Quantum fluctuations inhibit symmetry breaking in the Hamiltonian mean-field model

Ryan Plestid [1,2,*] and James Lambert [1,†]
[1] Department of Physics & Astronomy, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4M1
[2] Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario, Canada N2L 2Y5

(Received 11 September 2019; revised manuscript received 3 December 2019; published 31 January 2020)

It is widely believed that mean-field theory is exact for a wide range of classical long-range interacting systems. Is this also true once quantum fluctuations have been accounted for? As a test case we study the Hamiltonian mean-field (HMF) model for a system of bosons which is predicted (according to mean-field theory) to undergo a second-order quantum phase transition at zero temperature. The ordered phase is characterized by a spontaneously broken O(2) symmetry, which, despite occurring in a one-dimensional model, is not ruled out by the Mermin-Wagner theorem due to the presence of long-range interactions. Nevertheless, a spontaneously broken symmetry implies gapless Goldstone modes whose large fluctuations can restore broken symmetries. In this work we study the influence of quantum fluctuations by projecting the Hamiltonian onto the continuous subspace of symmetry-breaking mean-field states. We find that the energetic cost of gradients in the center-of-mass wave function inhibits the breaking of the O(2) symmetry, but that the energetic cost is very small, scaling as $O(1/N^2)$. Nevertheless, for any finite N, no matter how large, this implies that the ground state has a restored O(2) symmetry. Implications for the finite-temperature phases, as well as the classical limit, of the HMF model are discussed.

DOI: 10.1103/PhysRevE.101.012136

I. INTRODUCTION

Systems with long-range interactions lie beyond the scope of traditional statistical mechanics [1–3]. They can exhibit ensemble inequivalence [1,2,4–6], divergent relaxation time scales (scaling as $t \gtrsim \log N$ for N particles) [7], and non-ergodic dynamics that lead to late-time states that disagree with the microcanonical ensemble [8–15] (preferring instead Lynden-Bell [16] or core-halo statistics [3,17,18]). While these features were first appreciated in the context of self-gravitating systems [16,19], it has become increasingly clear that peculiarities of gravitational systems (such as negative specific heat [20] and the gravothermal heat catastrophe [21]) are special cases of a broader statistical theory of long-range interacting systems [1–3].

One important feature of long-range interactions is that fluctuations can be suppressed to such a degree that continuous symmetries can be spontaneously broken even in one-dimensional systems [1,2,22–26]. This can be understood in the context of lattice models by considering the coordination number of each lattice site. Long-range interactions lead to large coordination numbers, which is equivalent to considering the lattice in some effective dimension $d_{\rm eff} > d$. Given that fluctuations are well known to be suppressed in high-dimensional systems, it is not surprising that long-range interactions can achieve the same effect. Mathematically, the presence of long-range interactions invalidates the Mermin-Wagner theorem [27], and its inapplicability is what allows for spontaneous symmetry breaking in a low-dimensional long-range interacting system [2,24,26].

There is extensive literature concerning the validity of mean-field theory for long-range interacting systems. For instance, in the classical literature, it has been rigorously proven [28], i.e., with bounded error, that a long-range interacting system's exact dynamics is well approximated by a mean-field collisionless-Boltzmann, i.e., Vlasov, equation (see, e.g., [3] or [7]). This approximation is valid on timescales $t \lesssim O(\log N)$. Therefore, in the $N \to \infty$ limit, it is often said that the collisionless-Boltzmann equation, i.e., mean-field theory, is exact [3,7,28]. Similar claims exist for equilibrium physics. For instance, Lieb was able to rigorously bound the difference between a self-gravitating bosonic star's ground-state energy and its Hartree energy and showed that this difference vanishes in the thermodynamic limit [29]. Similarly, it is well known that long-range interacting spin models have mean-field critical exponents [23], and it was conjectured that their free energy is also identical to that derived via a mean-field (meaning all-to-all interacting) model [30]. Subsequent studies supported this idea for long-range interacting spin systems [31–34], while more recent work has revealed disagreements between the all-to-all and power-law decaying models in a limited region of parameter space [35–37].

The success of all-to-all models in describing the thermodynamics of long-range interacting systems has led to them being an essential building block upon which modern statistical theories of long-range interacting systems are built. Examples include (for a review see [2]) the Emery-Blume-Griffiths model [4], the mean-field ϕ^4 model [38–40], and the Hamiltonian mean-field (HMF) model [41].

Since its proposal [41] the HMF model has been perhaps the most influential toy model in the long-range interacting

*plestird@mcmaster.ca
†lambej3@mcmaster.ca

community. Originally proposed as a simplified model of self-gravitating systems, it has emerged as a paradigmatic starting point and tool for understanding generic features of long-range interacting systems. While all-to-all, i.e., mean-field, models were motivated above by appealing to equilibrium physics, they turn out to also capture dynamical behavior. The HMF model can be used as a tool for understanding chaos [42–45], violent relaxation [10,17,46–49], core-halo statistics [3,17,18], and other quasistationary states [10–12,46] in long-range interacting systems. The HMF model exhibits a second-order phase transition associated with the spontaneous breaking of a continuous [O(2)] symmetry [1,41]. The HMF model's canonical partition function can be calculated exactly in the classical limit and exhibits an ensemble equivalence to the microcanonical ensemble.

The HMF model describes particles of unit mass, on a circle of unit radius, interacting via a pairwise cosine potential. When quantized for N bosons, the HMF model is defined by the Hamiltonian [50]

$$\hat{H}_{\text{HMF}} = \sum_i \frac{\chi^2}{2}\frac{\partial^2}{\partial\theta_i^2} - \frac{1}{N}\sum_{i<j}\cos(\theta_i - \theta_j). \tag{1}$$

Here χ is a dimensionless Planck constant (for a ring of radius R, with particles of mass m and a prefactor of ϵ multiplying the cosine interaction $\chi = \hbar/\sqrt{mR^2\epsilon}$) and we have chosen the case of the attractive HMF model as indicated by the negative sign of the potential. The $\frac{1}{N}$ scaling in front of the cosine interaction, known as the Kac prescription [51], preserves the extensivity of the Hamiltonian and is a consequence of the peculiar thermodynamic limit for long-range interacting systems ($N \to \infty$ with system size and χ held fixed [2,50,52]). Equation 1 can also be interpreted as describing a lattice of O(2) quantum rotors interacting with one another via all-to-all interactions with θ_i labeling the angle of each rotor on the lattice.

As mentioned above, classically (in the limit $\chi \to 0$), the model undergoes a thermal clustering transition [1] characterized by the order parameter

$$\mathbf{M} = \langle\cos\theta\rangle\hat{\mathbf{x}} + \langle\sin\theta\rangle\hat{\mathbf{y}}, \tag{2}$$

which, for $\mathbf{M} \neq 0$, implies a spontaneously broken O(2) symmetry. This transition is second order and is driven by thermal fluctuations. At high temperatures $T > T_c$, the system is homogeneous and $\mathbf{M} = 0$. For low temperatures $T < T_c$, the system spontaneously breaks its underlying O(2) symmetry.

In addition to mimicking certain dynamical features of self-gravitating bosons [50], Eq. (1) is also closely related to a handful of quantum systems that can be realized in the laboratory. For instance, cold atoms loaded into optical cavities can realize the generalized HMF model, which is identical to Eq. (1) up to terms of the form $\sum_{i<j}\cos[\theta_i - \theta_j]$ [53,54]. If, in the rotor interpretation of the model, the sum in Eq. (1) is restricted to be nearest-neighbor rotors, then it can be shown that this nearest-neighbor quantum rotor model offers a low-energy description of bosons in an optical lattice [55], i.e., a coupled set of Bose-Josephson junctions. Likewise, a spin-S Heisenberg ladder with antiferromagnetic coupling can realize the O(3) nearest-neighbor quantum rotor model [55]. We would expect an infinite-range rotor model such as Eq. (1) to reproduce the physics of rotors with long-range, i.e., polynomially decaying $1/|r_i - r_j|^\alpha$, with $\alpha < 1$, couplings as can be engineered in trapped ion systems [56–58].

Chavanis undertook the first study of the model's bosonic [50] (and fermionic [59]) equilibrium phase diagram. Using a Hartree ansatz, one finds that for $\chi < \sqrt{2}$ the lowest-Hartree-energy state also spontaneously breaks the O(2) symmetry, becoming a δ function in the limit $\chi \to 0$. For $\chi > \sqrt{2}$ it is found that the gradient energy for a single-particle wave function is no longer compensated for by the gain in interaction energy; this leads to a homogeneous ground state. Both of these behaviors connect smoothly with the model's limiting cases. For $\chi \to 0$ this agrees with the $T \to 0$ prediction of the classical HMF model. For $\chi \to \infty$ we recover an ideal Bose gas in a finite volume, the ground state of which is indeed a homogeneous product state.

Recently, the HMF model has been considered as a quantum dynamical system. The quantum analog of certain classical behaviors, such as violent relaxation, and the formation of quasistationary states has been studied [48]. Interestingly, classical instabilities related to the formation of biclusters [10,11] have been found to be stabilized by quantum (kinetic) pressure [48]. The HMF model's Gross-Pitaevskii equation has also been found to admit exactly solvable solitary wave solutions [60]. In fact, the Hartree states considered by Chavanis [50] may be considered as a special case of these solutions.

In this paper we make use of the exact solutions of [60] to systematically study whether quantum effects beyond mean-field theory can modify the HMF model's symmetry-breaking pattern at zero temperature. In particular, the mean-field, i.e., Hartree, prediction of a spontaneously broken [O(2)] symmetry suggests a highly degenerate ground state; if there is one ground state $|\Theta\rangle$ with its center of mass at Θ, then there must be a continuous manifold of such states $\{|\Theta'\rangle\}$ with $\Theta' \in [-\pi, \pi)$. This is reminiscent, for instance, of spinor Bose-Einstein condensates, whose exact ground state is a continuous quantum superposition of mean-field solutions [61,62]; we term these states continuous cat states (CCSs). In our example, such states would correspond to fluctuations of the center of mass or, equivalently, of a low-lying Goldstone excitation related to the broken O(2) symmetry.

We focus on computing matrix elements of the Hamiltonian between different Hartree states $\langle\Theta|\hat{H}_{\text{HMF}}|\Theta'\rangle$. Translational invariance, as well as parity, ensures that these matrix elements can depend only on the difference $|\Theta - \Theta'|$. Then, since the Hartree states tend towards δ functions, we can expect a δ expansion (in terms of derivatives of the Dirac δ function) to provide a good approximation of their behavior. Projecting the Hamiltonian onto this subset of states and using this expansion, we may then infer whether or not quantum fluctuations of the center of mass raise, or lower, the energy.

Viewing the HMF model as archetypal of long-range interacting systems, it is natural to study how the model's phase diagram is modified by quantum effects. Mapping out the phase diagram for the HMF model in the χ-T plane is a natural, and important, addition to the cannon of literature surrounding the HMF model. In this paper we take the first step towards this goal by studying the role of quantum fluctuations at zero temperature.

The rest of the paper is dedicated to calculating the energetic cost (or profit) of center-of-mass fluctuations as sketched above. In Sec. II we review the Hartree analysis for the HMF model [50,60], which will serve as a starting point for our analysis. In Sec. III we calculate matrix elements of the Hamiltonian between different CCSs. In Sec. IV we develop a large-N asymptotic series for the energy of a given CCS. Then, in Sec. V we obtain explicit expressions for the energy at leading order in χ; this allows us to determine the symmetry-breaking properties of the ground state. In Sec. VI we summarize our results and suggest future directions for the quantum HMF model.

