x_k} \frac{\delta^2}{\delta\theta_j(\boldsymbol{x})\delta\theta_k(\boldsymbol{x})} + \nu \int d\boldsymbol{x}\, \theta_j(\boldsymbol{x}) \triangle \frac{\delta}{\delta\theta_j(\boldsymbol{x})}.$$

Note that a set of dilation transforms $\boldsymbol{\theta}(\boldsymbol{x}) \to \lambda^{d-1}\boldsymbol{\theta}(\lambda\boldsymbol{x})$, $\boldsymbol{u}(\boldsymbol{x}) \to \lambda\boldsymbol{u}(\lambda\boldsymbol{x})$ leaves $\int \boldsymbol{u}\cdot\boldsymbol{\theta} d\boldsymbol{x}$ invariant. Also note that functional derivatives transform as

$$\frac{\delta}{\delta\theta_j(\boldsymbol{x})} \to \lambda \frac{\delta}{\delta\theta_j(\lambda\boldsymbol{x})}$$

to ensure

$$\left[\frac{\delta}{\delta\theta_j(\boldsymbol{x})}, \theta_k(\boldsymbol{x}')\right] = \delta_{jk}\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

where $\delta(\cdot)$ denotes the Dirac delta function. On the other hand, the operator L transforms as

$$L \to i\lambda^{d+1} \int d\boldsymbol{x}\, \theta_j^{\perp}(\lambda\boldsymbol{x}) \frac{\partial}{\partial x_k} \frac{\delta^2}{\delta\theta_j(\lambda\boldsymbol{x})\delta\theta_k(\lambda\boldsymbol{x})} - \nu\lambda^d \int d\boldsymbol{x}\, \theta_j(\lambda\boldsymbol{x}) \triangle \frac{\delta}{\delta\theta_j(\lambda\boldsymbol{x})} = \lambda^2 L,$$

where the last line follows by $\boldsymbol{x} \to \lambda^{-1}\boldsymbol{x}$. The Hopf equation then takes the form

$$\left(\frac{\partial}{\partial t} - \lambda^2 L\right)\Phi[\lambda^{d-1}\boldsymbol{\theta}(\lambda\boldsymbol{x}), t] = 0,$$

or

$$\left(\frac{\partial}{\partial t} - L\right)\Phi[\lambda^{d-1}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t] = 0,$$

after rescaling t. Hence we conclude the following.

Property 2'. If $\Phi[\boldsymbol{\theta}(\boldsymbol{x}), t]$ is a solution to (16), so is $\Phi[\lambda^{d-1}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t]$.

It is important to observe that, unlike the deterministic case, the invariance property depends on the spatial dimension d. In particular, statistics of the velocity field attains criticality when and only when $d = 1$ in the sense that the argument of $\Phi[\cdot]$ takes a simplified form as $\theta(t^{1/2}x)$. It is noted that, with the choice of velocity, $\theta(x)$ of the characteristic functional for the 1D Burgers equation acquires the same physical dimension as that of $1/\nu$. We may contrast the situation with the deterministic counterpart of invariance Property 1' for the Navier-Stokes equations, which is *not* critical with the use of the velocity variable. We would have used the vector potential to achieve criticality for the deterministic Navier-Stokes equations [67].

In Property 2', if we consider the Hopf functional of the vorticity, we have $\Phi[\lambda^{d-2}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t]$ for the scaled functional. Likewise, if we consider the Hopf functional of the vorticity gradient, we have $\Phi[\lambda^{d-3}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t]$ for the scaled functional. Thus, by choosing the variable suitably we can make the prefactor of $\boldsymbol{\theta}(\lambda\boldsymbol{x})$ vanish, achieving statistical criticality, which renders the analysis of the Hopf equation the simplest possible one. It is noted that, with the choice of vorticity gradient, $\boldsymbol{\theta}(\boldsymbol{x})$ of the characteristic functional for the 3D Navier-Stokes equations acquires the same physical dimension as that of $1/\nu$.

To be more specific, we derive the dynamically scaled Hopf equation following the two different ways as above.

Method 1. We apply dynamic scaling to the Navier-Stokes equations and move onto the statistical description:

$$\frac{\partial \Phi}{\partial \tau} = i \int \theta_j^{\perp}(\boldsymbol{\xi}) \frac{\partial}{\partial \xi_k} \frac{\delta^2 \Phi}{\delta\theta_j(\boldsymbol{\xi})\delta\theta_k(\boldsymbol{\xi})} d\boldsymbol{\xi} + \nu \int \theta_j(\boldsymbol{\xi}) \triangle_{\xi} \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} d\boldsymbol{\xi} + a \int \theta_j(\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\nabla + 1) \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} d\boldsymbol{\xi}, \tag{17}$$

or

$$\frac{\partial \Phi}{\partial \tau} = i \int \theta_j^{\perp}(\boldsymbol{\xi}) \frac{\partial}{\partial \xi_k} \frac{\delta^2 \Phi}{\delta\theta_j(\boldsymbol{\xi})\delta\theta_k(\boldsymbol{\xi})} d\boldsymbol{\xi} + \nu \int \theta_j(\boldsymbol{\xi}) \triangle_{\xi} \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} d\boldsymbol{\xi} - a \int \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} (\boldsymbol{\xi}\cdot\nabla + d - 1)\theta_j(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{18}$$

It should be noted that when $d = 1$ the final term simplifies (that is, reduces to the drift term only).

Method 2. We start with the statistical formulation and apply dynamic scaling to it.

We apply the following set of transformations,

$$\Phi = \Psi[\boldsymbol{\zeta}, \tau],$$

$$\boldsymbol{\zeta} = [2a(t+t_*)]^{\frac{d-1}{2}} \boldsymbol{\theta}[\sqrt{2a(t+t_*)}\boldsymbol{x}], \quad \tau = \frac{1}{2a}\log\frac{t+t_*}{t_*},$$

to the Hopf equation. Setting $\boldsymbol{y} = \sqrt{2a(t+t_*)}\boldsymbol{x}$, we have

$$\frac{\partial \zeta_j}{\partial t} = \frac{1}{2(t+t_*)}\left((d-1)\zeta_j + y_k \frac{\partial \zeta_j}{\partial y_k}\right)$$

and

$$\frac{\partial \Phi}{\partial t} = \int \frac{\delta\Phi}{\delta\zeta_j}\frac{\partial\zeta_j}{\partial t} dy + \frac{\partial\Phi}{\partial\tau}\frac{\partial\tau}{\partial t} = \int \frac{\delta\Phi}{\delta\zeta_j} \frac{1}{2(t+t_*)}\left((d-1)\zeta_j + y_k\frac{\partial\zeta_j}{\partial y_k}\right) d\boldsymbol{y} + \frac{\partial\Phi}{\partial\tau}\frac{1}{2a(t+t_*)},$$

which equals

$$= \frac{1}{2a(t+t_*)} \int \left(i\zeta_j^{\perp} \frac{\partial}{\partial y_k} \frac{\delta^2\Phi}{\delta\zeta_j(\boldsymbol{y})\delta\zeta_k(\boldsymbol{y})} + \nu\zeta_j \triangle \frac{\delta\Phi}{\delta\zeta_j(\boldsymbol{y})}\right) d\boldsymbol{y}$$

by virtue of a substitution $L \to \frac{1}{\lambda^2}L$. Hence we find

$$\frac{\partial\Phi}{\partial\tau} + a\int \frac{\delta\Phi}{\delta\zeta_j}\left((d-1)\zeta_j + y_k\frac{\partial\zeta_j}{\partial y_k}\right) d\boldsymbol{y} = \int \left(i\zeta_j^{\perp} \frac{\partial}{\partial y_k} \frac{\delta^2\Phi}{\delta\zeta_j(\boldsymbol{y})\delta\zeta_k(\boldsymbol{y})} + \nu\zeta_j \triangle \frac{\delta\Phi}{\delta\zeta_j(\boldsymbol{y})}\right) d\boldsymbol{y},$$

which agrees with the above result (18) when a replacement $\boldsymbol{\zeta}(\boldsymbol{y}) \to \boldsymbol{\theta}(\boldsymbol{\xi})$ is made.

