Breakdown of effective temperature, power law interactions, and self-propulsion in a momentum-conserving active fluid

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The simplest extensions of single-particle dynamics in a momentum-conserving active fluid—an active suspension of two colloidal particles or a single particle confined by a wall—exhibit strong departures from Boltzmann behavior, resulting in either a breakdown of an effective temperature description or a steady state with nonzero-entropy production rate. This is a consequence of hydrodynamic interactions that introduce multiplicative noise in the stochastic description of particle positions. This results in fluctuation-induced interactions that depend on distance as a power law. We find that the dynamics of activated colloids in a passive fluid, with stochastic forcing localized on the particle, is different from that of passive colloids in an active fluctuating fluid.

I. INTRODUCTION

Fluctuations of a dilute active suspension (e.g., bacterial bath [1,2]) have often been described in terms of an equilibrium system with a (large) effective temperature [3,4], with theoretical rationalisations provided by studies of the dynamics of a single particle in an active fluctuating fluid or a single active particle embedded in a passive fluid [4,5].

However, as we find here, even the simplest extensions— the stochastic dynamics of *two* particles embedded in an unbounded isotropic active fluctuating gel or a particle in an isotropic active fluctuating gel bound within confining walls—do not allow for an effective temperature description, since the corresponding steady-state probability distribution shows strong departures from the Boltzmann form. This is a consequence of a drift that arises from hydrodynamic interactions, that introduces a multiplicative noise in the stochastic description of particle positions. The form of the drift can only be decided *after solving the full hydrodynamics problem*. This has important implications for current discussions on active contributions to pressure, osmotic pressure and surface tension in momentum-conserving active fluids [4,6,7].

Indeed, deviations from Boltzmann behavior and consequent breakdown of an effective temperature description have been systematically analyzed in a *dry* system of active Ornstein-Uhlenbeck particles (AOUP) [8], where momentum is not conserved. Within a systematic perturbation expansion in the active noise correlation time τ_n, the nonequilibrium nature of the steady-state distribution first shows up at order

τ_n (characterized by non-Boltzmann probability distribution but zero-entropy dissipation), while the full nonequilibrium aspect with nonzero-entropy production shows up at order $\tau_n^{3/2}$ [8]. Moreover, there have been several studies on the form of the effective interaction between passive particles (both fixed and mobile) embedded in a bath of dry active particles. For instance, passive colloids in a suspension of self-propelled particles interact through a nonequilibrium analog of depletion forces [9–11]. The range of these interactions depends on the shape of tracers—being short-range between passive spheres [9], and long-range between two parallel walls [11]. In addition, passive spheres can have large interactions due to density fluctuations of dry active particles [12].

In this paper, we ask, what is the nature of effective interactions and departures from Boltzmann distribution, in momentum-conserving active systems? We find that the simplest extensions of single-particle dynamics, viz., that of colloid particle-wall and colloid particle-particle interactions embedded in a three-dimensional active fluctuating gel, exhibits a clear non-Boltzmann steady-state distribution, characterized by an effective attractive potential ($\propto 1/r$, for particle-wall separated by r and $\propto 1/r^4$, for particle-particle). This is a consequence of the active (fluctuation-dissipation relation violating) fluctuations and hydrodynamic interactions that introduce a multiplicative noise. A dimer of unequal-sized spherical particles embedded in an active fluctuating gel self-propels with a velocity proportional to the fluctuation amplitude. Interestingly, for this momentum-conserving active system, both the breakdown of the effective temperature description and finite entropy production rate, *appear even in the limit* $\tau_n \to 0$. We next study the statistics of fluctuations of activated particles in a passive fluid, i.e., particles directly subject to a stochastic driving force. We find that the dynamics of activated particles in a passive medium is *not* the same as the dynamics of passive particles in an active medium—for

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instance, the effective particle-particle interaction is repulsive and long ranged ($\propto 1/r^2$). We proceed to demonstrate these results below.

Consider an incompressible, isotropic, actively fluctuating viscoelastic gel, described by a local stress,

$$(1 + \tau_v \partial_t)\sigma_{ij} = -p + \eta(\partial_i v_j + \partial_j v_i) + \sigma_{ij}^n, \qquad (1)$$

where $i, j \in (x, y, z)$, τ_v is a Maxwell time, η is the viscosity, p is the pressure which includes, *a priori*, the isotropic component of the mean and fluctuating active stress, and σ_{ij}^n is the fluctuating component of the active deviatoric stress, with zero mean and correlation [13,14]

$$\langle \sigma_{ij}^n(\mathbf{r}, t)\sigma_{kl}^n(\mathbf{r}', t') \rangle$$
$$= 2\pi\delta(\mathbf{r} - \mathbf{r}')\Delta(t - t')\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right], \qquad (2)$$

with $\Delta(t - t') = \Lambda\tau_n^{-1}e^{-|t-t'|/\tau_n}$. For simplicity, we have taken the variance of the anisotropic stress fluctuation to be a scalar $\Delta(t - t')$. In general, $\Delta_{ijkl}(t - t')$ is a fourth-rank tensor, which can arise from fluctuations of the nematic order parameter [13]. Since the temporal correlations of the noise are unrelated to the drag ($\tau_v \neq \tau_n$), this system does not satisfy the generalized Stokes-Einstein relation at a microscopic scale. This is a minimal active system, the nonequilibrium component is included as active fluctuations that violate fluctuation dissipation relation. Note that the stress in Eq. (1) should also have a stochastic contribution due to the thermal fluctuations, which we take to be much smaller than the active fluctuations.

Throughout this paper, we work at timescales larger than τ_n and τ_v, thus Eq. (1) becomes

$$\sigma_{ij} = -p + \eta(\partial_i v_j + \partial_j v_i) + \sigma_{ij}^n. \qquad (3)$$

with $\Delta(t - t') = 2\Lambda\delta(t - t')$, obtained by taking the limit $\tau_n \to 0$. The dynamics in the Stokes limit is $\nabla \cdot \boldsymbol{\sigma} = 0$, along with the incompressibility condition $\nabla \cdot \boldsymbol{v} = 0$. We emphasize that at long times, the stress as given by Eq. (3) is identical to that of a passive viscous fluid, however, as we show in the following, the stochastic dynamics of embedded colloids is, in general, not the same as that of colloids in a passive fluid.

The generalized Langevin dynamics of a spherical colloidal particle of radius a embedded in an unbounded fluid defined by Eq. (1) is obtained by integrating out the stress Eq. (1) and using no-slip boundary condition at its surface (see Appendix of Ref. [13]),

$$\tau_m\ddot{\mathbf{R}} + \int_{-\infty}^{t} dt' \gamma(t - t')\dot{\mathbf{R}}(t') = -\mu\nabla \cdot \mathsf{U} + \sqrt{2\lambda\mu}\,\boldsymbol{\theta}(t), \qquad (4)$$

where $\lambda = \Lambda/\eta$, $\mu = 1/6\pi\eta a$, $\tau_m = m\mu$, $\gamma(t) = \tau_v \exp(-t/\tau_v)$ is the friction coefficient, U is an externally applied potential, and $\boldsymbol{\theta}(t)$ is a Gaussian noise with correlation

$$\langle \theta_i(t)\theta_j(t') \rangle = \delta_{ij}\tau_n \exp[-(t - t')/\tau_n]. \qquad (5)$$

In the limit $\tau_m, \tau_v, \tau_n \to 0$, Eq. (4) reduces to

$$\partial_t\mathbf{R} = -\mu\nabla \cdot \mathsf{U} + \sqrt{2\lambda\mu}\,\boldsymbol{\vartheta}, \qquad (6)$$

where $\boldsymbol{\vartheta}(t)$ is a unit variance Gaussian white noise. As expected, the overdamped Langevin dynamics obtained by

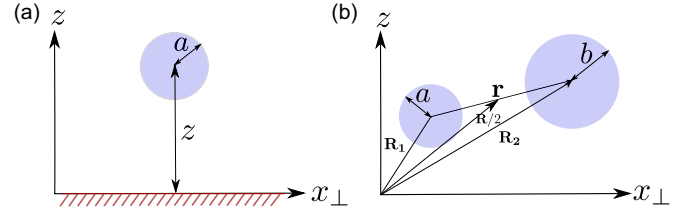FIG. 1. Schematic of (a) a sphere of radius a at distance z from a fixed wall at $z = 0$, and (b) two spheres of radius a and b centered at $\mathbf{R_1}$ and $\mathbf{R_2}$, respectively, in an unbound fluctuating viscous fluid.

integrating out the fluid stress Eq. (3) and using no-slip boundary condition at the surface [15–17] is identical to Eq. (6). Henceforth, wherever required, we use the standard form of Langevin equations corresponding to Eq. (3).

