Active uniaxially ordered suspensions on disordered substrates

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Multiple experiments on active systems consider oriented active suspensions on substrates or in chambers tightly confined along one direction. The theories of polar and apolar phases in such geometries were considered in A. Maitra *et al.* [Phys. Rev. Lett. **124**, 028002 (2020)] and A. Maitra *et al.* [Proc. Natl. Acad. Sci. USA **115**, 6934 (2018)], respectively. However, the presence of quenched random disorder due to the substrate cannot be completely eliminated in many experimental contexts possibly masking the predictions from those theories. In this paper, I consider the effect of quenched orientational disorder on the phase behavior of both polar and apolar suspensions on substrates. I show that polar suspensions have long-range order in two dimensions with anomalous number fluctuations, while their apolar counterparts have only short-range order, albeit with a correlation length that can increase with activity, and even more violent number fluctuations than active nematics without quenched disorder. These results should be of value in interpreting experiments on active suspensions on substrates with random disorder.

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I. INTRODUCTION

Biological systems and their artificial analogs convert chemical energy into mechanical work at the scale of *individual* constituents [1,2]. This breaking of detailed balance at the microscopic scale results in macroscopic stresses and currents [3–7] that have been shown to singularly modify the phase behavior of such "active systems" relative to their passive, equilibrium counterparts. This includes the prediction of long-range order in two dimensions for motile XY spins on a substrate [8–11], the generic destruction of apolar phases in infinite, momentum-conserved systems in any dimension [3,12,13], and the anomalous scaling of number fluctuations in a region which, being larger than $\sqrt{N}$, where N is the mean number in that region, violates the law of large numbers [14,15]. These predictions have been confirmed in experiments [3,16] and have been shown to be responsible for various biological phenomena ranging from flocking of birds [17] to spatiotemporally chaotic flows in bacterial fluids [18] to crawling of cell layers [19] to spontaneous rotation of the cellular nucleus [20].

Most experimentally relevant biological active systems and many of their artificial counterparts should be viewed as a suspension of active particles in a fluid medium. Furthermore, many such experiments are conducted in systems which are confined between a substrate and a cover slip. Boundary conditions fundamentally change the behaviors of active systems [20,21]. The theories of homogeneous active polar and apolar suspensions on substrates or in confined channels were considered in Refs. [22] and [23], respectively. There it was shown that, contrary to naive expectations, polar fluids in confined channels have long-range order due to a singular suppression of polarization fluctuations due to the long-range effect of fluid incompressibility and have normal (i.e., obeying law of large numbers) number fluctuations [22]. It was further demonstrated in Ref. [23] that the effective elastic modulus of active apolar suspension not only need not become negative (signalling an instability) at any activity but could also *increase* with activity. An apolar suspension, unlike its polar counterpart has only quasi-long-range order and has giant number fluctuations. However, often in experiments, the substrates and the cover slips are not perfectly uniform but, due to imperfections, may have local, quenched anisotropies. Such local anisotropies provide a local "easy direction" for the orientation of the active particles. Therefore, it is important to understand how the conclusions regarding order and number fluctuations are modified in the presence of such orientational quenched random-field disorder. It has been known that systems that break a continuous symmetry in the presence of quenched, random fields cannot have long-range order below four dimensions in equilibrium [24]. This can be understood by employing a classic argument due to Y. Imry and S-K. Ma [24]: For random field disorder, the average value of the random field vanishes, but its mean-squared fluctuations in region of size L scales as $\sim L^d$, where d is the dimension. Therefore, an ordered state gains bulk energy $\mathcal{O}(L^{d/2})$ by breaking up into domains. In all systems that break a continuous symmetry, the energy required to form a domain wall between two domains of size L scales as $\sim L^{d-2}$. Therefore, when $L^{d/2} \geqslant L^{d-2}$, i.e., $d \leqslant 4$, the ordered state must break up into domains of finite size, destroying the ordered state for arbitrarily small disorder strength. Of course, a parallel energetic argument is not available for orientational ordering in *active* systems in the presence of quenched disorder and, in particular, the conclusion that no ordered state exists below four dimensions may be modified. The effect of quenched random-field disorder has earlier been examined in active motile systems, but *without* a suspending fluid medium, by Refs. [25–27], where it was shown that it

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may be possible for such systems to have quasi-long-range order even in *two* dimensions contrary to expectations borne out of the Imry-Ma argument and with a distinct form of the disorder statistics by Ref. [28] (both analytically and experimentally) and Refs. [29,30] (numerically). The effect of disorder has also been considered for the dynamics of run-and-tumble particles in one dimension [31].