We make a set of two observations.

(1) For the Hopf equation, critical scale invariance is attained when we use a characteristic functional of the dth derivatives of the vector potential.

(2) For the corresponding scaled Navier-Stokes equations (i.e., the Leray equations), the dissipative term takes the form of the linear Fokker-Planck operator.

This in principle provides a method of deriving long-time asymptotics of the Hopf functional for the Navier-Stokes equations, just as for the Burgers equations (see below).

V. SOURCE-TYPE SOLUTIONS AND THEIR IMPLICATION ON HOPF FUNCTIONALS

To illustrate physical applications of the ideas developed here, we revisit the Burgers equations in one and two spatial dimensions. A source-type solution [68] is a forward self-similar solution, which starts from the Dirac delta function (in a variable attaining statistical criticality) and ends in a near-Gaussian universal profile associated with the heat kernel. We will show how it is crucial to choose a dependent variable to have the source-type solutions in the analysis of statistical solutions and the role they play in the determination of Hopf functionals in the late stage. Source-type solutions for the Burgers equations have been studied extensively [69–72], where their existence has been established in multidimensions, but an explicit functional form has been given only in one dimension.

A. One-dimensional Burgers equation

We will find a steady solution of (11) by taking the long-time limit $\tau \to \infty$. By the Cole-Hopf transform, the scaled velocity can be written

$$U(\xi,\tau) = -2\nu \frac{\partial_\xi \int_{\mathbb{R}^1} \psi_0(\lambda\eta) \exp\left(-\frac{a}{2\nu}\frac{(\xi-\eta)^2}{1-e^{-2a\tau}}\right) d\eta}{\int_{\mathbb{R}^1} \psi_0(\lambda\eta) \exp\left(-\frac{a}{2\nu}\frac{(\xi-\eta)^2}{1-e^{-2a\tau}}\right) d\eta},$$

where ψ_0 is the initial velocity potential. The numerator can be written by taking the derivative under the integral sign and integration by parts:

$$\int_{\mathbb{R}^1} \lambda\psi_0'(\lambda\eta) \exp\left(-\frac{a}{2\nu}\frac{(\xi-\eta)^2}{1-e^{-2a\tau}}\right) d\eta.$$

Now, noting $\lambda = e^{a\tau}$,

$$\lambda\psi_0'(\lambda\eta) \to K\delta(\eta) \quad \text{as} \quad \tau \to \infty,$$

where $K = \int_{-\infty}^{\infty} \psi_0'(\xi)d\xi = \psi_0(\infty) - \psi_0(-\infty)$. The numerator tends to

$$K \exp\left(-\frac{a}{2\nu}\xi^2\right),$$

whereas the denominator tends to its indefinite integral. Hence $U(\xi) = \lim_{\tau\to\infty} U(\xi,\tau)$ is given by

$$U(\xi) = -2\nu \frac{K \exp\left(-\frac{a}{2\nu}\xi^2\right)}{C + K\int_0^\xi \exp\left(-\frac{a}{2\nu}\eta^2\right) d\eta},$$

where C is a constant. Fixing C as $U(0) = -2\nu K/C$, we find

$$U(\xi) = \frac{U(0)\exp\left(-\frac{a}{2\nu}\xi^2\right)}{1 - \frac{U(0)}{2\nu}\int_0^\xi \exp\left(-\frac{a}{2\nu}\eta^2\right) d\eta}$$
$$= -2\nu \frac{\partial}{\partial\xi} \log\left\{1 - \frac{U(0)}{2\nu}\int_0^\xi \exp\left(-\frac{a}{2\nu}\eta^2\right) d\eta\right\}.$$

See Appendix D for an alternative derivation.

In the literature [69–71] it is often written $U(0) = F(M)$ for some function F, but we need to be more specific here. Defining $M = \int_{-\infty}^{\infty} U(\xi)d\xi$, it can be readily computed that

$$U(0) = \sqrt{\frac{8a\nu}{\pi}} \tanh \frac{M}{4\nu}.$$

In fact, we know that $K[= -\frac{C}{2\nu}U(0)]$ is invariant because of the conservation of the L^1 norm of the velocity under time evolution of the Burgers equation. This fact will be used below.

B. Higher-dimensional Burgers equations

Let us consider the two-dimensional case. Again using the Cole-Hopf transform, the scaled velocity reads

$$U_i(\boldsymbol{\xi},\tau) = -2\nu \frac{\partial_{\xi_i} \int_{\mathbb{R}^2} \psi_0(\lambda\boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu}\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|^2}{1-e^{-2a\tau}}\right) d\boldsymbol{\eta}}{\int_{\mathbb{R}^2} \psi_0(\lambda\boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu}\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|^2}{1-e^{-2a\tau}}\right) d\boldsymbol{\eta}}.$$

We know that $\lim_{\tau\to\infty} U_i(\boldsymbol{\xi},\tau)$ exists, however it is impossible to tell what it is from the expression above. The numerator can be written

$$\int_{\mathbb{R}^2} \lambda\partial_i\psi_0(\lambda\boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu}\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|^2}{1-e^{-2a\tau}}\right) d\boldsymbol{\eta},$$

where $\partial_i = \frac{\partial}{\partial(\lambda\eta_i)}$ denotes differentiation with respect to the argument. It is important to realize that, in two dimensions, to make use of

$$\lambda^2 f(\lambda\boldsymbol{\eta}) \to K\delta(\boldsymbol{\eta}) \quad \text{as} \quad \tau \to \infty$$

for localized f with $K = \int f d\boldsymbol{\eta}$, we must take higher derivatives:

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j} \int_{\mathbb{R}^2} \psi_0(\lambda\boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu}\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|^2}{1-e^{-2a\tau}}\right) d\boldsymbol{\eta}.$$

It then has the definite limit

$$\int_{\mathbb{R}^2} \lambda^2\partial_i\partial_j\psi_0(\lambda\boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu}\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|^2}{1-e^{-2a\tau}}\right) d\boldsymbol{\eta}$$
$$\to M_{ij} \exp\left(-\frac{a|\boldsymbol{\xi}|^2}{2\nu}\right),$$

as $\tau \to \infty$, where $M_{ij} = \int_{\mathbb{R}^2} \frac{\partial^2\psi_0}{\partial\eta_i\eta_j} d\boldsymbol{\eta}$ $(i,j = 1,2)$ denote constants. This is why we ought to choose the second derivatives of the velocity potential to achieve criticality.

