Non-Markovian harmonic oscillator across a magnetic field and time-dependent force fields

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We study the non-Markovian Brownian motion of an electrically charged harmonic oscillator through the action of both a constant magnetic field and time-dependent force fields. The generalized Langevin equation with a friction memory kernel is used to derive the generalized phase-space Fokker-Planck equation for the harmonic oscillator in the absence and in the presence of time-dependent force fields. To achieve our goal, the characteristic function method is applied to obtain, in an accurate way, the theoretical description of the problem. We explicitly calculate the correlation and cross-correlation functions for the position and velocity vectors. We show that the relevant physics behind the theory is contained in the generalized diffusion coefficient, which accounts for the natural coupling between both the harmonic oscillator and magnetic field. Our theoretical results are compared with those previously reported in the literature.

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I. INTRODUCTION

The non-Markovian Brownian motion continues to be a topic of increasing interest in the study of physical, chemical, and biological systems in which the thermal interaction between the system and its surroundings takes place through the friction memory effects. In the literature, there exists an important number of works related to non-Markovian Brownian motion characterized by a generalized Langevin equation (GLE) with Gaussian statistical properties of the noise. This has been corroborated in recent contributions as, for instance, non-Markovian intracellular transport with subdiffusion [1], the fluctuation-dissipation theorem for particle-bath systems in a harmonic field [2], charge-particle transport in semiconductors [3], test particles in a gas for Markovian and non-Markovian Langevin dynamics [4], work fluctuation and its optimal extraction from a non-Markovian bath [5], non-Markovian fluctuation relations [6–10], and others recent contributions [11–18]. Also, works on the study of colloidal motion in viscoelastic media [19–22], the dynamics of protein filaments [23–25], anomalous diffusion in disordered media [26,27], and molecular transport in cells [28,29] have been reported.

It is our purpose in the present contribution to describe the non-Markovian Brownian motion of a charged harmonic oscillator, under the action of a magnetic field and time-dependent force fields. We use the characteristic function method to exactly obtain the generalized phase-space Fokker-Planck equation for the charged harmonic oscillator described by a GLE with a general friction memory kernel. The friction memory kernel is assumed to be symmetric and its Laplace transform must exist. It is shown that upon the derivation of the non-Markovian Fokker-Planck equation, the coupling effect between the rotational character of the magnetic field and the harmonic oscillator arises in a natural way in the generalized diffusion coefficient, which is the relevant physics behind the theoretical description. In our contribution, we recognize a prolonged algebra involved in the calculations; however, the method of characteristic function provides an accurate and exact theoretical approach. Furthermore, it has recently been proven that in the large time limit, the GLE associated with the harmonic oscillator in a magnetic field without the time-dependent force fields is stationary, if the noise correlation function satisfies the fluctuation-dissipation relation of the second kind [8]. It is worthwhile to comment that very few works related to non-Markovian Brownian motion in a magnetic field has been reported in the literature [8,14–16,18]. For this reason, we consider that our present contribution can be useful to explore others alternative non-Markovian fluctuation relations, as well as the study of other non-Markovian systems where the influence of the magnetic field is important.

In 1976, a clever method to derive the generalized Fokker-Planck equation (FPE) for a free particle and for a particle bounded by a harmonic potential for simple non-Markovian systems was reported by Adelman [30]. Twenty years later, the study of the statistical properties of linear oscillators driven by both internal and external Gaussian colored noise was given in Ref. [31]. By means of the explicit solution of the GLE with a general friction memory kernel, the generalized phase-space FPE for the harmonic oscillator could also be derived using the characteristic function method. It was shown that the generalized phase-space FPE is exactly the same as the one derived by Adelman [30].

Over three years ago, the characteristic function method was also used to study the Brownian motion of a charged particle in the presence of a magnetic field and time-dependent force fields [15]. The consistency of the results reported in [15] was corroborated in two limiting cases, namely, in the case of a free particle and in the Markovian limit. In the former, the velocity-space and phase-space FPE exactly reduce to those obtained by Adelman

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[30], whereas in the latter, they also reduce to the expected results [32–36].

Very recently, in Ref. [14], an alternative method for deriving the non-Markovian phase-space FPE for a charged particle bounded by a harmonic potential and under the action of a constant magnetic field was proposed. However, before obtaining the non-Markovian phase-space FPE for such a system, the authors applied the proposed method to obtain the generalized FPE in the following cases: the velocity-space FPE for a free particle, the phase-space FPE for a particle bounded by a harmonic potential, the velocity-space FPE for a free particle in a magnetic field, and, finally, the generalized phase-space FPE for a harmonic oscillator in a magnetic field. Comments on the method in each case will be explicitly given in the last section of our present contribution. In principle, the generalized phase-space FPE for a harmonic oscillator in a magnetic field is the original contribution reported in [14]; however, notable differences are found when such a non-Markovian FPE is compared with the one derived in our present contribution, in which we use the method of characteristic function.

Our paper is organized as follows: To easily follow the mathematics in our current contribution, we consider it necessary to include a brief study, in Sec. II, of the problem of a harmonic oscillator in a similar way as in Ref. [31]. In Sec. III, we derive the generalized phase-space FPE for the harmonic oscillator across a magnetic field in the absence of time-dependent force fields, which in turn is compared with previous results reported in [15,30,31]. Section IV focuses on the derivation of the generalized phase-space FPE for the charged harmonic oscillator across a magnetic field, taking into account the presence of time-dependent force fields. The comparison of the results reported in Ref. [14] and the corresponding comments are explicitly given in Sec. V. Our concluding remarks are given in Sec. VI and, at the end of our work, two appendices are given for an explicit algebra.

II. GENERALIZED PHASE-SPACE FPE FOR A HARMONIC OSCILLATOR

A. Generalized Langevin equation

In this section, we summarize the calculations given in Ref. [31] to derive the generalized phase-space FPE for a harmonic oscillator of mass $m = 1$ embedded in a thermal bath of temperature T. The non-Markovian dynamics involving memory thermal interaction with its surroundings is, in general, characterized by a GLE containing a general friction memory kernel. For the one-dimensional (1D) harmonic oscillator, the GLE can be written as

$$\ddot{x} + \int_0^t \gamma(t-t')\,\dot{x}(t')\,dt' - \omega^2 x = f(t), \qquad (1)$$

where $\gamma(t)$ is the friction memory kernel considered as a general and symmetric function of time, $\omega^2 = k/m$ is the characteristic frequency of the harmonic oscillator, and $f(t)$ is the internal noise per unit mass, which satisfies the fluctuation-dissipation relation of the second kind [37],

$$\langle f(t)f(t')\rangle = k_B T\,\gamma(t-t'), \qquad (2)$$

with k_B being the Boltzmann's constant. This relation guarantees that the Langevin dynamics (1) becomes stationary in the large time limit [38]. The solution of Eqs. (1) can be obtained using Laplace transformation and reads

$$x(t) = \langle x(t)\rangle + \int_0^t H_0^a(t-t')f(t')dt', \qquad (3)$$

where the average value is given for nonrandom initial conditions by

$$\langle x(t)\rangle = x_0\chi_0^a(t) + v_0 H_0^a(t), \qquad (4)$$

with $x_0 = x(0)$, $v_0 = v(0)$. The function $\chi_0^a(t)$ reads

$$\chi_0^a(t) = 1 - \omega^2\int_0^t H_0^a(t')dt', \qquad (5)$$

and $H_0^a(t)$ is the inverse Laplace transform of $\hat{H}_0^a(s)$, such that

$$\hat{H}_0^a(s) = \frac{1}{s^2 + s\hat{\gamma}(s) + \omega^2}. \qquad (6)$$

Also, $\hat{\gamma}(s)$ is the Laplace transform of $\gamma(t)$.

B. Generalized phase-space FPE

The generalized phase-space FPE associated with GLE (1) was derived in [31] using the characteristic function. The non-Markovian FPE for the conditional probability density $P(x, v, t|x_0, v_0)$ is shown to be

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial x} - \tilde{\omega}^2(t)\,x\frac{\partial P}{\partial v} = \tilde{\beta}(t)\,\frac{\partial(vP)}{\partial v} + k_B T\tilde{\beta}(t)\frac{\partial^2 P}{\partial v^2} + k_B T[\tilde{\omega}^2(t) - \omega^2]\frac{\partial}{\partial v}\frac{\partial P}{\partial x}, \qquad (7)$$

where $\tilde{\beta}(t)$ is the friction function and $\tilde{\omega}^2(t)$ is the frequency function, both defined by

$$\tilde{\beta}(t) = -\frac{1}{\Delta_{ho}}(\mathcal{A}\dddot{\mathcal{C}} - \ddot{\mathcal{A}}\mathcal{C}) = -\frac{d\ln\Delta_{ho}}{dt}, \qquad (8)$$

$$\tilde{\omega}^2(t) = \frac{1}{\Delta_{ho}}(\dot{\mathcal{A}}\ddot{\mathcal{C}} - \dot{\mathcal{C}}\ddot{\mathcal{A}}), \qquad (9)$$

$$\Delta_{ho} = \mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C}, \qquad (10)$$

$$\mathcal{A} = \chi_0^a(t), \quad \dot{\mathcal{A}} = -\omega^2 H_0^a(t), \quad \ddot{\mathcal{A}} = -\omega^2\dot{H}_0^a(t), \qquad (11)$$

$$\mathcal{C} = H_0^a(t), \quad \dot{\mathcal{C}} = \dot{H}_0^a(t), \quad \ddot{\mathcal{C}} = \ddot{H}_0^a(t), \qquad (12)$$

where the "overdot" means time derivative. For a 3D isotropic harmonic oscillator, the generalized phase-space FPE for the conditional probability $\mathbb{P}(\mathbf{r}, \mathbf{v}, t|\mathbf{r}_0, \mathbf{v}_0)$, becomes

$$\frac{\partial\mathbb{P}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\mathbb{P} - \tilde{\omega}^2(t)\,\mathbf{r}\cdot\nabla_{\mathbf{v}}\mathbb{P} = \tilde{\beta}(t)\,\nabla_{\mathbf{v}}\cdot(\mathbf{v}\mathbb{P}) + k_B T\tilde{\beta}(t)\,\nabla_{\mathbf{v}}^2\mathbb{P} + k_B T[\tilde{\omega}^2(t) - \omega^2]\,[\nabla_{\mathbf{v}}\cdot\nabla_{\mathbf{r}}\mathbb{P}], \qquad (13)$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{v} = (v_x, v_y.v_z)$. Equation (13) is exactly the same as that reported by Adelman [30].

III. GENERALIZED PHASE-SPACE FPE FOR A HARMONIC OSCILLATOR ACROSS A MAGNETIC FIELD

In this section, we use the method of characteristic function to derive the generalized phase-space FPE for a harmonic oscillator across a magnetic field. The study is given without the presence of time-dependent force fields. We show the exactitude of the method as well as its consistency when compared with other results in some limiting cases.