The steady-state probability distribution of the position of the colloidal sphere obtained from the Fokker-Planck equation corresponding to Eq. (6) has a Boltzmann form $P(z) \propto e^{-\mathsf{U}/\lambda}$, with an effective temperature $k_B T_{\text{eff}} \equiv \lambda$.

II. PARTICLE-WALL INTERACTION

The Langevin dynamics of a spherical colloid of radius a at a distance z from a fixed wall at $z = 0$ [Fig. 1(a)], is obtained by integrating out Eq. (3) and using the no-slip boundary condition at the surface of the colloid [17],

$$\partial_t z = -H_z\partial_z U + \sqrt{2\lambda H_z}\vartheta_z, \qquad (7)$$

$$\partial_t x_\perp = -H_{x_\perp}\partial_{x_\perp}U + \sqrt{2\lambda H_{x_\perp}}\vartheta_{x_\perp}, \qquad (8)$$

where $x_\perp \equiv (x, y)$, $H_z(H_{x_\perp})$ is the mobility in the longitudinal (transverse) direction to the wall, $U(z, x_\perp)$ is the particle-wall potential, and ϑ_z and $\vartheta_{x_\perp}$ are zero mean Gaussian white noise with correlation

$$\langle \vartheta_i(t)\vartheta_j(t') \rangle = 2\lambda H_i^{-1}\delta_{ij}\delta(t - t'), \qquad (9)$$

where $i, j \in (x, y, z)$. The mobilities $H_i(z)$ are functions of the separation z from the wall, which can be evaluated as power series in a/z (Appendix B and Ref. [18]). The appearance of a multiplicative and correlated noise is typical of a stochastic dynamics with hydrodynamic interactions. As is well known [19,20], this Langevin equation is meaningless unless supplemented with a stochastic calculus convention for the multiplicative noise. The choice of convention depends on the fast timescales that have been integrated out—viscoelastic relaxation time (τ_v), particle inertial relaxation time ($\tau_m \sim m/\eta a$, where m is the particle mass and a is the particle size), and noise correlation time (τ_n). In Ref. [21,22] it was shown that for an active noise with $\tau_n \gg \sqrt{\tau_m\tau_v}$, the appropriate convention is Stratonovich [20,23]. For the cell cortex, for instance, the timescales are $\tau_v \sim 1$–10 s [21,24], $\tau_m \sim 10^{-9}$ s, and $\tau_n \sim 10$ s [21], which makes $\sqrt{\tau_m\tau_v} \sim 0.1$ ms $\ll \tau_n$. In the following, we work in the limit $\tau_n \gg \sqrt{\tau_m\tau_v}$, we believe this limit is reasonable for many biological systems. For the appropriate choice of noise convention when this limit does not hold, we refer to Refs. [21,22].

The Fokker-Planck equation corresponding to Eqs. (7) and (8), interpreted in Stratonovich convention is (Appendix A)

$$\partial_t P = \partial_z \left(H_z \partial_z U + \tfrac{1}{2}\lambda(\partial_z H_z) + \lambda H_z \partial_z \right) P$$
$$+ \partial_{x_\perp}(H_{x_\perp}\partial_{x_\perp}U + \lambda H_{x_\perp}\partial_{x_\perp})P, \qquad (10)$$

which leads to a steady-state probability distribution $P(z, x_\perp) \propto e^{-\Phi/\lambda}$, with an effective potential,

$$\Phi = U + \tfrac{1}{2}\lambda \log H_z. \qquad (11)$$

This effective interaction between the wall and the particle, as a result of the active noise and hydrodynamics, is long-range (goes as a/z, for large z), attractive, and anisotropic. The additional fluctuation term along with the applied potential cannot be captured by a simple effective temperature definition. Nevertheless, the steady state has zero current and obeys time reversal symmetry, making this a *nonthermal equilibrium model* [8]. We emphasize, however, that unlike in Ref. [8], this deviation from Boltzmann measure occurs *even in the limit $\tau_n \to 0$*. Further, in contrast to thermal fluctuations where hydrodynamic interactions only affect dynamics, active fluctuations in a fluid affect *both* the dynamics and the steady state.

III. PRESSURE

The contribution to the force on the wall due to the bare colloid-wall potential $U(z)$ is

$$F_p = \int_0^\infty dz P(z)\partial_z U(z) \propto \int_0^\infty dz \frac{1}{\sqrt{H_z}}e^{-U/\lambda}\partial_z U(z). \quad (12)$$

The observation that the force on the wall depends on the form of wall-particle interaction, is directly related to the fact that the probability distribution is non-Boltzmann. This is analogous to the situation in dry active particle systems [25], where, apart from the kinetic contribution, this would have sufficed to give the pressure. However, in momentum-conserving Stokesian fluid systems, the net force due to particle-wall interactions is *balanced* by the force due to the embedding fluid.

Consider a suspension of N colloidal particles in a semi-infinite active fluid confined by a wall. Assuming particles do not interact directly with each other via forces such as van der Waals, the total force on the Ith particle reads

$$f_i^{\text{tot}} = -\int_{S_p} \sigma_{ij} dS_j + \int_{V_p} dv\, g_i^p, \qquad (13)$$

where σ_{ij} is the fluid stress acting on the particle I, S_p is the surface with normal pointing out of the fluid, g_1^p is the force density on the colloid due to wall-particle interaction, and V_p is the volume of the particle I. Force balance on the particle I: $f_i^{\text{tot}} = 0$ implies

$$\int_{S_p} \sigma_{ij} dS_j = \int_{V_p} dv\, g_i^p. \qquad (14)$$

The dynamics of the Stokesian fluid is given by

$$g_i^f + \partial_j \sigma_{ij} = 0, \qquad (15)$$

where g_i^f is the force density on the fluid due to interaction between fluid particles and the wall. Integrating this relation

over a volume V bounded by two parallel surfaces S_f and S_f', the first at the wall, the second in the fluid at a distance such that the body forces g_i^p and g_i^f vanish:

$$\int_{S_f} \sigma_{ij} dS_j + \int_{S_f'} \sigma_{ij} dS_j + \int_{V-NV_p} dV g_i^f = -\sum_{I=1}^N \int_{S_p} \sigma_{ij} dS_j, \qquad (16)$$

where N are the number of colloidal particles within the volume V and the surface normals point outside the considered fluid volume. Substituting Eq. (14) into Eq. (16) gives

$$\int_{S_f} \sigma_{ij} dS + \int_{S_f'} \sigma_{ij} dS_j = -\int_{V-V_p} dV g_i^f - \sum_{I=1}^N \int_{V_p} dv\, g_i^p. \qquad (17)$$

Now, the force exerted by the suspension on the wall is

$$F_i = -\int_{S_f} \sigma_{ij} dS_j - \int_{V-V_p} dV g_i^f - \sum_{I=1}^N \int_{V_p} dv\, g_i^p, \qquad (18)$$

where the first term is the force on the confining wall due to the fluid, the second term is the force on the wall due to the fluid particles, which is equal and opposite to the force on the fluid due to the wall by Newton third law, similarly, the third term is due to the interaction between the colloidal particles and the wall. Substituting Eq. (17) into Eq. (18) we get

$$F_i = \int_{S_f'} \sigma_{ij} dS_j. \qquad (19)$$

Since S_f' is an arbitrary surface chosen to be far from the wall, we see that there is no net added force on the wall even though the particles feel an effective attraction towards it. The total force exerted on the wall by the system particles + fluid vanishes exactly. This point appears to have been disregarded in recent theoretical studies of the active contribution to osmotic pressure and surface tension in momentum-conserving active fluids [4,6,7].