The argument above identifies a choice of the velocity potential, e.g.,

$$\phi = \log\left[1 - \frac{M_{12}}{2\nu}\int_0^{\xi_1}\int_0^{\xi_2} \exp\left(-\frac{a}{2\nu}(\xi^2+\eta^2)\right) d\xi d\eta\right],$$

giving rise to $\frac{\partial U_1}{\partial \xi_2} = -2\nu \frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2}$. To summarize, the source-type solutions can be written using the velocity gradient, rather than the velocity, as

$$\frac{\partial U_1}{\partial \xi_2} = \frac{M_{12}}{[1 - R(\xi_1, \xi_2)]^2} \exp\left(-\frac{a}{2\nu}\left(\xi_1^2 + \xi_2^2\right)\right) = -2\nu \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \log[1 - R(\xi_1, \xi_2)],$$

where

$$R(\xi_1, \xi_2) = \frac{M_{12}}{2\nu} \int_0^{\xi_1} \exp\left(-\frac{a\xi^2}{2\nu}\right) d\xi \int_0^{\xi_2} \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta.$$

Observe that the velocity gradient has a near-Gaussian form, slightly moderated by $R(\xi_1, \xi_2)$, which is small because of the late stage of evolution.

The idea can be extended to any dimension, noting that we have in d dimensions

$$\lambda^d f(\lambda \boldsymbol{\eta}) \to K\delta(\boldsymbol{\eta}) \text{ as } \tau \to \infty$$

with $K = \int f d\boldsymbol{\eta}$. This explains why we need to choose the dth derivative of the vector (or tensor) potential to achieve criticality.

C. Late-stage behavior of the Hopf functional for the Burgers equation

We consider an initial-value problem of the Hopf equation (2) for the Burgers equation and its scaled counterpart (13). Needless to mention, the Burgers equation and the scaled Burgers equation describe the same initial-value problem. Likewise, the Hopf equation for the Burgers equation and its scaled counterpart describes the same statistical initial-value problem, with a common initial probability measure of the velocity field. To realize this, it helps to recall that the exponent of the characteristic functional, in abridged notations, reads

$$\int_{-\infty}^{\infty} u(x,t)\theta(x)dx = \int_{-\infty}^{\infty} \frac{1}{\lambda(t)} U\left(\frac{x}{\lambda(t)}, \frac{t}{\lambda(t)^2}\right)\theta(x)dx = \int_{-\infty}^{\infty} U(\xi,\tau)\zeta(\xi)d\xi,$$

where $x = \lambda\xi$, $t = \lambda^2\tau$, $\zeta(\xi) = \theta(\lambda\xi)$.

Once a source-type solution is obtained we can determine an asymptotic form of the Hopf functional in the late stage. Let us illustrate how this works using the 1D Burgers equation.

We recall that by substituting the Cole-Hopf solution

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\left(-\frac{1}{2\nu}\int^y u_0(x')dx' - \frac{(x-y)^2}{4\nu t}\right)dy}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\nu}\int^y u_0(x')dx' - \frac{(x-y)^2}{4\nu t}\right)dy}$$

into the definition of the Hopf functional

$$\Phi[\theta(x), t] = \left\langle \exp\left(i\int_{-\infty}^{\infty} u(x,t)\theta(x)dx\right)\right\rangle$$

we can in principle obtain the time-dependent Hopf functional. However, the above expression is obscure in that we must carry out infinite-dimensional (functional) integration because $\langle \dots \rangle = \int \dots d\mu(u_0)$ denotes an average over initial velocity, which is generally a formidable task.

We will show that if we focus on the late stage of the decaying process we can obtain an asymptotic form of the Hopf functional up to a quadrature. This is done by combining the dynamically scaled Hopf equation and the source-type solution in the late stage. In taking the limit of $\tau \to \infty$ almost all the information of the initial data will be lost through the emergence of the delta function; the information will be squeezed into that of K alone:

$$\Phi[\theta(\xi)] = \int \exp\left(i\int_{-\infty}^{\infty} \frac{\sqrt{\frac{8a\nu}{\pi}} \tanh\frac{K}{4\nu} \exp\left(-\frac{a}{2\nu}\xi^2\right)}{1 - \frac{1}{2\nu}\sqrt{\frac{8a\nu}{\pi}} \tanh\frac{K}{4\nu} \int_0^{\xi} \exp\left(-\frac{a}{2\nu}\eta^2\right)d\eta}\theta(\xi)d\xi\right)d\mu(K), \tag{19}$$

where $d\mu(K) = P(K)dK$ denotes the probability measure of the L^1 norm of the initial velocity, with $P(K)$ its density. Note that the above average is just a definite integral (a quadrature), the meaning of which is clear and the denominator of which never hits zero, because

$$\frac{1}{2\nu}\sqrt{\frac{8a\nu}{\pi}} \tanh\frac{K}{4\nu} \int_0^{\xi} \exp\left(-\frac{a}{2\nu}\eta^2\right)d\eta < \frac{1}{2\nu}\sqrt{\frac{8a\nu}{\pi}} \frac{1}{2}\sqrt{\frac{2\pi\nu}{a}} = 1.$$

Note also that the above expression generalizes the Hopf-Titt solution. For small $|K/\nu|$, in fact, we have

$$\Phi[\theta(\xi)] \approx \int \exp\left(iK\int_{-\infty}^{\infty} \tilde{g}(\xi)\theta(\xi)d\xi\right)d\mu(K),$$

where $\tilde{g}(\xi) = \sqrt{\frac{a}{2\pi\nu}} \exp\left(-\frac{a}{2\nu}\xi^2\right)$. Because we have a source-type solution as $\tau \to \infty$ with $U_0 = K\delta(\cdot)$, the exponent can be written [73]

$$i\int_{-\infty}^{\infty} (\tilde{g} * U_0)(\xi)\,\theta(\xi)d\xi = i\int_{-\infty}^{\infty} U_0(\xi)(\tilde{g} * \theta)(\xi)d\xi = i\exp\left(\frac{\nu}{2a}D\right)\int_{-\infty}^{\infty} U_0(\xi)\theta(\xi)d\xi.$$

Hence, we find

$$\Phi[\theta(\xi)] \approx \exp\left(\frac{\nu}{2a}D\right)\Phi_0[\theta(\xi)],$$

which in the original variables corresponds to

$$\Phi[\theta(x)] \approx \exp(\nu t D)\Phi_0[\theta(x)].$$

TABLE I. Self-similar (source-type) solutions.

Equations	Deterministic criticality achieved with	Statistical criticality achieved with	Source-type solutions
1D Burgers	Velocity potential ϕ	Velocity u	Known explicitly
n-D Burgers	Velocity potential ϕ	$\partial^n \phi$	Existence known, made explicit here
2D Navier-Stokes	Stream function ψ	Vorticity ω	Known explicitly as the Burgers vortex
3D Navier-Stokes	Vector potential $\boldsymbol{\psi}$	Vorticity gradient $\nabla \times \boldsymbol{\omega}$	Existence known, explicit form unknown

This is nothing but the Hopf-Titt solution. Thus the expression (19) is an improvement of the Hopf-Titt solution.

D. Source-type solutions for the Navier-Stokes equations

Let us clarify how the results obtained for the Burgers equations can carry over to the Navier-Stokes equations. As confirmed in Sec. IV, the derivation of the Hopf equation for the dynamically scaled Navier-Stokes equations stands parallel to that of the Burgers equations. It is also known that there exist self-similar solutions of the 3D Navier-Stokes equations for small data [74–76]. See also, e.g., [77,78] for recent developments. A self-similar profile is known to exist, but its functional form is unknown.