II. MEAN-FIELD THEORY

Mean-field theory for the bosonic HMF model at zero temperature is equivalent to a product-state ansatz for the ground state. Taking $|\Psi\rangle = \bigotimes|\psi\rangle$, with $|\psi\rangle$ a single-particle state, leads to an energy functional $\mathcal{E}[\psi] = \langle\Psi|\hat{H}_{\rm HMF}|\Psi\rangle$. Minimizing this energy with respect to the single-particle wave functions $\delta\mathcal{E}/\delta\psi = 0$ then leads to a self-consistent eigenvalue problem [60]

$$-\frac{\chi^2}{2}\partial_\theta^2\psi_H + M\cos\theta\,\psi_H = \mu\psi_H, \tag{3}$$

where μ is the chemical potential and M is the aforementioned order parameter of Eq. (2); in the Hartree theory, M must be determined self-consistently. Equation (3) is exactly soluble and its solutions can be expressed in terms of Mathieu functions [60,63]

$$\psi_H(\theta) = \frac{1}{\sqrt{\pi}}\mathrm{ce}_0\left(\frac{\theta-\pi}{2}; q[\chi]\right), \tag{4}$$

where $q(\chi)$ is the depth parameter of the Mathieu equation [63], whose dependence on χ can be determined by solving the self-consistency condition

$$q = \frac{4M}{\chi^2}. \tag{5}$$

In this context, the magnetization may be thought of as a function of q and is defined via the integral

$$M(q) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\mathrm{ce}_0\left(\frac{\theta-\pi}{2}; q\right)\right]^2\cos\theta. \tag{6}$$

Solving Eq. (3), one finds that for $\chi > \sqrt{2}$ the magnetization vanishes, $M = 0$, and that the lowest-energy wave function is homogeneous, i.e., $\psi_H = 1/\sqrt{2\pi}$ [50]. Furthermore, a Bogoliubov theory of fluctuations about this ground state can be constructed and it can be easily checked that the quantum depletion of the ground state $\sum_{k\neq 0}\langle a_k^\dagger a_k\rangle_{T=0}$ (with a_k the atomic ladder operator) is finite, being given by $\sum_{k\neq 0}\sinh^2\theta_k$, with $\sinh^2\theta_k = \frac{1}{2}(\sqrt{1-2\delta_{k,\pm 1}/\chi^2} - 1)$.

For $\chi < \sqrt{2}$ one finds instead that $M \neq 0$ and the ground-state wave function begins to acquire nonzero curvature, with the explicit wave function being give by Eq. (4). The transition between the spatially homogeneous ground state and the spatially localized ground state can be viewed as a quantum phase transition associated with the spontaneous breaking of translational invariance; the transition is predicted to be second order.

This simplified analysis then predicts that there is a degenerate manifold of ground states, given by $|\Theta; N\rangle = \bigotimes|\psi_H;\Theta\rangle$, where Θ labels the wave function's center of mass (c.m.), such that $\psi_H(\theta-\Theta) = \langle\theta|\psi_H;\Theta\rangle$ is peaked at $\theta = \Theta$. For this kind of mean-field analysis to be self-consistent, however, we require that quantum fluctuations of the c.m. are small *a posteriori*. Because the clustered phase is characterized by a spontaneously broken continuous symmetry, we must then consider fluctuations of gapless excitations corresponding to the shift symmetry $\Theta \to \Theta + \Delta\Theta$.

III. CENTER-OF-MASS FLUCTUATIONS

We can study the importance of c.m. fluctuations by considering a CCS

$$|\mathfrak{f}; N\rangle = \int d\Theta\,\mathfrak{f}(\Theta)|\Theta; N\rangle, \tag{7}$$

where $\mathfrak{f}$ is the c.m. wave function such that $|\mathfrak{f}; N\rangle$ is a coherent superposition of product states centered about Θ. The product states

$$|\Theta; N\rangle = \bigotimes_{i=1}^{N}|\psi_H;\Theta\rangle \tag{8}$$

are composed of single-particle wave functions, centered at Θ, $\psi_H(\theta-\Theta) = \langle\theta|\psi_H;\Theta\rangle$, that minimize the Hartree, i.e., mean-field, energy.

The case of $\mathfrak{f} \propto \delta(\Theta)$ corresponds to a Hartree state (localized about a single c.m.), whereas if $\mathfrak{f}(\Theta)$ is independent of Θ then this state has a restored translational invariance. To test whether or not quantum fluctuations restore translational symmetry we can compute the average energy of a CCS. We are therefore interested in minimizing the energy per particle

$$E[\mathfrak{f}] = \frac{1}{N}\langle\mathfrak{f}; N|\hat{H}_{\rm HMF}|\mathfrak{f}; N\rangle. \tag{9}$$

Because of the system's translational invariance, we can guarantee that the resulting functional can be diagonalized in momentum space

$$E[\mathfrak{f}] = \sum_k \hat{E}(k)|\hat{\mathfrak{f}}(k)|^2, \tag{10}$$

where $\mathfrak{f}(\Theta) = \sum_k e^{ik\Theta}\hat{\mathfrak{f}}(k)/\sqrt{2\pi}$. Studying the variational problem $\delta E/\delta\mathfrak{f} = 0$ is equivalent to minimizing $\hat{E}(k)|\mathfrak{f}(k)|^2$ subject to the constraint that $\langle\mathfrak{f}; N|\mathfrak{f}; N\rangle = 1$; we can therefore conclude without any loss of generality that the minimum-energy c.m. wave function will be of the form $\mathfrak{f}_k(\Theta) = e^{ik\Theta}$, with $k \in \mathbb{Z}$.

It will be useful to introduce the functions

$$\mathcal{T}_H(y) = \frac{\chi^2}{2}\int d\theta\,\partial\psi_H^*\left(\theta - \frac{1}{2}y\right)\partial\psi_H\left(\theta + \frac{1}{2}y\right) \tag{11}$$

and

$$\mathcal{V}_H(y) = \frac{1}{2}\int d\theta\,d\theta'\,\psi_H^*\left(\theta - \frac{1}{2}y\right)\psi_H^*\left(\theta' - \frac{1}{2}y\right) \times\cos(\theta-\theta')\psi_H\left(\theta + \frac{1}{2}y\right)\psi_H\left(\theta' + \frac{1}{2}y\right), \tag{12}$$

with $y = \Theta - \Theta'$. Throughout our analysis we will find that the Hamiltonian's matrix elements between two Hartree states $\langle\Theta_2; N|\hat{H}|\Theta_1; N\rangle$ can be expressed in terms of derivatives of the above functions evaluated at zero separation. With this in mind we introduce the notation

$$\mathcal{V}_H^{(n)} = \partial_y^n \mathcal{V}_H(y)|_{y=0}, \quad \mathcal{T}_H^{(n)} = \partial_y^n \mathcal{T}_H(y)|_{y=0}. \tag{13}$$

Explicitly, the functions $\mathcal{V}_H(y)$ and $\mathcal{T}_H(y)$ are related to the the matrix elements of the kinetic, $\hat{T} = \sum_i \frac{1}{2m}\hat{p}_i^2$, and potential, $\hat{V} = -\frac{1}{N}\sum_{ij}\cos(\hat{\theta}_i - \hat{\theta}_j)$, operators via

$$\frac{1}{N}\langle\Theta_1; N|\hat{T}|\Theta_2; N\rangle = \mathcal{T}_H(y)\mathcal{O}(y; N-1), \tag{14}$$

$$\frac{1}{N}\langle\Theta_1; N|\hat{V}|\Theta_2; N\rangle = -\mathcal{V}_H(y)\left[\frac{N(N-1)}{N^2}\right]\mathcal{O}(y; N-2), \tag{15}$$

where we define the overlap

$$\mathcal{O}(y;\aleph) = \langle\Theta_1;\aleph|\Theta_2;\aleph\rangle = [\langle\psi_H;\Theta_1|\psi_H;\Theta_2\rangle]^{\aleph} \tag{16}$$

(with $\aleph = N$, $N-1$, or $N-2$), in terms of the coordinate difference $y = \Theta_1 - \Theta_2$. Introducing the c.m. coordinate $x = \frac{1}{2}(\Theta_1 + \Theta_2)$, we can write

$$E[\mathfrak{f}] = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dx\,dy\,\mathfrak{f}^*\left(x - \frac{1}{2}y\right)\mathfrak{f}\left(x + \frac{1}{2}y\right)\left[\mathcal{T}_H(y)\mathcal{O}(y; N-1) - \mathcal{V}_H(y)\left(1 - \frac{1}{N}\right)\mathcal{O}(y; N-2)\right], \tag{17}$$

from which we can immediately see that

$$\hat{E}(k) = \int_{-\pi}^{\pi} dy\, e^{-iky}\left[\mathcal{T}_H(y)\mathcal{O}(y; N-1) - \mathcal{V}_H(y)\left(1 - \frac{1}{N}\right)\mathcal{O}(y; N-2)\right]. \tag{18}$$

Note that in both Eqs. (17) and (18) all of the derivatives from the full many-body Hamiltonian (1) appear in the functions $\mathcal{T}_H(y)$ and $\mathcal{V}_H(y)$ and they do not act directly on the c.m. wave function.

IV. LARGE-N EXPANSION

We are ultimately interested in the thermodynamic limit ($N \to \infty$ with χ held fixed [50,52]) and in particular whether subleading corrections in $1/N$ can modify the symmetry-breaking pattern at zero temperature. To study this limit we develop an expansion that relies on the $\aleph$-body overlap $\mathcal{O}(y;\aleph)$ being tightly peaked for $\aleph \gg 1$. Because $\mathcal{O}(y;\aleph) = [\langle\psi_H;\Theta_1|\psi_H;\Theta_2\rangle]^{\aleph}$ can be written as an exponentiated single-particle overlap, this will be true even for moderately peaked single-particle overlaps. In the clustered phase, provided $\chi \lesssim 1$, the overlap between two Hartree states $|\Theta_1;\aleph\rangle$ and $|\Theta_2;\aleph\rangle$ admits a δ expansion of the form

$$\mathcal{O}(y;\aleph) = \frac{1}{|C(\aleph,\chi)|^2}\left[\delta(y) + \sum_{p>0}\frac{\mathcal{K}_p(\chi)}{\aleph^p}\delta^{(2p)}(y)\right]. \tag{19}$$

We use this to develop a systematic expansion in $1/N$ by considering a perturbative expansion of the c.m. wave function

$$\mathfrak{f} = C(N,\chi)\left(\mathfrak{f}_0 + \frac{1}{N}\mathfrak{f}_1 + \frac{1}{N^2}\mathfrak{f}_2 + \cdots\right). \tag{20}$$

The multiplicative constant $C(N,\chi)$ is chosen such that $\langle\mathfrak{f}_0,\mathfrak{f}_0\rangle = \int \mathfrak{f}_0^*(x)\mathfrak{f}_0(x)dx = 1$, which ensures that $\langle\mathfrak{f}; N|\mathfrak{f}; N\rangle = 1$ at leading order.[1] To maintain this normalization order by order in $1/N$ we impose, on the c.m. wave function, the constraints

$$\langle\mathfrak{f}_0,\mathfrak{f}_0\rangle = 1, \tag{21}$$

$$2\,\mathrm{Re}\langle\mathfrak{f}_1,\mathfrak{f}_0\rangle = \mathcal{K}_1\langle\mathfrak{f}_0^{(1)},\mathfrak{f}_0^{(1)}\rangle, \tag{22}$$

$$\langle\mathfrak{f}_1,\mathfrak{f}_1\rangle + 2\,\mathrm{Re}\langle\mathfrak{f}_0,\mathfrak{f}_2\rangle = \mathcal{K}_1 2\,\mathrm{Re}\langle\mathfrak{f}_1^{(1)},\mathfrak{f}_0^{(1)}\rangle - \mathcal{K}_2\langle\mathfrak{f}_0^{(2)},\mathfrak{f}_0^{(2)}\rangle, \tag{23}$$

which can be derived using Eq. (19) and the identity (A5). Note that, as above, the inner product $\langle\mathfrak{f}_i,\mathfrak{f}_j\rangle = \int dx\,\mathfrak{f}_i^*(x)\mathfrak{f}_j(x)$ is the L^2 inner product and should not be confused with the state overlap $\langle\mathfrak{f}|\mathfrak{f}\rangle$.