IV. TWO PARTICLES EMBEDDED IN ACTIVE FLUID

We now consider the dynamics of two spherical colloids of radius a and b, centered at $\mathbf{R}_1$, and $\mathbf{R}_2$, respectively [Fig. 1(b)], in an unbounded active fluctuating fluid. Since for spherical colloids, the translational motion of the center of mass is decoupled from the rotational motion, we will consider only the hydrodynamic coupling between the translational degree of freedom.

The Langevin dynamics for the centers of the spherical colloids follows as before and is given by [2,18]

$$\frac{d\mathbf{R}_\alpha}{dt} = \sum_\beta \mathsf{H}_{\alpha\beta} \cdot (\mathbf{f}_\beta + \boldsymbol{\vartheta}_\beta), \qquad (20)$$

where α and β are particle labels, $\mathsf{H}_{\alpha\beta}$ is the 3×3 hydrodynamic interaction tensor coupling the translation motion of particle β with that of particle α (see Appendix B for the form), and $\mathbf{f}_\beta = -\nabla_{\mathbf{R}_\beta} U$ is the deterministic force from an externally applied potential U. The stochastic force $\boldsymbol{\vartheta}_\beta$ on the particle β, is a three-dimensional vector of Gaussian white

noise with correlation

$$\langle \vartheta_{i\alpha}(t)\vartheta_{j\beta}(s)\rangle = 2\lambda \mathsf{H}_{\alpha\beta}^{-1}\delta_{ij}\delta(t-s). \qquad (21)$$

The Fokker-Planck equation corresponding to Eq. (20), interpreted in Stratonovich convention is Ref. [26] (see Appendix A)

$$\partial_t P = \nabla_{\mathbf{R}_\alpha} \cdot \left[-\mathsf{H}_{\alpha\beta} \cdot \mathbf{f}_\beta + \frac{\lambda}{2}\left(\nabla_{\mathbf{R}_\beta} \cdot \mathsf{H}_{\alpha\beta}\right) + \lambda \mathsf{H}_{\alpha\beta} \cdot \nabla_{\mathbf{R}_\beta} \right] P. \qquad (22)$$

Due to incompressibility, $\nabla_{\mathbf{R}_\beta} \cdot \mathsf{H}_{\alpha\beta}$ is identically zero in Oseen approximation of $\mathsf{H}_{\alpha\beta}$ [17], the first nonzero contribution is at order $1/r^4$. In terms of variables, $\mathbf{R} \equiv \mathbf{R_1} + \mathbf{R_2}$ and $\mathbf{r} \equiv \mathbf{R_2} - \mathbf{R_1}$, Eq. (22) is

$$\partial_t P = \nabla_{\mathbf{R}} \cdot (-\mathsf{M}_{11} \cdot \tilde{\mathbf{f}}_1 - \mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_2 + \lambda \mathsf{M}_{11} \cdot \nabla_{\mathbf{R}})P$$
$$+ \lambda \nabla_{\mathbf{R}} \cdot \left(\frac{1}{2}\nabla_{\mathbf{r}} \cdot \mathsf{M}_{12} + \mathsf{M}_{12} \cdot \nabla_{\mathbf{r}}\right)P$$
$$+ \nabla_{\mathbf{r}} \cdot (\lambda \mathsf{M}_{12} \cdot \nabla_{\mathbf{R}})P$$
$$+ \nabla_{\mathbf{r}} \cdot \left(-\mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_1 - \mathsf{M}_{22} \cdot \tilde{\mathbf{f}}_2\right.$$
$$\left.+ \frac{\lambda}{2}\nabla_{\mathbf{r}} \cdot \mathsf{M}_{22} + \lambda \mathsf{M}_{22} \cdot \nabla_{\mathbf{r}}\right)P, \qquad (23)$$

where $\tilde{\mathbf{f}}_1 = -\nabla_{\mathbf{R}}U, \tilde{\mathbf{f}}_2 = -\nabla_{\mathbf{r}}U$, and the mobility matrix

$$\mathsf{M} = \begin{bmatrix} \mathsf{H}_{11} + \mathsf{H}_{22} + 2\mathsf{H}_{12} & \mathsf{H}_{22} - \mathsf{H}_{11} \\ \mathsf{H}_{22} - \mathsf{H}_{11} & \mathsf{H}_{11} + \mathsf{H}_{22} - 2\mathsf{H}_{12} \end{bmatrix}. \qquad (24)$$

Let us first look at equal sized colloids, $a = b$. In this case, the off-diagonal block matrix $\mathsf{M}_{12} = 0$, and Eq. (23) reduces to

$$\partial_t P = \nabla_{\mathbf{R}} \cdot (-\mathsf{M}_{11} \cdot \tilde{\mathbf{f}}_1 + \lambda \mathsf{M}_{11} \cdot \nabla_{\mathbf{R}})P$$
$$+ \nabla_{\mathbf{r}} \cdot \left(-\mathsf{M}_{22} \cdot \tilde{\mathbf{f}}_2 + \frac{\lambda}{2}\nabla_{\mathbf{r}} \cdot \mathsf{M}_{22} + \lambda \mathsf{M}_{22} \cdot \nabla_{\mathbf{r}}\right)P. \qquad (25)$$

Now, the necessary and sufficient condition for the steady-state solution of the Fokker-Plank equation of the from

$$\partial_t P = \sum_{\alpha=1}^{N} \nabla_\alpha \cdot (-\mathbf{F}_\alpha + \mathsf{D}_{\alpha\beta}\nabla_\beta)P, \qquad (26)$$

to have zero probability current is that it satisfies the potential condition [20], defined as

$$\frac{\partial Z_{\beta j}}{\partial R_{\alpha i}} = \frac{\partial Z_{\alpha i}}{\partial R_{\beta j}}, \qquad (27)$$

where $i, j \in (x, y, z)$ and

$$Z_{\alpha i} = \mathsf{D}_{\alpha i, \beta j}^{-1}F_{\beta j}. \qquad (28)$$

We see that at steady state, Eq. (25) satisfies the potential condition Eq. (27). Using this, we find that the steady-state distribution has the form $P \propto e^{-\Phi/\lambda}$, where

$$\Phi(\mathbf{r}, \mathbf{R}) = U - \frac{1}{2}\lambda\left[\frac{15b^4}{8r^4} + \mathcal{O}\left(\frac{1}{r^5}\right)\right]. \qquad (29)$$

As in the colloid-wall interaction, the steady-state distribution is non-Boltzmann with a fluctuation-induced particle-particle interaction that is attractive, though short-ranged.

For spheres of unequal size, $a \neq b$, the steady-state distribution does not obey the potential condition given by Eq. (27); thus the steady state has a finite probability current and an associated entropy production rate, resulting in a finite propulsion velocity at steady state. We emphasize that unlike AOUP [8], this nonequilibrium steady state with finite entropy dissipation occurs *even in the limit of* $\tau_n \to 0$.

To see this, we describe the two particles as a dimer, characterized by the separation r, the orientation $\hat{\mathbf{r}}$, and the center-of-mass position $\mathbf{R}$ [27]. In general, it is difficult to obtain an analytic expression for the steady-state distribution $P(r, \hat{\mathbf{r}}, \mathbf{R})$. However, in a well defined limit where there is a timescale separation, we obtain analytic expressions for the steady-state distribution, fluctuation-induced potential and mean propulsion velocity.