A connection of the late-stage behavior of the Hopf functional to the source-type solution with a suitably chosen dependent variable also holds, if smoothness of solutions is assumed for the 3D Navier-Stokes equations. It should be noted that this connection itself is valid, even though the universal profile is not known explicitly. See Table I for a comparison of studies on source-type solutions of the Burgers and Navier-Stokes equations. No method of exact linearization is known for the Navier-Stokes equations, but the existence of forward self-similar solutions and the conservation of its L^1 norm of the suitably chosen unknown suffice to apply the current methods to the the Navier-Stokes equations.

The source-type solution for the two-dimensional (2D) Navier-Stokes equations is known explicitly as the Burgers vortex [79]:

$$\Omega(\xi) = \frac{a\Gamma}{2\pi\nu}\exp\left(-\frac{a|\boldsymbol{\xi}|^2}{2\nu}\right),$$

where $\Gamma = \int_{\mathbb{R}^2}\Omega(\xi)d\boldsymbol{\xi}$ denotes a circulation invariant. Following the current approach, the late-stage Hopf functional is clearly given by

$$\Phi[\theta(\xi)] = \int \exp\left[i\frac{a\Gamma}{2\pi\nu}\int_{-\infty}^{\infty}\exp\left(-\frac{a}{2\nu}\xi^2\right)\theta(\xi)d\xi\right]d\mu(\Gamma).$$

However, the Burgers vortex solves not only the linearized equation but also the fully nonlinear equations accidentally, because of its radial symmetry (dependent on $|\boldsymbol{\xi}|$ only). For this reason the above Hopf functional is precisely equivalent to, and not an improvement of, the Hopf-Titt solution for the 2D Navier-Stokes equations.

For the 3D Navier-Stokes equations the existence of self-similar solutions is known (in velocity) in a number of function spaces. The corresponding vorticity gradient gives the source-type solution implicitly. With this variable it is near Gaussian and we have the scaled vorticity curl

$$\boldsymbol{X}(\boldsymbol{\xi}) \ \text{ close to } \ {}'\exp\left(-\frac{a}{2\nu}|\boldsymbol{\xi}|^2\right)',$$

where the effects of nonlinear terms and incompressibility should be taken into account. We can in principle write

$$\boldsymbol{X}(\boldsymbol{\xi}) = \boldsymbol{F}\left[\exp\left(-\frac{a}{2\nu}|\boldsymbol{\xi}|^2\right);\boldsymbol{K}\right],$$

where $\boldsymbol{F}$ denotes a *near-identity* nonlocal functional and $\boldsymbol{K} = \int_{\mathbb{R}^3}\boldsymbol{X}(\boldsymbol{\xi})d\boldsymbol{\xi}$ is an invariant. With this understanding the late-stage Hopf functional can be written

$$\Phi[\boldsymbol{\theta}(\xi)] = \int \exp\left\{i\int_{-\infty}^{\infty}\boldsymbol{F}\left[\exp\left(-\frac{a}{2\nu}|\boldsymbol{\xi}|^2\right);\boldsymbol{K}\right]\boldsymbol{\theta}(\xi)d\xi\right\}d\mu(\boldsymbol{K}).$$

This motivates a determination of the self-similar profile $\boldsymbol{X}(\boldsymbol{\xi})$, at least approximately.

VI. SUMMARY AND OUTLOOK

We have studied basic issues of statistical solutions of the Navier-Stokes equations. After presenting the main ideas using the Burgers equations, we have shown how those ideas carry over to the Navier-Stokes equations.

First, we have introduced the exponential operator G, which enables us to write the Hopf equations as an integral equation (in time). By applying the Duhamel principle on this basis, a successive approximation is formulated. The leading-order approximation is presented for the Burgers equation for an illustrative purpose. This can in principle be extended to the the Navier-Stokes equations, while the algebra involved would be lengthy because of the incompressibility condition.

Second, we have seen that the suitable choice of dependent variables, that achieve critical scale invariance of statistical solutions of the Burgers and Navier-Stokes equations, depends on spatial dimensions. It should be noted that this recognition holds valid for the Navier-Stokes equations for which exact linearization is not available. It may be in order to summarize statistical criticality, comparing to more conventional dependent variables for the 3D Navier-Stokes equations (Table II).

Furthermore, we have seen that when we choose a dependent variable satisfying statistical criticality the variable will take a near-Gaussian form in the long-time limit for the corresponding deterministic problem. On this basis, the late-stage evolution of statistical solutions of the Burgers and Navier-Stokes equations can be determined most conveniently by using the source-type solutions of the deterministic problem.

TABLE II. Characteristic functionals for the scaled Navier-Stokes equations. To attain criticality, the temporal prefactor in front of $\boldsymbol{\theta}(\cdot)$ must be absent, i.e., t^0 in the scale-invariant form.

Variables	General forms	Scale-invariant forms
Vector potential $\boldsymbol{\psi}$	$\Phi[\lambda^d\theta(\lambda\boldsymbol{x}), \lambda^{-2}t]$	$\Phi[t^{\frac{d}{2}}\boldsymbol{\theta}(t^{1/2}\boldsymbol{x})]$
Velocity $\boldsymbol{u}$	$\Phi[\lambda^{d-1}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t]$	$\Phi[t^{\frac{d-1}{2}}\boldsymbol{\theta}(t^{1/2}\boldsymbol{x})]$
Vorticity $\boldsymbol{\omega}$	$\Phi[\lambda^{d-2}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t]$	$\Phi[t^{\frac{d-2}{2}}\boldsymbol{\theta}(t^{1/2}\boldsymbol{x})]$
Vorticity gradient $\nabla\times\boldsymbol{\omega}$	$\Phi[\lambda^{d-3}\boldsymbol{\theta}(\lambda\boldsymbol{x}), \lambda^{-2}t]$	$\Phi[t^{\frac{d-3}{2}}\boldsymbol{\theta}(t^{1/2}\boldsymbol{x})]$

We have employed the Cole-Hopf transform for the Burgers equations for illustration. The method itself, however, can also be applied to the Navier-Stokes equations, where self-similar solutions of the 3D Navier-Stokes equations are known to exist, but not explicitly.

In two dimensions it is the vorticity the statistics of which satisfies statistical criticality. In fact, the late evolution of the 2D Navier-Stokes solutions is dominated by a collection of Burgers vortices (source-type solutions; see, e.g., [80]), which are steady solutions of the two-dimensional linear Fokker-Planck equation. The late-stage asymptotics for the Hopf functional is equivalent to the Hopf-Titt solution.

In three dimensions it is the vorticity gradient the statistics of which satisfies a critical condition. We can still argue that the late-stage asymptotics for the Hopf functional is given modulo a near-identity functional of the Gaussian function. It is of interest to determine their functional form, at least approximately, as a 3D analog of the Burgers vortex in two dimensions. This is left for future study.

Finally, we note that criticality for the statistical equations is achieved with a dynamical variable, the dissipative term of which is given by the linear Fokker-Planck operator, and is closely related to the self-adjoint property (duality) of the dissipative operators in the long-time limit.