A. Generalized Langevin equation

We now consider the above harmonic oscillator of mass $m = 1$ with a charge q across a constant magnetic field pointing along the z axis, that is, $\mathbf{B} = (0, 0, B)$. As proposed in [39,40], the diffusion process arises from local fluctuations of the electric field, which induce collisions between particles in a Brownian motionlike manner. These local fluctuations of the electric field $[\mathbf{E}_{\rm int}(t)]$ are then identified as the internal noise responsible for the thermal diffusion of the charged particles. The non-Markovian dynamics of the charged Brownian harmonic oscillator involving memory thermal interaction with its surroundings can be written as

$$\ddot{\mathbf{r}} + \int_0^t \gamma(t-t')\,\dot{\mathbf{r}}(t')\,dt' + \omega^2\mathbf{r} - \frac{q}{c}\,\dot{\mathbf{r}}\times\mathbf{B} = \mathbf{f}(t), \tag{14}$$

where $\mathbf{r} = (x, y, z)$ is the position vector, and $\mathbf{f}(t) = q\mathbf{E}_{\rm int}(t)$ is the internal noise per unit mass which satisfies the fluctuation-dissipation relation of the second kind [37] given by

$$\langle f_i(t)f_j(t')\rangle = k_{_B}T\,\delta_{ij}\,\gamma(t-t'). \tag{15}$$

It should be mentioned that the validity of this relation guarantees that the stochastic process (14) is stationary in the large time limit. The statement has been explicitly proven in Ref. [8]. Due to the orientation of the magnetic field, the GLE (14) can be written in terms of the components as follows:

$$\ddot{x} - \Omega\dot{y} + \omega^2 x + \int_0^t \gamma(t-t')\,\dot{x}(t')\,dt' = f_x(t), \tag{16}$$

$$\ddot{y} + \Omega\dot{x} + \omega^2 y + \int_0^t \gamma(t-t')\,\dot{y}(t')\,dt' = f_y(t), \tag{17}$$

$$\ddot{z} + \omega^2 z + \int_0^t \gamma(t-t')\,\dot{z}(t')\,dt' = f_z(t), \tag{18}$$

where $\Omega = qB/c$ is the cyclotron frequency. Clearly, along the z axis, the GLE is magnetic field independent and also independent of the process in the (x, y) plane. The solution of Eqs. (16)–(18) can be calculated using Laplace transformation, leading, respectively, to the following solutions:

$$\begin{aligned} x(t) = \langle x(t)\rangle &+ \int_0^t H_0(t-t')f_x(t')dt' \\ &- \Omega^2\int_0^t \mathcal{H}_2(t-t')f_x(t')dt' \\ &+ \Omega\int_0^t \mathcal{H}_1(t-t')f_y(t')dt', \end{aligned} \tag{19}$$

$$\begin{aligned} y(t) = \langle y(t)\rangle &+ \int_0^t H_0(t-t')f_y(t')dt' \\ &- \Omega^2\int_0^t \mathcal{H}_2(t-t')f_y(t')dt' \\ &- \Omega\int_0^t \mathcal{H}_1(t-t')f_x(t')dt', \end{aligned} \tag{20}$$

$$z(t) = \langle z(t)\rangle + \int_0^t H_0(t-t')f_z(t')dt', \tag{21}$$

where the average values are given for nonrandom initial conditions by

$$\begin{aligned} \langle x(t)\rangle = {}& x_0[\chi_0(t) + \Omega^2\omega^2\chi_2(t)] - y_0\Omega\omega^2 H_1(t) \\ &+ v_{x0}[H_0(t) - \Omega^2\mathcal{H}_2(t)] + v_{y0}\Omega\mathcal{H}_1(t), \end{aligned} \tag{22}$$

$$\begin{aligned} \langle y(t)\rangle = {}& y_0[\chi_0(t) + \Omega^2\omega^2\chi_2(t)] + x_0\Omega\omega^2 H_1(t) + v_{y0}[H_0(t) \\ &- \Omega^2\mathcal{H}_2(t)] - v_{x0}\Omega\mathcal{H}_1(t), \end{aligned} \tag{23}$$

$$\langle z(t)\rangle = z_0\chi_0(t) + v_{z0}H_0(t), \tag{24}$$

with $x_0 = x(0)$, $y_0 = y(0)$, $z_0 = z(0)$, $v_{x0} = v_x(0)$, $v_{y0} = v_y(0)$, $v_{z0} = v_z(0)$, and the functions $\chi_0(t)$ and $\chi_2(t)$ defined by

$$\chi_0(t) = 1 - \omega^2\int_0^t H_0(t')dt', \tag{25}$$

$$\chi_2(t) = \int_0^t \mathcal{H}_2(t')dt'. \tag{26}$$

The functions $H_0(t)$, $\mathcal{H}_1(t)$, and $\mathcal{H}_2(t)$, are, respectively, the inverse Laplace transform of $\hat{H}_0(s)$, $\hat{\mathcal{H}}_1(s)$, and $\hat{\mathcal{H}}_2(s)$, which are denoted by $H_0(t) = \mathcal{L}^{-1}\{\hat{H}_0(s)\}$, $\mathcal{H}_1(t) = \mathcal{L}^{-1}\{\hat{\mathcal{H}}_1(s)\}$, and $\mathcal{H}_2(t) = \mathcal{L}^{-1}\{\hat{\mathcal{H}}_2(s)\}$, where $\hat{\mathcal{H}}_1(s) = s\hat{H}_1(s)$ and $\hat{\mathcal{H}}_2(s) = s^2\hat{H}_2(s)$, such that

$$\hat{H}_0(s) = \frac{1}{s^2 + s\hat{\gamma}(s) + \omega^2}, \tag{27}$$

$$\hat{H}_1(s) = \frac{1}{(s^2 + s\hat{\gamma}(s) + \omega^2)^2 + (\Omega s)^2}, \tag{28}$$

$$\hat{H}_2(s) = \frac{1}{(s^2 + s\hat{\gamma}(s) + \omega^2)[(s^2 + s\hat{\gamma}(s) + \omega^2)^2 + (\Omega s)^2]}, \tag{29}$$

and $\hat{\gamma}(s)$ is the Laplace transform of $\gamma(t)$.

B. The functions $H_0(t)$, $H_1(t)$, $\mathcal{H}_1(t)$, and $\mathcal{H}_2(t)$ at time $t = 0$

Let us now determine the values of the functions $H_0(t)$, $H_1(t)$, $\mathcal{H}_1(t)$, and $\mathcal{H}_2(t)$ at time $t = 0$, which are required in this work. We begin our analysis with the solution given in Eq. (24) for the $z(t)$ variable. At time $t = 0$, it reads

$$z_0 = z_0\chi_0(0) + v_{z0}H_0(0), \tag{30}$$

which is true only if $\chi_0(0) = 1$ and $H_0(0) = 0$. In the case of Eqs. (22) and (23), we get

$$x_0 = x_0[\chi_0(0) + \Omega^2\omega^2\chi_2(0)] - y_0\Omega\omega^2 H_1(0) + v_{x0}[H_0(0) - \Omega^2\mathcal{H}_2(0)] + v_{y0}\Omega\mathcal{H}_1(0), \quad (31)$$

$$y_0 = y_0[\chi_0(0) + \Omega^2\omega^2\chi_2(0)] + x_0\Omega\omega^2 H_1(0) + v_{y0}[H_0(0) - \Omega^2\mathcal{H}_2(0)] - v_{x0}\Omega\mathcal{H}_1(0). \quad (32)$$

However, because $\chi_0(0) = 1$ and $\chi_2(0) = 0$, Eqs. (31) and (32) reduce to

$$\Omega\omega^2 H_1(0)y_0 = v_{x0}[H_0(0) - \Omega^2\mathcal{H}_2(0)] + v_{y0}\Omega\mathcal{H}_1(0), \quad (33)$$

$$\Omega\omega^2 H_1(0)x_0 = -v_{y0}[H_0(0) - \Omega^2\mathcal{H}_2(0)] + v_{x0}\Omega\mathcal{H}_1(0). \quad (34)$$

Again, they are identically satisfied if $H_1(0) = 0$, $\mathcal{H}_1(0) = 0$, and $H_0(0) = \Omega^2\mathcal{H}_2(0)$. But, $H_0(0) = 0$ and thus $\mathcal{H}_2(0) = 0$. Also, for the velocities $v_x(t)$, $v_y(t)$, and $v_z(t)$, the time derivative of the functions $H_0(t)$, $H_1(t)$, $\mathcal{H}_1(t)$, and $\mathcal{H}_2(t)$ at time $t = 0$ is required. In this case, we take the time derivative of Eqs, (19)–(21) and, after evaluating at time $t = 0$, we obtain

$$v_{x0} = x_0[-\omega^2 H_0(0) + \Omega^2\omega^2\mathcal{H}_2(0)] - y_0\Omega\omega^2\dot{H}_1(0) + v_{x0}[\dot{H}_0(0) - \Omega^2\dot{\mathcal{H}}_2(0)] + v_{y0}\Omega\dot{\mathcal{H}}_1(0), \quad (35)$$

$$v_{y0} = y_0[-\omega^2 H_0(0) + \Omega^2\omega^2\mathcal{H}_2(0)] + x_0\Omega\omega^2\dot{H}_1(0) + v_{y0}[\dot{H}_0(0) - \Omega^2\dot{\mathcal{H}}_2(0)] - v_{x0}\Omega\dot{\mathcal{H}}_1(0), \quad (36)$$

$$v_{z0} = z_0[-\omega^2 H_0(0)] + v_{z0}\dot{H}_0(0). \quad (37)$$

From Eq. (37), we easily conclude that $\dot{H}_0(0) = 1$. From Eqs. (35) and (36), we thus have that $\dot{H}_1(0) = 0$, $\dot{\mathcal{H}}_1(0) = 0$, and $\dot{H}_0(0) - \Omega^2\dot{\mathcal{H}}_2(0) = 1$, but because $\dot{H}_0(0) = 1$, we conclude that $\dot{\mathcal{H}}_2(0) = 0$. It is now clear from Eqs. (19)–(21) that

$$v_x(t) = \langle v_x(t)\rangle + \int_0^t \dot{H}_0(t-t')f_x(t')dt'\Omega^2 - \int_0^t \dot{\mathcal{H}}_2(t-t')f_x(t')dt' + \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')f_y(t')dt', \quad (38)$$

$$v_y(t) = \langle v_y(t)\rangle + \int_0^t \dot{H}_0(t-t')f_y(t')dt' - \Omega^2\int_0^t \dot{\mathcal{H}}_2(t-t')f_y(t')dt' - \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')f_x(t')dt', \quad (39)$$

$$v_z(t) = \langle v_z(t)\rangle + \int_0^t \dot{H}_0(t-t')f_z(t')dt', \quad (40)$$

with

$$\langle v_x(t)\rangle = x_0[-\omega^2 H_0(t) + \Omega^2\omega^2\mathcal{H}_2(t)] - y_0\Omega\omega^2\dot{H}_1(t) + v_{x0}[\dot{H}_0(t) - \Omega^2\dot{\mathcal{H}}_2(t)] + v_{y0}\Omega\dot{\mathcal{H}}_1(t), \quad (41)$$

$$\langle v_y(t)\rangle = y_0[-\omega^2 H_0(t) + \Omega^2\omega^2\mathcal{H}_2(t)] + x_0\Omega\omega^2\dot{H}_1(t) + v_{y0}[\dot{H}_0(t) - \Omega^2\dot{\mathcal{H}}_2(t)] - v_{x0}\Omega\dot{\mathcal{H}}_1(t), \quad (42)$$

$$\langle v_z(t)\rangle = -\omega^2 z_0 H_0(t) + v_{z0}\dot{H}_0(t). \quad (43)$$