In this paper, I consider the experimentally relevant case of polar and apolar suspension on substrates (or confined in narrow channels) in the presence of random, quenched orientational disorder. I demonstrate that, within a linearized theory, polar suspensions on substrates have *long-range order* in *two dimensions* due to the stabilization of orientational fluctuations caused by the nonlocal constraint of fluid incompressibility. Unlike their counterparts with only annealed noise, however [22], the number fluctuations are predicted to be singularly larger than in passive systems and to violate the law of large numbers. In contrast, apolar fluids, which are not motile on average, do not even have quasi-long-range order in two dimensions—thus, a monodomain nematic phase is not possible even in the presence of infinitesimal random quenched disorder beyond an active Imry-Ma length scale. However, this length scale is itself predicted to grow with activity, signifying that apolar suspensions that are not destabilized at high activities can have large Imry-Ma scales and therefore, at moderate disorder strengths, a monodomain nematic phase may be observed in experiments. Further, I show that number fluctuations in regions much smaller than the Imry-Ma length, i.e., at scales where a monodomain nematic exists, are *enormous*, much larger than the corresponding system with annealed noise. I now demonstrate how I obtain these results in detail, first considering a polar suspension with quenched disorder and then an apolar one.

II. POLAR SUSPENSIONS ON DISORDERED SUBSTRATES

The equations of motion of a polar suspension on a substrate in the *absence* of quenched disorder have been considered in detail in Ref. [22]. I adapt these equations to describe a situation in which the presence of impurities on the substrate create local "easy directions" for the polarization field of the suspension $\mathbf{p}$ which acts as a quenched noise for $\mathbf{p}$. The concentration of the active particles is denoted by c and the center-of mass velocity of the fluid and the active particles by $\mathbf{u}$. The joint density of the fluid and the active particles together is incompressible leading to the constraint $\nabla\cdot\mathbf{u}=0$. The dynamical equations are

$$\begin{aligned}\partial_t\mathbf{p}+\mathbf{u}\cdot\nabla\mathbf{p}+\lambda_a\mathbf{p}\cdot\nabla\mathbf{p}-\boldsymbol{\omega}\times\mathbf{p}\\ =\Lambda\mathbf{u}-\lambda\mathbf{p}\cdot\mathbf{U}-\frac{\delta\mathcal{H}}{\delta\mathbf{p}}+\boldsymbol{\xi}_Q,\end{aligned}\tag{1}$$

where $\mathbf{U}$ is the symmetric part of $\nabla\mathbf{u}$, $\boldsymbol{\omega}=(1/2)\nabla\times\mathbf{u}$, λ_a denotes the strength of active self-advection, and λ is the flow-alignment parameter. The Λ term describes the fact that on a substrate, polarization responds to the local velocity and not only its gradient [2,22,32,33]. The free energy $\mathcal{H}$, which would have controlled the statics and dynamics in the absence of activity is

$$\mathcal{H}=\int_{\mathbf{x}}\left[\frac{\alpha(c)}{2}|\mathbf{p}|^2+\frac{\beta}{4}|\mathbf{p}|^4+\frac{K}{2}|\nabla\mathbf{p}|^2+\gamma\mathbf{p}\cdot\nabla c+c\ln c\right],\tag{2}$$

where the first two terms would describe the ordering transition to a polar state in a passive system, the elastic modulus K suppresses the heterogeneities of the polarization, the γ-term describes the tendency of the polarity to align along or opposite to concentration gradients [34], while the last term is characteristic of an ideal solution (setting $k_BT=1$). Finally, $\boldsymbol{\xi}_Q$ denotes a quenched random field whose statistics I take to be Gaussian [25–27]:

$$\langle\xi_{Q_i}(\mathbf{x})\xi_{Q_j}(\mathbf{x}')\rangle=T_Q\delta_{ij}\delta(\mathbf{x}-\mathbf{x}'),\tag{3}$$

where $\mathbf{x}$ is the spatial coordinate, T_Q is the strength of the disorder, and the angular brackets imply an average over the quenched disorder. The equation for the velocity field on a substrate reads

$$\begin{aligned}\Gamma\mathbf{u}=\upsilon\mathbf{p}-\nabla\Pi+\zeta_1\mathbf{p}\cdot\nabla\mathbf{p}+\zeta_2\mathbf{p}\nabla\cdot\mathbf{p}\\ -\Lambda\frac{\delta\mathcal{H}}{\delta\mathbf{p}}-\lambda\nabla\cdot\left[\mathbf{p}\frac{\delta\mathcal{H}}{\delta\mathbf{p}}\right]^S-\nabla\cdot\left[\mathbf{p}\frac{\delta\mathcal{H}}{\delta\mathbf{p}}\right]^A,\end{aligned}\tag{4}$$

where Π enforces the incompressibility condition $\nabla\cdot\mathbf{u}=0$ and superscripts S and A denote symmetric and antisymmetric parts of a tensor. $\upsilon\mathbf{p}$ describes the active motility and ζ_1 and ζ_2 are active terms at the next order in gradients [23]. The remaining terms are required by Onsager symmetry to obtain an equilibrium system in the limit $\upsilon=\zeta_1=\zeta_2=\lambda_a=0$. I have not considered active and passive higher order in gradients contributions to the force-density explicitly since they are subdominant at long wavelengths to the terms in (4) and (1). The equation for the concentration fluctuations, $\delta c=c-c_0$, where c_0 is the mean concentration of active particles, is