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APPENDIX A: FORMAL DERIVATION OF THE FORMULA FOR OPERATOR *G*

We give a formal derivation on the action of the operator G:

$$\Phi[\theta(x),t] = \exp(\nu t D)\Phi_0[\theta] - \frac{i}{2}\int_0^t \exp(\nu(t-s)D)\int \theta(x)\frac{\partial}{\partial x}\frac{\delta^2\Phi}{\delta\theta(x)^2}[\theta(x),s]dxds,$$

where

$$D \equiv \int dx\theta(x)\frac{\partial^2}{\partial x^2}\frac{\delta}{\delta\theta(x)}.$$

Proof. Consider the Liouville equation (also known as the Hopf-Foias equation)

$$\frac{d}{dt}\int e^{i(\theta,u(t))}d\mu(u_0) = i\int(\theta,\triangle u(t))e^{i(\theta,u(t))}d\mu(u_0) - \frac{i}{2}\int(\theta,\partial_x u(t)^2)e^{i(\theta,u(t))}d\mu(u_0),$$

where

$$(\theta,u(t)) = \int_{-\infty}^{\infty}\theta(x)u(x,t)dx.$$

Setting $\theta(t) = e^{(T-t)\triangle}\theta_0$, we write

$$\frac{d}{dt}\int e^{i(\theta(t),u(t))}d\mu(u_0) = i\int[(\dot\theta(t),u(t)) + (\theta(t),\dot u(t))]e^{i(\theta(t),u(t))}d\mu(u_0)$$
$$= -\frac{i}{2}\int e^{i(\theta(t),u(t))}(\theta,\partial_x u(t)^2)d\mu(u_0).$$

Integrating with respect to time in $0 \leqslant t \leqslant T$, we get

$$\int e^{i(\theta(T),u(T))}d\mu(u_0) = \int e^{i(\theta(0),u(0))}d\mu(u_0) - \frac{i}{2}\int_0^T ds\int e^{i(\theta(s),u(s))}(\theta(s),\partial_x u(s)^2)d\mu(u_0),$$

that is,

$$\int e^{i(\theta_0,u(T))}d\mu(u_0) = \int e^{i(e^{T\triangle}\theta_0,u(0))}d\mu(u_0) - \frac{i}{2}\int_0^T ds\int e^{i(e^{(T-s)\triangle}\theta_0,u(s))}(e^{(T-s)\triangle}\theta_0,\partial_x u(s)^2)d\mu(u_0).$$

Replacing $T \to t$ and $\theta_0 \to \theta$, we find

$$\int e^{i(\theta,u(t))}d\mu(u_0) = \int e^{i(e^{t\triangle}\theta,u(0))}d\mu(u_0) - \frac{i}{2}\int_0^t ds \int e^{i(e^{(t-s)\triangle}\theta,u(s))}(e^{(t-s)\triangle}\theta, \partial_x u(s)^2)d\mu(u_0).$$

Now, the left-hand side is $\Phi[\theta, t]$, and the first term on the right-hand side is $\Phi(e^{t\triangle}\theta)$. Noting that in the second term on the right-hand side

$$\partial_x u(s)^2\Phi = -\partial_x \frac{\delta^2\Phi}{\delta\theta^2},$$

we obtain the desired result by noting that the first term on the right-hand side equals $\int e^{i(\theta,e^{t\triangle}u(0))}d\mu(u_0)$ by self-adjointness of the heat semigroup. ■

APPENDIX B: LEADING-ORDER APPROXIMATION

The zeroth-order functional is

$$\begin{aligned}\tilde{\Phi}[\theta] &= \Phi_0[\exp(\nu t\triangle)\theta]\\ &= \exp\left(-\frac{1}{2}\iint dx'dx''Q(x',x'')\frac{1}{\sqrt{4\pi\nu t}}\int e^{-\frac{(x'-y')^2}{4\nu t}}\theta(y')dy'\frac{1}{\sqrt{4\pi\nu t}}\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy''\right)\\ &= \exp\left(-\frac{1}{8\pi\nu t}\iint dx'dx''Q(x',x'')\int e^{-\frac{(x'-y')^2}{4\nu t}}\theta(y')dy'\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy''\right).\end{aligned} \tag{B1}$$

It is straightforward to compute

$$\frac{\delta\tilde{\Phi}}{\delta\theta(x)} = -\frac{1}{4\pi\nu t}\iint dx'dx''Q(x',x'')e^{-\frac{(x'-x)^2}{4\nu t}}\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy'' \times \tilde{\Phi}, \tag{B2}$$

$$\begin{aligned}\frac{\delta^2\tilde{\Phi}}{\delta\theta(x)^2} &= -\frac{1}{4\pi\nu t}\iint dx'dx''Q(x',x'')e^{-\frac{(x'-x)^2}{4\nu t}-\frac{(x''-x)^2}{4\nu t}} \times \tilde{\Phi}\\ &\quad+\left(\frac{1}{4\pi\nu t}\iint dx'dx''Q(x',x'')e^{-\frac{(x'-x)^2}{4\nu t}}\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy''\right)^2 \times \tilde{\Phi},\end{aligned} \tag{B3}$$

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\delta^2\tilde{\Phi}}{\delta\theta(x)^2} &= -\frac{1}{4\pi\nu t}\iint dx'dx''Q(x',x'')\frac{x'+x''-2x}{2\nu t}e^{-\frac{(x'-x)^2}{4\nu t}-\frac{(x''-x)^2}{4\nu t}} \times \tilde{\Phi}\\ &\quad+2\left(\frac{1}{4\pi\nu t}\iint dx'dx''Q(x',x'')e^{-\frac{(x'-x)^2}{4\nu t}}\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy''\right)\\ &\quad\times\left(\frac{1}{4\pi\nu t}\iint dx'dx''Q(x',x'')\frac{x'-x}{2\nu t}e^{-\frac{(x'-x)^2}{4\nu t}}\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy''\right) \times \tilde{\Phi}.\end{aligned} \tag{B4}$$

Hence we find

$$\begin{aligned}\int dx\theta(x)\frac{\partial}{\partial x}\frac{\delta^2\tilde{\Phi}}{\delta\theta(x)^2}(t) &= \frac{-1}{4\pi\nu t}\iiint dxdx'dx''\theta(x)Q(x',x'')\frac{x'+x''-2x}{2\nu t}e^{-\frac{(x'-x)^2}{4\nu t}-\frac{(x''-x)^2}{4\nu t}} \times \tilde{\Phi}\\ &\quad+\frac{2}{(4\pi\nu t)^2}\int dx\,\theta(x)\iint dx'dx''Q(x',x'')e^{-\frac{(x'-x)^2}{4\nu t}}\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy''\\ &\quad\times\iint dx'dx''Q(x',x'')\frac{x'-x}{2\nu t}e^{-\frac{(x'-x)^2}{4\nu t}}\int e^{-\frac{(x''-y'')^2}{4\nu t}}\theta(y'')dy'' \times \tilde{\Phi}\\ &= \frac{-1}{4\pi\nu t}\iiint dxdx'dx''\theta(x)Q(x',x'')\frac{x'+x''-2x}{2\nu t}e^{-\frac{(x'-x)^2}{4\nu t}-\frac{(x''-x)^2}{4\nu t}} \times \Phi_0[e^{\nu t\triangle}\theta]\\ &\quad+\frac{1}{2\pi\nu t}\int dx\theta(x)\iint dx'dx''Q(x',x'')e^{-\frac{(x'-x)^2}{4\nu t}}e^{\nu t\triangle}\theta(x'')\\ &\quad\times\iint dx'dx''Q(x',x'')\frac{x'-x}{2\nu t}e^{-\frac{(x'-x)^2}{4\nu t}}e^{\nu t\triangle}\theta(x'') \times \Phi_0[e^{\nu t\triangle}\theta],\end{aligned}$$

which for convenience we may write in a compact form:

$$= -\Phi_0[e^{\nu t\triangle}\theta] \int dx\theta(x) \frac{\partial}{\partial x}\left\{ e^{\nu t(\triangle'+\triangle'')}Q(x',x'') - \left(\int dx'' e^{\nu t\triangle'}Q(x',x'')\, e^{\nu t\triangle}\theta(x'')\right)^2\right\}.$$