Taking U to be a function of r alone, $\tilde{\mathbf{f}}_1 = 0$ and $\tilde{\mathbf{f}}_2 = -U'(r)\hat{\mathbf{r}}$. With this, the probability flux for $\mathbf{r}$ in Eq. (25) is now independent of $\mathbf{R}$, hence, we can integrate out $\mathbf{R}$ to obtain the marginal dynamics of $\mathbf{r}$. This allows us to solve for the steady-state marginal distribution. The steady state for the marginal distribution: $P(r)$, obtained by integrating Eq. (14) in the main text over $\mathbf{R}$, with no flux boundary condition gives

$$\nabla_{\mathbf{r}} \log P(\mathbf{r}) = \frac{1}{\lambda}\tilde{\mathbf{f}}_2 - \frac{1}{2}\mathsf{M}_{22}^{-1} \cdot (\nabla_{\mathbf{r}} \cdot \mathsf{M}_{22}). \qquad (30)$$

The mobility tensor M defined in Eq. (15) decomposed as sum of projection along $\hat{\mathbf{r}}$ (denoted by superscript n) and perpendicular to $\hat{\mathbf{r}}$ (denoted by superscript q) is

$$\mathsf{M}_{\alpha\beta} = m_{\alpha\beta}^q(r)(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + m_{\alpha\beta}^n(r)\hat{\mathbf{r}}\hat{\mathbf{r}}. \qquad (31)$$

From this we see that the inverse is

$$\mathsf{M}_{\alpha\beta}^{-1} = \frac{1}{m_{\alpha\beta}^q}(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + \frac{1}{m_{\alpha\beta}^n}\hat{\mathbf{r}}\hat{\mathbf{r}}, \qquad (32)$$

and its divergence is

$$\nabla_{\mathbf{r}} \cdot \mathsf{M}_{\alpha\beta} = \left[\frac{\partial}{\partial r}m_{\alpha\beta}^n + 2\frac{\left(m_{\alpha\beta}^n - m_{\alpha\beta}^q\right)}{r}\right]\hat{\mathbf{r}}. \qquad (33)$$

Using Eqs. (32) and (33) we get

$$\mathsf{M}_{22}^{-1} \cdot \left(\nabla_{\mathbf{r}} \cdot \mathsf{M}_{22}\right) = \frac{1}{m_{22}^n}\left[\frac{\partial}{\partial r}m_{22}^n + 2\frac{\left(m_{22}^n - m_{22}^q\right)}{r}\right]\hat{\mathbf{r}}. \qquad (34)$$

The form of M is given in Appendix B. We see that in Oseen and Rotne-Pragar approximation of M the right hand side of Eq. (34) is zero. The first nonzero contribution comes when M is of order $1/r^4$, at which the self mobilities of the particles are also modified. Substituting M to the order $1/r^4$ (see Appendix B) in Eq. (34) gives

$$\mathsf{M}_{22}^{-1} \cdot (\nabla_{\mathbf{r}} \cdot \mathsf{M}_{22}) = \frac{15ab(a^3 + b^3)}{2(a+b)}\frac{1}{r^5}\hat{\mathbf{r}}. \qquad (35)$$

Substituting this expression in Eq. (30) and using $\tilde{\mathbf{f}} = -U'(r)\hat{\mathbf{r}}$ gives

$$\nabla_{\mathbf{r}} \log P(\mathbf{r}) = -\frac{1}{\lambda}U'(r)\hat{\mathbf{r}} - \frac{15ab(a^3 + b^3)}{4(a+b)}\frac{1}{r^5}\hat{\mathbf{r}}. \qquad (36)$$

Integrating this gives

$$\log P(\mathbf{r}) \propto -\frac{1}{\lambda} U(r) + \frac{15ab(a^3 + b^3)}{16(a+b)} \frac{1}{r^4}, \quad (37)$$

which gives $P(r) \propto e^{-\Phi(\mathbf{r})/\lambda}$, where

$$\Phi = U(r) - \frac{1}{2}\lambda \left[\frac{15ab(a^3 + b^3)}{8(a+b)} \frac{1}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) \right]. \quad (38)$$

Note that while we can define an effective potential for the marginal dynamics of r, there is no effective potential description in the full Fokker-Planck description that includes $\hat{\mathbf{r}}$ and $\mathbf{R}$. If we now assume that the dynamics of r is *fast*, we can decompose the probability distribution as

$$P(\mathbf{R}, \hat{\mathbf{r}}, r) = P(r) \int dr\, r^2 P(\mathbf{R}, \hat{\mathbf{r}}, r) = P(r)\tilde{P}(\mathbf{R}, \hat{\mathbf{r}}), \quad (39)$$

where we have defined $\tilde{P}(\mathbf{R}, \hat{\mathbf{r}}) \equiv \int dr\, r^2 P(\mathbf{R}, \hat{\mathbf{r}}, r)$.

Decomposing the derivative as radial and rotational derivative

$$\nabla_{\mathbf{r}} = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{r}} \times \mathcal{R}, \quad (40)$$

where $\mathcal{R}$ is the rotation operator that rotates the $\mathbf{r}$ keeping r fixed. In spherical coordinates it reads

$$\hat{\mathbf{r}} \times \mathcal{R} = \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi}. \quad (41)$$

Integrating out r from Eq. (25) we get [27]

$$\partial_t \tilde{P} = \nabla_{\mathbf{R}} \cdot \left[-v_0 \hat{\mathbf{r}} + \lambda \left\langle \frac{\mathsf{M}_{12}}{r} \right\rangle \cdot \hat{\mathbf{r}} \times \mathcal{R} + \lambda \langle \mathsf{M}_{11} \rangle \cdot \nabla_{\mathbf{R}} \right] \tilde{P}$$
$$+ \lambda \hat{\mathbf{r}} \times \mathcal{R} \cdot \left[\left\langle \frac{\mathsf{M}_{12}}{r} \right\rangle \cdot \nabla_{\mathbf{R}} + \left\langle \frac{\mathsf{M}_{22}}{r^2} \right\rangle \cdot \hat{\mathbf{r}} \times \mathcal{R} \right] \tilde{P}, \quad (42)$$

where the averages are over the distributions $P(r)$, $\langle \phi(r) \rangle = \int dr\, r^2 \phi(r) P(r)$, and the self-propelled velocity

$$v_0 = \int r^2 dr\, \hat{\mathbf{r}} \cdot \left(\mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_2 - \frac{\lambda}{2} \nabla_{\mathbf{r}} \cdot \mathsf{M}_{12} - \lambda \mathsf{M}_{12} \cdot \nabla_{\mathbf{r}} \right) P(r). \quad (43)$$

Note that if the fluctuations are thermal, the middle term in Eq. (43) is not present, the probability distribution has the Boltzmann form $P(r) \propto e^{-U/\lambda}$, and the velocity vanishes as shown Appendix C. To evaluate the average, we consider the bare interparticle potential to be a stiff spring, $U = k(r-l)^2/2$, and $l \gg a, b$. In this limit, the relaxation timescale of r is set by M_{22}, k, and λ; $kl^2/2\lambda \gg 1$ ensures that the r dynamics is fast [27]. Using the Laplace approximation [28], we obtain, to leading order in $1/l$, the propulsion velocity

$$v_0 = \frac{5ab}{4\pi\eta}(b-a)\frac{\lambda}{l^5} + \mathcal{O}\left(\frac{1}{l^6}\right), \quad (44)$$

directed along $\hat{\mathbf{r}}$, if $b > a$. Orientation decorrelation will lead to diffusion over times longer than the orientation correlation time of $\hat{\mathbf{r}}$ (τ_R). The enhancement of the diffusion constant due to this self-propulsion in units of diffusion of a sphere of radius l is $v_0^2 \tau_R/D_t = (1-\delta)^2 \delta^2 \epsilon^6$, where $D_t = \lambda/6\pi\eta l$, $\tau_R \sim 3\eta l^3/\lambda$, $\delta = a/b$, and $\epsilon = b/l$. Since $\delta < 1$ and $\epsilon \ll 1$, this enhancement is very small.