$$\partial_t\delta c+\mathbf{u}\cdot\nabla\delta c=-\nabla\cdot(\upsilon_p c\mathbf{p}-D_c\nabla\delta c),\tag{5}$$

to lowest order in fields and gradients, where the term with the coefficient υ_p describes an active current proportional to the polarization and D_c is the isotropic diffusivity. I have ignored a further active contribution to the current, allowed in active uniaxial systems on substrates, $\propto(\mathbf{pp}\cdot\mathbf{u})$ since it can be shown to not qualitatively affect the fluctuation spectrum of a stable polar phase. The impurity field is assumed to *only* affect the local polarization. There are additional *annealed* noises in these equations, but in this paper I will only consider *equal-time* correlations whose dominant contribution arises from the quenched component (the equal-time correlations in the absence of quenched noise but with annealed noise was calculated in Ref. [22]).

As shown in Ref. [22], for $\tilde{\alpha}=\alpha(c_0)-\Lambda\upsilon/(\Gamma+\Lambda^2)=\alpha-w<0$, the disordered state (with $p_0=0$) is unstable and an ordered state with $p_0^2=\sqrt{\tilde{\alpha}/\beta}$ sets in. I take the direction of ordering to be along $\hat{x}$. The uniform fluid velocity $\mathbf{u}_0=(w/\Lambda)p_0\hat{x}$ in this state [22]. I now consider the statistics of small fluctuations of the polarization $\delta\mathbf{p}=(p_0+\delta p)(\cos\theta\hat{x}+\sin\theta\hat{y})-p_0\hat{x}$ and concentration δc about this state in the presence of quenched disorder. Since it was shown in Ref. [22] that a stable aligned phase is only possible for

$w > 0$, this is the case I will consider. Fourier transforming in space and eliminating the incompressible velocity field and the fast δp field, I obtain the coupled, linearized equation for θ_q and δc_q. Only retaining up to $\mathcal{O}(q)$ terms in the θ_q equation (since, this will be sufficient for calculating the static structure factor in the limit of small wave vectors), I obtain

$$\partial_t \theta_q = -i\frac{\gamma}{p_0} q_y \delta c - i\Lambda_a q_x \theta - w\frac{q_y^2}{q^2}\theta_q + \frac{1}{p_0}\xi_Q, \qquad (6)$$

where the expression for Λ_a is given in the Appendix and ξ_Q is the projection of $\boldsymbol{\xi}_Q$ transverse to $\mathbf{p}$. The concentration equation is

$$\partial_t \delta c_q = -D_c q^2 \delta c - i v_p c_0 p_0 q_y \theta - i\Big(v_p + \frac{w}{\Lambda}\Big) p_0 q_x \delta c. \quad (7)$$

The steady-state solution of (6) and (7) are simply obtained by setting $\partial_t \theta_q = \partial_t \delta c_q = 0$. The disorder-averaged correlation function for angular fluctuation can be calculated straightforwardly from the solution of the steady-state equations, which in the limit of small wave vectors is

$$\langle|\theta_q|^2\rangle = \frac{T_Q}{p_0^2\big[\big(q_y^4/q^4\big)w^2 + q_x^2\Lambda_a^2\big]} \approx \frac{T_Q q^4}{p_0^2\big(q_y^4 w^2 + q_x^6\Lambda_a^2\big)}. \qquad (8)$$

This implies that $\langle|\theta_q|^2\rangle \sim q^0$ for most directions of the wave vector space but diverge as $\sim 1/q^2$ for $q_y^4 \lesssim q_x^6$, i.e., the fluctuations are strongest for wave vectors $q_y^4 \sim q_x^6 \Rightarrow q_y \sim q_x^{3/2}$. Note that this is somewhat different from the equal-time correlator in the presence of *only* annealed noise [22]: $\sim q_x^2/(wq_y^2 + Kq_x^4)$. However, it is clear that the real space angular fluctuations $\langle\theta(\mathbf{x})^2\rangle = \int d^2q/(2\pi)^2\langle|\theta_q|^2\rangle$ converges in the infrared, $\sim 1/\sqrt{L}$, as the system size $L \to \infty$ (again, in contrast to the case with annealed noise where it scaled as $\sim 1/L$), signifying the presence of long-range order in two dimensions. More formally, by examining the behavior of the equal-time correlator under the transformation $x \to bx$, $y \to b^\zeta y$, $t \to b^z t$, $\theta \to b^\chi \theta$, I can calculate the linear roughness χ, anisotropy ζ, and dynamical z exponents. Since the linear fluctuations are governed by Λ_a (which can be replaced by its isotropic value for this argument), w, and T_Q, choosing exponent values such that these quantities remain invariant under the rescaling transformation will ensure that $\langle|\theta_q|^2\rangle$ remains unchanged. Under rescaling, $\Lambda_a \to b^{z-1}\Lambda_a$, $w \to b^{z-2\zeta+2}w$, and $T_Q \to b^{(2z-2\chi-\zeta-1)}T_Q$, which implies that $z = 1$, $\zeta = 3/2$, and $\chi = -1/4$. The linear anisotropy exponent is consistent with the fact that the fluctuations are strongest in the regime $q_y \sim q_x^{3/2}$ and the roughness exponent implies that the real space angular fluctuations $\langle\theta(\mathbf{x})^2\rangle \sim L^{2\chi}$ decay as $1/\sqrt{L}$ for $L \to \infty$, as discussed earlier. The negativity of the roughness exponent thus implies that the polar, moving state has long-range order despite the presence of disorder.