C. Generalized phase-space FPE in a magnetic field

We now proceed to explicitly calculate the generalized phase-space FPE for a harmonic oscillator. First of all, we must comment that along the z axis, the GLE (18) is exactly the same as studied in Sec. II, and thus we just pay attention to the coupled process (16) and (17) taking place in the (x, y) plane. Due to the fact that the coupled process is Gaussian, the phase-space conditional probability density (CPD) defined by $P(\mathbf{x}, \mathbf{u}, t|\mathbf{x}_0, \mathbf{u}_0) \equiv P(\mathbf{R}, \mathbf{S})$, with $\mathbf{x} = (x, y)$ and $\mathbf{u} = (v_x, v_y)$, satisfies the Gaussian distribution function,

$$P(\mathbf{R}, \mathbf{S}) = \frac{1}{4\pi^2\sqrt{\det\boldsymbol{\sigma}(t)}}\exp\left[-\frac{1}{2}\bar{\mathbf{y}}^T\cdot\boldsymbol{\sigma}^{-1}(t)\cdot\bar{\mathbf{y}}\right], \quad (44)$$

where the components of the vector $\bar{\mathbf{y}}$ read $\bar{y}_i = \xi_i - \langle\xi_i\rangle$, with $\xi_i = x, y, v_x, v_y$, and $\boldsymbol{\sigma}(t) \equiv \sigma_{ij}(t)$ is the variance and covariance matrix defined by $\sigma_{ij}(t) = \langle[\xi_i - \langle\xi_i\rangle][\xi_j - \langle\xi_j\rangle]\rangle$. It is easy to check from the solutions given by Eqs. (19), (20), (38), and (39) that $\sigma_{xx}(t) = \sigma_{yy}(t)$, $\sigma_{v_xv_x}(t) = \sigma_{v_yv_y}(t)$, $\sigma_{xv_x}(t) = \sigma_{yv_y}(t)$, $\sigma_{xv_y}(t) = -\sigma_{yv_x}(t)$, $\sigma_{xy}(t) = \sigma_{yx}(t) = 0$, and $\sigma_{v_xv_y}(t) = \sigma_{v_yv_x}(t) = 0$. By defining $F \equiv \sigma_{xx}(t)$, $G \equiv \sigma_{v_xv_x}(t)$, $H \equiv \sigma_{xv_x}(t)$, and $I \equiv \sigma_{xv_y}(t)$, we thus have

$$F = \langle[x - \langle x\rangle]^2\rangle = \frac{2}{\beta}\bigg[\int_0^t H_0(t')dt'\int_0^{t'} H_0(t'')\gamma(t'-t'')dt'' + \Omega^2\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt'' + \Omega^4\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt'' - \Omega^2\int_0^t H_0(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt'' - \Omega^2\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'} H_0(t'')\gamma(t'-t'')dt''\bigg]. \quad (45)$$

$$G = \langle[v_x - \langle v_x\rangle]^2\rangle = \frac{2}{\beta}\bigg[\int_0^t \dot{H}_0(t')dt'\int_0^{t'}\dot{H}_0(t'')\gamma(t'-t'')dt'' + \Omega^2\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt'' + \Omega^4\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt'' - \Omega^2\int_0^t \dot{H}_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt'' - \Omega^2\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\dot{H}_0(t'')\gamma(t'-t'')dt''\bigg]. \quad (46)$$

$$
\begin{aligned}
H = \langle [x-\langle x\rangle][v_x-\langle v_x\rangle]\rangle = \frac{1}{\beta}\bigg[&\int_0^t H_0(t')dt'\int_0^{t'}\dot{H}_0(t'')\gamma(t'-t'')dt'' \\
&+\int_0^t \dot{H}_0(t')dt'\int_0^{t'}H_0(t'')\gamma(t'-t'')dt''-\Omega^2\int_0^t H_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt'' \\
&-\Omega^2\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}H_0(t'')\gamma(t'-t'')dt''-\Omega^2\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\dot{H}_0(t'')\gamma(t'-t'')dt'' \\
&-\Omega^2\int_0^t \dot{H}_0(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt''+\Omega^2\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt'' \\
&+\Omega^2\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt''+\Omega^4\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt'' \\
&+\Omega^4\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt''\bigg].
\end{aligned} \tag{47}
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$$
\begin{aligned}
I = \langle [x-\langle x\rangle][v_y-\langle v_y\rangle]\rangle = \frac{1}{\beta}\bigg[&-\Omega\int_0^t H_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt''-\Omega\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}H_0(t'')\gamma(t'-t'')dt'' \\
&+\Omega^3\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt''+\Omega^3\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt'' \\
&-\Omega^3\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt''-\Omega^3\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt'' \\
&+\Omega\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\dot{H}_0(t'')\gamma(t'-t'')dt''+\Omega\int_0^t \dot{H}_0(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt''\bigg].
\end{aligned} \tag{48}
$$

With the help of Eqs. (27)–(29), we can calculate the matrix elements $\sigma_{ij}(t)$. After a very long algebra, we can shown that [15]

$$
\begin{aligned}
\beta F = &-[H_0(t)-\Omega^2\mathcal{H}_2(t)]^2-\Omega^2\mathcal{H}_1^2(t)+2\int_0^t H_0(t')dt'-2\Omega^2\int_0^t\mathcal{H}_2(t')dt'-\omega^2\Omega^2H_1^2(t)-\omega^2\left(\int_0^t H_0(t')dt'\right)^2 \\
&+2\omega^2\Omega^2\left(\int_0^t H_0(t')dt'\right)\left(\int_0^t \mathcal{H}_2(t')dt'\right)-\omega^2\Omega^4\left(\int_0^t \mathcal{H}_2(t')dt'\right)^2,
\end{aligned} \tag{49}
$$

$$
\beta G = \left[1-\Omega^2\dot{\mathcal{H}}_1^2(t)\right]-[\dot{H}_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)]^2-\omega^2\Omega^2\mathcal{H}_1^2(t)-\omega^2[H_0(t)-\Omega^2\mathcal{H}_2(t)]^2, \tag{50}
$$

$$
\begin{aligned}
\beta H = &\, H_0(t)[1-\dot{H}_0(t)]-\Omega^2\mathcal{H}_2(t)-\Omega^4\mathcal{H}_2(t)\dot{\mathcal{H}}_2(t)+\Omega^2[H_0(t)\dot{\mathcal{H}}_2(t)+\dot{H}_0(t)\mathcal{H}_2(t)]-\Omega^2\mathcal{H}_1(t)\dot{\mathcal{H}}_1(t)-\omega^2\Omega^2H_1(t)\dot{H}_1(t) \\
&-\omega^2H_0(t)\int_0^t H_0(t')dt'+\omega^2\Omega^2H_0(t)\int_0^t\mathcal{H}_2(t')dt'+\omega^2\Omega^2\mathcal{H}_2(t)\int_0^t H_0(t')dt'-\omega^2\Omega^4\mathcal{H}_2(t)\int_0^t\mathcal{H}_2(t')dt',
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\beta I = &-\Omega\mathcal{H}_1(t)+\Omega[H_0(t)\dot{\mathcal{H}}_1(t)-\dot{H}_0(t)\mathcal{H}_1(t)]+\Omega^3[\mathcal{H}_1(t)\dot{\mathcal{H}}_2(t)-\dot{\mathcal{H}}_1(t)\mathcal{H}_2(t)] \\
&-\omega^2\Omega H_1(t)[H_0(t)-\Omega^2\mathcal{H}_2(t)]+\omega^2\Omega\mathcal{H}_1(t)\left[\int_0^t H_0(t')dt'-\Omega^2\int_0^t\mathcal{H}_2(t')dt'\right].
\end{aligned} \tag{52}
$$

It can also be corroborated that $\frac{1}{2}\dot{F} = H$. The variance and covariance matrix thus becomes

$$
\boldsymbol{\sigma}(t) = \begin{pmatrix} F & 0 & H & I \\ 0 & F & -I & H \\ H & -I & G & 0 \\ I & H & 0 & G \end{pmatrix}, \tag{53}
$$

and its inverse is

$$
\boldsymbol{\sigma}^{-1}(t) = \frac{1}{FG-H^2-I^2}\begin{pmatrix} G & 0 & -H & -I \\ 0 & G & I & -H \\ -H & I & F & 0 \\ -I & -H & 0 & F \end{pmatrix}. \tag{54}
$$

Thus, according to Eq. (44), the phase-space CPD can be written as

$$P(\mathbf{R},\mathbf{S}) = \frac{1}{4\pi^2(FG-H^2-I^2)}\exp\left\{-\frac{[F|\mathbf{S}|^2 - 2H\mathbf{R}\cdot\mathbf{S} - 2I(\mathbf{R}\times\mathbf{S})_z + G|\mathbf{R}|^2]}{2(FG-H^2-I^2)}\right\}, \tag{55}$$

where the vectors are $\mathbf{R}=(R_1,R_2)$ and $\mathbf{S}=(S_1,S_2)$, $\mathbf{R}\cdot\mathbf{S}$ is the scalar product, and $(\mathbf{R}\times\mathbf{S})_z$ is the z component of the cross product $\mathbf{R}\times\mathbf{S}$, such that

$$R_1 = x-\langle x\rangle = x - x_0[\chi_0(t)+\Omega^2\omega^2\chi_2(t)] + y_0\Omega\omega^2 H_1(t) - v_{x0}[H_0(t)-\Omega^2\mathcal{H}_2(t)] - v_{y0}\Omega\mathcal{H}_1(t), \tag{56}$$

$$R_2 = y-\langle y\rangle = y - y_0[\chi_0(t)+\Omega^2\omega^2\chi_2(t)] - x_0\Omega\omega^2 H_1(t) - v_{y0}[H_0(t)-\Omega^2\mathcal{H}_2(t)] + v_{x0}\Omega\mathcal{H}_1(t), \tag{57}$$

$$S_1 = v_x-\langle v_x\rangle = v_x - x_0[-\omega^2 H_0(t)+\Omega^2\omega^2\mathcal{H}_2(t)] + y_0\Omega\omega^2 \dot H_1(t) - v_{x0}[\dot H_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)] - v_{y0}\Omega\dot{\mathcal{H}}_1(t), \tag{58}$$

$$S_2 = v_x-\langle v_x\rangle = v_y - y_0[-\omega^2 H_0(t)+\Omega^2\omega^2\mathcal{H}_2(t)] - x_0\Omega\omega^2 \dot H_1(t) - v_{y0}[\dot H_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)] + v_{x0}\Omega\dot{\mathcal{H}}_1(t). \tag{59}$$

The generalized phase-space FPE can be calculated with the help of the characteristic function, which now is given by

$$C(\boldsymbol{\eta},t) = \exp\left[\sum_{i=1}^{4} i\langle\xi_i\rangle\eta_i - \frac{1}{2}\sum_{i,j=1}^{4}\sigma_{ij}\eta_i\eta_j\right], \tag{60}$$

where $\boldsymbol{\eta}=(\eta_1,\eta_2,\eta_3,\eta_4)$ and the vector $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3,\xi_4)=(x,y,v_x,v_y)$. The details of how the generalized phase-space FPE is calculated are given in Appendix A. The method is exact and shows that the exact generalized phase-space FPE is given by

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \mathbf{u}\cdot\nabla_{\mathbf{x}}P + \mathcal{Q}_1(t)\,\mathbf{x}\cdot\nabla_{\mathbf{u}}P - \mathcal{P}_2(t)\,I\,\nabla^2_{\mathbf{x}}P\\ &= \mathcal{P}_1(t)\,[\mathbf{x}\times\nabla_{\mathbf{x}}P]_z + \mathcal{P}_2(t)\,[\mathbf{u}\times\nabla_{\mathbf{x}}P]_z\\ &\quad -\mathcal{R}_1(t)\,[\mathbf{x}\times\nabla_{\mathbf{u}}P]_z - \mathcal{R}_3(t)\,[\mathbf{u}\times\nabla_{\mathbf{u}}P]_z\\ &\quad -\mathcal{Q}_3(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P - \mathcal{S}_1(t)\,\nabla^2_{\mathbf{u}}P - \mathcal{S}_2(t)\,\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P\\ &\quad +\mathcal{S}_3(t)\,[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{u}}P]_z,\end{aligned} \tag{61}$$

where each $[\mathbf{a}\times\nabla_{\mathbf{b}}P]_z$ represents the z component of the cross product $\mathbf{a}\times\nabla_{\mathbf{b}}P$, and all coefficients are explicitly defined in Appendix A. As we will show below, the generalized diffusion coefficient $\mathcal{P}_2(t)\,I$ accounts for the coupling effect between the magnetic field and harmonic oscillator. The consistency of Eq. (61) must be proven when it is compared with previous results reported in the literature.