V. ACTIVATING THE PARTICLE BY A FLUCTUATING FORCE

Now consider two colloidal spheres of radii a and b, embedded in an unbounded passive fluid, each of which experiences a stochastic force, *localized on the colloids*. To make the discussion simple, we set the stress fluctuations of the embedding medium $\sigma_{ij}^n = 0$; the dynamics of the colloids is then given by Eq. (20), with $\boldsymbol{\vartheta}_\beta = 0$, and the applied force on the particles is a sum of deterministic and stochastic components, $\mathbf{f}_\beta \equiv \mathbf{f}_\beta + \boldsymbol{\xi}_\beta$. The fluctuating force on the two colloids is taken to be isotropic, zero mean Gaussian white with correlations,

$$\langle \xi_{\alpha i}(t) \xi_{\beta j}(s) \rangle = 2\Lambda \delta_{\alpha\beta} \delta_{ij} \delta(t-s). \quad (45)$$

Since the fluctuation is external and the dissipation is in the fluid bath, the system is always active. Hence, for any choice of noise convention the fluctuation dissipation relation is not satisfied. For consistency with the approximation for passive fluid, we take the noise correlation time to be the slowest timescale leading to Stratonovich noise convention. The corresponding Fokker-Planck equation (in Stratonovich convention) is now (see Appendix A)

$$\partial_t P = \nabla_{\mathbf{R}} \cdot (-\mathsf{M}_{11} \cdot \tilde{\mathbf{f}}_1 - \mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_2 + \Lambda \mathsf{D}_{11} \cdot \nabla_{\mathbf{R}})P$$
$$+ \Lambda \nabla_{\mathbf{R}} \cdot \left(\tfrac{1}{2} \nabla_{\mathbf{r}} \cdot \mathsf{D}_{12} + \mathsf{D}_{12} \cdot \nabla_{\mathbf{r}} \right)P$$
$$+ \nabla_{\mathbf{r}} \cdot (\Lambda \mathsf{D}_{12} \cdot \nabla_{\mathbf{R}})P$$
$$+ \nabla_{\mathbf{r}} \cdot \left(-\mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_1 - \mathsf{M}_{22} \cdot \tilde{\mathbf{f}}_2 \right.$$
$$\left. + \tfrac{1}{2} \Lambda \nabla_{\mathbf{r}} \cdot \mathsf{D}_{22} + \Lambda \mathsf{D}_{22} \cdot \nabla_{\mathbf{r}} \right)P, \quad (46)$$

where M is given by Eq. (24), and the diffusion matrix $\mathsf{D}_{\alpha\beta}$ is 3×3 matrix given in terms of $\mathsf{H}_{\alpha\beta}$ by $\mathsf{D}_{11} = (\mathsf{H}_{11} + \mathsf{H}_{12})^2 + (\mathsf{H}_{22} + \mathsf{H}_{12})^2$, $\mathsf{D}_{12} = \mathsf{D}_{21} = \mathsf{H}_{22}^2 - \mathsf{H}_{11}^2$, and $\mathsf{D}_{22} = (\mathsf{H}_{11} - \mathsf{H}_{12})^2 + (\mathsf{H}_{22} - \mathsf{H}_{12})^2$.

Once again, Eq. (46) does not satisfy the potential condition given by Eq. (27) and hence does not have a zero probability current steady state, *even when the spheres are of the same size*. This proves that the dynamics of particles in an active medium is fundamentally different from the dynamics of activated particles. In this context, we refer to recent experiments [29,30] in which two spheres embedded in a fluid are held in two optical traps. A fluctuating force is applied on one sphere by moving the position of its laser trap randomly. This has been modeled as a two temperature system [29–31], where the static particle feels the bath temperature and the particle in the fluctuating trap, a higher temperature. Our study demonstrates the inadequacy of such an effective temperature approach, and in principle (at least numerically) provides a full solution to the steady-state distribution.

We now consider the case when U is a function of r alone implying $\tilde{\mathbf{f}}_1 = 0$ and $\tilde{\mathbf{f}}_2 = -U'\hat{\mathbf{r}}$. For this case Eq. (46) reduces to

$$\partial_t P = \nabla_{\mathbf{R}} \cdot \left(-\mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_2 + \tfrac{1}{2} \nabla_{\mathbf{r}} \cdot \mathsf{D}_{12} + \Lambda \mathsf{D}_{11} \cdot \nabla_{\mathbf{R}} \right)P$$
$$+ \Lambda \nabla_{\mathbf{R}} \cdot (\mathsf{D}_{12} \cdot \nabla_{\mathbf{r}})P + \nabla_{\mathbf{r}} \cdot (\Lambda \mathsf{D}_{12} \cdot \nabla_{\mathbf{R}})P$$
$$+ \nabla_{\mathbf{r}} \cdot \left(-\mathsf{M}_{22} \cdot \tilde{\mathbf{f}}_2 + \tfrac{1}{2}\Lambda \nabla_{\mathbf{r}} \cdot \mathsf{D}_{22} + \Lambda \mathsf{D}_{22} \cdot \nabla_{\mathbf{r}} \right)P. \quad (47)$$

The steady state for the marginal distribution $P(r)$ obtained by integrating Eq. (47) over $\mathbf{R}$, with no flux boundary condition, gives

$$\nabla_{\mathbf{r}} \log P(\mathbf{r}) = \frac{1}{\Lambda} \mathsf{D}_{22}^{-1} \cdot \mathsf{M}_{22} \cdot \tilde{\mathbf{f}}_2 - \frac{1}{2} \mathsf{D}_{22}^{-1} \cdot \nabla_{\mathbf{r}} \cdot \mathsf{D}_{22}. \quad (48)$$

The tensor D decomposed as sum of projection along $\hat{\mathbf{r}}$ (denoted by superscript n) and perpendicular to $\hat{\mathbf{r}}$ (denoted by superscript q) is

$$\mathsf{D}_{\alpha\beta} = d^q_{\alpha\beta}(r)(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + d^n_{\alpha\beta}(r)\hat{\mathbf{r}}\hat{\mathbf{r}}. \quad (49)$$

The inverse is

$$\mathsf{D}_{\alpha\beta}^{-1} = \frac{1}{d^q_{\alpha\beta}}(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + \frac{1}{d^n_{\alpha\beta}}\hat{\mathbf{r}}\hat{\mathbf{r}}, \quad (50)$$

and the divergence is

$$\nabla_{\mathbf{r}} \cdot \mathsf{D}_{\alpha\beta} = \left[\frac{\partial}{\partial r} d^n_{\alpha\beta} + 2 \frac{(d^n_{\alpha\beta} - d^q_{\alpha\beta})}{r} \right] \hat{\mathbf{r}}. \quad (51)$$

Using Eqs. (50) and (51), we get

$$\mathsf{D}_{22}^{-1} \cdot (\nabla_{\mathbf{r}} \cdot \mathsf{D}_{22}) = \frac{1}{d^n_{22}} \left[\frac{\partial}{\partial r} d^n_{22} + 2 \frac{(d^n_{22} - d^q_{22})}{r} \right] \hat{\mathbf{r}}. \quad (52)$$

Taking the diffusion tensor D (see Appendix D) to the order $1/r$ and substituting it in Eq. (52) gives

$$\mathsf{D}_{22}^{-1} \cdot (\nabla_{\mathbf{r}} \cdot \mathsf{D}_{22}) = -\frac{9(a^2 b^2)}{4(a^2 + b^2)} \frac{1}{r^3} \hat{\mathbf{r}}, \quad (53)$$

and substituting the diffusion tensor in the first term on the right-hand side of Eq. (48), we obtain

$$\mathsf{D}_{22}^{-1} \cdot \mathsf{M}_{22} \cdot \tilde{\mathbf{f}}_2 = -\frac{m^n_{22}}{d^n_{22}} U'(r) \hat{\mathbf{r}}. \quad (54)$$

Substituting Eqs. (53) and (54) into Eq. (48) gives

$$\nabla_{\mathbf{r}} \log P(\mathbf{r}) = -\frac{1}{\Lambda} \frac{m^n_{22}}{d^n_{22}} U'(r) \hat{\mathbf{r}} + \frac{9(a^2 b^2)}{8(a^2 + b^2)} \frac{1}{r^3} \hat{\mathbf{r}}. \quad (55)$$

From this the effective potential $\Phi \equiv -\Lambda \log P$ upon integration of Eq. (55) is

$$\Phi = \int dr r^2 \frac{m^n_{22}}{d^n_{22}} U' + \Lambda \frac{9(a^2 b^2)}{16(a^2 + b^2)} \frac{1}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right). \quad (56)$$

Note that in this case m_{22}/d_{22} depends on r, it is not possible to define an effective free energy, keeping the energy U and a constant effective temperature. As in the colloid-colloid interaction in an active fluid, the steady-state distribution is non-Boltzmann with a fluctuation-induced particle-particle interaction. However, in contrast, the interaction is repulsive, long-ranged, and depends on the form of the interaction potential U. This effect is similar to that of effective colloid-wall interactions due interplay between hydrodynamic and electrostatic interactions [32].