These normalization constraints play an important role in the calculation of the energy as discussed in Appendix B. Due to nontrivial correlations between $\mathfrak{f}_1$ and $\mathfrak{f}_0$, expanding $\hat{E}(k)$ directly will not tell us how the energy $E[\mathfrak{f}]$ depends on the wave function $\mathfrak{f}$. Rather, one must expand $\mathfrak{f}$ and $\hat{E}(k)$ concurrently,

$$\hat{E}(k) = \frac{1}{|C(N,\chi)|^2}\left[\hat{E}_0 + \frac{1}{N}\hat{E}_1 + \frac{1}{N^2}\hat{E}_2 + \cdots\right], \tag{24}$$

where $\hat{E}_0 = E_H$, with $E_H = \mathcal{T}_H^{(0)} - \mathcal{V}_H^{(0)}$ the Hartree energy. We find that $\hat{E}$ is given by

$$\hat{E}(k)|\mathfrak{f}(k)|^2 = \hat{E}_0|\mathfrak{f}_0|^2 + \frac{1}{N}[\hat{E}_1|\mathfrak{f}_0|^2 + 2\hat{E}_0\mathrm{Re}\mathfrak{f}_0^*\mathfrak{f}_1] + \frac{1}{N^2}[\hat{E}_2|\mathfrak{f}_0|^2 + \hat{E}_1|\mathfrak{f}_1|^2 + 2\hat{E}_0\mathrm{Re}\mathfrak{f}_0^*\mathfrak{f}_2]. \tag{25}$$

By using Eqs. (21)–(23), the expression (25) can be simplified such that $E[\mathfrak{f}] = \sum_k \hat{E}(k)|\mathfrak{f}(k)|^2$ can be written as

$$E[\mathfrak{f}] = E_0 + \frac{E_2}{N^2}\langle\mathfrak{f}_0^{(1)},\mathfrak{f}_0^{(1)}\rangle + O\left(\frac{1}{N^3}\right) \tag{26}$$

[1] The physical state overlap $\langle\mathfrak{f}|\mathfrak{f}\rangle$ differs from the L^2 inner product $\langle\mathfrak{f},\mathfrak{f}\rangle$ at $O(1/N)$, i.e., $\langle\mathfrak{f}|\mathfrak{f}\rangle = \langle\mathfrak{f},\mathfrak{f}\rangle + O(1/N)$.

or at the same level of accuracy

$$E[\mathfrak{f}] = E_0 + \frac{E_2}{N^2}\langle \mathfrak{f}^{(1)}, \mathfrak{f}^{(1)}\rangle + O\left(\frac{1}{N^3}\right). \tag{27}$$

Equation (27) controls the symmetry breaking in the HMF model. Naively, the term E_2 is irrelevant in the thermodynamic limit [being $O(1)$]; however, because the leading-order term predicts a degenerate ground state, the small $O(1/N^2)$ perturbation $\hat{E}_2$ dictates the symmetry-breaking pattern of the ground state. The sign of E_2 dictates whether inhomogeneity, i.e., nonzero values of k, raises or lowers the energy of a CCS and is consequently indicative of whether or not quantum fluctuations can destroy the localized (magnetized) phase. The full details of our calculation can be found in Appendix B; however, for the sake of brevity we simply quote the leading-order contribution for each quantity

$$E_0 = E_H + \frac{1}{N}\left[\mathcal{T}_H^{(2)} - \mathcal{V}_H^{(2)} - \frac{1}{2}\mathcal{T}_H^{(0)}\right] + O\left(\frac{1}{N^2}\right) \tag{28}$$

and

$$E_2 = \left[\mathcal{K}_1^2 - 6\mathcal{K}_2\right]\left[\mathcal{T}_H^{(2)} - \mathcal{V}_H^{(2)}\right] - \mathcal{K}_1\left[\mathcal{T}_H^{(0)} - 2\mathcal{V}_H^{(0)}\right]. \tag{29}$$

The fact that gradient corrections vanish at $O(1/N)$ is a consequence of a cancellation between the $\hat{E}_1|\mathfrak{f}_0|^2$ and $2\hat{E}_0\mathrm{Re}\mathfrak{f}_0^*\mathfrak{f}_1$ in Eq. (25). This cancellation is not accidental and is discussed in greater detail in Appendix B 4.

V. STRONG-COUPLING REGIME

To determine whether these fluctuations can restore translational invariance, we can study a point in parameter space deep within the clustered phase $\chi \lesssim 1$ and see if quantum fluctuations can lead to a translationally invariant c.m. wave function, i.e., $\mathfrak{f} = 1/\sqrt{2\pi}$. For this to occur $\hat{E}_2$ must be positive such that $k = 0$ is energetically preferred.

Although left implicit until now, the parameters $\mathcal{K}_1(\chi)$ and $\mathcal{K}_2(\chi)$ are themselves functions of χ, as are the derivatives of the CCS energies $\mathcal{V}_H^{(n)}(\chi)$ and $\mathcal{T}_H^{(n)}(\chi)$. These functions are determined exactly in terms of the integrals in Eqs. (11) and (12) involving the Hartree ground state $\psi_H(\theta;\chi)$ [whose χ dependence is determined by Eq. (3)]. To test whether quantum fluctuations of the c.m. can restore the spontaneously broken symmetry, it is sufficient to restrict our attention to small but finite values of χ satisfying $\chi \leqslant \sqrt{2}$; in practice, we focus on $\chi \lesssim 1$ such that a small-χ asymptotic expansion is justified.

Both $\mathcal{K}_1$ and $\mathcal{K}_2$ are determined by $\mathcal{O}(y; N,\chi)$. As argued in Appendix D, for small values of χ this can be well approximated by (see Appendix C)

$$\mathcal{O}(y;\aleph) \approx \left[\frac{I_0(\sqrt{q}\cos\frac{y}{2})}{I_0(\sqrt{q})}\right]^{\aleph}, \tag{30}$$

where $I_0(z)$ is the modified Bessel function of the first kind and q is an auxiliary depth parameter related to the mean-field magnetization M and χ via $q = 4M/\chi^2$. We are interested in finding a δ expansion for $\mathcal{O}(y)$ and are thus interested in integrals of the form $\int_\pi^\pi \mathcal{O}(y)f(y)dy$. For $1 \lesssim y \lesssim \pi$ the overlap is exponentially small, i.e., $O(e^{-\sqrt{q}})$, so we can neglect this contribution to the integral. For moderate values of y we can then use the large argument expansion of the modified Bessel functions $I_0(z) \sim e^{-z}/\sqrt{2\pi z}$ leading to

$$\mathcal{O}(y;\aleph) \sim \exp\left\{\aleph\left[\frac{4}{\chi}\left(1 - \frac{1}{8}\chi\right)\sin^2\frac{y}{4} - \frac{1}{2}\log\cos\frac{y}{2}\right]\right\}. \tag{31}$$

Using this exponential form, the integrals we are interested in studying can then be approximated using Watson's lemma

$$\int e^{-\aleph G(y)} f(y)dy \sim \sqrt{\frac{2\pi}{\aleph G^{(2)}}}\sum_p \frac{f^{(2p)}}{(2p)!![\aleph G^{(2)}]^p}, \tag{32}$$

where the bracketed superscripts denote the $2p$th derivative of the function evaluated at $y = 0$. For $\mathcal{O}(y;\aleph)$ we have

$$G^{(2)} = \frac{1}{2\chi} - \frac{3}{16}. \tag{33}$$

We can then read off the overall prefactor of Eq. (20),

$$|C(N,\chi)|^2 = \sqrt{\frac{2\pi}{NG^{(2)}}} = 2\sqrt{\frac{\pi\chi}{N}}\left[1 + \frac{3\chi}{32}\right] \tag{34}$$

and the coefficients $\mathcal{K}_1$ and $\mathcal{K}_2$, which are given at next to leading order by

$$\mathcal{K}_1 \sim \chi + \frac{3\chi^2}{8}, \quad \mathcal{K}_2 \sim \frac{\chi^2}{2} + \frac{3\chi^3}{8}. \tag{35}$$

Next, using Eq. (C6) for the Mathieu functions, we can derive the small-χ behavior of the CCS functionals and their derivatives,

$$\mathcal{T}_H^{(0)} \sim \frac{\chi}{4}, \quad \mathcal{T}_H^{(2)} \sim -\frac{3}{8}, \tag{36}$$

$$\mathcal{V}_H^{(0)} \sim \frac{1}{2} - \frac{\chi}{4}, \quad \mathcal{V}_H^{(2)} \sim -\frac{1}{2\chi} + \frac{3}{8}. \tag{37}$$

Note that we need the subleading corrections to $\mathcal{V}_H^{(0)}$ and $\mathcal{V}_H^{(2)}$ because they are the same order as $\mathcal{T}_H^{(0)}$ and $\mathcal{T}_H^{(2)}$.

Including these terms, we find that $O(\chi)$ contribution vanishes, but the $O(\chi^2)$ contribution does not. We finally arrive at

$$\hat{E}_2 \sim \frac{3\chi^2}{8} + O(\chi^3). \tag{38}$$

This tell us that that curvature of the c.m. wave function is energetically unfavorable such that the system prefers a homogeneous CCS over a clumped one. Thus, quantum fluctuations corresponding to Goldstone modes restore the spontaneously broken translational invariance. For small values of χ the lowest-energy state, at all finite values of N (no matter how large), is given by

$$|\mathrm{GS}\rangle_{\mathrm{CCS}} = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} d\Theta|\Theta\rangle + O\left(\frac{1}{N}\right). \tag{39}$$

As was alluded to earlier, this is reminiscent of spinor Bose-Einstein condensates, whose exact ground state is known to be a CCS that is formally identical to Eq. (39) [61,62].

VI. DISCUSSION AND CONCLUSION

Quantum fluctuations of Goldstone modes can play an important role in determining the zero-temperature behavior of a long-range interacting system. In the example studied here, properties of the ground state such as its symmetry-breaking pattern are left undetermined at the level of mean-field theory due to a high level of degeneracy in the energy spectrum. Previous work on Bose stars suggests that this degeneracy is a generic consequence of long-range interactions [29]. In the case of the HMF model, we find that this degeneracy is only lifted at $O(1/N^2)$ for any finite N (no matter how large). At zero temperature this has the striking consequence of leading to a restored O(2) symmetry in the ground state.

At finite N, the system is gapped, $\Delta = 3\chi^2/8N^2$, with excitations corresponding to departures from a homogeneous c.m. wave function. In the $N \to \infty$ limit the system becomes gapless such that $|\mathrm{GS}\rangle_{\mathrm{CCS}}$ becomes embedded in a highly degenerate manifold of states, almost all of which break the model's underlying O(2) symmetry. This is reminiscent of the behavior of spin-1/2 chains, where a rotationally invariant singlet ground state is separated at finite N from a triplet excitation that breaks rotation invariance. In the $N \to \infty$ limit the gap closes and the singlet becomes embedded in a degenerate ground-state manifold whose low-lying excitations are triplets [64] in analogy with the clumping excitations in the HMF.