To calculate the number fluctuations, I now consider the concentration fluctuations

$$\langle|\delta c_q|^2\rangle = \frac{T_Q c_0^2 q_y^2 v_p^2 \Lambda^2}{-2c_0 p_0 q_x^2 q_y^2 v_p \gamma \Lambda (w + v_p\Lambda)\Lambda_a + p_0^2 q_x^2 (w + v_p\Lambda)^2\big[\big(q_y^4/q^4\big)w^2 + q_x^2\Lambda_a^2\big] + q_y^4\Lambda^2(D_c w + c_0 v_p\gamma)^2}. \qquad (9)$$

This goes as q^0 for most directions of the wave-vector space. However, for $q_x \lesssim q_y^2$ (i.e., around $\mathbf{q} \approx q_y\hat{y}$), it diverges as $\sim 1/q_y^2$ and has another peak at around $q_y^2 \sim q_x^3$, where it diverges as $1/q_x$. The strongest divergence is for wave vectors $\mathbf{q} \approx q_y\hat{y}$ whose contribution must dominate the integral of $\langle|\delta c_q|^2\rangle$ over all wave vectors. In the direction $q_x \lesssim q_y^2$, concentration is the only hydrodynamic field, with a linear diffusive equation of motion and driven by a conserving but quenched noise. In this regime, the angular fluctuations that enter the concentration equation (7) can be simply replaced by $\theta = \xi_Q/p_0 w - (i\gamma/p_0 w)q_y\delta c$. The first leads to the quenched conserved noise, while the second just modifies the diffusivity, implying a $1/q^2$ divergence of the static structure factor in this regime of the wave-vector space. For $q \approx q_y$, $\langle|\delta c_q|^2\rangle$ may be approximated as

$$\langle|\delta c_q|^2\rangle \approx \frac{T_Q c_0^2 v_p^2 \Lambda^2}{p_0^2\delta\phi^2(w + v_p\Lambda)^2 w + q^2\Lambda^2(D_c w + c_0 v_p\gamma)^2}, \qquad (10)$$

where $\delta\phi$ is the angle between $\mathbf{q}$ and $\hat{y}$ (note the unusual definition). Now, performing an integral over the angular variable by noting that this integral will be dominated by the contribution around $\delta\phi \approx 0$ and therefore the range of the integral can be extended to infinity, I obtain $\langle|\delta c_q|^2\rangle \sim 1/q$. This $1/q$ divergence of the static structure factor of concentration fluctuations implies that the rms number fluctuations $\sqrt{\langle\delta N^2\rangle}$ in a region of size ℓ scales as $\langle N\rangle^{3/4}$, where $\langle N\rangle$ is the mean number of particles in the region, instead of $\sqrt{\langle N\rangle}$, as it would in equilibrium systems not at a critical point [3]. Further, such anomalous number fluctuation are also predicted to be *absent* in polar suspensions in the *absence* of quenched disorder [22], where $\sqrt{\langle\delta N^2\rangle}$ scales as $\sqrt{\langle N\rangle}$ as in passive systems. Furthermore, the concentration dynamics is unaffected by any nonlinearity in this regime because the allowed nonlinearities $\partial_x\delta c^2 \ll \partial_y^2\delta c^2$, since this is the regime of the wave-vector space under consideration, and $\partial_y^2\delta c^2 \ll \partial_y^2\delta c$, since a $1/q$ divergence of the concentration static structure factor implies that the real-space concentration fluctuations vanish at large scales as $1/L$.

The conclusions reached on the basis of the linear theory may, however, be modified by relevant nonlinearities. I have already argued above that the dynamics is essentially linear for $\mathbf{q} \approx q_y\hat{y}$. In the regime $q_y \sim q_x^{3/2}$, where angular fluctuations are the largest and concentration fluctuations are also divergent at small q, albeit not as strongly as along $\hat{y}$, the size of linear concentration fluctuations is determined by v_p (the choice of the linear dynamical, anisotropy and roughness exponents fixes the other quantities). Therefore, additionally rescaling $\delta c \to b^{\chi_c}\delta c$ and demanding that the concentration fluctuations remain invariant implies that $v_p \to b^{(z-\chi_c+\chi-\zeta)}v_p$