Integrating out r from Eq. (47) leads to the similar form of self-propulsion velocity as Eq. (43). To order $1/r^3$, M_{12} is a constant and self-propulsion velocity is

$$v_0 = -N \int dr r^2 \left(m^n_{12} U' - d^n_{12} \frac{\partial}{\partial r} \Phi \right) e^{-\Phi/\Lambda}. \quad (57)$$

Using Eq. (56) we get

$$\frac{\partial}{\partial r} \Phi = \frac{m^n_{22}}{d^n_{22}} U' - \Lambda \frac{9(a^2 b^2)}{8(a^2 + b^2)} \frac{1}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \quad (58)$$

and

$$\frac{m^n_{22}}{d^n_{22}} = \frac{(6\pi\eta ba)(a + b)}{a^2 + b^2} \left[1 + \frac{6b^2 a^2}{(a^2 + b^2)(a + b)} \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \right]. \quad (59)$$

Thus, to leading order in $1/r$, we obtain

$$v_0 = -N \int dr r^2 m^n_{12} \left(1 - \frac{d^n_{12} m^n_{22}}{d^n_{22} m^n_{12}} \right) U' e^{-U/\Lambda'}, \quad (60)$$

where $\Lambda' = \Lambda(a^2 + b^2)/6\pi\eta ab(a + b)$. As before, taking $U = k(r - l)^2/2$, with k large and $l \gg a, b$, such that $kl^2/2\Lambda \gg 1$, gives the mean self-propulsion velocity of the dimer to be

$$v_0 \sim \frac{a - b}{18\pi^2 \eta^2 ab(a + b)} \frac{\Lambda}{l},$$

directed along $\hat{\mathbf{r}}$. Note that this leading order contribution is due to the interaction potential between the dimer. In contrast, the leading order contribution in the fluctuation fluid case was from the fluctuation induced interaction. The long-time dynamics of the dimer is diffusive, thus resembling an active Brownian particle [33]. The enhancement in diffusion constant over the bare diffusion $D_t \sim k_B T/6\pi\eta l$ is

$$\frac{v_0^2 \tau_R}{D_t} \sim \frac{(1 - \delta)^2}{(1 + \delta^2)^2 \epsilon^2} \frac{\Lambda'^2}{k_B T^2}, \quad (61)$$

where the rotational correlation time is $\tau_R \sim 3\eta l^3/\lambda$, $\delta = a/b$, $\epsilon = b/l$, and $\Lambda' = \Lambda(a^2 + b^2)/6\pi\eta ab(a + b)$. Since both $k_B T/\Lambda', \epsilon \ll 1$, this enhancement can now be large compared to the bare diffusion. For $\Lambda' \sim k_B T$, $a = 2$ nm, $b = 3$ nm, $l = 10$ nm, and $\eta = 10^{-3}$ Pa s we get $v_0 = 5$ mm/s and $v_0^2 \tau_R \sim 25 \, \mu m^2/s$, a value which is comparable to thermal diffusivity D_t.

One-dimensional example of self-propulsion

To obtain an intuitive understanding of the self-propulsion described above, consider a simple example in which two spherical colloids interacting via a harmonic potential are confined along the x axis. The center of the spheres are positioned at $x_1(t)$ and $x_2(t)$ with $x_1 < x_2$. This dimer is activated by an external force f acting on the colloid at x_1. The one-dimensional Langevin dynamics for this dimer, with the hydrodynamic interaction tensor [18] (Appendix B) can be obtained from Eq. (20),

$$\dot{x}_1 = \left(\frac{1}{6\pi\eta a} - \frac{1}{4\pi\eta r} \right) k(r - l) + \frac{f}{6\pi\eta a}, \quad (62)$$

$$\dot{x}_2 = -\left(\frac{1}{6\pi\eta a} - \frac{1}{4\pi\eta r} \right) k(r - l) + \frac{f}{4\pi\eta a}, \quad (63)$$

where $r = x_2 - x_1$. In terms of the separation r and center of mass $R = x_1 + x_2$ we get

$$\dot{r} = -\left(\frac{1}{6\pi\eta a} - \frac{1}{4\pi\eta r}\right)[2k(r-l)+f], \quad (64)$$

$$\dot{R} = \left(\frac{1}{6\pi\eta a} + \frac{1}{4\pi\eta r}\right)f. \quad (65)$$

At steady state, $r = l - f/2k$ and the center of mass velocity is

$$\dot{R} = \frac{f}{6\pi\eta a} + \frac{f}{4\pi\eta(l - f/2k)}. \quad (66)$$

Thus, we see that the magnitude of $\dot{R}$ are not equal for $f \to -f$. For positive f the harmonic spring is compressed leading to a larger speed in comparison to the case when f is negative and the spring is stretched leading to a lower speed. Now consider a periodic symmetric driving where f switches direction after fixed time intervals of duration Δt. The average force applied is zero, however, the dimer will have a net positive velocity given by

$$\langle\dot{R}\rangle = \frac{f^2/k}{8\pi\eta(l^2 - (f/2k)^2)}. \quad (67)$$

This mechanism is similar to that proposed in Refs. [34,35] for the self-propulsion of an elastic dimer on a frictional substrate. This setup can be experimentally realized by connecting a paramagnetic and a diamagnetic bead by a polymer and applying a magnetic field gradient which periodically switches direction.

VI. DISCUSSION

We have seen that the interplay between nonequilibrium fluctuations and hydrodynamics, even in the simplest extension of a single particle embedded in a momentum-conserving fluid, viz., two particles embedded in an unbounded active gel or a particle in an active gel bound within confining walls, brings out the inadequacy of the effective temperature description, since the corresponding steady-state probability distribution shows strong departures from the equilibrium Boltzmann form. This leads to a nonequilibrium effective "Casimir-like" power law interaction [36–41]. Furthermore, both in an active fluid or in a passive fluid activated by stochastic forcing, unequal size particles exhibit short-time ballistic motion. This shows that the effective equilibrium limits, that have been obtained for "dry" active systems, do not hold for "wet" active systems.

The effective interaction between activated particles can be experimentally verified by more precise measurements of the particle position in the optical trap setups used in the experiments reported in Ref. [30].

We expect the interplay between hydrodynamic interactions and active fluctuations will also lead to effective interaction between self-propelled particles in a passive fluid. Since the effective interactions are sensitive to the precise origin of the fluctuation, the form of effective interaction may be different for different models of self-propelled particles.

There are various extensions of this work that will be useful to explore. So far in this work we only look at incompressible fluid, it would be of interest to analyze the behavior for compressible fluids.

The form of the effective interaction and self-propulsion velocity was computed in the far-field approximation of the hydrodynamic interaction. We expect the effective interaction and self-propulsion to exist in lubrication limit as well.