As outlined above, the calculations in this paper shed light on the HMF model's low-lying energy excitation spectrum at finite N. In addition, our results demonstrate that for any finite values of N and χ, the HMF model's ground state *does not* spontaneously break its O(2) symmetry. In the language of the canonical ensemble, our calculation corresponds to taking the $T \to 0$ limit first (singling out the ground state), followed by taking $N \to \infty$. The absence of symmetry breaking for the ground state is interesting, because the HMF model's classical partition function can be calculated *exactly* in the $N \to \infty$ limit and exhibits a thermally driven second-order phase transition [2,41]; at low (but finite) temperatures the system breaks the O(2) symmetry. This result corresponds to the $\chi \to 0$ limit being taken first, followed by $N \to \infty$. Thus we have an apparent noncommutativity of limits

$$\lim_{T\to 0}\lim_{N\to\infty}\lim_{\chi\to 0}\rho_{\mathrm{HMF}} \neq \lim_{\chi\to 0}\lim_{N\to\infty}\lim_{T\to 0}\rho_{\mathrm{HMF}}, \tag{40}$$

where ρ_{HMF} is the equilibrium state of the system. The left-hand side corresponds to the classical result and the right-hand side corresponds to the calculation presented in this paper. If we are concerned with the equilibrium physics of the HMF model in the thermodynamic limit, then the "correct" limit is given by $\lim_{N\to\infty}\rho_{\mathrm{HMF}}$ with T and χ held fixed. Then, after having taken the $N \to \infty$ limit, we may investigate limits such as $\chi \to 0$ and $T \to 0$.

There is a clear subtlety in the definition and appropriate ordering of the various limits. We identify a list of three possibilities that could explain the "mismatch" in Eq. (40).

(i) The limits of $\chi \to 0$ and $N \to \infty$ do not commute such that $\chi \to 0$ is a singular limit, and the classically ordered phase exists only for $\chi = 0$.

(ii) The $\chi \to 0$ and $T \to 0$ limits do not commute. For example, The HMF model could exhibit a reentrant phase wherein at finite temperature, for small values of χ, the O(2) symmetry is broken.

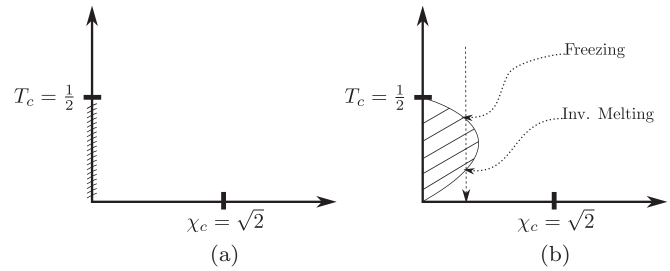

FIG. 1. Two possible resolutions of our result and the exact classical calculation. (a) The limit $\chi \to 0$ could be singular such that symmetry breaking (hashed lines) is only possible for $\chi = 0$. (b) Alternatively, a reentrant phase could appear at finite temperature. We identify this possibility as analogous to inverse melting, as indicated by the line of decreasing temperature at fixed χ. The parameters corresponding to classical and quantum (mean-field) symmetry breaking are marked with thick black lines.

(iii) The $T \to 0$ limit does not commute with $N \to \infty$ such that the $T \to 0$ limit is singular. This could occur if the lowest-energy excitations are not entropically significant.

Schematic phase diagrams for the first two of these scenarios are sketched in Fig. 1. The third of these possibilities is not sketched because if indeed the $T \to 0$ limit is singular, then neither our study nor that of [50] provides any reliable information about the low-temperature region of the phase diagram in the strong-coupling regime.

One might also wonder why we have not included the conclusion of [50] that there is a symmetry-breaking phase in a "bubble" near the bottom left corner of the T-χ plane. This possibility is excluded because we have explicitly demonstrated that no matter how small χ is made, the ground state of the many-body system does not spontaneously break the underlying O(2) symmetry. Thus, quantum fluctuations cannot be treated as small as is implicitly assumed in a mean-field analysis such as the Gross-Pitaevskii-equation-based treatment of [50].

In Fig. 1(b), interpreting the O(2) symmetry as a translational invariance for particles on a ring, this is reminiscent of inverse melting which is known to exist in certain spin models [65,66]. Viewing deformations of the c.m. wave function as low-lying excitations [all of which break the O(2) symmetry], it is conceivable that at finite temperatures it could be entropically favorable to macroscopically excite these degrees of freedom and break (melt) the O(2) symmetry.

The third possibility would imply that the properties of the ground state do not encapsulate the low-temperature behavior of the canonical ensemble. For example, suppose that there are two populations of relevant excitations about the ground state at finite N. One family (type-I) has an energy of $\mathcal{E}_1 \sim O(1/N)$ but an entropy of $S_1 \sim O(1)$, while the other family (type-II) has an energy of $\mathcal{E}_2 \sim O(1)$ but an entropy of $S_2 \sim O(\log N)$, i.e., there are polynomially many type-II excitations. Then, minimizing the free energy $F = E - TS$ will result in two different regimes: At low temperatures one should focus on minimizing the energy such that type-II excitations are Boltzmann suppressed and type-I excitations dominate the system's behavior, while at high temperatures, due to the large density of states, type-II excitations will dominate. The crossover temperature between these two regimes will be $T_{\rm cross} \sim O(1/\log N)$. Notice in the $N \to \infty$ limit that $T_{\rm cross} \to 0$ such that only at $T = 0$ are type-I excitations relevant. In fact, the scenario outlined above is a plausible characterization of the HMF model. For instance, single-particle excitations out of the condensate will have an energy per particle of $O(1/N)$, but there should be at least of $O(N)$ of them. In contrast, there will be $O(1)$ ways to deform the c.m. wave function, and the energetic cost per particle of doing so is $3\chi^2/8N^2$, as shown explicitly in Eqs. (27) and (38).

The determination of which of these three possibilities is borne out by the HMF model is beyond the scope of this paper, but a definitive answer to which of these three scenarios actually takes place is clearly the most pressing question relating to the quantum HMF model's equilibrium physics. We therefore advocate for a numerically exact exploration of the HMF model's phase diagram in the χ-T plane; for example, a path-integral Monte Carlo (PIMC) study should be capable of providing a definitive answer. The possible noncommutativity between the $N \to \infty$ limit and the $T \to 0$ and $\chi \to 0$ limits presents a possible hurdle in extending (necessarily) finite-N computational results to the thermodynamic limit. We therefore advocate, in addition to a full PIMC study, that a finite-N scaling theory appropriate to the HMF model be developed. In particular, it will be essential to estimate (or calculate) the excitation spectrum of the model and also the density of states.

In summary, we have shown definitively that Eq. (39) has a lower energy than a naive product state. We interpret these quantum fluctuations of the center of mass as the destruction of symmetry breaking due to Goldstone modes. The energetic cost to excite a nonhomogeneous center-of-mass wave function vanishes in the thermodynamic limit, suggesting that finite-temperature effects could substantially alter our predictions. While we have provided an analytic study of the HMF model's ground state, our approach is necessarily approximate and we have only included c.m. fluctuations. It is possible that these are the lowest-lying excitations about the ground state while still being entropically dominated by other types of excitations. If this is the case then the thermodynamic limit is highly nontrivial and any application of numerical techniques, e.g., PIMC, would need to be supplemented with an analytic understanding of how to properly take the $N \to \infty$ limit. Developing an appropriate finite-N scaling ansatz, coupled with a full numerical investigation into the finite-temperature properties of the system, is a natural extension of this work and is the most important next step in the study of the HMF model.

ACKNOWLEDGMENTS

We would like to thank Dr. Sung-Sik Lee for suggestions, discussions, and encouragement. This work was supported by funds from the National Science and Engineering Research Council of Canada. Support from the Perimeter Institute for Theoretical Physics is also acknowledged. Research at the Perimeter Institute was supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research and Innovation.

APPENDIX A: δ-FUNCTION IDENTITIES

In Appendix B we frequently encounter integrals of the form

$$\int dx\, dy\, \delta^{(2n)}(y) g(x - y/2) f(x + y/2) h(y), \tag{A1}$$

and in this Appendix we provide a short derivation of a useful identity (A9). We may first, however, study the simpler case of

$$\int dx\, dy\, \delta^{(2)}(y) g(x - y/2) f(x + y/2). \tag{A2}$$

In this case we must integrate by parts twice to pull the derivative off of the δ function. This gives

$$\begin{aligned}\int dx\, dy\, \delta(y) \frac{1}{4}[g'' f + f'' g - 2g' f'] \\ = \int dx \frac{1}{4}[g''(x) f(x) + f''(x) g(x) - 2g'(x) f'(x)] \\ = -\int dx\, g'(x) f'(x).\end{aligned} \tag{A3}$$

This generalizes naturally. If we define $\partial^m[g(x - \frac{1}{2}y) f(x + \frac{1}{2}y)] = \mathcal{G}_m(\alpha, \beta)$, then the generalized identity is

$$\begin{aligned}\int dx\, \mathcal{G}_{2m}(x, x) = (-1)^m \int dx\, g^{(m)}(x) f^{(m)}(x) \\ = \langle \bar{g}^{(m)}, f^{(m)} \rangle.\end{aligned} \tag{A4}$$

Applying this result to a δ function leads to

$$\begin{aligned}\int dx\, dy\, \delta^{(2n)}(y) g\left(x - \frac{1}{2}y\right) f\left(x + \frac{1}{2}y\right) \\ = \int dx\, dy\, \delta(y) \partial_y^{2n}\left[g\left(x + \frac{1}{2}y\right) f\left(x - \frac{1}{2}y\right)\right] \\ = (-1)^n \int dx\, g^{(n)}(x) f^{(n)}(x).\end{aligned} \tag{A5}$$

Finally, when including an additional function in the integrand, we simply distribute the derivatives and find

$$\begin{aligned}\int dx\, dy\, \delta^{(2n)}(y) g\left(x - \frac{1}{2}y\right) f\left(x + \frac{1}{2}y\right) h(y) \\ = \int dx\, dy\, \delta(y) \partial_y^{2n}\left[g\left(x + \frac{1}{2}y\right) f\left(x - \frac{1}{2}y\right) h(y)\right] \\ = \int dx\, dy\, \delta(y) \sum_m \binom{2n}{m} \mathcal{G}_m(\alpha, \beta) \partial_y^{2n-m} h(y).\end{aligned} \tag{A6}$$

Because $h(y)$ is an even function, all of the odd derivatives vanish, leading to

$$\int dx\, dy\, \delta(y) \sum_m \binom{2n}{2m} \mathcal{G}_m(\alpha, \beta) \partial_y^{2(n-m)} h(y). \tag{A7}$$

Now we can perform the integration over y,

$$\begin{aligned}\sum_m \binom{2n}{2m} \int dx\, dy\, \delta(y) \mathcal{G}_{2m}(\alpha, \beta) h^{(2m-2n)}(y) \\ = \sum_m \binom{2n}{2m} \left[\partial_y^{2(n-m)} h(y)\right]_{y=0} \int dx\, \mathcal{G}_{2m}(x, x).\end{aligned} \tag{A8}$$

Now using Eq. (A4), we arrive at

$$\int dx\, dy\, \delta^{(2n)}(y) g\left(x - \frac{1}{2}y\right) f\left(x + \frac{1}{2}y\right) h(y) = \sum_m \binom{2n}{2m} (-1)^m \langle \bar{g}^{(m)}, f^{(m)} \rangle h_0^{(2n-2m)}, \quad \text{(A9)}$$

where $h_0^{(2n-2m)} = \partial_y^{2(n-m)} h(y)|_{y=0}$. In calculations throughout this paper $f(x + \frac{1}{2}y) = \mathfrak{f}_a(x + \frac{1}{2}y)$ and $g(x - \frac{1}{2}y) = \mathfrak{f}_b^*(x - \frac{1}{2}y)$ such that $\langle \bar{g}^{(m)}, f^{(m)} \rangle = \langle \mathfrak{f}_b^{(m)}, \mathfrak{f}_a^{(m)} \rangle$.

APPENDIX B: LARGE-N ASYMPTOTICS FOR THE ENERGY

In Eqs. (28) and (29) we quote results for ground-state energy shift E_0 and the c.m. wave-function gradient energy E_2. In this Appendix we derive these results.