(i) *Absence of the magnetic field.* In this limit, we must exactly obtain the same non-Markovian phase-space FPE for the harmonic oscillator reported in [30,31]. In the absence of the magnetic field, $\Omega=0$, and the quantities defined in Appendix A become $\mathcal{A}=\chi_0(t)$, $\mathcal{B}=0$, $\mathcal{C}=H_0(t)$, and $\mathcal{D}=0$, which also means that $\mathcal{A}=\chi_0(t)=\chi_0^a(t)$ and $\mathcal{C}=H_0(t)=H_0^a(t)$, given by Eqs. (11) and (12) in Sec. II. This allows one to conclude that

$$\mathcal{P}_1(t) = \frac{1}{\Delta_m}[\dot{\mathcal{A}}\hat c_x(t) - \dot{\mathcal{C}}\hat c_{v_x}(t)] = 0, \tag{62}$$

$$\mathcal{P}_2(t) = \frac{1}{\Delta_m}[-\dot{\mathcal{A}}\hat d_x(t) - \dot{\mathcal{C}}\hat d_{v_x}(t)] = 0, \tag{63}$$

$$\begin{aligned}\mathcal{Q}_1(t) &= \frac{1}{\Delta_m}[\ddot{\mathcal{A}}\hat a_x(t) + \ddot{\mathcal{C}}\hat a_{v_x}(t)]\\ &= \frac{1}{\Delta_m}[(\ddot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{A}}\ddot{\mathcal{C}})(\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C})],\end{aligned} \tag{64}$$

$$\mathcal{Q}_2(t) = \frac{1}{\Delta_m}[\ddot{\mathcal{A}}\hat c_x(t) - \ddot{\mathcal{C}}\hat c_{v_x}(t)] = 0, \tag{65}$$

$$\begin{aligned}\mathcal{Q}_3(t) &= \frac{1}{\Delta_m}[-\ddot{\mathcal{A}}\hat b_x(t) + \ddot{\mathcal{C}}\hat b_{v_x}(t)]\\ &= \frac{1}{\Delta_m}(\ddot{\mathcal{C}}\mathcal{A} - \ddot{\mathcal{A}}\mathcal{C})(\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C}),\end{aligned} \tag{66}$$

$$\mathcal{Q}_4(t) = \frac{1}{\Delta_m}[-\ddot{\mathcal{A}}\hat d_x(t) - \ddot{\mathcal{C}}\hat d_{v_x}(t)] = 0. \tag{67}$$

However, due to the fact that $\mathcal{Q}_1(t)=\mathcal{R}_2(t)$, $\mathcal{Q}_2(t)=-\mathcal{R}_1(t)=0$, $\mathcal{Q}_3(t)=\mathcal{R}_4(t)$, $\mathcal{Q}_4(t)=-\mathcal{R}_3(t)=0$, and the determinant $\Delta_m=(\mathcal{A}\dot{\mathcal{C}}-\dot{\mathcal{A}}\mathcal{C})^2$, it can be shown that $\Delta_m=\Delta_{ho}^2$, where $\Delta_{ho}=(\mathcal{A}\dot{\mathcal{C}}-\dot{\mathcal{A}}\mathcal{C})$ is the same as Eq. (10) given in Sec. II. Hence,

$$\mathcal{Q}_1(t) = \mathcal{R}_2(t) = \frac{1}{\Delta_{ho}}(\ddot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{A}}\ddot{\mathcal{C}}), \tag{68}$$

$$\mathcal{Q}_3(t) = \mathcal{R}_4(t) = \frac{1}{\Delta_{ho}}(\ddot{\mathcal{C}}\mathcal{A} - \ddot{\mathcal{A}}\mathcal{C}) = \frac{d\ln\Delta_{ho}}{dt}. \tag{69}$$

Also the time-dependent coefficients $\mathcal{S}_1(t)$, $\mathcal{S}_2(t)$, and $\mathcal{S}_3(t)$, appearing in Eq. (61) and given, respectively, by Eqs. (A41)–(A43), can be reduced to the following expressions:

$$\begin{aligned}\mathcal{S}_1(t) &= \mathcal{Q}_1(t)H - \mathcal{Q}_2(t)I + \mathcal{Q}_3(t)G - \tfrac{1}{2}\dot G\\ &= \mathcal{Q}_1(t)H + \mathcal{Q}_3(t)G - \tfrac{1}{2}\dot G,\end{aligned} \tag{70}$$

$$\begin{aligned}\mathcal{S}_2(t) &= G + [\mathcal{Q}_4(t) - \mathcal{P}_1(t)]I + \mathcal{Q}_1(t)F + \mathcal{Q}_3(t)H - \dot H\\ &= G + \mathcal{Q}_1(t)F + \mathcal{Q}_3(t)H - \dot H,\end{aligned} \tag{71}$$

$$\begin{aligned}\mathcal{S}_3(t) &= \mathcal{Q}_2(t)F - \mathcal{P}_2(t)G + [\mathcal{Q}_4(t) - \mathcal{P}_1(t)]H\\ &\quad -\mathcal{Q}_3(t)I + \dot I = 0.\end{aligned} \tag{72}$$

This is because $I=0$ [see Eq. (52)], $\mathcal{P}_1(t)=\mathcal{P}_2(t)=\mathcal{Q}_2(t)=\mathcal{Q}_4(t)=0$, so that when the magnetic field is absent, the non-Markovian Fokker-Planck equation (61) reduces to

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \mathbf{u}\cdot\nabla_{\mathbf{x}}P + \mathcal{Q}_1(t)\,\mathbf{x}\cdot\nabla_{\mathbf{u}}P\\ &= -\mathcal{Q}_3(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P - \left[\mathcal{Q}_1(t)H + \mathcal{Q}_3(t)G - \frac{1}{2}\dot G\right]\nabla^2_{\mathbf{u}}P\\ &\quad - [G + \mathcal{Q}_1(t)F + \mathcal{Q}_3(t)H - \dot H]\,\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P.\end{aligned} \tag{73}$$

This generalized phase-space FPE should be consistent with those reported in Refs. [30,31]. To see that this is indeed the case, we now proceed to identify the time-dependent coefficients which multiply the Laplacian $\nabla^2_{\mathbf{u}}P$ and dot product $\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P$. For such a purpose, we can see that when $\Omega = 0$, the expressions for F, G, and H, given by Eqs. (49)–(51), are simply

$$\beta F = -H_0^2(t) + 2\int_0^t H_0(t')dt' - \omega^2\left[\int H_0(t')dt'\right]^2$$
$$= -\mathcal{C}^2 + 2\int_0^t \mathcal{C}\,dt' - \omega^2\left(\int \mathcal{C}\,dt'\right)^2, \tag{74}$$

$$\beta G = 1 - \dot{H}_0^2(t) - \omega^2 H_0^2(t) = 1 - \dot{\mathcal{C}}^2 - \omega^2\mathcal{C}^2, \tag{75}$$

$$\beta H = H_0(t) - \omega^2 H_0(t)\int_0^t H_0(t')\,dt' - H_0(t)\dot{H}_0(t)$$
$$= \mathcal{C}\mathcal{A} - \mathcal{C}\dot{\mathcal{C}}, \tag{76}$$

where we recall that $\mathcal{A} = \chi_0(t)$ and $\mathcal{C} = H_0(t)$. Therefore,

$$\mathcal{S}_1(t) = \mathcal{Q}_1(t)H + \mathcal{Q}_3(t)G - \frac{1}{2}\dot{G}$$
$$= \frac{1}{\beta\Delta_{ho}}[(\ddot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{A}}\ddot{\mathcal{C}})(\mathcal{C}\mathcal{A} - \mathcal{C}\dot{\mathcal{C}})$$
$$+ (\ddot{\mathcal{C}}\mathcal{A} - \ddot{\mathcal{A}}\mathcal{C})(1 - \dot{\mathcal{C}}^2 - \omega^2\mathcal{C}^2)$$
$$+ (\dot{\mathcal{C}}\ddot{\mathcal{C}} + \omega^2\mathcal{C}\dot{\mathcal{C}})(\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C})]. \tag{77}$$

Using the fact that $\dot{\mathcal{A}} = -\omega^2\mathcal{C}$, it can be shown that

$$\mathcal{S}_1(t) = \frac{1}{\beta}\frac{\ddot{\mathcal{C}}\mathcal{A} - \ddot{\mathcal{A}}\mathcal{C}}{\Delta_{ho}} = \frac{1}{\beta}\frac{\ddot{\mathcal{C}}\mathcal{A} - \ddot{\mathcal{A}}\mathcal{C}}{\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C}} = \frac{1}{\beta}\frac{d\ln\Delta_{ho}}{dt}, \tag{78}$$

and thus

$$\mathcal{S}_1(t) = \frac{1}{\beta}\mathcal{Q}_3(t) = k_B T\frac{d\ln\Delta_{ho}}{dt}. \tag{79}$$

For the time-dependent coefficient $\mathcal{S}_2(t)$, we have

$$\mathcal{S}_2(t) = G + \mathcal{Q}_1(t)F + \mathcal{Q}_3(t)H - \dot{H}$$
$$= \frac{1}{\beta\Delta_{ho}}\Bigg\{1 - \dot{\mathcal{C}}^2 - \omega^2\mathcal{C}^2 + (\ddot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{A}}\ddot{\mathcal{C}})$$
$$\times\left[-\mathcal{C}^2 + 2\int_0^t \mathcal{C}\,dt' - \omega^2\left(\int \mathcal{C}\,dt'\right)^2\right]$$
$$+ (\ddot{\mathcal{C}}\mathcal{A} - \ddot{\mathcal{A}}\mathcal{C})(\mathcal{C}\mathcal{A} - \mathcal{C}\dot{\mathcal{C}})$$
$$- (\dot{\mathcal{C}}\mathcal{A} + \mathcal{C}\dot{\mathcal{A}} - \mathcal{C}\ddot{\mathcal{C}} - \dot{\mathcal{C}}^2)(\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C})\Bigg\}. \tag{80}$$

After some easy algebra, it can be shown that

$$-\mathcal{S}_2(t) = \frac{1}{\beta\,\omega^2}\left[-\frac{\ddot{\mathcal{A}}\dot{\mathcal{C}} - \ddot{\mathcal{C}}\dot{\mathcal{A}}}{\Delta_{ho}} - \omega^2\right]$$
$$= \frac{k_B T}{\omega^2}[-\mathcal{Q}_1 - \omega^2]. \tag{81}$$

If we want to compare with Adelman's result, we must identify the following coefficients: according to Eq. (68), we can write

$$-\mathcal{Q}_1(t) = \frac{1}{\Delta_{ho}}[\dot{\mathcal{A}}\ddot{\mathcal{C}} - \dot{\mathcal{C}}\ddot{\mathcal{A}}] = \tilde{\omega}^2(t), \tag{82}$$

which is the same as Eq. (9) given in Sec. II. In a similar way, $-\mathcal{S}_2(t) = (k_B T/\omega^2)[\tilde{\omega}^2(t) - \omega^2]$, the coefficient $-\mathcal{Q}_3(t) = -\frac{d\ln\Delta_{ho}}{dt} = -\tilde{\beta}(t)$, and $-\mathcal{S}_1(t) = -k_B T\frac{d\ln\Delta_{ho}}{dt} = k_B T\,\tilde{\beta}(t)$. Therefore, in the absence of the magnetic field, Eq. (73) reduces to

$$\frac{\partial P}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}P - \tilde{\omega}^2(t)\,\mathbf{x}\cdot\nabla_{\mathbf{u}}P$$
$$= \tilde{\beta}(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P + k_B T\tilde{\beta}(t)\,\nabla^2_{\mathbf{u}}P$$
$$+ \frac{k_B T}{\omega^2}[\tilde{\omega}^2(t) - \omega^2]\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P. \tag{83}$$

Along the z direction, the non-Markovian FPE is the same as Eq. (7) of Sec. II, so that the generalized FPE for the 3D probability distribution function, $\mathbb{P}(\mathbf{r}, \mathbf{v}, t|\mathbf{r}_0, \mathbf{v}_0) \equiv P(\mathbf{x}, \mathbf{u}, t|\mathbf{x}_0, \mathbf{u}_0)P(z, v_z, t|z_0, v_{z0})$, is the same as given by Eq. (13) of Sec. II, where $P(\mathbf{x}, \mathbf{u}, t|\mathbf{x}_0, \mathbf{u}_0)$ satisfies Eq. (83). We conclude that in the absence of the magnetic field, the generalized phase-space FPE given by Eq. (61) is totally consistent with the results reported in [30,31].