APPENDIX A: FOKKER-PLANCK FROM LANGEVIN

Following Ref. [26] we derive the Fokker-Planck equation from a multivariate overdamped Langevin equation with multiplicative noise, for general choice of stochastic calculus. The Langevin equations are

$$\dot{x}_i = F_i(\mathbf{x}) + G_{ij}(\mathbf{x})\vartheta_j, \quad (A1)$$

where $i \in (1, ..., N)$ and $j \in (1, ..., M)$, ϑ_j is a zero mean Gaussian white noise with correlation

$$\langle\vartheta_k(t)\vartheta_j(t')\rangle = C_{kj}(\mathbf{x})\delta(t-t'), \quad (A2)$$

where $(k, j) \in (1, ..., M)$. Integrating Eq. (A1) over a small time interval Δt gives

$$\Delta x_i = \int_t^{t+\Delta t} dt' F_i(\mathbf{x}) + \int_t^{t+\Delta t} dt' G_{ij}(\mathbf{x})\vartheta_j. \quad (A3)$$

The first term on the right-hand side is unambiguously approximated for small Δt using a straightforward Taylor expansion of F_i, in contrast, the limit of the second term is not well defined [20,42]. Unlike deterministic calculus, there are multiple choices for stochastic calculus. This amounts to choosing the time between t and $t + \Delta t$ at which $\mathbf{x}$ in $G(\mathbf{x})$ is evaluated. If $G(\mathbf{x})$ is evaluated at t, it is Ito calculus [20]; at $t + \Delta t/2$, it is Stratonovich calculus [20,23]; and at $t + \Delta t$, it is Hanggi-Klimontovich calculus [43,44]. Using the definition introduced in Ref. [42], $\mathbf{x}(t)$ evaluated at any generic point between t and $t + \Delta t$, parameterized by $\epsilon \in (0, 1)$ gives

$$\Delta x_i = F_i(\mathbf{x}_t)\Delta t + G_{ij}(\mathbf{x}_t + \epsilon\Delta\mathbf{x})\int_t^{t+\Delta t} dt'\vartheta_j(t'). \quad (A4)$$

Taylor expanding $G(\mathbf{x})$ around $\mathbf{x}(t)$

$$\Delta x_i = F_i(\mathbf{x}_t)\Delta t + G_{ij}(\mathbf{x}_t)\int_t^{t+\Delta t} dt'\vartheta_j(t')$$
$$+ \epsilon\frac{\partial G_{ij}(\mathbf{x}_t)}{\partial x_k}\Delta x_k\int_t^{t+\Delta t} dt'\vartheta_j(t'). \quad (A5)$$

Δx_k has a term of order $\sqrt{\Delta t}$, hence the ϵ term has a contribution of order Δt, substituting Δx_k back in the equation and keeping terms to order Δt we get

$$\Delta x_i = F_i(\mathbf{x}_t)\Delta t + G_{ij}(\mathbf{x}_t)\int_t^{t+\Delta t} dt'\vartheta_j(t')$$
$$+ \epsilon\frac{\partial G_{ij}(\mathbf{x}_t)}{\partial x_k}G_{kl}(\mathbf{x}_t)\int_t^{t+\Delta t} dt'\vartheta_l(t')\int_t^{t+\Delta t} dt''\vartheta_j(t''). \quad (A6)$$

The first and the second moment of Δx are

$$\langle \Delta x_i \rangle = F_i(\mathbf{x}_t)\Delta t + \epsilon \frac{\partial G_{ij}(\mathbf{x}_t)}{\partial x_k} G_{kl}(\mathbf{x}_t)C_{lj}(\mathbf{x}_t)\Delta t, \quad \text{(A7)}$$

$$\langle \Delta x_i \Delta x_l \rangle = G_{ij}G_{lk}C_{jk}\Delta t. \quad \text{(A8)}$$

The corresponding Fokker-Plank equation is [20]

$$\partial_t P = \frac{\partial}{\partial x_i}\left(-F_i - \epsilon \frac{\partial G_{ik}}{\partial x_j}S_{jk} + \frac{1}{2}\frac{\partial}{\partial x_j}G_{ik}S_{jk}\right)P, \quad \text{(A9)}$$

where $S_{ij} = C_{ik}G_{jk}$. Thus, we see that different choices of stochastic calculus (ϵ) leads to different Fokker-Planck equations and hence different physics.

For a given problem, the relevant value of ϵ depends on the fast timescales which have been integrated out. Furthermore the existence of a simple convention choice is not always guaranteed [21]. In this paper, the effective description of a viscoelastic gel is obtained by integrating out: the viscoelastic relaxation time (τ_v), the inertial relaxation time ($\tau_m = m/\eta$), and the noise correlation time (τ_n). In Refs. [21,22] it was shown that for an exponentially correlated noise with $\tau_n \gg \sqrt{\tau_m \tau_v}$ $\epsilon = 1/2$ (Stratonovich convention) is the right value. For thermal noise, in general, no simple convention works. But if $\frac{G_{ik}}{\partial x_j}S_{jk} = G_{ik}\frac{S_{jk}}{\partial x_j}$, $\epsilon = 1$ (Hanggi-Klimontovich convention) is the right value (see Ref. [42] for a detailed discussion on noise convention for thermal fluctuations). These are the convention choices used throughout this paper.

The Langevin equations in the main text are of the form given by Eq. (A1) and the corresponding Fokker-Planck will be given by Eq. (A9) by making the following identifications in different cases

(a) *Wall-particle.* Comparing Eq. (A1) with Eqs. (7) and (8) and noise correlation given by Eq. (9), we get $\mathbf{F} = -\nabla U \hat{z}$, $\mathsf{G} = \mathsf{H}$, and $\mathsf{C} = 2\lambda \mathsf{H}^{-1}$, which gives $\mathsf{S} = 2\lambda \mathsf{H}^{-1} \cdot \mathsf{H} = 2\lambda \mathsf{I}$, where I is 3×3 identity matrix. Using these values in Eq. (A9) leads to Eq. (8).

(b) *Fluctuating fluid.* Comparing Eq. (A1) with Eq. (20) and noise correlation given by Eq. (21), we get $\mathsf{G} = \mathsf{H}$, and $\mathsf{C} = 2\lambda \mathsf{H}^{-1}$, and $\mathsf{S} = \mathsf{C} \cdot \mathsf{H} = 2\lambda \mathsf{I}$, where I is 6×6 identity matrix. Using the above values in Eq. (A9), we get Eq. (13).

(c) *Fluctuating force.* Comparing Eq. (A1) and Eq. (11) and the noise correlation given by Eq. (45), we get $\mathsf{G} = \mathsf{H}$, $\mathsf{C} = 2\Lambda \mathsf{I}$, and $\mathsf{S} = \mathsf{C} \cdot \mathsf{H} = 2\Lambda \mathsf{H}$, where I is the 6×6 identity matrix. Using the above values in Eq. (A9) and changing of variables to $\mathbf{R}$ and $\mathbf{r}$ we get Eq. (46).