We begin by considering the kinetic energy

$$T[\mathfrak{f}] = \frac{1}{N} \langle \mathfrak{f}; N | \hat{T} | \mathfrak{f}; N \rangle = \int dx\, dy\, \mathfrak{f}^*\left(x - \frac{1}{2}y\right) \mathfrak{f}\left(x + \frac{1}{2}y\right) \mathcal{T}_H(y) \mathcal{O}(y; N-1).$$

Note that the overlap has had a particle removed since we are computing the expectation value of a single-particle operator. Because of our c.m. wave-function normalization, this means we will find an overall prefactor of $|C(N)|^2/|C(N-1)|^2 = \sqrt{N/(N-1)}$, leading to

$$T[\mathfrak{f}] = \sqrt{\frac{N}{N-1}} \int dx\, dy\, \mathcal{T}_H(y) \times \sum_{a,b} \frac{1}{N^{a+b}} \mathfrak{f}_a^*\left(x - \frac{1}{2}y\right) \mathfrak{f}_b\left(x + \frac{1}{2}y\right) \times \sum_p \frac{\mathcal{K}_p}{N^p} \frac{1}{(1 - \frac{1}{N})^p} \delta^{(2p)}(y),$$

where we have used $|C(N)/C(N-1)|^2 = \sqrt{N/(N-1)}$. It is convenient to ignore the prefactor and work with the integral defined above directly. To simplify our analysis we introduce a rescaled kinetic energy

$$\tilde{T}[\mathfrak{f}] = \sqrt{\frac{N}{N-1}} T[\mathfrak{f}] \quad \text{(B1)}$$

such that

$$\tilde{T}[\mathfrak{f}]z = \int dx\, dy \sum_{a,b} \frac{1}{N^{a+b}} \mathfrak{f}_a^*\left(x - \frac{1}{2}y\right) \mathfrak{f}_b\left(x + \frac{1}{2}y\right) \times \mathcal{T}_H(y) \sum_p \frac{\mathcal{K}_p}{N^p} \frac{1}{(1 - \frac{1}{N})^p} \delta^{(2p)}(y). \quad \text{(B2)}$$

If we next consider the potential energy a similar expression may be defined. Starting with

$$V[\mathfrak{f}] = \int dx\, dy\, \mathfrak{f}^*\left(x - \frac{1}{2}y\right) \mathfrak{f}\left(x + \frac{1}{2}y\right) \times \mathcal{V}_H\left(1 - \frac{1}{N}\right) \langle \Theta_1; N-2 | \Theta_2; N-2 \rangle, \quad \text{(B3)}$$

we have

$$V[\mathfrak{f}] = \int dx\, dy \sum_{a,b} \frac{1}{N^{a+b}} \mathfrak{f}_a^*\left(x - \frac{1}{2}y\right) \mathfrak{f}_b\left(x + \frac{1}{2}y\right) \times \mathcal{V}_H(y)\left(1 - \frac{1}{N}\right) \left|\frac{C(N)}{C(N-2)}\right|^2 \times \sum_p \frac{\mathcal{K}_p}{N^p} \frac{1}{(1 - \frac{2}{N})^p} \delta^{(2p)}(y). \quad \text{(B4)}$$

As before we may use $|C(N)/C(N-2)|^2 = \sqrt{\frac{N}{N-2}}$ and introduce the function

$$\tilde{V}[\mathfrak{f}] = \sqrt{\frac{N(N-1)^2}{N^2(N-2)}} \tilde{V}[\mathfrak{f}] \quad \text{(B5)}$$

such that

$$\tilde{V}[\mathfrak{f}] = \int dx\, dy \sum_{a,b} \frac{1}{N^{a+b}} \mathfrak{f}_a^*\left(x - \frac{1}{2}y\right) \mathfrak{f}_b\left(x + \frac{1}{2}y\right) \times \mathcal{V}_H(y) \sum_p \frac{\mathcal{K}_p}{N^p} \frac{1}{(1 - \frac{2}{N})^p} \delta^{(2p)}(y). \quad \text{(B6)}$$

Notice that the expressions for $\tilde{T}$ and $\tilde{V}$ are nearly identical beyond cosmetic changes such as $\mathcal{T}_H \leftrightarrow \mathcal{V}_H$, save for one exception. The sum over p has a factor of $1/(1 - m/N)^p$, where $m = 1$ for $\tilde{T}$ and $m = 2$ for $\tilde{V}$; this effect enters first at $O(1/N^2)$ via the term

$$\frac{1}{N^2} m \mathcal{K}_1 \delta^{(2)}(y) \quad \text{for } m = 1 \text{ or } 2. \quad \text{(B7)}$$

At this level of accuracy we therefore have (omitting the explicit arguments of $x \pm \frac{1}{2}y$ for the sake of brevity)

$$\tilde{T}[\mathfrak{f}] = \int dx\, dy \left[\mathfrak{f}_0^*\mathfrak{f}_0 + \frac{1}{N}(\mathfrak{f}_1^*\mathfrak{f}_0 + \mathfrak{f}_0^*\mathfrak{f}_1) + \frac{1}{N^2}(\mathfrak{f}_2^*\mathfrak{f}_0 + \mathfrak{f}_0^*\mathfrak{f}_2 + \mathfrak{f}_1^*\mathfrak{f}_1)\right] \times \mathcal{T}_H(y)\left[\delta(y) + \frac{1}{N}\mathcal{K}_1\delta^{(2)}(y) + \frac{1}{N^2}[\mathcal{K}_2\delta^{(4)}(y) + \mathcal{K}_1\delta^{(2)}(y)\delta^{(2)}(y)]\right], \quad \text{(B8)}$$

$$\tilde{V}[\mathfrak{f}] = \int dx\, dy \left[\mathfrak{f}_0^*\mathfrak{f}_0 + \frac{1}{N}(\mathfrak{f}_1^*\mathfrak{f}_0 + \mathfrak{f}_0^*\mathfrak{f}_1) + \frac{1}{N^2}(\mathfrak{f}_2^*\mathfrak{f}_0 + \mathfrak{f}_0^*\mathfrak{f}_2 + \mathfrak{f}_1^*\mathfrak{f}_1)\right] \times \mathcal{V}_H(y)\left[\delta(y) + \frac{1}{N}\mathcal{K}_1\delta^{(2)}(y) + \frac{1}{N^2}[\mathcal{K}_2\delta^{(4)}(y) + 2\mathcal{K}_1\delta^{(2)}(y)\delta^{(2)}(y)]\right]. \quad \text{(B9)}$$

1. Kinetic energy

At leading order the only contribution to the kinetic energy is given by

$$\tilde{T}_0 = \int dx\, dy\, \mathfrak{f}_0^* \mathfrak{f}_0 \delta(y) \mathcal{T}_H(y) = \mathcal{T}_H^{(0)}. \tag{B10}$$

At next leading order we have

$$\begin{aligned}\tilde{T}_1 &= \int dx\, dy (\mathfrak{f}_1^* \mathfrak{f}_0 + \mathfrak{f}_0^* \mathfrak{f}_1) \delta(y) \\ &\quad + \int dx\, dy\, \mathfrak{f}_0^* \mathfrak{f}_0 \mathcal{K}_1 \delta^{(2)}(y) \\ &= 2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1 \rangle + \mathcal{K}_1 \sum_{n=0}^{1} \binom{2}{2n} (-1)^n \langle \mathfrak{f}_0^{(n)}, \mathfrak{f}_0^{(n)} \rangle \mathcal{T}_H^{(2-2n)} \\ &= 2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1 \rangle \mathcal{T}_H^{(0)} - \mathcal{K}_1 \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle \mathcal{T}_H^{(0)} + \mathcal{K}_1 \mathcal{T}_H^{(2)} \\ &= \mathcal{K}_1 \mathcal{T}_H^{(2)},\end{aligned}$$

where we have used Eq. (A9), and in going to the final equality, we have imposed the normalization condition (22). At next to next to leading order we have

$$\begin{aligned}\tilde{T}_2 &= \int dx\, dy (\mathfrak{f}_0^* \mathfrak{f}_2 + \mathfrak{f}_2^* \mathfrak{f}_0 + \mathfrak{f}_1^* \mathfrak{f}_1) \delta(y) \mathcal{T}_H(y) \\ &\quad + \int dx\, dy (\mathfrak{f}_0^* \mathfrak{f}_1 + \mathfrak{f}_1^* \mathfrak{f}_0) \mathcal{K}_1 \delta^{(2)}(y) \mathcal{T}_H(y) \\ &\quad + \int dx\, dy\, \mathfrak{f}_0^* \mathfrak{f}_0 \mathcal{K}_2 \delta^{(4)}(y) \mathcal{T}_H(y) \\ &\quad + \int dx\, dy\, \mathfrak{f}_0^* \mathfrak{f}_0 \mathcal{K}_1 \delta^{(2)}(y) \mathcal{T}_H(y)\end{aligned} \tag{B11}$$

using Eq. (A9) and

$$\begin{aligned}\tilde{T}_2 &= 2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_2 \rangle \mathcal{T}_H^{(0)} + \langle \mathfrak{f}_1, \mathfrak{f}_1 \rangle \mathcal{T}_H^{(0)} \\ &\quad + 2\mathcal{K}_1 \mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1 \rangle \mathcal{T}_H^{(2)} - 2\mathcal{K}_1 \mathrm{Re}\langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_1^{(1)} \rangle \mathcal{T}_H^{(0)} \\ &\quad + \mathcal{K}_2 \mathcal{T}_H^{(4)} - 6\mathcal{K}_2 \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle \mathcal{T}_H^{(2)} + \mathcal{K}_2 \langle \mathfrak{f}_0^{(2)}, \mathfrak{f}_0^{(2)} \rangle \mathcal{T}_H^{(0)} \\ &\quad + \mathcal{K}_1 \mathcal{T}_H^{(2)} - \mathcal{K}_1 \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle \mathcal{T}_H^{(0)}.\end{aligned} \tag{B12}$$

Summing all of the terms and imposing the normalization conditions from Eqs. (21)–(23), we find

$$\begin{aligned}\tilde{T}_2 &= \mathcal{K}_2 \mathcal{T}_H^{(4)} + \mathcal{K}_1 \mathcal{T}_H^{(2)} \\ &\quad + \big([\mathcal{K}_1^2 - 6\mathcal{K}_2] \mathcal{T}_H^{(2)} - \mathcal{K}_1 \mathcal{T}_H^{(0)} \big) \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle.\end{aligned} \tag{B13}$$

In conclusion, we find

$$\tilde{T}_0 = \mathcal{T}_H^{(0)}, \tag{B14}$$

$$\tilde{T}_1 = \mathcal{K}_1 \mathcal{T}_H^{(2)}, \tag{B15}$$

$$\begin{aligned}\tilde{T}_2 &= \mathcal{K}_2 \mathcal{T}_H^{(4)} + \mathcal{K}_1 \mathcal{T}_H^{(2)} \\ &\quad + \big([\mathcal{K}_1^2 - 6\mathcal{K}_2] \mathcal{T}_H^{(2)} - \mathcal{K}_1 \mathcal{T}_H^{(0)} \big) \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle.\end{aligned} \tag{B16}$$

Using $T = (1 - \frac{1}{2N} + \frac{3}{8N^2})\tilde{T}$, we then find

$$T_0 = \tilde{T}_0, \tag{B17}$$

$$T_1 = \tilde{T}_1 - \tfrac{1}{2}\tilde{T}_0, \tag{B18}$$

$$T_2 = \tilde{T}_2 - \tfrac{1}{2}\tilde{T}_1 + \tfrac{3}{8}\tilde{T}_0. \tag{B19}$$