must be held fixed. This yields the linear exponent $\chi_c = -3/4$ (χ_c is smaller than $-1/2$—the value which would have been expected based on $\int_{\mathbf{q}} \langle |\delta c_q|^2 \rangle \sim 1/L$—because, for $q_y \sim q_x^{3/2}$, the small wave-vector divergence of $\langle |\delta c_q|^2 \rangle$ is weaker than for $q \approx q_y$). As in the corresponding annealed noise problem [22,35] and as discussed in the Appendix, the nonlinearities in the θ_q equation have the forms $(q_y/q_x)(\theta^2)_q$, $(\theta^3)_q$ (both with coefficient w), $q_y(\theta^2)_q$, $\delta c_{q-k} k_x \theta_k$, and $q_y(\delta c^2)_q$, while those in the concentration equation have the form $q_x(\delta c^2)_q$, $q_y \theta_{q-k} \delta c_k$, and $q_x(\theta^2)_q$. Using the linear exponents obtained above, I find that the only relevant nonlinearities in the θ_q equation are $(q_y/q_x)(\theta^2)_q$ and $(\theta^3)_q$, both of which must rescale the same way as the linear $(q_y^2/q^2)\theta_q$ term, while concentration equation has only one relevant nonlinearity $q_x(\theta^2)_q$. In the annealed noise model, if one ignored the concentration fluctuations, one could obtain the exact static exponents [22,35] through a mapping to a two-dimensional smectic and, ultimately, to a $1+1$-dimensional KPZ equation [36,37]. However, an equivalent mapping is not available here. Nevertheless, these nonlinearities must change the linear exponents discussed earlier. This can be seen by a simple argument: Within the linear theory, the nonlinearities $(q_y/q_x)(\theta^2)_q$ and $(\theta^3)_q$ grow, while the linear term $(q_y^2/q^2)\theta_q$ does not. However, due to rotation invariance, the coefficient of all three terms must rescale the same way (which implies $\chi = 1 - \zeta$), implying that the exponents calculated above have to be modified by fluctuations. This parallels the situation with only annealed noise [22,35], where the linear exponents ($\chi = -1/2$, $\zeta = 2$) do not satisfy this relationship, but the exact exponents ($\chi = -1/2$, which remains unchanged and $\zeta = 3/2$) do. While I have not been able to calculate the exact exponents for this model, it is reasonable to assume that nonlinearities will not destroy the long-range order in this system with quenched disorder as well. If that is the case, the concentration fluctuations are bound to remain anomalous, as predicted by the linear theory.

III. APOLAR SUSPENSIONS ON DISORDERED SUBSTRATES

I now contrast the behavior of polar suspensions on substrates with quenched disorder with apolar suspensions. The deterministic parts of the equations of motion for the apolar suspension are the equivalent to those in Ref. [23]. I denote angular fluctuations about a homogeneous state, with concentration c_0, ordered nematically along the $\hat{x}$, by the angle field $\theta(\mathbf{x})$, the concentration fluctuations by $\delta c = c - c_0$, and an overdamped velocity field by $\mathbf{u}$. As in the polar model, I ignore time-dependent, annealed fluctuations since their contribution to the static correlations are subdominant to the quenched disorder considered here. The equation of motion for angular fluctuations is

$$\dot{\theta} = \frac{1-\lambda}{2}\partial_x u_y - \frac{1+\lambda}{2}\partial_y u_x - \Gamma_\theta \frac{\delta \mathcal{H}}{\delta \theta} + \xi_Q, \tag{11}$$

where $|\lambda| > 1$ describes particles with a tendency to align under a shear flow, ξ_Q is a quenched orientational disorder which has the correlation $\langle \xi_Q(\mathbf{x}) \xi_Q(\mathbf{x}') \rangle = T_Q \delta(\mathbf{x} - \mathbf{x}')$ as in the polar system, and

$$\mathcal{H} = \int d^2\mathbf{r} \left[\frac{K}{2}(\nabla\theta)^2 + \gamma c \partial_x \partial_y \theta + c \ln c \right], \tag{12}$$

where $K > 0$ characterizes the tendency of the particles to align and γ denotes a symmetry allowed coupling between the concentration and orientation fields. The adsorbed equation for the flow velocity is

$$\Gamma \mathbf{u} = -\nabla \Pi + \mathbf{f}^p + \mathbf{f}^a, \tag{13}$$

where Γ is the friction coefficient against the substrate and the pressure Π serves as a Lagrange multiplier enforcing the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ for the suspension as a whole. Onsager symmetry and Eq. (11) yield the density of passive forces

$$\mathbf{f}^p = -\frac{1+\lambda}{2}\partial_y \left(\frac{\delta \mathcal{H}}{\delta \theta} \right) \hat{\mathbf{x}} + \frac{1-\lambda}{2}\partial_x \left(\frac{\delta \mathcal{H}}{\delta \theta} \right) \hat{\mathbf{y}}, \tag{14}$$

while the active force density $\mathbf{f}^a$, to lowest order in gradients, is

$$\mathbf{f}^a = -\left(\zeta_1^Q \Delta\mu + \zeta_2^Q \Delta\mu\right)\partial_y \theta \hat{x} - \left(\zeta_1^Q \Delta\mu - \zeta_2^Q \Delta\mu\right)\partial_x \theta \hat{y}, \tag{15}$$