(ii) *Absence of a harmonic potential*. In this limiting case, Eq. (61) must also be consistent with the non-Markovian FPE reported in [15] for a free particle in a magnetic field. In this case, $\omega^2 = 0$ and now the quantities $\mathcal{A} = 1$, $\mathcal{B} = 0$, $\mathcal{C} = H_0(t) - \Omega^2\mathcal{H}_2(t)$, and $\mathcal{D} = \Omega\,\mathcal{H}_1(t)$, and thus, according to Appendix A we have

$$\mathcal{P}_1(t) = \frac{1}{\Delta_m}[\dot{\mathcal{A}}\hat{c}_x(t) - \dot{\mathcal{C}}\hat{c}_{v_x}(t) + \dot{\mathcal{D}}\hat{c}_{v_y}(t)]$$
$$= \frac{1}{\Delta_m}\{\dot{\mathcal{A}}[\dot{\mathcal{A}}\dot{\mathcal{C}}\mathcal{D} - \dot{\mathcal{A}}\mathcal{C}\dot{\mathcal{D}}] - \dot{\mathcal{C}}[\dot{\mathcal{A}}^2\mathcal{D} - \mathcal{A}\dot{\mathcal{A}}\dot{\mathcal{D}}]$$
$$+ \dot{\mathcal{D}}[\dot{\mathcal{A}}^2\mathcal{C} - \mathcal{A}\dot{\mathcal{A}}\dot{\mathcal{C}}]\} = 0, \tag{84}$$

$$\mathcal{P}_2(t) = \frac{1}{\Delta_m}[-\dot{\mathcal{A}}\hat{d}_x(t) - \dot{\mathcal{C}}\hat{d}_{v_x}(t) + \dot{\mathcal{D}}\hat{d}_{v_y}(t)]$$
$$= \frac{1}{\Delta_m}\{-\dot{\mathcal{A}}[-\mathcal{A}\mathcal{C}\dot{\mathcal{D}} + \mathcal{A}\dot{\mathcal{C}}\mathcal{D}] - \dot{\mathcal{C}}[\mathcal{A}^2\dot{\mathcal{D}} - \mathcal{A}\dot{\mathcal{A}}\mathcal{D}]$$
$$+ \dot{\mathcal{D}}[\mathcal{A}^2\dot{\mathcal{C}} - \mathcal{A}\dot{\mathcal{A}}\mathcal{C}]\} = 0, \tag{85}$$

$$\mathcal{Q}_1(t) = \frac{1}{\Delta_m}[\ddot{\mathcal{C}}\hat{a}_{v_x}(t) + \ddot{\mathcal{D}}\hat{a}_{v_y}(t)] = 0, \tag{86}$$

$$\mathcal{Q}_2(t) = \frac{1}{\Delta_m}[-\ddot{\mathcal{C}}\hat{c}_{v_x}(t) + \ddot{\mathcal{D}}\hat{c}_{v_y}(t)] = 0, \tag{87}$$

$$\mathcal{Q}_3(t) = \frac{1}{\Delta_m}[\ddot{\mathcal{C}}\hat{b}_{v_x}(t) + \ddot{\mathcal{D}}\hat{b}_{v_y}(t)] = \frac{1}{\Delta_m}[\dot{\mathcal{C}}\ddot{\mathcal{C}} + \dot{\mathcal{D}}\ddot{\mathcal{D}}], \tag{88}$$

$$\mathcal{Q}_4(t) = \frac{1}{\Delta_m}[-\ddot{\mathcal{C}}\hat{d}_{v_x}(t) + \ddot{\mathcal{D}}\hat{d}_{v_y}(t)] = \frac{1}{\Delta_m}[\dot{\mathcal{C}}\ddot{\mathcal{D}} - \dot{\mathcal{D}}\ddot{\mathcal{C}}], \tag{89}$$

where the determinant now reads $\Delta_m = \dot{\mathcal{C}}^2 + \dot{\mathcal{D}}^2$. We have in mind that $\mathcal{Q}_1(t) = \mathcal{R}_2(t) = 0$, $\mathcal{Q}_2(t) = -\mathcal{R}_1(t) = 0$, $\mathcal{Q}_3(t) = \mathcal{R}_4(t)$, and $\mathcal{Q}_4(t) = -\mathcal{R}_3(t)$. The functions of time $\mathcal{S}_1(t)$, $\mathcal{S}_2(t)$, and $\mathcal{S}_3(t)$ also reduce to

$$\mathcal{S}_1(t) = \mathcal{Q}_1(t)H - \mathcal{Q}_2(t)I + \mathcal{Q}_3(t)G - \tfrac{1}{2}\dot{G}$$
$$= \mathcal{Q}_3(t)G - \tfrac{1}{2}\dot{G}, \tag{90}$$

$$\begin{aligned}\mathcal{S}_2(t) &= G + [\mathcal{Q}_4(t) - \mathcal{P}_1(t)]I + \mathcal{Q}_1(t)F + \mathcal{Q}_3(t)H - \dot{H}\\ &= G + \mathcal{Q}_3(t)H + \mathcal{Q}_4(t)I - \dot{H},\end{aligned} \tag{91}$$

$$\begin{aligned}\mathcal{S}_3(t) &= \mathcal{Q}_2(t)F - \mathcal{P}_2(t)G + [\mathcal{Q}_4(t) - \mathcal{P}_1(t)]H\\ &\quad - \mathcal{Q}_3(t)I + \dot{I}\\ &= \mathcal{Q}_4(t)H - \mathcal{Q}_3(t)I + \dot{I}.\end{aligned} \tag{92}$$

Therefore, in the absence of harmonic potential, the generalized phase-space Fokker-Planck Eq. (61) reduces to

$$\begin{aligned}&\frac{\partial P}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}P\\ &= -\mathcal{Q}_3(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P + \mathcal{Q}_4(t)\,[\mathbf{u}\times\nabla_{\mathbf{u}}P]_z\\ &\quad - \left[\mathcal{Q}_3(t)G - \frac{1}{2}\dot{G}\right]\nabla_{\mathbf{u}}^2P - [G + \mathcal{Q}_3(t)H + \mathcal{Q}_4(t)I - \dot{H}]\\ &\quad \times\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P + [\mathcal{Q}_4(t)H - \mathcal{Q}_3(t)I + \dot{I}]\,[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{u}}P]_z.\end{aligned} \tag{93}$$

To compare with the non-Markovian FPE reported in [15], we require the identification of the following time-dependent parameters: The determinant calculated in Ref. [15] is given by $\Delta_{xv}(t) = \dot{\mathcal{A}}^2(t) + \dot{\mathcal{B}}^2(t)$, where $\mathcal{A}(t) = \mathcal{H}_0(t) - \Omega^2\mathcal{H}_2(t)$ and $\mathcal{B}(t) = \Omega\mathcal{H}_1(t)$. In our present contribution and according to Appendix A, we can see that the parameters $\mathcal{C} = \mathcal{A}(t)$ and $\mathcal{D} = \mathcal{B}(t)$, and thus $\Delta_m = \dot{\mathcal{C}}^2 + \dot{\mathcal{D}}^2 = \Delta_{xv}(t)$. Also, in Ref. [15], the time-dependent parameters $\mathcal{F}(t)$ and $\mathcal{G}(t)$ have been defined by

$$\mathcal{F}(t) = \frac{\dot{\mathcal{A}}(t)\ddot{\mathcal{A}}(t) + \dot{\mathcal{B}}(t)\ddot{\mathcal{B}}(t)}{\Delta_{xv}(t)}, \tag{94}$$

$$\mathcal{G}(t) = \frac{\dot{\mathcal{A}}(t)\ddot{\mathcal{B}}(t) - \dot{\mathcal{B}}(t)\ddot{\mathcal{A}}(t)}{\Delta_{xv}(t)}. \tag{95}$$

However, we see from Eqs. (88) and (89) that $\mathcal{Q}_3(t) = \mathcal{F}(t)$ and $\mathcal{Q}_4(t) = \mathcal{G}(t)$, and therefore the following sum can be written as

$$\begin{aligned}&-\mathcal{Q}_3(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P + \mathcal{Q}_4(t)\,[\mathbf{u}\times\nabla_{\mathbf{u}}P]_z\\ &= -\mathcal{F}(t)\left(\frac{\partial v_xP}{\partial v_x} + \frac{\partial v_yP}{\partial v_y}\right) + \mathcal{G}(t)\left(v_x\frac{\partial P}{\partial v_y} - v_y\frac{\partial P}{\partial v_x}\right)\\ &= \nabla\cdot\Gamma(t)\,\mathbf{u}P,\end{aligned} \tag{96}$$

where $\Gamma(t)$ is an antisymmetric matrix defined by

$$\Gamma(t) = -\begin{pmatrix}\mathcal{F}(t) & \mathcal{G}(t)\\ -\mathcal{G}(t) & \mathcal{F}(t)\end{pmatrix}. \tag{97}$$

Finally, in the absence of the harmonic oscillator, the non-Markovian FPE (93) reduces to

$$\begin{aligned}\frac{\partial P}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}P &= \nabla\cdot\Gamma(t)\,\mathbf{u}P - \left[\mathcal{F}(t)G - \frac{1}{2}\dot{G}\right]\nabla_{\mathbf{u}}^2P\\ &\quad - [G + \mathcal{F}(t)H + \mathcal{G}(t)I - \dot{H}]\,\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P\\ &\quad + [\mathcal{G}(t)H - \mathcal{F}(t)I + \dot{I}]\,[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{u}}P]_z,\end{aligned} \tag{98}$$

which is exactly the same as Eq. (114) reported in Ref. [15], when the the time-dependent force fields are absent. Therefore, also in this limiting case, Eq. (61) is totally consistent with the result reported in Ref. [15]. As a consequence of both limiting cases, it is now clear that the generalized diffusion coefficient $\mathcal{P}_2(t)I$ accounts for the coupling effect between the magnetic field and harmonic oscillator. This is because, in the absence only of the magnetic field, the time-dependent coefficient $I = 0$, and thus $\mathcal{P}_2(t)I = 0$. However, in the case of only a free particle in a magnetic field, the time-dependent coefficient $\mathcal{P}_2(t) = 0$, and thus also $\mathcal{P}_2(t)I = 0$. The comparison with the non-Markovian FPE reported in Ref. [14] will be studied in the last section of this work.