2. Potential energy

The calculation for $\tilde{V}_n$ largely parallels that of $\tilde{T}_n$,

$$\begin{aligned}\tilde{V}_0 &= \int dx\, dy\, \mathfrak{f}_0^* \mathfrak{f}_0 \mathcal{V}_H(y) \delta(y) \\ &= \mathcal{V}_H^{(0)} \langle \mathfrak{f}_0, \mathfrak{f}_0 \rangle = \mathcal{V}_H^{(0)},\end{aligned} \tag{B20}$$

$$\begin{aligned}\tilde{V}_1 &= \int dx\, dy\, \mathfrak{f}_0^* \mathfrak{f}_0 \mathcal{V}_H(y) \mathcal{K}_1 \delta^{(2)}(y) \\ &\quad + [\mathfrak{f}_0^* \mathfrak{f}_1 + \mathfrak{f}_1^* \mathfrak{f}_0^*] \mathcal{V}_H(y) \delta(y) \\ &= \mathcal{V}_H^{(2)} + 2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1 \rangle \mathcal{V}_H^{(0)} - \mathcal{K}_1 \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle \mathcal{V}_H^{(0)} \\ &= \mathcal{V}_H^{(2)},\end{aligned} \tag{B21}$$

where we have used the c.m. wave-function normalization constraint (22). We then find

$$\begin{aligned}\tilde{V}_2 &= \int dx\, dy\, \mathfrak{f}_0^* \mathfrak{f}_0 \mathcal{V}_H(y) [\mathcal{K}_2 \delta^{(4)}(y) + 2\mathcal{K}_1 \delta^{(2)}(y)] \\ &\quad + \int dx\, dy [\mathfrak{f}_0^* \mathfrak{f}_1 + \mathfrak{f}_1^* \mathfrak{f}_0^*] \mathcal{V}_H(y) \mathcal{K}_1 \delta^{(2)}(y) \\ &\quad + \int dx\, dy [\mathfrak{f}_1^* \mathfrak{f}_1 + \mathfrak{f}_0^* \mathfrak{f}_2 + \mathfrak{f}_2^* \mathfrak{f}_0] \mathcal{V}_H(y) \delta(y).\end{aligned} \tag{B22}$$

Notice the factor of $2\mathcal{K}_1 \delta^{(2)}(y)$ in contrast to the factor of $\mathcal{K}_1 \delta^{(2)}(y)$ found in Eq. (B11).

As before, we will address each term in the calculation separately,

$$\begin{aligned}&2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_2 \rangle \mathcal{V}_H^{(0)} + \langle \mathfrak{f}_1, \mathfrak{f}_1 \rangle \mathcal{V}_H^{(0)} + 2\mathcal{K}_1 \big[\mathcal{V}_H^{(2)} - \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle \mathcal{V}_H^{(0)} \big] \\ &\quad + \mathcal{K}_2 \big[\mathcal{V}_H^{(4)} - 6 \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle \mathcal{V}_H^{(2)} + \langle \mathfrak{f}_0^{(2)}, \mathfrak{f}_0^{(2)} \rangle \mathcal{V}_H^{(0)} \big] \\ &\quad + \mathcal{K}_1 \big[(2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1 \rangle) \mathcal{V}_H^{(2)} - \big(2\,\mathrm{Re}\langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_1^{(1)} \rangle \big) \mathcal{V}_H^{(0)} \big].\end{aligned} \tag{B23}$$

Adding all of these terms together and making use of the normalization conditions (21)–(23), we find

$$\begin{aligned}\tilde{V}_2 &= \mathcal{K}_2 \mathcal{V}_H^{(4)} + 2\mathcal{K}_1 \mathcal{V}_H^{(2)} \\ &\quad + \big([\mathcal{K}_1^2 - 6\mathcal{K}_2] \mathcal{T}_H^{(2)} - \mathcal{K}_1 \mathcal{T}_H^{(0)} \big) \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle.\end{aligned} \tag{B24}$$

This leads to

$$\tilde{V}_0 = \mathcal{V}_H^{(0)}, \tag{B25}$$

$$\tilde{V}_1 = \mathcal{K}_1 \mathcal{V}_H^{(2)}, \tag{B26}$$

$$\begin{aligned}\tilde{V}_2 &= \mathcal{K}_2 \mathcal{V}_H^{(4)} + 2\mathcal{K}_1 \mathcal{V}_H^{(2)} \\ &\quad + \big([\mathcal{K}_1^2 - 6\mathcal{K}_2] \mathcal{V}_H^{(2)} - 2\mathcal{K}_1 \mathcal{V}_H^{(0)} \big) \langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)} \rangle.\end{aligned} \tag{B27}$$

Finally, we can use the formula $V = (1 + \frac{1}{N^2})\tilde{V} + O(1/N^3)$ to find

$$V_0 = \tilde{V}_0, \quad V_1 = \tilde{V}_1, \quad V_2 = \tilde{V}_2 + \tilde{V}_0. \tag{B28}$$

3. Total energy

Recall that $E[\mathfrak{f}] = T[\mathfrak{f}] - V[\mathfrak{f}]$. Let us focus first on the shift of the ground-state energy. We find, at leading order,

$$\delta E_0 \approx \frac{1}{N} \left[\mathcal{T}_H^{(2)} - \mathcal{V}_H^{(2)} - \frac{1}{2} \mathcal{T}_H^{(0)} \right]. \tag{B29}$$

For the gradient energy of the c.m. wave function, we find (again at leading order)

$$\hat{E}_2 \approx \left[\mathcal{K}_1^2 - 6\mathcal{K}_2\right]\left[\mathcal{T}_H^{(2)} - \mathcal{V}_H^{(2)}\right] - \mathcal{K}_1\left[\mathcal{T}_H^{(0)} - 2\mathcal{V}_H^{(0)}\right]. \quad \text{(B30)}$$

As emphasized in the main text, this is the mean result of our work and demonstrates that quantum fluctuations of the c.m. can lower the energy of a CCS state.

4. Cancellations due to normalization conditions

In the preceding section we found that terms such as $\langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)}\rangle$ were absent at $O(1/N)$ and likewise terms such as $\langle \mathfrak{f}_0^{(2)}, \mathfrak{f}_0^{(2)}\rangle$ were absent at $O(1/N^2)$. In this section we outline that this is not an accidental cancellation, but is a direct consequence of the normalization conditions (21)–(23).

To derive Eqs. (21)–(23) we demand that $\langle \mathfrak{f}; N|\mathfrak{f}; N\rangle = 1$ and that this normalization is maintained order by order in $1/N$. The exact expression for the overlap is given by

$$\langle \mathfrak{f}|\mathfrak{f}\rangle = \int dx\, dy\, \mathfrak{f}\left(x - \frac{1}{2}y\right)\mathfrak{f}^*\left(x + \frac{1}{2}y\right)\mathcal{O}(y; N). \quad \text{(B31)}$$

At leading order, using the δ expansion of $\mathcal{O}(y; N)$, this is equivalent to demanding that

$$\langle \mathfrak{f}_0, \mathfrak{f}_0\rangle := \int_{-\pi}^{\pi} \mathfrak{f}_0^*(x)\mathfrak{f}_0(x)dx = 1, \quad \text{(B32)}$$

which is Eq. (21). At $O(1/N)$ we find instead

$$\langle \mathfrak{f}|\mathfrak{f}\rangle = \langle \mathfrak{f}_0, \mathfrak{f}_0\rangle + \frac{1}{N}\left[2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1\rangle - \mathcal{K}_1\langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)}\rangle\right]. \quad \text{(B33)}$$

By requiring that this correction at $O(1/N)$ vanish, we arrive at Eq. (22). Similarly, at $O(1/N^2)$ we have

$$\begin{aligned}\langle \mathfrak{f}|\mathfrak{f}\rangle &= \langle \mathfrak{f}_0, \mathfrak{f}_0\rangle + \frac{1}{N}\left[(2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1\rangle) - \mathcal{K}_1\langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)}\rangle\right] \\ &\quad + \frac{1}{N^2}\left[\mathcal{K}_2\langle \mathfrak{f}_0^{(2)}, \mathfrak{f}_0^{(2)}\rangle - \mathcal{K}_1\left(2\,\mathrm{Re}\langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_1^{(1)}\rangle\right) \\ &\quad + 2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_2\rangle + \langle \mathfrak{f}_1, \mathfrak{f}_1\rangle\right]. \end{aligned} \quad \text{(B34)}$$

Our third normalization condition (23) then follows from the requirement that the bracketed term of $O(1/N^2)$ must vanish.

Importantly, this exact same combination of terms is guaranteed to appear in our calculations of $E[\mathfrak{f}]$. This is most clearly illustrated at $O(1/N)$. Let us consider just the term

$$\int dx\, dy\, \mathcal{K}_1\delta^{(2)}(y)\mathfrak{f}_0\mathfrak{f}_0^*\mathcal{V}_H(y) = \mathcal{K}_1\mathcal{V}_H^{(2)} - \mathcal{K}_1\mathcal{V}_H^{(0)}\langle \mathfrak{f}_0^{(1)}, \mathfrak{f}_0^{(1)}\rangle. \quad \text{(B35)}$$

Notice that when the derivatives act on the function $\mathfrak{f}_0$ it gives the same result as the normalization condition, but with an overall prefactor of $\mathcal{V}_H^{(0)}$. The same prefactor will also appear in the term

$$\begin{aligned}&\int dx\, dy\, \delta(y)[\mathfrak{f}_0(\alpha)\mathfrak{f}_1^*(\beta) + \mathfrak{f}_1(\alpha)\mathfrak{f}_0^*(\beta)]\mathcal{V}_H(y) \\ &\quad = \mathcal{V}_H^{(0)}(2\,\mathrm{Re}\langle \mathfrak{f}_0, \mathfrak{f}_1\rangle),\end{aligned} \quad \text{(B36)}$$

where we have used $\alpha = x - \frac{1}{2}y$ and $\beta = x + \frac{1}{2}y$ for shorthand. Upon addition of these two terms, we will have the combination that corresponds to Eq. (22). This happens when *all* of the derivatives from the δ expansion act on $\mathfrak{f}_0$; this leaves no derivatives left over to act on $\mathcal{V}_H(y)$ and this ensures that the prefactor appearing in front of $\mathcal{K}_n\langle \mathfrak{f}_0^{(n)}, \mathfrak{f}_0^{(n)}\rangle$ is $\mathcal{V}_H^{(0)}$. This is why the gradient corrections to the c.m. wave-function energy appear at $O(1/N^2)$ as opposed to $O(1/N)$ as may be naively expected. The same cancellation occurs at $O(1/N^2)$ but precludes terms of the form $\langle \mathfrak{f}_0^{(2)}, \mathfrak{f}_0^{(2)}\rangle$.