where ζ_1^Q and ζ_2^Q are two independent phenomenological constants and $\Delta\mu$ denotes the strength of the overall activity in the system [23]. The evolution of the concentration fluctuations δc is governed by a conservation equation,

$$\partial_t \delta c = D_c \nabla^2 \frac{\delta \mathcal{H}}{\delta c} + \zeta_c \Delta\mu \partial_x \partial_y \theta = D_c \nabla^2 \delta c + \zeta_c \Delta\mu \partial_x \partial_y \theta, \tag{16}$$

where the active term $\zeta_c \Delta\mu$ couples orientation fluctuations with concentration fluctuations and is a feature of active nematics [3,14]. On eliminating the fluid velocity $\mathbf{u}$, the Fourier-transformed equation for angular fluctuations become

$$\partial_t \theta_q = -q^2 \left[\frac{\Delta\mu \{q^2 - \lambda(q_x^2 - q_y^2)\}\{\zeta_2^Q q^2 - \zeta_1^Q (q_x^2 - q_y^2)\}}{2\Gamma q^4} + \Gamma_\theta K \right] \theta_q + q_x q_y \Gamma_\theta \gamma \delta c_q + \xi_Q = -q^2 \bar{K} \theta_q + q_x q_y \Gamma_\theta \gamma \delta c_q + \xi_Q, \tag{17}$$

where $\bar{K}$ is the nematic stiffness renormalized by activity. It was shown in Ref. [23] that $\bar{K}$ can be positive even at arbitrarily high $\Delta\mu$ when $\zeta_2 > |\zeta_1|$ and $|\lambda| < 1$ [23]. A negative $\bar{K}$ would signal a linear instability of the ordered state. Here I concentrate on $\bar{K} > 0$. The static correlators for angular and concentration fluctuations are calculated from (17) and the Fourier-transformed version of (16) after setting $\partial_t \theta = \partial_t \delta c = 0$ and averaging over the quenched noise. The static structure

factor of angular correlations, averaged over the quenched disorder is

$$\langle|\theta_q|^2\rangle = \frac{T_Q D_c^2 q^4}{\left(D_c \bar{K} q^4 + q_x^2 q_y^2 \gamma \Gamma_\theta \Delta\mu \zeta_c\right)^2} \propto \frac{1}{q^4}. \quad (18)$$

This implies that the angular fluctuations diverge at least as strongly as $1/q^4$ in *all* directions of the wave-vector space. If $\gamma\zeta_c < 0$, then there is a possibility that along some special directions the denominator goes to 0, but in that case higher order in wave vector terms have to be retained in the equations of motion and the divergence of angular fluctuations at small wave vectors would be *stronger* than $1/q^4$. This implies that the real space angular correlation function diverges strongly (as L^2) with the system size L, implying that nematic order is impossible on disordered substrates even for active suspensions (i.e., they only have short-ranged order) just like their passive counterparts. However, the active analog of the Imry-Ma length scale $\xi_{\rm IM}$ scales with the activity $\Delta\mu$, for large activity. To see this, note (as pointed out in Ref. [23]) that for large $\Delta\mu$, the active term in $\bar{K}$, $\propto \Delta\mu$ dominates the passive one $\propto \Gamma_\theta K$, $\bar{K} \sim \Delta\mu$. Thus, the denominator of (18) $\sim \Delta\mu^2$. The rms angular fluctuations can be calculated as

$$\langle\theta(\mathbf{x})^2\rangle = \int_{|\mathbf{q}|\geqslant 1/L} \frac{d\mathbf{q}}{4\pi^2} \langle|\theta_q|^2\rangle \sim \frac{T_Q}{\Delta\mu^2} L^2, \quad (19)$$

where L is the system size. Setting the distortion to be $\mathcal{O}(1)$, I obtain the Imry-Ma length scale $\xi_{\rm IM} \sim \Delta\mu/\sqrt{T_Q}$. This implies that the domain size beyond which the active nematic (which is linearly stable) loses order *increases* with increasing activity, which is another manifestation of active stabilization discussed in Ref. [23]. Thus, it is possible to experimentally tune activity to obtain arbitrarily large nematic domains.

The concentration fluctuations also scale as $\sim 1/q^4$ in all directions for length scales below $\xi_{\rm IM}$ (above this scale, the nematic state itself is destroyed and a description in terms of the angle field is invalidated):

$$\langle|\delta c_q|^2\rangle = \frac{T_Q q_x^2 q_y^2 \Delta\mu^2 \zeta_c^2}{\left(D_c \bar{K} q^4 + q_x^2 q_y^2 \gamma \Gamma_\theta \Delta\mu \zeta_c\right)^2}. \quad (20)$$