IV. GENERALIZED PHASE-SPACE FPE FOR A HARMONIC OSCILLATOR ACROSS A MAGNETIC FIELD AND TIME-DEPENDENT FORCE FIELDS

A. Generalized Langevin equation

In this section, we study the influence of additional time-dependent force fields in the non-Markovian Brownian motion of the harmonic oscillator studied in the preceding section. In general, the time-dependent force fields account for electrical $q\mathbf{E}_{\text{ext}}(t)$ and mechanical $\mathbf{F}_{\text{mec}}(t)$ forces. The external forces per unit mass are thus defined by $\mathbf{a}(t) = [q\mathbf{E}_{\text{ext}}(t) + \mathbf{F}_{\text{mec}}(t)]/m$, and thus the GLE in terms of its components now reads

$$\ddot{x} - \Omega\dot{y} + \omega^2x + \int_0^t\gamma(t-t')\,\dot{x}(t')\,dt' - a_x(t) = f_x(t), \tag{99}$$

$$\ddot{y} + \Omega\dot{x} + \omega^2y + \int_0^t\gamma(t-t')\,\dot{y}(t')\,dt' - a_y(t) = f_y(t), \tag{100}$$

$$\ddot{z} + \omega^2z + \int_0^t\gamma(t-t')\,\dot{z}(t')\,dt' - a_z(t) = f_z(t). \tag{101}$$

The solution of each equation is exactly the same as obtained in Sec. III, except for the mean values $\langle x(t)\rangle$, $\langle y(t)\rangle$, $\langle z(t)\rangle$, $\langle v_x(t)\rangle$, $\langle v_y(t)\rangle$, and $\langle v_z(t)\rangle$, which are given by

$$\begin{aligned}\langle x(t)\rangle &= x_0[\chi_0(t) + \Omega^2\omega^2\chi_2(t)] - y_0\Omega\omega^2H_1(t)\\ &\quad + v_{x0}[H_0(t) - \Omega^2\mathcal{H}_2(t)] + v_{y0}\Omega\mathcal{H}_1(t)\\ &\quad + \int_0^t H_0(t-t')a_x(t')dt' - \Omega^2\int_0^t\mathcal{H}_2(t-t')a_x(t')dt'\\ &\quad + \Omega\int_0^t\mathcal{H}_1(t-t')a_y(t')dt',\end{aligned} \tag{102}$$

$$\begin{aligned}\langle y(t)\rangle &= y_0[\chi_0(t) + \Omega^2\omega^2\chi_2(t)] + x_0\Omega\omega^2H_1(t)\\ &\quad + v_{y0}[H_0(t) - \Omega^2\mathcal{H}_2(t)] - v_{x0}\Omega\mathcal{H}_1(t)\\ &\quad + \int_0^t H_0(t-t')a_y(t')dt' - \Omega^2\int_0^t\mathcal{H}_2(t-t')a_y(t')dt'\\ &\quad - \Omega\int_0^t\mathcal{H}_1(t-t')a_x(t')dt',\end{aligned} \tag{103}$$

$$\langle z(t)\rangle = z_0\chi_0(t) + v_{z0}H_0(t) + \int_0^t H_0(t-t')a_z(t')dt', \tag{104}$$

$$\begin{aligned}\langle v_x(t)\rangle &= x_0[-\omega^2H_0(t) + \Omega^2\omega^2\mathcal{H}_2(t)] - y_0\Omega\omega^2\dot{H}_1(t)\\ &\quad + v_{x0}[\dot{H}_0(t) - \Omega^2\dot{\mathcal{H}}_2(t)] + v_{y0}\Omega\dot{\mathcal{H}}_1(t)\\ &\quad + \int_0^t\dot{H}_0(t-t')a_x(t')dt' - \Omega^2\int_0^t\dot{\mathcal{H}}_2(t-t')\\ &\quad \times a_x(t')dt' + \Omega\int_0^t\dot{\mathcal{H}}_1(t-t')a_y(t')dt',\end{aligned} \tag{105}$$

$$\begin{aligned}\langle v_y(t)\rangle &= y_0[-\omega^2 H_0(t)+\Omega^2\omega^2\mathcal{H}_2(t)]+x_0\Omega\omega^2\dot{H}_1(t)\\ &\quad+v_{y0}[\dot{H}_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)]-v_{x0}\Omega\dot{\mathcal{H}}_1(t)\\ &\quad+\int_0^t \dot{H}_0(t-t')a_y(t')dt'-\Omega^2\int_0^t \dot{\mathcal{H}}_2(t-t')\\ &\quad\times a_y(t')dt'-\Omega\int_0^t \dot{\mathcal{H}}_1(t-t')a_x(t')dt', \end{aligned}\tag{106}$$

$$\langle v_z(t)\rangle = -\omega^2 z_0 H_0(t)+v_{z0}\dot{H}_0(t)+\int_0^t \dot{H}_0(t-t')a_z(t')dt'.\tag{107}$$

Also, the process along the z axis is independent of the process in the (x, y) plane. Again, we just pay attention to the planar process. In this case, the conditional probability density $P(\mathbf{R},\mathbf{S})$ is the same as Eq. (55), but now

$$\begin{aligned}R_1 &= x-\langle x\rangle\\ &= x-x_0[\chi_0(t)+\Omega^2\omega^2\chi_2(t)]\\ &\quad+y_0\Omega\omega^2 H_1(t)-v_{x0}[H_0(t)-\Omega^2\mathcal{H}_2(t)]-v_{y0}\Omega\mathcal{H}_1(t)\\ &\quad-\int_0^t H_0(t-t')a_x(t')dt'+\Omega^2\int_0^t \mathcal{H}_2(t-t')a_x(t')dt'\\ &\quad-\Omega\int_0^t \mathcal{H}_1(t-t')a_y(t')dt,\end{aligned}\tag{108}$$

$$\begin{aligned}R_2 &= y-\langle y\rangle\\ &= y-y_0[\chi_0(t)+\Omega^2\omega^2\chi_2(t)]\\ &\quad-x_0\Omega\omega^2 H_1(t)-v_{y0}[H_0(t)-\Omega^2\mathcal{H}_2(t)]+v_{x0}\Omega\mathcal{H}_1(t)\\ &\quad-\int_0^t H_0(t-t')a_y(t')dt'+\Omega^2\int_0^t \mathcal{H}_2(t-t')a_y(t')dt'\\ &\quad+\Omega\int_0^t \mathcal{H}_1(t-t')a_x(t')dt',\end{aligned}\tag{109}$$

$$\begin{aligned}S_1 &= v_x-\langle v_x\rangle\\ &= v_x-x_0[-\omega^2 H_0(t)+\Omega^2\omega^2\mathcal{H}_2(t)]\\ &\quad+y_0\Omega\omega^2\dot{H}_1(t)-v_{x0}[\dot{H}_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)]-v_{y0}\Omega\dot{\mathcal{H}}_1(t)\\ &\quad-\int_0^t \dot{H}_0(t-t')a_x(t')dt'+\Omega^2\int_0^t \dot{\mathcal{H}}_2(t-t')a_x(t')dt'\\ &\quad-\Omega\int_0^t \dot{\mathcal{H}}_1(t-t')a_y(t')dt,\end{aligned}\tag{110}$$

$$\begin{aligned}S_2 &= v_x-\langle v_x\rangle\\ &= v_y-y_0[-\omega^2 H_0(t)+\Omega^2\omega^2\mathcal{H}_2(t)]\\ &\quad-x_0\Omega\omega^2\dot{H}_1(t)-v_{y0}[\dot{H}_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)]+v_{x0}\Omega\dot{\mathcal{H}}_1(t)\\ &\quad-\int_0^t H_0(t-t')a_y(t')dt'+\Omega^2\int_0^t \mathcal{H}_2(t-t')a_y(t')dt'\\ &\quad+\Omega\int_0^t \mathcal{H}_1(t-t')a_x(t')dt'.\end{aligned}\tag{111}$$

Also, the functions $H_0(t)$, $\mathcal{H}_1(t)$, $\mathcal{H}_2(t)$, F, G, H, and I are exactly the same as those given in Sec. III.

B. Generalized phase-space FPE including both the magnetic field and time-dependent force fields

To derive the generalized phase-space Fokker-Planck equation, we again use the method of the characteristic function, which is explicitly given in Appendix B. The concluding result is the generalized phase-space FPE for the harmonic oscillator in the presence of both a constant magnetic field and time-dependent force fields, which can be written as

$$\begin{aligned}&\frac{\partial P}{\partial t}+\mathcal{P}_1(t)\,[\mathbf{q}\times\nabla_{\mathbf{x}}P]_z+\mathcal{P}_2(t)\,[\mathbf{p}\times\nabla_{\mathbf{x}}P]_z\\ &\quad-\mathcal{Q}_1(t)\,\mathbf{q}\cdot\nabla_{\mathbf{u}}P+\mathcal{Q}_2(t)\,[\mathbf{q}\times\nabla_{\mathbf{u}}P]_z\\ &\quad+[\dot{\mathbf{p}}-\mathcal{Q}_3(t)\mathbf{p}]\cdot\nabla_{\mathbf{u}}P+\mathcal{Q}_4(t)\,[\mathbf{p}\times\nabla_{\mathbf{u}}P]_z\\ &\quad+\mathbf{u}\cdot\nabla_{\mathbf{x}}P+\mathcal{Q}_1(t)\,\mathbf{x}\cdot\nabla_{\mathbf{u}}P-\mathcal{P}_2(t)\,I\,\nabla_{\mathbf{x}}^2P\\ &=\mathcal{P}_1(t)\,[\mathbf{x}\times\nabla_{\mathbf{x}}P]_z+\mathcal{P}_2(t)\,[\mathbf{u}\times\nabla_{\mathbf{x}}P]_z\\ &\quad-\mathcal{R}_1(t)\,[\mathbf{x}\times\nabla_{\mathbf{u}}P]_z-\mathcal{R}_3(t)\,[\mathbf{u}\times\nabla_{\mathbf{u}}P]_z\\ &\quad-\mathcal{Q}_3(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P-\mathcal{S}_1(t)\,\nabla_{\mathbf{u}}^2P-\mathcal{S}_2(t)\,\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P\\ &\quad+\mathcal{S}_3(t)\,[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{u}}P]_z,\end{aligned}\tag{112}$$

where the vectors $\mathbf{p}$ and $\mathbf{q}$ are defined in Appendix B, and all the time-dependent coefficients are the same as defined in Appendix A. Clearly, the additional contributions to Eq. (61) are given by the second term up to the sixth term in the left-hand side of Eq. (112). Evidently, in the absence of the time-dependent force fields, Eq. (112) reduces to Eq. (61).

For a particle in the presence only of both a magnetic field and time-dependent force fields, the time-dependent coefficients $\mathcal{P}_1(t)=\mathcal{P}_2(t)=0$, $\mathcal{Q}_1(t)=\mathcal{R}_2(t)=0$, $\mathcal{Q}_2(t)=-\mathcal{R}_1(t)=0$, $\mathcal{Q}_3(t)=\mathcal{R}_4(t)$, and $\mathcal{Q}_4=-\mathcal{R}_3(t)$. Hence, Eq. (112) reduces to

$$\begin{aligned}&\frac{\partial P}{\partial t}+[\dot{\mathbf{p}}-\mathcal{F}(t)\mathbf{p}]\cdot\nabla_{\mathbf{u}}+\mathcal{G}_4(t)\,[\mathbf{p}\times\nabla_{\mathbf{u}}P]_z+\mathbf{u}\cdot\nabla_{\mathbf{x}}P\\ &\quad=\nabla\cdot\Gamma(t)\,\mathbf{u}P-\big[\mathcal{F}(t)G-\frac{1}{2}\dot{G}\big]\,\nabla_{\mathbf{u}}^2P\\ &\qquad-[G+\mathcal{F}(t)H+\mathcal{G}(t)I-\dot{H}]\,\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P\\ &\qquad+[\mathcal{G}(t)H-\mathcal{F}(t)I+\dot{I}]\,[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{u}}P]_z,\end{aligned}\tag{113}$$

which is exactly the same result obtained in [15] (see Eq. (114) in Ref. [15]).

For a particle bounded by a harmonic potential and in the presence only of the time-dependent force fields, we have that $\mathcal{P}_1(t)=\mathcal{P}_2(t)=0$, $\mathcal{Q}_2(t)=-\mathcal{R}_1(t)=0$, $\mathcal{Q}_4(t)=-\mathcal{R}_3(t)=0$, $\mathcal{Q}_1(t)=-\tilde{\omega}^2(t)$, $-\mathcal{Q}_3(t)=-\mathcal{S}_1(t)=\tilde{\beta}(t)$, $\mathcal{S}_2(t)=(k_BT/\omega^2)[\tilde{\omega}^2-\omega^2]$, and $\mathcal{S}_3=0$. Therefore, Eq. (112) reduces to

$$\begin{aligned}&\frac{\partial P}{\partial t}+\tilde{\omega}^2(t)\,\mathbf{q}\cdot\nabla_{\mathbf{u}}P+[\dot{\mathbf{p}}+\tilde{\beta}(t)\mathbf{p}]\cdot\nabla_{\mathbf{u}}P+\mathbf{u}\cdot\nabla_{\mathbf{x}}P-\tilde{\omega}^2(t)\,\mathbf{x}\cdot\nabla_{\mathbf{u}}P\\ &\quad=\tilde{\beta}(t)\,\nabla_{\mathbf{u}}\cdot\mathbf{u}P+k_BT\tilde{\beta}(t)\,\nabla_{\mathbf{u}}^2P+\frac{k_BT}{\omega^2}\,[\tilde{\omega}^2(t)-\omega^2]\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}}P.\end{aligned}\tag{114}$$

In the absence of time-dependent force fields, $\mathbf{p}$ and $\mathbf{q}$ are equal to zero, and therefore Eq. (114) becomes exactly the same result obtained by Adelman [30], as expected.