APPENDIX C: MANY-BODY OVERLAP FUNCTIONS

In the main text we claimed that the functions $\mathcal{O}(y; \aleph)$ could be expanded in the large-$\aleph$ limit in a δ expansion

$$\mathcal{O}(y; \aleph) = \frac{1}{|C(\aleph, \chi)|^2}\left[\delta(y) + \sum_{p>0}\frac{\mathcal{K}_p(\chi)}{\aleph^p}\delta^{(2p)}(y)\right]. \quad \text{(C1)}$$

In this Appendix we will justify this claim by making use of the properties of the Hartree wave functions $\psi_H(\theta)$. The results obtained in this Appendix will allow us to obtain explicit expressions for $\mathcal{K}_1$ and $\mathcal{K}_2$ in Appendix D. As noted before, the $\aleph$-body overlap can be rewritten as an exponentiated overlap of the Hartree states

$$\mathcal{O}(y; \aleph) = \left[\langle \psi_H; x - \tfrac{1}{2}y|\psi_H; x + \tfrac{1}{2}y\rangle\right]^{\aleph}, \quad \text{(C2)}$$

where

$$\begin{aligned}&\left\langle \psi_H; x - \frac{1}{2}y|\psi_H; x + \frac{1}{2}y\right\rangle \\ &\quad = \int d\theta\, \psi_H^*\left(\theta - \left[x - \frac{1}{2}y\right]\right)\psi_H\left(\theta - \left[x + \frac{1}{2}y\right]\right) \\ &\quad = \int d\theta\, \psi_H\left(\theta + \frac{1}{2}y\right)\psi_H\left(\theta - \frac{1}{2}y\right)\end{aligned} \quad \text{(C3)}$$

and we have used the fact that $\psi_H(\theta)$ is real. The forms of the Hartree wave functions are known; they are given by appropriately scaled and shifted Mathieu functions, with an auxiliary parameter $q(\chi)$ that can be determined exactly

$$\psi_H(\theta) = \frac{1}{\sqrt{\pi}}\mathrm{ce}_0\left[\frac{1}{2}(\theta - \pi); q(\chi)\right]. \quad \text{(C4)}$$

Thus, we have

$$\begin{aligned}&\left\langle \psi_H; x - \frac{1}{2}y|\psi_H; x + \frac{1}{2}y\right\rangle \\ &\quad = \frac{1}{\pi}\int d\theta\, \mathrm{ce}_0\left(\frac{1}{2}\theta; q\right)\mathrm{ce}_0\left[\frac{1}{2}(\theta + y); q\right].\end{aligned} \quad \text{(C5)}$$

Now for $\chi \ll 1$ we have that $q \sim 1/\chi^2$ such that q is very large. In this regime the Mathieu functions are well approximated by parabolic cylinder functions D_n via the Sips expansion [63]

$$\mathrm{ce}_0(z; q) \sim C_0(q)[U_0(\xi; q) + V_0(\xi; q)], \quad \text{(C6)}$$

$$C_0(q) \sim \left[\frac{\pi\sqrt{q}}{2}\right]^{1/4}\left[1 + \frac{1}{8\sqrt{q}}\right]^{-1/2}, \quad \text{(C7)}$$

$$U_0(\xi; q) \sim D_0(\xi) - \frac{1}{4\sqrt{q}}D_4(\xi), \quad \text{(C8)}$$

$$V_0(\xi; q) \sim -\frac{1}{16\sqrt{q}}D_2(\xi) \quad \text{(C9)}$$

such that

$$\mathrm{ce}_0(z;q) \sim \left[\frac{\pi\sqrt{q}}{2}\right]^{1/4} D_0(\xi) + O\left(\frac{1}{\sqrt{q}}\right). \tag{C10}$$

Introducing the variables $\zeta = 2q^{1/4}\sin\frac{\theta}{2}$, we then find

$$\psi_H(\theta;\chi) \sim \left[\frac{\sqrt{q}}{2\pi}\right]^{1/4} D_0(\zeta) + O\left(\frac{1}{\sqrt{q}}\right)$$
$$= \left[\frac{q}{(2\pi)^2}\right]^{1/8} e^{-\sqrt{q}\sin^2(\theta/2)} + O\left(\frac{1}{\sqrt{q}}\right). \tag{C11}$$

Using the leading-order behavior for ψ_H, the overlap can be expressed as a Bessel function

$$\left\langle \psi_H; x - \frac{1}{2}y | \psi_H; x + \frac{1}{2}y \right\rangle$$
$$= \int_0^{2\pi} d\theta\, \psi_H^*\left(\theta - \frac{1}{2}y;\chi\right)\psi_H\left(\theta + \frac{1}{2}y;\chi\right)$$
$$\sim \left[\frac{q}{(2\pi)^2}\right]^{1/4} \int_0^{2\pi} d\theta\, e^{-\sqrt{q}\sin^2[(x-y/2)/2]} e^{-\sqrt{q}\sin^2[(x+y/2)/2]}$$
$$= \left[\frac{q}{(2\pi)^2}\right]^{1/4} \int_0^{2\pi} d\theta\, e^{\sqrt{q}[1-\cos x\cos(y/2)]}$$
$$= \frac{I_0\left(\sqrt{q}\cos\frac{y}{2}\right)}{\sqrt{2\pi q^{1/2} e^{\sqrt{q}}}}, \tag{C12}$$

where $I_0(z)$ is the modified Bessel function of the first kind [63]. At the same order of accuracy we can instead write

$$\left\langle \psi_H; x - \frac{1}{2}y | \psi_H; x + \frac{1}{2}y \right\rangle \sim \frac{I_0\left(\sqrt{q}\cos\frac{y}{2}\right)}{I_0(\sqrt{q})}, \tag{C13}$$

which is exact for $y = 0$. For most values of y we can use a large-argument expansion for the Bessel function $I_0(z) \sim e^z/\sqrt{2\pi z}$. For values of y such that $\sqrt{q}\cos\frac{y}{2} \sim O(1)$ it follows that $I_0(y) \sim O(1)$ and so the overlap is $O(q^{1/4}e^{-\sqrt{q}})$.

When considering integrals on the interval $y \in [-\pi, \pi]$ it is therefore justifiable to neglect contributions from this exponentially suppressed region. Then, on the remainder of the interval, we can use the large-argument expansion of the Bessel function as a global approximation. This allows us to rewrite the overlap as

$$\left\langle \psi_H; x - \frac{1}{2}y | \psi_H; x + \frac{1}{2}y \right\rangle$$
$$\sim \exp\left[2\sqrt{q}\sin^2\frac{y}{4} - \frac{1}{2}\log\cos\frac{y}{2} + O\left(\frac{1}{\sqrt{q}}\right)\right]. \tag{C14}$$

By extension the $\aleph$-body overlap assumes the form

$$\mathcal{O}(y;\aleph) \sim \exp\left\{\aleph\left[2\sqrt{q}\sin^2\frac{y}{4} - \frac{1}{2}\log\cos\frac{y}{2}\right]\right\}, \tag{C15}$$

where we have neglected terms of $O(1/\sqrt{q})$ or smaller. Trading q for χ via $q \sim 4\chi^{-2}(1-\chi/4)$, we find at the same order of accuracy

$$\mathcal{O}(y;\aleph) \sim \exp\left\{\aleph\left[\left(\frac{4}{\chi} - \frac{1}{2}\right)\sin^2\frac{y}{4} - \frac{1}{2}\log\cos\frac{y}{2}\right]\right\}. \tag{C16}$$

APPENDIX D: SMALL-χ EXPANSIONS

As noted in the main text, $\mathcal{T}_H(y)$'s leading-order behavior as a function of χ is important. We would like to compute $\mathcal{T}_H^{(0)}$ and $\mathcal{T}_H^{(2)}$ and we will make use of the Sips expansion for the ground-state wave functions (C6),

$$\psi_H(x) \sim \left[\frac{\sqrt{q}}{2\pi}\right]^{1/4}\left[D_0(\zeta) - \frac{1}{16\sqrt{q}}\mathfrak{D}(\zeta)\right], \tag{D1}$$

where $\zeta = 2q^{1/4}\sin\frac{\theta}{2}$ and

$$\mathfrak{D}(\zeta) = D_0(\zeta) + D_2(\zeta) + \tfrac{1}{4}D_4(\zeta). \tag{D2}$$

We are interested in

$$\mathcal{T}_H(y) = \frac{\chi^2}{2}\int dx \left[\frac{d}{dx}\psi_H\left(x - \frac{1}{2}y\right)\right]\left[\frac{d}{dx}\psi_H\left(x + \frac{1}{2}y\right)\right]. \tag{D3}$$

It will be useful to have the identities

$$\frac{d\zeta_\pm}{dx} = q^{1/4}\cos\left(\frac{x \pm \frac{1}{2}y}{2}\right) = q^{1/4}\left(1 - \frac{\zeta_\pm^2}{4\sqrt{q}}\right)^{1/2}, \tag{D4}$$

$$\frac{d^2\zeta_\pm}{dx^2} = -\frac{q^{1/4}}{2}\left(\frac{x \pm \frac{1}{2}y}{2}\right) = -\frac{q^{1/4}}{4}\zeta_\pm, \tag{D5}$$

where $\zeta_\pm = \zeta(x \pm \frac{1}{2}y)$, with which we can reexpress Eq. (D3) as

$$\mathcal{T}_H(y) = -\frac{\chi^2}{2}\sqrt{q}\int \frac{d\zeta}{q^{1/4}\sqrt{1-\frac{\zeta^2}{4\sqrt{q}}}}\left(1 - \frac{\zeta_-^2}{4\sqrt{q}}\right)^{1/2}$$
$$\times \left(1 - \frac{\zeta_+^2}{4\sqrt{q}}\right)^{1/2}\psi_H'(\zeta_-)\psi_H'(\zeta_+). \tag{D6}$$

At leading order in $1/\sqrt{q}$ we have

$$\mathcal{T}_H(y) \sim -\frac{\chi^2}{2}\sqrt{q}\int \frac{d\zeta}{\sqrt{2\pi}} D_0'(\zeta_-)D_0'(\zeta_+). \tag{D7}$$

This leads immediately to the result

$$\mathcal{T}_H^{(0)} \sim -\frac{\chi^2}{2}\sqrt{q}\int \frac{d\zeta}{\sqrt{2\pi}} D_0'(\zeta)D_0'(\zeta)$$
$$= -\frac{\sqrt{q}}{8}\chi^2. \tag{D8}$$

Next, to calculate $\mathcal{T}_H^{(2)}$ we must act with $\frac{d^2}{dy^2}$ on Eq. (D7). A useful identity is

$$\frac{d^2}{dy^2}[f(\zeta_-)g(\zeta_+) + f(\zeta_+)g(\zeta_-)]$$
$$= \left[\frac{d^2\zeta_+}{dy^2} + \frac{d^2\zeta_-}{dy^2}\right][f'g + g'f] + 4\frac{d\zeta_+}{dy}\frac{d\zeta_-}{dy}f'g'$$
$$+ \left[\left(\frac{d\zeta_+}{dy}\right)^2\left(\frac{d\zeta_-}{dy}\right)^2\right][f''g + g''f], \tag{D9}$$

which holds when $y = 0$. We can insert this identity underneath the integral after acting with the derivative operator. This will give us an integral representation for $\mathcal{T}_H^{(2)} := \mathcal{T}_H''(y=0)$. Using the explicit forms of the derivatives

$$\frac{d\zeta_\pm}{dy} = \pm\frac{q^{1/4}}{2}\left(1-\frac{\zeta_\pm^2}{4\sqrt{q}}\right)^{1/2}, \tag{D10}$$

$$\frac{d^2\zeta_\pm}{dy^2} = -\frac{\zeta}{16}, \tag{D11}$$

we find

$$\begin{aligned}\mathcal{T}_H^{(2)} &\sim -q\frac{\chi^2}{2}\int\frac{d\zeta}{\sqrt{2\pi}}2[D_0'D_0' - 2D_0'''D_0'] \\ &= 2q\chi^2\int\frac{d\zeta}{\sqrt{2\pi}}D_0''D_0'' \\ &= \frac{3q\chi^2}{32},\end{aligned} \tag{D12}$$

where we have used the leading-order approximation for $d\zeta_\pm/dy$ and neglected the contribution from terms proportional to $d^2\zeta_\pm/d^2y$ because they are subleading. To obtain the second equality we integrated by parts; however at higher orders in $1/\sqrt{q}$ one needs to be careful to keep track of factors of ζ^2 in the integrand.