This can be used to calculate the mean-squared number fluctuations $\langle\delta N^2\rangle = \langle N^2\rangle - \langle N\rangle^2$ in a box of size $\ell < \xi_{\rm IM} \langle\delta N^2\rangle = \int_{\ell\times\ell} d^2r d^2r' \langle\delta c(\mathbf{r})\delta c(\mathbf{r}')\rangle$. Since $\langle\delta c(\mathbf{r})\delta c(\mathbf{r}')\rangle \sim |\mathbf{r}-\mathbf{r}'|^2$ in all directions, below the Imry-Ma length scale $\langle\delta N^2\rangle \sim \ell^{2+4} \Rightarrow \sqrt{\langle\delta N^2\rangle} \sim \ell^3$. Since $\langle N\rangle \sim \ell^2$, this implies $\langle\delta N^2\rangle \sim \langle N\rangle^{3/2}$. This implies that the number fluctuations in active nematic suspensions on a dirty substrate, in a region in which the nematic is ordered (i.e., in a region smaller than the Imry-Ma length scale), are expected to be even larger than in active nematics on a clean substrate. Beyond the Imry-Ma scale, the nematic order itself is destroyed and the number fluctuations should be normal and obey the law of large numbers.

IV. CONCLUSIONS

In this paper, I have demonstrated that active polar fluids can have long-range order with anomalous number fluctuations even in the presence of weak random-field disorder. The presence of long-range order in a two-dimensional system with disorder is a further consequence of the active Anderson-Higgs mechanism [38,39] discussed in detail in Ref. [22] where it was shown that due to the interplay of activity and the constraint of incompressibility, the Nambu-Goldstone mode [40,41] associated with the broken rotation symmetry becomes gapped, with the gap vanishing only precisely along the ordering direction. The calculation here suggests that the long-range ordered state should be observed in biological experiments, such as in crawling cell layers [19] in which a certain amount of disorder may be expected. Furthermore, the theory described here should be quantitatively testable even in weakly compressible systems such as the one composed of motile rods in a bath of immotile beads [32] since, as shown in Ref. [22], such systems behave as incompressible two-dimensional polar suspensions up to large length scales. In this artificial system, the strength of quenched disorder may be tuned by fixing a subset of the polar particles in space. It is interesting to note that in Ref. [42] anomalous number fluctuations are observed in a polar suspension with long-range order, which could, perhaps indicate the presence of quenched disorder. However, the presence of Toner-Tu waves [8] in that experiment, which should be damped in the kind of incompressible system I consider here, probably indicates that the constraint of incompressibility is not experimentally relevant there due to the large chamber thickness and, instead it should be modelled as a *compressible* polar flock. However, a more strongly confined variant of Ref. [28] should observe the effects I predict.

The theory for the apolar suspensions on substrates should also be testable in a variety of biological and nonbiological experiments. First, though I constructed a theory of apolar suspensions, the orientational and density fluctuations should have the same form in *dry* active nematics without a suspending fluid medium, unlike in the polar system. In this context, the prediction of increasing Imry-Ma scale with activity as well as number fluctuations which are *even larger* than in disorder-free active nematics can be tested by introducing a low density of fixed, randomly oriented apolar rods in the experiment described in Ref. [43]. Turning now to experiments on bacteria, a highly ordered nematic state was observed in a suspension of *Escherichia coli* [44]. Introduction of disorder would imply that the quasi-long-range nematic state observed in that experiment would become short range with a Imry-Ma scale that increases with bacterial swimming speed, which should be proportional to activity. Finally, the predictions of this paper can also be tested in experiments on living liquid crystal, composed of swimming bacteria in a passive, ordered nematic fluid [45] by introducing a small degree of quenched disorder in the passive nematic.

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APPENDIX : POLAR SUSPENSION WITH QUENCHED DISORDER DETAILS

As discussed in the main text, the Fourier transformed linear equations of motion for a polar suspension in the

presence of disorder are

$$\partial_t\theta_q = -i\frac{\gamma}{p_0}q_y\delta c - i\Lambda_a q_x\theta - w\frac{q_y^2}{q^2}\theta_q + \frac{1}{p_0}\xi_Q, \tag{A1}$$

where

$$\Lambda_a = \Big(\lambda_a + \frac{w}{\Lambda}\Big)p_0 - \frac{wp_0}{2\Lambda}\left(1 - \lambda\frac{q_x^2 - q_y^2}{q^2}\right) - \frac{\Lambda p_0^2}{\Gamma}\left[\left(\zeta_1 + w\frac{1-\lambda}{2}\right)\frac{q_x^2}{q^2} - \left(\zeta_2 - w\frac{1+\lambda}{2}\right)\frac{q_y^2}{q^2}\right] \tag{A2}$$

and ξ_Q is the projection of $\boldsymbol{\xi}_Q$ transverse to $\mathbf{p}$. Only the value of Λ_a for $q \approx q_x$ affects the equal-time angular correlator in the limit of small wave vectors. We assume that $\Lambda_a|_{q=q_x} > 0$ in this paper. The concentration equation is

$$\partial_t\delta c_q = -D_c q^2\delta c - iv_p c_0 p_0 q_y\theta - i\Big(v_p + \frac{w}{\Lambda}\Big)p_0 q_x\delta c. \tag{A3}$$