V. COMMENTS ON THE METHOD REPORTED IN REF. [14]

In this section, it is necessary to compare the results obtained in Ref. [14] with those obtained in our present contribution and those reported in previous works. The method proposed in Ref. [14] to obtain the non-Markovian FPEs is basically related to the structure of the correlation matrix defined by $\mathbf{A}(t)$. If the matrix is diagonal, then the non-Markovian FPE is constructed taking into account the structure of its corresponding Markovian FPE, except by the time-dependent coefficients. However, if the matrix is not diagonal, then the non-Markovian FPE is constructed on the basis of the Markovian FPE plus extra contributions which are "intuitively" proposed, due to the off-diagonal elements of matrix $\mathbf{A}(t)$. We proceed to analyze and clarify each case studied in [14].

A. Generalized velocity-space FPE for a free particle

As studied in Sec. II of Ref. [14], the proposed method to obtain the generalized velocity-space FPE for a free particle seems to be fine. This non-Markovian FPE is similar to its corresponding Markovian one, except by the time-dependent coefficients. We can check this below.

As written in Ref. [14], the Markovian velocity-space FPE for a free particle reads

$$\frac{\partial P}{\partial t} = \gamma_0 \nabla \cdot (\mathbf{u}P) + \gamma_0 k_B T \nabla^2 P. \tag{115}$$

In Sec. II of Ref. [14], it is shown that the correlation matrix $\mathbf{A}(t)$ is diagonal with elements $A_{11} = A_{22} = A_{33} = A(t)$, where $A(t) = k_B T[1 - \chi^2(t)]$ and $\chi(t) = \frac{1}{3k_B T}\langle \mathbf{u}(t) \cdot \mathbf{u}_0 \rangle$, with $\mathbf{u}(t)$ the 3D velocity vector. Due to this fact and taking into account the structure of the Markovian FPE (115), the authors construct the non-Markovian FPE as follows:

$$\frac{\partial P}{\partial t} = \beta(t) \nabla \cdot (\mathbf{u}P) + H(t) \nabla^2 P. \tag{116}$$

As can be seen, this equation is similar in structure to Eq. (115), except by the time-dependent coefficients. Once this is done, the authors show that the non-Markovian velocity-space FPE reads

$$\frac{\partial P}{\partial t} = -\frac{\dot{\chi}(t)}{\chi(t)} \nabla \cdot \mathbf{u}P + \frac{1}{6}\chi^2(t)\frac{d}{dt}[\chi^{-2}(t)A(t)]\nabla^2 P, \tag{117}$$

where

$$\beta(t) = -\frac{\dot{\chi}(t)}{\chi(t)}, \quad H(t) = \frac{1}{6}\chi^2(t)\frac{d}{dt}[\chi^{-2}(t)A(t)], \tag{118}$$

which is consistent with the generalized velocity-space FPE obtained by Adelman in [30] (see Eq. (2.11) in Ref. [30]) and the one derived in [15] (see Eq. (37) given in Ref. [15]).

B. Generalized phase-space FPE for a harmonic oscillator

In this case, the Markovian FPE is given by Eq. (36) in Sec. III of Ref. [14], and reads

$$\frac{\partial P}{\partial t} = -\mathbf{u} \cdot \frac{\partial P}{\partial \mathbf{x}} + \omega^2 \mathbf{x} \cdot \frac{\partial P}{\partial \mathbf{u}} + \gamma_0 \frac{\partial}{\partial \mathbf{u}} \cdot \mathbf{u}P + \gamma_0 k_B T \frac{\partial^2 P}{\partial \mathbf{u}^2}. \tag{119}$$

However, the correlation matrix $\mathbf{A}(t)$ is not diagonal and, due to this fact, the proposed non-Markovian phase-space FPE is now constructed taking into account the same terms of Eq. (119) plus two extra terms given by $\frac{\partial^2 P}{\partial \mathbf{x}^2}$ and $\frac{\partial}{\partial \mathbf{u}} \cdot \frac{\partial P}{\partial \mathbf{x}}$. Therefore, the non-Markovian phase-space FPE is proposed to have the following structure with time-dependent coefficients:

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\mathbf{u} \cdot \frac{\partial P}{\partial \mathbf{x}} + H_1(t)\mathbf{x} \cdot \frac{\partial P}{\partial \mathbf{u}} + H_2(t)\frac{\partial}{\partial \mathbf{u}} \cdot \mathbf{u}P \\ &\quad + H_3(t)\frac{\partial^2 P}{\partial \mathbf{x}^2} + H_4(t)\frac{\partial^2 P}{\partial \mathbf{u}^2} + H_5(t)\frac{\partial}{\partial \mathbf{u}} \cdot \frac{\partial P}{\partial \mathbf{x}}.\end{aligned} \tag{120}$$

In principle, the Laplacian term $\frac{\partial^2 P}{\partial \mathbf{x}^2}$ is due to one diagonal element of matrix $\mathbf{A}(t)$, and the other one given by $\frac{\partial}{\partial \mathbf{u}} \cdot \frac{\partial P}{\partial \mathbf{x}}$ is due to the off-diagonal elements of the same matrix. In the same Sec. III of Ref. [14], it can be checked that the time-dependent diffusion coefficient $H_3(t) = \frac{1}{6}[\dot{A}_{11} - 2A_{12}] = 0$ (see Eqs. (27) and (30) in Ref. [14]). Therefore, the extra diffusion term $H_3(t)\frac{\partial^2 P}{\partial \mathbf{x}^2}$ does not contribute in the non-Markovian phase-space FPE (120). After some algebra, the authors conclude that the non-Markovian phase-space FPE is given by (see Eq. (47) in Ref. [14])

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\mathbf{u} \cdot \frac{\partial P}{\partial \mathbf{x}} + \tilde{\omega}^2(t)\mathbf{x} \cdot \frac{\partial P}{\partial \mathbf{u}} + \tilde{\beta}(t)\frac{\partial}{\partial \mathbf{u}} \cdot \mathbf{u}P \\ &\quad + \frac{1}{6}[\dot{A}_{22} + 2\tilde{\omega}^2(t)A_{12} + 2\tilde{\beta}(t)A_{22}]\frac{\partial^2 P}{\partial \mathbf{u}^2} \\ &\quad + \frac{1}{3}[\dot{A}_{12} + \tilde{\omega}^2(t)A_{11} + \tilde{\beta}(t)A_{22} - A_{22}]\frac{\partial}{\partial \mathbf{u}} \cdot \frac{\partial P}{\partial \mathbf{x}}.\end{aligned} \tag{121}$$

The authors say that this equation was derived earlier in Ref. [31] using the characteristic function method. In reality, it was derived by Adelman [30] 10 years before the one reported in Ref. [31]. The equation derived in [31] is precisely Eq. (7) reported in Sec. II of our present contribution using the characteristic function method.

Some important details that we would like to comment on here are the following: the generalized phase-space FPE for the harmonic oscillator derived in [31], which is the same as Eq. (7) of our present contribution, is exactly the same as derived by Adelman [30], in 3D. However, in Eq. (121), the authors of Ref. [14] did not verify that the sum of the time-dependent coefficients $\frac{1}{6}[\dot{A}_{22} + 2\tilde{\omega}^2(t)A_{12} + 2\tilde{\beta}(t)A_{22}]$ must be equal to friction function $\tilde{\beta}(t)$ given by Eq. (8) in Sec. II of our current work. Nor did they verify that the sum $\frac{1}{3}[\dot{A}_{12} + \tilde{\omega}^2(t)A_{11} + \tilde{\beta}(t)A_{22} - A_{22}]$ must be equal to $[\tilde{\omega}^2 - \omega^2]$, where $\tilde{\omega}(t)$ is given by Eq. (9) in Sec. II of this work.

We must recall that in Sec. III C of our present contribution, we have shown that in the absence of the magnetic field, the generalized phase-space FPE (61) reduces to Eq. (83), which in turn is exactly the same as Eq. (13) given in Sec. II of this work and also derived by Adelman [30].

C. Generalized velocity-space FPE for a free particle in a magnetic field

As shown in Sec. IV of Ref. [14], the Markovian velocity-space FPE for a free particle in a magnetic is given by

$$\frac{\partial P}{\partial t} = \gamma_0\left[\frac{\partial u_x P}{\partial u_x} + \frac{\partial u_y P}{\partial u_y}\right] - \Omega\left[\frac{\partial u_y P}{\partial u_x} - \frac{\partial u_x P}{\partial u_y}\right] + \gamma_0 k_B T\left[\frac{\partial^2 P}{\partial u_x^2} + \frac{\partial^2 P}{\partial u_y^2}\right]. \tag{122}$$

Because the correlation matrix $\mathbf{A}(t)$ is diagonal, the non-Markovian velocity-space FPE is constructed in a similar way but with time-dependent coefficients, that is,

$$\frac{\partial P}{\partial t} = \beta_1(t)\left[\frac{\partial u_x P}{\partial u_x} + \frac{\partial u_y P}{\partial u_y}\right] - \beta_2(t)\left[\frac{\partial u_y P}{\partial u_x} - \frac{\partial u_x P}{\partial u_y}\right] + H(t)\left[\frac{\partial^2 P}{\partial u_x^2} + \frac{\partial^2 P}{\partial u_y^2}\right], \tag{123}$$

which is the same given by Eq. (79) in [14]. This equation was verified in the Markovian limit, however, curiously it was not compared with the non-Markovian FPE (31) derived in Ref. [15], where the method of characteristic function is used. Neither, in the absence of the magnetic field, did the authors verify the consistence of Eq. (123) with the case of a free particle studied by the same authors in Sec. II of Ref. [14].

It must be commented that in the Markovian equation (122), the cross derivatives ($\frac{\partial u_y P}{\partial u_x} - \frac{\partial u_x P}{\partial u_y}$), which is the same as $-[\mathbf{u} \times \nabla_{\mathbf{u}} P]_z$, arise as a consequence of the magnetic field.