Here $e^{\nu t(\triangle'+\triangle'')}Q(x',x'')$ and $e^{\nu t\triangle'}Q(x',x'')$ denote functions of x in the relevant arguments, that is,

$$e^{\nu t(\triangle'+\triangle'')}Q(x',x'') = \frac{1}{4\pi\nu t}\iint \exp\left(-\frac{(y-x)^2+(z-x)^2}{4\nu t}\right)Q(y,z)dydz,$$

$$e^{\nu t\triangle'}Q(x',x'') = \frac{1}{\sqrt{4\pi\nu t}}\int \exp\left(-\frac{(y-x)^2}{4\nu t}\right)Q(y,x'')dy.$$

Thus we find

$$G\tilde{\Phi} = \frac{i}{2}\int_0^t dse^{\nu(t-s)D}\int dx\theta(x)\frac{\partial}{\partial x}\frac{\delta^2\tilde{\Phi}}{\delta\theta(x)^2}(s)$$
$$= -\frac{i}{2}\Phi_0[e^{\nu t\triangle}\theta]\int_0^t ds\int dxe^{\nu(t-s)\triangle}\theta(x)\frac{\partial}{\partial x}\left\{e^{\nu s(\triangle'+\triangle'')}Q(x',x'') - \left(\int dx'' e^{\nu s\triangle'}Q(x',x'')\, e^{\nu t\triangle}\theta(x'')\right)^2\right\}.$$

Spelling this out more explicitly, we obtain (9).

APPENDIX C: DERIVATION OF THE ERROR ESTIMATE

We shall estimate

$$I \equiv \left\| \int_0^t ds \int e^{\nu(t-s)\triangle}\frac{\partial\theta}{\partial x}\left\{e^{\nu s(\triangle'+\triangle'')}Q(x',x'') - \left(\int dx'' e^{\nu s\triangle'}Q(x',x'')\, e^{\nu t\triangle}\theta(x'')\right)^2\right\}dx\right\|.$$

We first divide it in two parts to consider

$$I_1 = \int e^{\nu(t-s)\triangle}\frac{\partial\theta}{\partial x}e^{\nu s(\triangle'+\triangle'')}Q(x',x'')dx,$$
$$I_2 = \int e^{\nu(t-s)\triangle}\frac{\partial\theta}{\partial x}\left(\int e^{\nu s\triangle'}Q(x',x'')\, e^{\nu t\triangle}\theta(x'')dx''\right)^2 dx.$$

By the Hölder inequality

$$\int |f(x)g(x)|dx \leqslant \|f\|_{L^p}\|g\|_{L^q},\ \text{for } \frac{1}{p}+\frac{1}{q}=1,$$

and an estimate for the heat-kernel (see, e.g., [76]),

$$\|e^{\nu t\triangle}\theta\|_{L^p} \leqslant \frac{1}{(4\pi\nu t)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\|\theta\|_{L^q},$$

we bound I_1 as

$$I_1 \leqslant \frac{C_1}{\{\nu(t-s)\}^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}(\nu s)^{\frac{1}{2}(\frac{1}{u}-\frac{1}{q})}}\left\|\frac{\partial\theta}{\partial x}\right\|_{L^q} \|\|Q(x,y)\|_{L^r(dx)}\|_{L^u(dy)},$$

where $1\leqslant q\leqslant p\leqslant\infty, 1\leqslant r\leqslant q\leqslant\infty, 1\leqslant u\leqslant r\leqslant\infty$, $\frac{1}{p}+\frac{1}{q}=1$, and C_1 is a nondimensional constant. Taking $p=\infty, q=1, r=u=1$, and noting $\int_0^t\frac{ds}{\sqrt{s(t-s)}}=\pi(<\infty)$, we find

$$I_1 \leqslant \frac{C}{\nu}\left\|\frac{\partial\theta}{\partial x}\right\|_{L^1}\|Q(x,y)\|_{L^1(\mathbb{R}^2)},$$

where $C(=\pi C_1)$ is another constant. Similarly, we bound I_2 as

$$I_2 \leqslant \frac{C_2}{\{\nu(t-s)\}^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\frac{1}{(\nu s)^{\frac{1}{2}(\frac{1}{r}-\frac{1}{2q})}}\frac{1}{(\nu t)^{\frac{1}{2}(\frac{1}{w}-\frac{1}{v})}}\left\|\frac{\partial\theta}{\partial x}\right\|_{L^q} \|\|Q(x,y)\|^2_{L^r(dx)}\|_{L^u(dy)}\|\theta\|^2_{L^w},$$

where $1\leqslant q\leqslant p\leqslant\infty, 1\leqslant r\leqslant 2q\leqslant\infty, 1\leqslant w\leqslant v\leqslant\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{u}+\frac{1}{v}=1$, and C_2 is a constant. Choosing, for example, $r=u=1, p=\infty, q=1, v=\infty, w=\infty$, we find

$$I_2 \leqslant \frac{C'}{\nu}\left\|\frac{\partial\theta}{\partial x}\right\|_{L^1}\|Q(x,y)\|^2_{L^1(\mathbb{R}^2)}\|\theta\|^2_{L^\infty},$$

with $C'(=\pi C_2)$. Combining those two results and writing $C=\max(C,C')$, we obtain (10). It is readily checked that those bounds are nondimensional, noting that the physical dimension of θ is the same as that of ν for the 1D Burgers equation.

APPENDIX D: SELF-SIMILAR DECAY

In the case of forward self-similarity, we take the length scale as $\lambda(t)=\sqrt{2a(t+t_*)}$ to write the Leray equation of the form

$$U\frac{\partial U}{\partial\xi} = \nu\frac{\partial^2 U}{\partial\xi^2} + a\left(\xi\frac{\partial U}{\partial\xi}+U\right).$$

This can be exactly solved as follows.

Upon integration we find, after taking a constant to be zero,

$$\frac{dU}{d\xi} = \frac{1}{2\nu}U(U-2a\xi).$$

By $V=1/U$ in this Bernoulli equation, we have

$$\frac{dV}{d\xi} = -\frac{1}{2\nu}(1-2a\xi V),$$

which is a linear inhomogeneous equation. Solving it we find

$$V(\xi) = e^{a\xi^2/(2\nu)}\left(V(0) - \frac{1}{2\nu}\int_0^{\xi} e^{-a\eta^2/(2\nu)}d\eta\right)$$

or

$$U(\xi) = \frac{e^{-a\xi^2/(2\nu)}}{U(0)^{-1} - \frac{1}{2\nu}\int_0^{\xi} e^{-a\eta^2/(2\nu)}d\eta}.$$

Note that by making $U(0)$ small enough such that $U(0) < \sqrt{\frac{8a\nu}{\pi}}$ we can make the denominator nonzero for all ξ. This means that the particular self-similar solution is valid for small initial data.