The steady-state solution of (A1) and (A3) are simply obtained by setting $\partial_t\theta_q = \partial_t\delta c_q = 0$ and solving the matrix equation

$$\begin{bmatrix} i\Lambda_a q_x + w\frac{q_y^2}{q^2} & i\frac{\gamma}{p_0}q_y \\ iv_p c_0 p_0 q_y & i\big(v_p + \frac{w}{\Lambda}\big)p_0 q_x + D_c q^2 \end{bmatrix}\begin{pmatrix}\theta_q \\ \delta c_q\end{pmatrix} = \begin{pmatrix}\frac{1}{p_0}\xi_Q \\ 0\end{pmatrix}. \tag{A4}$$

The solution of this equation then straightforwardly yields the disorder-averaged correlation functions for θ_q:

$$\langle|\theta_q|^2\rangle = \frac{T_Q q^4\big[D_c^2 q^4\Lambda^2 + p_0^2 q_x^2(w + v_p\Lambda)^2\big]}{-2c_0p_0q^4q_x^2q_y^2v_p\gamma\Lambda(w + v_p\Lambda)\Lambda_a + p_0^2q_x^2(w + v_p\Lambda)^2\big(q_y^4w^2 + q^4q_x^2\Lambda_a^2\big) + q^4\Lambda^2\big[q_y^4(D_cw + c_0v_p\gamma)^2 + D_c^2q^4q_x^2\Lambda_a^2\big]}. \tag{A5}$$

In the $q \to 0$ limit, this then yields

$$\langle|\theta_q|^2\rangle = \frac{T_Q}{p_0^2\big[\big(q_y^4/q^4\big)w^2 + q_x^2\Lambda_a^2\big]} \approx \frac{T_Q q^4}{p_0^2\big(q_y^4w^2 + q_x^6\Lambda_a^2\big)} \tag{A6}$$

as discussed in the main text.

Beyond the linear theory, the nonlinear equation of motion for θ_q is

$$\partial_t\theta_q = -i\frac{\gamma}{p_0}q_y\delta c - i\Lambda_a q_x\theta + w\left[\frac{q_y}{q_x}\left(-\frac{q_y}{q_x}\theta_q + \frac{\theta_k\theta_{q-k}}{2}\right) - \theta_{q-k}\left(-\frac{k_y}{k_x}\theta_k + \frac{\theta_{k-m}\theta_m}{2}\right)\right] - i\lambda_1\delta c_{q-k}k_x\theta_k - i\lambda_2 q_x\theta_{q-k}\theta_k - i\gamma_1 q_y\delta c_{q-k}\delta c_k - i\gamma_2\theta_{q-k}k_x\delta c_k + \frac{1}{p_0}\xi_Q, \tag{A7}$$

where we have used the fact that in the regime in which angular fluctuations are the largest, $q_y \sim q_x^{3/2}$, and therefore $q^2 \approx q_x^2$ for small wave vectors. Here λ_1, λ_2, γ_1, γ_2 are coefficients that may scale independently under a renormalization group treatment, unlike the nonlinearities with the coefficient w, in the square brackets, which, due to rotation invariance, must all scale the same way to preserve the form of the term in the square brackets. They will be anisotropic, just like Λ_a, but since, as discussed in the main text, all of them turn out to be irrelevant, we have not shown their (somewhat cumbersome) anisotropic forms nor expressed them in terms of coefficients introduced in (1) and (4). However, it is easy to see the origin of these nonlinearties: λ_2, within this theory, should arise from the same terms that contribute to Λ_a and therefore should be equal to Λ_a. The concentration nonlinearities can be understood as arising from the concentration dependence of γ, λ_a, ζ_1, ζ_2, υ. For instance, since every term in Λ_a can be a function of concentration, λ_1 should be $\partial\Lambda_a/\partial c|_{c=c_0}$ plus additional contributions from the projection of $\mathbf{p}$ orthogonal to itself to obtain an equation in terms of θ_q. γ_1 arises due to the concentration dependence of γ while γ_2 appears due to the projection of $\mathbf{p}$ transverse to itself.

The nonlinear concentration equation is

$$\partial_t\delta c_q = -D_c q^2\delta c - iv_p c_0 p_0 q_y\theta - i\Big(v_p + \frac{w}{\Lambda}\Big)p_0 q_x\delta c - i\lambda_c q_y\theta_{q-k}\delta c_k + iw_1 q_x\delta c_{q-k}\delta c_k + iw_2 q_x\theta_{q-k}\theta_k. \tag{A8}$$

The nonlinear coefficients λ_c and w_1 can rescale independently under a renormalization group treatment but w_2 must scale the same way as $v_p c_0 p_0$. It is simple to see the origin of these nonlinearities as well: λ_c should be v_p/c_0, w_1 appears from a concentration dependence of v_p or υ, and w_2 appears from expanding $\boldsymbol{\nabla}\cdot\mathbf{p}$ to second order in θ and therefore should have a coefficient $\propto v_p c_0 p_0$.

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