When calculating $\mathcal{V}_H^{(0)}$ and $\mathcal{V}_H^{(2)}$ we need to work beyond leading order, because the leading-order piece cancels in Eq. (29). We are interested in

$$\begin{aligned}\mathcal{V}_H(y) = \frac{1}{2}\int dx_1 dx_2 \psi_H\left(x_1+\frac{y}{2}\right)\psi_H\left(x_1-\frac{y}{2}\right) \\ \times\,\psi_H\left(x_2+\frac{y}{2}\right)\psi_H\left(x_2-\frac{y}{2}\right)\cos(x_1-x_2),\end{aligned} \tag{D13}$$

which can be rewritten as

$$\mathcal{V}_H(y) = \frac{1}{2}[I_C(y)^2 + I_S(y)^2], \tag{D14}$$

$$I_C(y) = \int dx\,\psi_H\left(x+\frac{y}{2}\right)\psi_H\left(x-\frac{y}{2}\right)\cos(x), \tag{D15}$$

$$I_S(y) = \int dx\,\psi_H\left(x+\frac{y}{2}\right)\psi_H\left(x-\frac{y}{2}\right)\sin(x). \tag{D16}$$

Importantly $I_S(0)=0$, $I_S'(0)=0$, and $I_C'(0)$ such that

$$\mathcal{V}_H^{(0)} = \tfrac{1}{2}I_C^2(0), \qquad \mathcal{V}_H^{(2)} = I_C(0)I_C''(0), \tag{D17}$$

so we can focus exclusively on the integral $I_C(y)$. Rewriting this in terms of ζ and keeping only terms to order $1/\sqrt{q}$, we arrive at

$$\begin{aligned}I_C(y) = \frac{1}{\sqrt{2\pi}}\bigg[\int D_0(\zeta_-)D_0(\zeta_+)\left(1-\frac{3\zeta^2}{8\sqrt{q}}\right)d\zeta \\ -\frac{1}{16\sqrt{q}}\int D_0(\zeta_-)\mathfrak{D}(\zeta_+) + D_0(\zeta_+)\mathfrak{D}(\zeta_-)d\zeta\bigg].\end{aligned} \tag{D18}$$

Evaluating at $y=0$ sets $\zeta_\pm=\zeta$ and we find

$$\begin{aligned}I_C(0) = \frac{1}{\sqrt{2\pi}}\bigg[\int D_0(\zeta)D_0(\zeta)\left(1-\frac{3\zeta^2}{8\sqrt{q}}\right)d\zeta \\ -\frac{1}{8\sqrt{q}}\int D_0(\zeta)\mathfrak{D}(\zeta)d\zeta\bigg] = 1-\frac{1}{2\sqrt{q}}.\end{aligned} \tag{D19}$$

To find $I_C''(0)$ we must act on Eq. (D18) with $\frac{d^2}{dy^2}$. Being careful to retain subleading terms, we find

$$\begin{aligned}I_C''(0) &= -\frac{\sqrt{q}}{\sqrt{2\pi}}\bigg[\int\frac{D_0'D_0' - D_0''D_0}{2}\left(1-\frac{5\zeta^2}{8\sqrt{q}}\right)d\zeta \\ &\quad + \frac{2}{\sqrt{2\pi}}\int D_0'D_0\frac{\zeta}{16}d\zeta - \frac{1}{8\sqrt{q}}\int D_0'\mathfrak{D}'d\zeta\bigg] \\ &= -\frac{\sqrt{q}}{4}\left(1-\frac{3}{4\sqrt{q}}\right).\end{aligned} \tag{D20}$$

Using $\mathcal{V}_H^{(0)} = \frac{1}{2}[I_C(0)]^2$, $\mathcal{V}_H^{(2)} = I_C''(0)I_C(0)$, and the small-$\chi$ behavior of q [60],

$$q \sim \frac{4}{\chi^2}\left[1-\frac{\chi}{4}+O(\chi^2)\right], \tag{D21}$$

we then find

$$\begin{aligned}\mathcal{V}_H^{(0)} &= \frac{1}{2}\left[1-\frac{1}{\sqrt{q}}+O\left(\frac{1}{q}\right)\right] \\ &= \frac{1}{2}\left[1-\frac{\chi}{2}+O(\chi^2)\right],\end{aligned} \tag{D22}$$

$$\begin{aligned}\mathcal{V}_H^{(2)} &= -\frac{\sqrt{q}}{4}\left[1-\frac{5}{16\sqrt{q}}+O\left(\frac{1}{q}\right)\right] \\ &= -\frac{1}{2\chi}\left[1-\frac{3\chi}{4}+O(\chi^2)\right],\end{aligned} \tag{D23}$$

$$\begin{aligned}\mathcal{T}_H^{(0)} &= \chi^2\frac{\sqrt{q}}{8}\left[1+O\left(\frac{1}{\sqrt{q}}\right)\right] \\ &= \frac{\chi}{4}[1+O(\chi)],\end{aligned} \tag{D24}$$

$$\begin{aligned}\mathcal{T}_H^{(2)} &= \chi^2\left(-\frac{3q}{32}\right)\left[1+O\left(\frac{1}{\sqrt{q}}\right)\right] \\ &= -\frac{3}{8}[1+O(\chi)].\end{aligned} \tag{D25}$$

[1] T. Dauxois, S. Ruffo, E. Arimondo, and M. Wilkens, *Dynamics and Thermodynamics of Systems with Long Range Interactions*, Lecture Notes in Physics Vol. 602 (Springer, Berlin, 2008).

[2] A. Campa, T. Dauxois, and S. Ruffo, Phys. Rep. **480**, 57 (2009).

[3] Y. Levin, R. Pakter, F. B. Rizzato, T. N. Teles, and F. P. C. Benetti, Phys. Rep. **535**, 1 (2014).

[4] J. Barré, D. Mukamel, and S. Ruffo, Phys. Rev. Lett. **87**, 030601 (2001).
[5] L. Casetti and M. Kastner, Physica A **384**, 318 (2007).
[6] M. Kastner, Phys. Rev. Lett. **104**, 240403 (2010).
[7] H. Spohn, in *Large Scale Dynamics of Interaction*, 1st ed., edited by E. H. Lieb, W. Beiglbock, T. Regge, R. P. Geroch, and W. Thirring (Springer, Berlin, 1992).
[8] S. Goldstein, S. Cuperman, and M. Lecar, Mon. Not. R. Astron. Soc. **143**, 209 (1969).
[9] M. Lecar and L. Cohen, in *Gravitational N-Body Problem: Proceedings of the Iau Colloquium No. 10 Held in Cambridge, England August 12–15, 1970*, edited by M. Lecar, Astrophysics and Space Science Library Vol. 31 (Springer Netherlands, Dordrecht, 1972), pp. 262–275.
[10] J. Barré, F. Bouchet, T. Dauxois, and S. Ruffo, Phys. Rev. Lett. **89**, 110601 (2002).
[11] J. Barré, F. Bouchet, T. Dauxois, and S. Ruffo, Eur. Phys. J. B **29**, 577 (2002).
[12] Y. Y. Yamaguchi, J. Barré, F. Bouchet, T. Dauxois, and S. Ruffo, Physica A **337**, 36 (2004).
[13] Y. Y. Yamaguchi, Phys. Rev. E **78**, 041114 (2008).
[14] M. Joyce and T. Worrakitpoonpon, Phys. Rev. E **84**, 011139 (2011).
[15] Y. Y. Yamaguchi and S. Ogawa, Phys. Rev. E **92**, 042131 (2015).
[16] D. Lynden-Bell, Mon. Not. R. Astron. Soc. **136**, 101 (1967).
[17] R. Pakter and Y. Levin, Phys. Rev. Lett. **106**, 200603 (2011).
[18] T. N. Teles, Y. Levin, and R. Pakter, Mon. Not. R. Astron. Soc.: Lett. **417**, L21 (2011).
[19] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 2011), pp. 289–291.
[20] D. Lynden-Bell, Physica A **263**, 293 (1999).
[21] D. Lynden-Bell and R. Wood, Mon. Not. R. Astron. Soc. **138**, 495 (1968).
[22] F. J. Dyson, Commun. Math. Phys. **12**, 91 (1969).
[23] M. E. Fisher, S.-k. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).
[24] P. Bruno, Phys. Rev. Lett. **87**, 137203 (2001).
[25] D. Peter, S. Müller, S. Wessel, and H. P. Büchler, Phys. Rev. Lett. **109**, 025303 (2012).
[26] M. F. Maghrebi, Z.-X. Gong, and A. V. Gorshkov, Phys. Rev. Lett. **119**, 023001 (2017).
[27] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).
[28] W. Braun and K. Hepp, Commun. Math. Phys. **56**, 101 (1977).
[29] E. H. Lieb and H.-T. Yau, Astrophys. J. **323**, 140 (1987).
[30] S. A. Cannas, A. C. N. de Magalhães, and F. A. Tamarit, Phys. Rev. B **61**, 11521 (2000).
[31] A. Campa, A. Giansanti, and D. Moroni, Phys. Rev. E **62**, 303 (2000).
[32] F. Tamarit and C. Anteneodo, Phys. Rev. Lett. **84**, 208 (2000).
[33] J. Barré, F. Bouchet, T. Dauxois, and S. Ruffo, J. Stat. Phys. **119**, 677 (2005).
[34] A. Campa and P.-H. Chavanis, J. Stat. Mech. (2010) P06001.
[35] T. Mori, Phys. Rev. E **82**, 060103(R) (2010).
[36] T. Mori, Phys. Rev. E **84**, 031128 (2011).
[37] T. Mori, J. Stat. Mech. (2013) P10003.
[38] I. Hahn and M. Kastner, Eur. Phys. J. B **50**, 311 (2006).
[39] A. Campa and S. Ruffo, Physica A **369**, 517 (2006).
[40] A. Campa, S. Ruffo, and H. Touchette, Physica A **385**, 233 (2007).
[41] M. Antoni and S. Ruffo, Phys. Rev. E **52**, 2361 (1995).
[42] V. Latora, A. Rapisarda, and S. Ruffo, Physica D **131**, 38 (1999).
[43] F. Ginelli, K. A. Takeuchi, H. Chaté, A. Politi, and A. Torcini, Phys. Rev. E **84**, 066211 (2011).
[44] T. Manos and S. Ruffo, Transp. Theory Stat. Phys. **40**, 360 (2011).
[45] L. H. M. Filho, M. A. Amato, and T. M. Rocha Filho, J. Stat. Mech. (2018) 033204.
[46] J. Barré, F. Bouchet, T. Dauxois, S. Ruffo, and Y. Y. Yamaguchi, Physica A **365**, 177 (2006).
[47] W. Ettoumi and M.-C. Firpo, J. Phys. A: Math. Theor. **44**, 175002 (2011).
[48] R. Plestid, P. Mahon, and D. H. J. O'Dell, Phys. Rev. E **98**, 012112 (2018).
[49] G. Giachetti and L. Casetti, J. Stat. Mech. (2019) 043201.
[50] P. Chavanis, J. Stat. Mech. (2011) P08003.
[51] M. Kac, G. E. Uhlenbeck, and P. Hemmer, J. Math. Phys. **4**, 216 (1963).
[52] J. Barré and F. Bouchet, C. R. Phys. **7**, 414 (2006).
[53] S. Schütz and G. Morigi, Phys. Rev. Lett. **113**, 203002 (2014).
[54] T. Keller, V. Torggler, S. B. Jäger, S. Schütz, H. Ritsch, and G. Morigi, New J. Phys. **20**, 025004 (2018).
[55] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 2000).
[56] K. Kim, M.-S. Chang, R. Islam, S. Korenblit, L.-M. Duan, and C. Monroe, Phys. Rev. Lett. **103**, 120502 (2009).
[57] J. Zeiher, J.-Y. Choi, A. Rubio-Abadal, T. Pohl, R. van Bijnen, I. Bloch, and C. Gross, Phys. Rev. X **7**, 041063 (2017).
[58] R. Islam, C. Senko, W. C. Campbell, S. Korenblit, J. Smith, A. Lee, E. E. Edwards, C.-C. J. Wang, J. K. Freericks, and C. Monroe, Science **340**, 583 (2013).
[59] P.-H. Chavanis, J. Stat. Mech. (2011) P08002.
[60] R. Plestid and D. H. J. O'Dell, Phys. Rev. E **100**, 022216 (2019).
[61] T.-L. Ho and S. K. Yip, Phys. Rev. Lett. **84**, 4031 (2000).
[62] Y. Castin and C. Herzog, C. R. Acad. Sci. Ser. IV **2**, 419 (2001).
[63] *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Cambridge University Press, Cambridge, 2010), Vol. 5.
[64] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer Science + Business Media, New York, 2012).
[65] N. Schupper and N. M. Shnerb, Phys. Rev. E **72**, 046107 (2005).
[66] M. Sellitto, Phys. Rev. B **73**, 180202(R) (2006).