In passing we comment on backward self-similar solutions. By reversing the sign of a, a possibility of blowup can be studied using backward self-similarity with the length scale $\sqrt{2a(t_* - t)}$. (Of course, it is known that there is no blowup). In this case, the steady equation is

$$U\frac{\partial U}{\partial \xi} + a\left(\xi\frac{\partial U}{\partial \xi} + U\right) = \nu\frac{\partial^2 U}{\partial \xi^2},$$

the smooth solution of which is $U \equiv 0$ only. The solution is nonetheless obtained as

$$U(\xi) = \frac{e^{a\xi^2/(2\nu)}}{U(0)^{-1} - \frac{1}{2\nu}\int_0^{\xi} e^{a\eta^2/(2\nu)}d\eta}.$$

Note that $U(\xi) \to \frac{1}{|\xi|}$ as $\xi \to \pm\infty$. More importantly, this has a singular point (a pole) somewhere, say at $\xi = \xi_*$, at which

$$U(0)^{-1} - \int_0^{\xi_*} e^{a\eta^2/(2\nu)}d\eta = 0.$$

Hence, $U(\xi) \propto 1/(\xi - \xi_*)$ near there and it is nonintegrable, $U \notin L^1$.

[1] E. Hopf, Statistical hydromechanics and functional calculus, J. Rat. Mech. Anal. **1**, 87 (1952)

[2] E. Hopf and E. W. Titt, On certain special solutions of the cfl-equation of statistical hydrodynamics, J. Rat. Mech. Anal. **2**, 587 (1953)

[3] A. S. Monin and A. M. Yaglom, *Statistical Fluid Dynamics: Mechanics of Turbulence* (MIT, Cambridge, MA, 1971), Vol. 1 ; *Statistical Fluid Dynamics: Mechanics of Turbulence* (MIT, Cambridge, MA, 1975), Vol. 2.

[4] M. I. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydromechanics* (Kluwer, Dordrecht, 1988).

[5] G. Rosen, Turbulence theory and functional integration. I and II, Phys. Fluids **3**, 519 (1960).

[6] G. Rosen, *Formulation of Classical and Quantum Dynamical Theory* (Academic, New York, 1969).

[7] J. C. Rotta, *Turbulente Strömungen* (Teubner, Stuttgart, 1972).

[8] T. Imamura, *Mathematics of Random Fields* (Iwanami, Tokyo, 1976), in Japanese.

[9] T. Butler, A Mathematical Example by Hopf with Features of turbulence, J. Math. Anal. Appl. **9**, 215 (1964).

[10] V. I. Tatarskii, Application of the methods of quantum field theory to the problem of degeneration of homogeneous turbulence, Soviet Physics JETP **15**, 961 (1962).

[11] I. Hosokawa, Functional approach to classical non-equilibrium statistical mechanics, J. Math. Phys. **8**, 221 (1967).

[12] E. A. Novikov, The convergence of a functional Taylor series for the characteristic functional of a random field, Uspekhi Mat. Nauk. **19**, 195 (1964).

[13] E. A. Novikov, The solution of certain equations with variational derivatives, Uspekhi Mat. Nauk. **16**, 135 (1961).

[14] R. Friedrich, A. Daitche, O. Kamps, J. Lülff, M. Voßkuhle, and M. Wilczek, The Lundgren-Monin-Novikov hierarchy: Kinetic equations for turbulence, Compt. Rend. Phys. **13**, 929 (2012).

[15] M. Wilczek, A. Daitche, and R. Friedrich, On the velocity distribution in homogeneous isotropic turbulence: Correlations and deviations from Gaussianity, J. Fluid Mech. **676**, 191 (2011).

[16] W. D. Thacker, A path integral for turbulence in incompressible fluids, J. Math. Phys. **38**, 300 (1997).

[17] D. Venturi, The numerical approximation of nonlinear functionals and functional differential equations, Phys. Rep. **732**, 1 (2018).

[18] E. Gkioulekas, Remarks on the Frisch framework of hydrodynamic turbulence and the quasi-Lagrangian formulation, arXiv:nlin/0506064v1 (2006).

[19] S. Molchanov, Wiener-Kolmogorov conception of the stochastic organization of nature, in *Proceedings of Symposia in Applied Mathematics* (American Mathematical Society, Providence, 1997), Vol. 52, pp. 1–36.

[20] M. J. Beran, *Statistical Continuum Theories* (Interscience, New York, 1968), Chap. 7.

[21] I. M. Gel'fand and A. M. Yaglom, Integration in functional spaces and its applications in quantum physics, J. Math. Phys. **1**, 48 (1960).

[22] M. D. Donsker and J. L. Lions, Frechet-Volterra variational equations, boundary value problems, and function space integrals, Acta Math. **108**, 147 (1962).

[23] A. Inoue, Strong and classical solutions of the Hopf equation, Tohoku Math. Journ. **39**, 115 (1987).

[24] A. Inoue, A tiny step towards a theory of functional derivative equations: A strong solution of the space-time Hopf equation, *The Navier-Stokes Equations II: Theory and Numerical Methods* (Springer, New York, 1992), pp. 246–261.

[25] A. Inoue, On weak, strong and classical solution of the Hopf equation an example of FDE of second order, Proc. Japan Acad. Ser. A, Mathematical Sciences **62**, 54 (1986).

[26] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions* (Cambridge University, Cambridge, England, 1992).

[27] P.-L. Chow, Stochastic partial differential equations in turbulence related problems, *Probabilistic Analysis and Related Topics* (Elsevier, Amsterdam, 1978), Vol. 1, pp. 1–43.

[28] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes Equations and Turbulence* (Cambridge University, Cambridge, England, 2001).