APPENDIX B: HYDRODYNAMIC INTERACTION TENSOR

1. Wall particle

For distances larger than the particle size ($z \gg a$), the mobilities can be calculated as a power series in the inverse of separation from the wall ($1/z$). To second order (Rotne-Prager approximation) the mobility longitudinal to the wall is [18,45]

$$H_z = \frac{1}{6\pi \eta a}\left(1 - \frac{9}{8}\frac{a}{z} + \frac{1}{2}\frac{a^3}{z^3}\right), \quad \text{(B1)}$$

and the mobility transverse to the wall is

$$\mathsf{H}_{x_\perp} = \frac{1}{6\pi \eta a}\left(1 - \frac{9}{16}\frac{a}{z} + \frac{1}{8}\frac{a^3}{z^3}\right). \quad \text{(B2)}$$

2. Two spheres

The hydrodynamic interaction tensor $\mathsf{H}_{\alpha\beta}$ coupling the translational degree of freedom is a 3×3 tensor. This tensor can be calculated as a power series in the inverse of separation ($1/r$) between the center of the two spheres. To the fourth power it is given by [18]

$$\mathsf{H}_{11} = \frac{1}{6\pi \eta a}(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + \left(\frac{1}{6\pi \eta a} - \frac{5}{8}\frac{b^3}{\pi \eta r^4}\right)\hat{\mathbf{r}}\hat{\mathbf{r}},$$

$$\mathsf{H}_{22} = \frac{1}{6\pi \eta b}(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + \left(\frac{1}{6\pi \eta b} - \frac{5}{8}\frac{a^3}{\pi \eta r^4}\right)\hat{\mathbf{r}}\hat{\mathbf{r}},$$

$$\mathsf{H}_{12} = \mathsf{H}_{21} = \left(\frac{1}{8\pi \eta r} + \frac{1}{24}\frac{(a^2 + b^2)}{\pi \eta r^3}\right)(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})$$

$$+ \left(\frac{1}{4\pi \eta r} + \frac{1}{12}\frac{(a^2 + b^2)}{\pi \eta r^3}\right)\hat{\mathbf{r}}\hat{\mathbf{r}}. \quad \text{(B3)}$$

The mobility tensor M defined in Eq. (15) of the main text decomposed as sum of projection along $\hat{\mathbf{r}}$ (denoted by superscript n) and perpendicular to $\hat{\mathbf{r}}$ (denoted by superscript q) is

$$\mathsf{M}_{\alpha\beta} = m^q_{\alpha\beta}(r)(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + m^n_{\alpha\beta}(r)\hat{\mathbf{r}}\hat{\mathbf{r}}. \quad \text{(B4)}$$

By substituting Eq. (B3) into Eq. (B4), we get

$$m^q_{11} = \frac{(a + b)}{6\pi \eta ab} + \frac{1}{4\pi \eta r} + \frac{1}{12}\frac{(a^2 + b^2)}{\pi \eta r^3}, \quad \text{(B5)}$$

$$m^n_{11} = \frac{(a + b)}{6\pi \eta ab} + \frac{1}{2\pi \eta r} - \frac{1}{6}\frac{(a^2 + b^2)}{\pi \eta r^3} - \frac{5}{8}\frac{a^3 + b^3}{\pi \eta r^4}, \quad \text{(B6)}$$

$$m^n_{12} = \frac{a - b}{6\pi \eta ab} + \frac{5}{8}\frac{(b^3 - a^3)}{\pi \eta r^4}, \quad \text{(B7)}$$

$$m^q_{12} = \frac{a - b}{6\pi \eta ab}, \quad \text{(B8)}$$

$$m^q_{22} = \frac{(a + b)}{6\pi \eta ab} - \frac{1}{4\pi \eta r} - \frac{1}{12}\frac{(a^2 + b^2)}{\pi \eta r^3}, \quad \text{(B9)}$$

$$m^n_{22} = \frac{(a + b)}{6\pi \eta ab} - \frac{1}{2\pi \eta r} + \frac{1}{6}\frac{(a^2 + b^2)}{\pi \eta r^3} - \frac{5}{8}\frac{a^3 + b^3}{\pi \eta r^4}. \quad \text{(B10)}$$

APPENDIX C: PASSIVE COLLOIDS IN ACTIVE FLUID—THERMAL FLUCTUATIONS

For thermal fluctuations, $\lambda = k_B T$, and the Fokker-Planck equation corresponding to Eq. (20) with noise correlation given by Eq. (21) in Hanggi-Klimontovich convention is

$$\partial_t P = \nabla_{\mathbf{R}} \cdot (-\mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_2 + \lambda \mathsf{M}_{12} \cdot \nabla_{\mathbf{r}} + \lambda \mathsf{M}_{11} \cdot \nabla_{\mathbf{R}})P$$

$$+ \nabla_{\mathbf{r}} \cdot (-\mathsf{M}_{22} \cdot \tilde{\mathbf{f}}_2 + \lambda \mathsf{M}_{12} \cdot \nabla_{\mathbf{R}} + \lambda \mathsf{M}_{22} \cdot \nabla_{\mathbf{r}})P. \quad \text{(C1)}$$

The marginal of steady-state distribution of r is $P(r) \propto e^{-U/\lambda}$. Integrating out separation r as for active fluctuation we get the self-propulsion velocity of the form

$$v_0 = \int r^2 dr \hat{\mathbf{r}} \cdot (\mathsf{M}_{12} \cdot \tilde{\mathbf{f}}_2 - \lambda \mathsf{M}_{12} \cdot \nabla_{\mathbf{r}})\mathsf{P}(\mathbf{r}). \quad \text{(C2)}$$

Expanding this we get

$$v_0 = -\int dr r^2 \left(m^n_{12} U' e^{-U/\lambda} + \lambda m^n_{12}\frac{\partial}{\partial r}e^{-U/\lambda}\right) = 0. \quad \text{(C3)}$$

As expected for thermal fluctuations the self-propulsion velocity is identically zero.

APPENDIX D: ACTIVATED PARTICLES IN A PASSIVE FLUID

The effective diffusion tensor for an activated particle in a passive fluid as defined in Eq. (46) is

$$
\mathsf{D} = \begin{bmatrix} \mathsf{L}_{11} + \mathsf{L}_{22} + \mathsf{L}_{12} + \mathsf{L}_{21} & \mathsf{L}_{22} - \mathsf{L}_{11} \\ \mathsf{L}_{22} - \mathsf{L}_{11} & \mathsf{L}_{11} + \mathsf{L}_{22} - \mathsf{L}_{12} - \mathsf{L}_{21} \end{bmatrix},
$$
(D1)

where L in terms of the hydrodynamic interaction tensor H is

$$
\mathsf{L} = \begin{bmatrix} \mathsf{H}_{11} \cdot \mathsf{H}_{11} + \mathsf{H}_{12} \cdot \mathsf{H}_{21} & \mathsf{H}_{11} \cdot \mathsf{H}_{12} + \mathsf{H}_{12} \cdot \mathsf{H}_{22} \\ \mathsf{H}_{21} \cdot \mathsf{H}_{11} + \mathsf{H}_{22} \cdot \mathsf{H}_{21} & \mathsf{H}_{12} \cdot \mathsf{H}_{21} + \mathsf{H}_{22} \cdot \mathsf{H}_{22} \end{bmatrix}.
$$
(D2)

Substituting in Eq. (D1) H from Eq. (B3) and keeping terms only to order $1/r$ gives

$$
\mathsf{D}_{11} = \left[\left(\frac{1}{6\pi\eta a} + \frac{1}{8\pi\eta r} \right)^2 + \left(\frac{1}{6\pi\eta b} + \frac{1}{8\pi\eta r} \right)^2 \right](\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})
$$
$$
+ \left[\left(\frac{1}{6\pi\eta a} + \frac{1}{4\pi\eta r} \right)^2 + \left(\frac{1}{6\pi\eta b} + \frac{1}{4\pi\eta r} \right)^2 \right]\hat{\mathbf{r}}\hat{\mathbf{r}},
$$
(D3)

$$
\mathsf{D}_{12} = \mathsf{D}_{21} = \left(\frac{1}{(6\pi\eta b)^2} - \frac{1}{(6\pi\eta a)^2} \right)(\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})
$$
$$
+ \left(\frac{1}{(6\pi\eta b)^2} - \frac{1}{(6\pi\eta a)^2} \right)\hat{\mathbf{r}}\hat{\mathbf{r}},
$$
(D4)

$$
\mathsf{D}_{22} = \left[\left(\frac{1}{6\pi\eta a} - \frac{1}{8\pi\eta r} \right)^2 + \left(\frac{1}{6\pi\eta b} - \frac{1}{8\pi\eta r} \right)^2 \right](\mathsf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})
$$
$$
+ \left[\left(\frac{1}{6\pi\eta a} - \frac{1}{4\pi\eta r} \right)^2 + \left(\frac{1}{6\pi\eta b} - \frac{1}{4\pi\eta r} \right)^2 \right]\hat{\mathbf{r}}\hat{\mathbf{r}}.
$$
(D5)

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