[29] P.-L. Chow, *Stochastic Partial Differential Equations*, 2nd ed. (Chapman and Hall, London, 2014).
[30] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach* (Cambridge University, Cambridge, England, 2007).
[31] K. O. Friedrichs, T. Seidman, B. Wendroff, H. N. Shapiro, and J. Schwartz, *Integration of Functionals* (New York University, New York, 1957).
[32] G. Birkhoff, J. Bona, and J. Kampé de Fériet, Statistical well-set Cauchy problems, in *Probabilistic Methods in Applied Mathematics*, edited by A. T. Bharucha-Reid (Elsevier, Amsterdam, 1973), Vol. 3.
[33] J. Kampe de Feriet, Random integrals of differential equations, in *Lectures on Modern Mathematics*, edited by T. L. Saaty (John Wiley & Sons, New York, 1965), Vol. 3, pp. 277–321.
[34] J. Kampe de Feriet, Random functions and statistical theory of turbulence, in *Theory of Random Functions*, edited by A. Blanc-Lapierre and R. Fortet, translated by J. Gani (Gordon and Breach, New York, 1968), Vol. 2.
[35] M. I. Freidlin, *Functional Integration and Partial Differential Equations* (Princeton University, Princeton, NJ, 1985).
[36] C. Foias, Statistical study of Navier-Stokes equations I, Rend. Sem. Mat. Univ. Padova **48**, 219 (1972).
[37] C. Foias, Statistical study of Navier-Stokes equations II, Rend. Sem. Mat. Univ. Padova **49**, 9 (1973).
[38] C. Foias, A functional approach to turbulence, Russ. Math. Surv. **29**, 293 (1974).
[39] G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics: Dimensional Analysis and Intermediate Asymptotics* (Cambridge University, Cambridge, England, 1996).
[40] Ya. B. Zel'dovich and Yu. P. Raizer, *Physics of Shock Waves* (Dover, New York, 2002).
[41] G. Rosen, Exact theory for the self-similarity and decay of homogeneous turbulence, J. Math. Phys. **23**, 2582 (1982).
[42] A. L. Dawtowicz, On the existence of an invariant measure for a quasi-linear partial differential equation, Prace Matematyczne **23**, 117 (1982).
[43] V. I. Gishlarkaev, Uniqueness of a solution to the Cauchy problem for the Hopf equation in the two-dimensional case, J. Math. Sci. **169**, 64 (2010).
[44] M. I. Vishik and A. I. Komec, On the solvability of the Cauchy problem for the Hopf equation corresponding to a nonlinear hyperbolic equation, Amer. Math. Soc. translations **118**, 161 (1982).
[45] T. Alankus, An exact representation of the space-time characteristic functional of turbulent Navier-Stokes flows with prescribed random initial states and driving forces, J. Stat. Phys. **54**, 859 (1989).
[46] T. Alankus, The generating functional for the probability density functions of Navier-Stokes turbulence, J. Stat. Phys. **53**, 1261 (1988).
[47] H. H. Shen and A. Wray, Stationary turbulent closure via the Hopf functional equation, J. Stat. Phys. **65**, 33 (1991).
[48] H.-P. Breuer, F. Petruccione, and F. Weber, On a Fourier space master equation for Navier-Stokes turbulence, Z. Phys. B **100**, 461 (1996).
[49] P. Biechele, Stochastische simulation der dreidimensionalen turbulenz, Ph.D. thesis, Albert Ludwigs Universität Freiburg, 2000.
[50] I. Hosokawa and K. Yamamoto, Numerical study of the burgers' model of turbulence based on the characteristic functional formalism, Phys. Fluids **13**, 1683 (1970).
[51] A. Karczewska, Statistical solutions for a stochastic Burger's equation, Statistics and Probability Letters **27**, 305 (1996).
[52] J. Bertoin, The inviscid Burgers equation with Brownian initial velocity, Commun. Math. Phys. **193**, 397 (1998).
[53] L. Carraro and J. Duchon, Equation de Burgers avec conditions initiales a accroissements independants et homogenes, Ann. Inst. Henri Poincare C **15**, 431 (1998).
[54] L. Frachebourg and Ph. A. Martin, Exact statistical properties of the Burgers equation, J. Fluid Mech. **417**, 323 (2000).
[55] M.-L. Chabanol and J. Duchon, Markovian solutions of inviscid burgers equation, J. Stat. Phys. **114**, 525 (2009).
[56] T. Grafke, R. Grauer, and T. Schäfer, Instanton filtering for the stochastic Burgers equation, J. Phys. A **46**, 062002 (2013).
[57] T. Grafke, R. Grauer and T. Schäfer, The instanton method and its numerical implementation in fluid mechanics, J. Phys. A **48**, 333001 (2015).
[58] T. Grafke, R. Grauer, T. Schäfer, and E. Vanden-Eijnden, Relevance of instantons in Burgers turbulence, Europhys. Lett. **109**, 34003 (2015).
[59] Strictly speaking, the second-order functional derivative should be understood as a kind of trace formula

$$\frac{\delta^2\Phi}{\delta\theta(x)\delta\theta(x)} = \lim_{y\to x}\frac{\delta^2\Phi}{\delta\theta(x)\delta\theta(y)}$$

by regarding

$$\frac{\delta^2\Phi}{\delta\theta(x)\delta\theta(x)}\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial x}$$

as a differential form in a distributional sense [23].
[60] H. D. Dahmen, G. Jona-Lasinio, and J. Tarski, The method of characteristics for functional-derivative equations, Il Nuovo Cimento A **10**, 513 (1972).
[61] M. Oberlack and M. Waclawczyk, On the extension of Lie group analysis to functional differential equations, Arch Mech. **58**, 597 (2006).
[62] M. Oberlack and M. Waclawczyk, Application of Lie group analysis to functional differential equations, arXiv:math-ph/0610080 (2006).
[63] Compare with $\exp(a\frac{d}{dx})f(x) = f(x+a)$, where $f(x)$ is a smooth function and a is a constant.
[64] Using the heat kernel

$$g_t = \frac{1}{(4\pi\nu t)^{3/2}}\exp\left(-\frac{|\boldsymbol{x}|^2}{4\nu t}\right),$$

we apply the Duhamel principle to the Navier-Stokes equations to obtain

$$\boldsymbol{u}(t) = g_t * \boldsymbol{u}_0 - \int_0^t g_{t-s} * \mathbb{P}\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})(\cdot,s)ds,$$

where $\mathbb{P} = \boldsymbol{I} - \nabla\triangle^{-1}\nabla\cdot$ denotes solenoidal projection and $*$ denotes convolution.
[65] F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Dover, New York, 1990).
[66] V. P. Maslov, *Théorie des Perturbations et Méthodes Asymptotiques* (Gauthier-Villars, Paris, 1972).
[67] K. Ohkitani, Near-invariance under dynamic scaling for the Navier-Stokes equations in critical spaces: A probabilistic approach to regularity problems, J. Phys. A **50**, 045501 (2017).

[68] It is a counterpart to a fundamental solution to linear problems, e.g., the heat equation.
[69] P. Biler, G. Karch, and W. A. Woyczyński, Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws, Ann. Inst. Henri Poincare **18**, 613 (2001).
[70] M. Escobedo and E. Zuazua, Large time behavior for convection-diffusion equations in R^N, J. Func. Anal. **100**, 119 (1991).
[71] M. Escobedo, J. L. Vazquez, and E. Zuazua, A diffusion-convection equation in several space dimensions, Indiana Univ. Math. J. **42**, 1413 (1993).
[72] T.-P. Liu and M. Pierre, Source-solutions and asymptotic behavior in conservation laws, J. Diff. Eq. **51**, 419 (1984).
[73] This means that the self-dual property of $\tilde{g}_\tau$ in scaled variables is restored after taking the $\tau \to \infty$ limit.
[74] M. Cannone and F. Planchon, Self-similar solutions for Navier-Stokes equations in $\mathbb{R}^3$, Commun. in P. D. E. **21**, 179 (1996).
[75] Y. Giga and T. Miyakawa, Navier-Stokes flow in R3 with measures as initial vorticity and Morrey spaces, Commun. in P. D. E. **14**, 577 (1989).
[76] M.-H. Giga, Y. Giga, and J. Saal, *Nonlinear Partial Differential Equations* (Birkhäuser, Boston, 2010).
[77] D. Chae and J. Wolf, Existence of discretely self-similar solutions to the Navier–Stokes equations for initial value in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, Ann. Inst. Henri Poincare **35**, 1019 (2018).
[78] Z. Bradshaw and T.-P. Tsai, Forward discretely self-similar solutions of the Navier-Stokes equations II, Ann. Inst. Henri Poincare **18**, 1095 (2017).
[79] In the original Burgers vortex the linear straining has been introduced to represent stretching by other vortices. Here the linear strain term appears as a result of dynamic scaling. The equations are identical.
[80] T. Gallay and C. E. Wayne, Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$, Arch. Rat. Mech. Anal. **163**, 209 (2002).

Correction: Section V contained text and mathematical errors and has been fixed. Specifically, in the first paragraph of Sec. V A, sentence 3 has been modified and sentence 4 deleted. In the second paragraph, sentences 1 and 3 have been modified. In Sec. V B, sentence 6 has been modified.