D. Generalized phase-space FPE for a harmonic oscillator in a magnetic field

In this section, we compare the generalized phase-space FPE for a harmonic oscillator in a magnetic field proposed in Ref. [14], with the one given by Eq. (61) derived in our present contribution. Notable differences between both equations will be shown below. In Sec. V of Ref. [14], the authors show that the correlation matrix $\mathbf{A}(t)$ is clearly not diagonal and the proposed generalized phase-space FPE contains the Markovian terms plus extra contributions coming from the diagonal and off-diagonal elements of matrix $\mathbf{A}(t)$. Such a non-Markovian FPE proposed by the authors in [14] is given by (see Eq. (117) in Ref. [14])

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial u_x P}{\partial x} - \frac{\partial u_y P}{\partial y} + H_1(t)\left[x\frac{\partial P}{\partial u_x} + y\frac{\partial P}{\partial u_y}\right] \\ &\quad + H_2(t)\left[\frac{\partial u_x P}{\partial u_x} + \frac{\partial u_y P}{\partial u_y}\right] - H_3(t)\left[\frac{\partial u_y P}{\partial u_x} + \frac{\partial u_x P}{\partial u_y}\right] \\ &\quad + H_4(t)\left[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right] + H_5(t)\left[\frac{\partial^2 P}{\partial u_x^2} + \frac{\partial^2 P}{\partial u_y^2}\right] \\ &\quad + H_6(t)\left[\frac{\partial}{\partial x}\frac{\partial P}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial P}{\partial u_y}\right] + H_7(t)\left[\frac{\partial}{\partial x}\frac{\partial P}{\partial u_y} + \frac{\partial}{\partial y}\frac{\partial P}{\partial u_x}\right].\end{aligned} \tag{124}$$

Upon the comparison of this Eq. (124) with our result given by Eq. (61) of the present contribution, we first note the following: the first two terms in the right-hand side of Eq. (124) are the same as

$$-\frac{\partial u_x P}{\partial x} - \frac{\partial u_y P}{\partial y} = -\mathbf{u} \cdot \nabla_{\mathbf{x}} P. \tag{125}$$

The sum of the derivatives which multiplies $H_1(t)$ reads

$$x\frac{\partial P}{\partial u_x} + y\frac{\partial P}{\partial u_y} = \mathbf{x} \cdot \nabla_{\mathbf{x}} P. \tag{126}$$

The sum which multiplies $H_2(t)$ is the same as

$$\frac{\partial u_x P}{\partial u_x} + \frac{\partial u_y P}{\partial u_y} = \nabla_{\mathbf{u}} \cdot \mathbf{u} P. \tag{127}$$

Also, the Laplacian that multiplies $H_4(t)$ and the one that multiplies $H_5(t)$ are the same as

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \nabla_{\mathbf{x}}^2 P, \quad \frac{\partial^2 P}{\partial u_x^2} + \frac{\partial^2 P}{\partial u_y^2} = \nabla_{\mathbf{u}}^2 P. \tag{128}$$

The sum which multiplies $H_6(t)$ reads

$$\frac{\partial}{\partial x}\frac{\partial P}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial P}{\partial u_y} = \nabla_{\mathbf{u}} \cdot \nabla_{\mathbf{x}} P = \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{u}} P. \tag{129}$$

However, an important point that has to be noted in Eq. (124) is that the sum $\frac{\partial u_y P}{\partial u_x} + \frac{\partial u_y P}{\partial u_y}$ which multiplies $H_3(t)$, in principle must be a difference, not a sum, and also has to be the difference of cross derivatives because it must come from the Markovian FPE [see the second term in above Eq. (122)] due to the presence of the magnetic field. Thus, the cross derivatives must be

$$\frac{\partial u_y P}{\partial u_x} - \frac{\partial u_x P}{\partial u_y} = -[\mathbf{u} \times \nabla_{\mathbf{u}} P]_z, \tag{130}$$

not as proposed by the authors. In a similar way, the sum of the cross derivatives, $\frac{\partial}{\partial x}\frac{\partial P}{\partial u_y} + \frac{\partial}{\partial y}\frac{\partial P}{\partial u_x}$, which multiplies $H_7(t)$ must also be a difference, not a sum, that is,

$$\frac{\partial}{\partial x}\frac{\partial P}{\partial u_y} - \frac{\partial}{\partial y}\frac{\partial P}{\partial u_x} = [\nabla_{\mathbf{x}} \times \nabla_{\mathbf{u}} P]_z, \tag{131}$$

which also must arise due to the magnetic field effects.

As can be seen only two cross derivatives due to the magnetic field effects have been proposed in Eq. (124), namely, the cross derivatives which multiply $H_3(t)$ and those which multiply $H_7(t)$. The questions are now the following: *Are those the only two cross derivatives which must appear in the non-Markovian FPE (124), due to the presence of the magnetic field? How many and which other cross derivatives must appear in Eq. (124)?*

Upon the comparison of Eq. (124) with Eq. (61) given in our work, we can note the following:

(i) In the first place, the two cross derivatives $[\mathbf{u} \times \nabla_{\mathbf{u}} P]_z$ given by Eq. (130) and $[\nabla_{\mathbf{x}} \times \nabla_{\mathbf{u}} P]_z$ given by Eq. (131) must be the correct cross derivatives in Eq. (124), not as written by the authors. Furthermore, in Eq. (61), there are three extra cross derivatives given by $\mathcal{P}_1(t)\,[\mathbf{x} \times \nabla_{\mathbf{x}} P]_z$, $\mathcal{P}_2(t)\,[\mathbf{u} \times \nabla_{\mathbf{x}} P]_z$, and $\mathcal{R}_1(t)\,[\mathbf{x} \times \nabla_{\mathbf{u}} P]_z$, which do not appear in Eq. (124). With the method of characteristic function, we are showing that these extra terms arise in a natural way due precisely to the presence of the magnetic field. Obviously, these terms disappear when the magnetic field is not taken into account, as must be. In the method proposed in [14], there is no way to know *a priori* which and how many cross derivatives must appear.

(ii) The other notable difference between Eqs. (61) and (124) is the following: in Eq. (124), the time-dependent diffusion coefficient $H_4(t) = \frac{1}{2}[\dot{A}'_{11} - 2A'_{12}] = 0$. This fact can be corroborated according to Eqs. (119) and (122) given in Ref. [14], and therefore the diffusion term $H_4(t)[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}] = H_4(t)\nabla^2_{\mathbf{x}} P$ given in Eq. (124) does not appear. This fact was not explicitly revealed in Sec. V of Ref. [14].

In our present contribution, we have shown that the time-dependent diffusion coefficient $\mathcal{P}_2(t)I$ is clearly different from zero. Furthermore, it accounts for the coupling effect between both the magnetic field and harmonic oscillator. This effect must appear due to the rotational character of the magnetic field and the nature of the harmonic oscillator. Therefore, on the derivation of the generalized phase-space FPE for a harmonic oscillator in the presence of the magnetic field by means of the characteristic function method, the diffusion term $\mathcal{P}_2(t)I\nabla^2_{\mathbf{x}} P$ is, in general, different from zero, contrary to what happens in Eq. (124). In conclusion, due to the aforementioned notable differences, the method of the characteristic function provides the accurate solution of the problem.

VI. CONCLUDING REMARKS

Using the characteristic function method, we have been able to derive the generalized phase-space FPE for a charged harmonic oscillator across a magnetic field and time-dependent force fields, as given by Eq. (112). In the absence of time-dependent force fields, the non-Markovian FPE is exactly given by Eq. (61). This equation has been compared with previous results reported in Refs. [15,30,31], showing a perfect consistence in each case.

However, when Eq. (61) is compared with Eq. (124) (which is the same Eq. (117) given in Ref. [14]), we have found notable differences with the method proposed in Ref. [14]. The lacking of additional cross derivatives in Eq. (124) is due to the fact that the correlation matrix $\mathbf{A}(t)$ obtained in Sec. V of Ref. [14] does not *a priori* guarantee which and how many cross derivatives must appear in the non-Markovian FPE (124). We emphasize that the cross derivatives arise due to the effects of the magnetic field, and when the characteristic function method is used they arise in a natural way as precisely shown in our work.

The other crucial and relevant difference is also the lacking of the diffusion term $\nabla^2_{\mathbf{x}} P$ in Eq. (124); however, in Eq. (61), it is taken into account. In fact, it is multiplied by the generalized diffusion coefficient $\mathcal{P}_2(t)I$ which arises in a natural way, and shows the existence of a coupling effect between both the magnetic field and harmonic oscillator.

For these reasons, we can conclude that the method of the characteristic function provides an effective and accurate theoretical approach of how to obtain the non-Markovian Fokker-Planck equations when a magnetic field is taken into account, compared with the method proposed in Ref. [14].

It must be pointed out that if the fluctuation-dissipation relation of the second kind given by Eq. (15) is assumed to be valid, then the GLE (14) becomes stationary in the large time limit [8]. For a classical derivation of the fluctuation-dissipation relation for macroscopic non-Markovian dynamics in the presence of time-dependent force fields, without the presence of a magnetic field, we can refer to the paper by Grabert *et al.* [41]. On the other hand, in our theoretical description, the solution of the generalized phase-space FPE (61) is given by the phase-space conditional probability density (55) for all time $t > 0$. In principle, in the large time limit, this probability density must converge to its corresponding stationary probability density under the condition that the time-dependent quantities F, G, H, and I given, respectively, by Eqs. (49), (50), (51), (52) are also convergent, in this limiting case. This can be achieved for an appropriate election of the friction memory kernel, which must be symmetric and well behaved, and its Laplace transform must exist. Therefore, the stationary probability density must also be the solution of the generalized FPE for which $\frac{\partial P_{st}}{\partial t} = 0$. Under these conditions, we can guarantee reliable results even in the stationary state. This is indeed the case for an exponentially correlated memory kernel (Ornstein-Uhlenbeck process), which is a stationary stochastic process. If the friction memory kernel does not satisfy these properties, the proposal cannot, in general, guarantee reliable results.

Finally, our theoretical results can be used to study others non-Markovian fluctuation relations and the non-Markovian Crooks fluctuation relation for generalized electrical and mechanical works. The topics are in progress.

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APPENDIX A: GENERALIZED PHASE-SPACE FPE FOR A HARMONIC OSCILLATOR IN A MAGNETIC FIELD

In this Appendix, we give the algebraic steps to derive the non-Markovian phase-space Fokker-Planck equation for a Brownian particle in a harmonic potential. We do this in terms of the characteristic function in the 2D case. Taking into account the solutions given in Sec. III, the characteristic function reads

$$C(\boldsymbol{\eta}, t) = \exp\left[\sum_{i=1}^{4} i\langle\xi_i\rangle\eta_i - \frac{1}{2}\sum_{i,j=1}^{4}\sigma_{ij}\eta_i\eta_j\right], \tag{A1}$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4)$, and we define the vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4) = (x, y, v_x, v_y)$. Accordingly, the characteristic function can also be written as $C(\boldsymbol{\eta}, t) = C_1(\boldsymbol{\eta}, t)C_2(\boldsymbol{\eta}, t)$, where

$$C_1(\boldsymbol{\eta}, t) = e^{-(I/2)[\eta_1\eta_4 - \eta_2\eta_3]}\widetilde{C}_1(\eta_1, \eta_3), \tag{A2}$$

$$C_2(\boldsymbol{\eta}, t) = e^{-(I/2)[\eta_1\eta_4 - \eta_2\eta_3]}\widetilde{C}_2(\eta_2, \eta_4), \tag{A3}$$

with

$$\widetilde{C}_1(\eta_1,\eta_3) = \exp\{i\langle x\rangle\eta_1 + i\langle v_x\rangle\eta_3 - \tfrac{1}{2}[F\eta_1^2 + 2H\eta_1\eta_3 + G\eta_3^2]\}, \tag{A4}$$

$$\widetilde{C}_2(\eta_2,\eta_4) = \exp\{i\langle y\rangle\eta_2 + i\langle v_y\rangle\eta_4 - \tfrac{1}{2}[F\eta_2^2 + 2H\eta_2\eta_4 + G\eta_4^2]\}, \tag{A5}$$

with F, G, H, and I the elements of matrix $\boldsymbol{\sigma}(t)$, given by Eq. (53). The CPD given by Eq. (44) can also be obtained from the inverse Fourier transform of the characteristic function (A1) as follows:

$$P(\mathbf{R},\mathbf{S}) = \frac{1}{(2\pi)^4}\int\cdots\int C(\boldsymbol{\eta},t)\,e^{-i\boldsymbol{\eta}\cdot\boldsymbol{\xi}}\,d\boldsymbol{\eta}, \tag{A6}$$

where the integrations limits are taken from $-\infty$ to $+\infty$ (not written). Also,

$$\frac{\partial P}{\partial t} = \frac{1}{(2\pi)^4}\int\cdots\int\left(C_1\frac{\partial C_2}{\partial t} + C_2\frac{\partial C_1}{\partial t}\right)e^{-i\boldsymbol{\eta}\cdot\boldsymbol{\xi}}\,d\boldsymbol{\eta}. \tag{A7}$$

From Eqs. (A2) and (A3), we get

$$\frac{\partial C_1}{\partial t} = -\frac{\dot{I}}{2}[\eta_1\eta_4 - \eta_2\eta_3]C_1 + e^{-(I/2)[\eta_1\eta_4-\eta_2\eta_3]}\frac{\partial\widetilde{C}_1}{\partial t}, \tag{A8}$$

$$\frac{\partial C_2}{\partial t} = -\frac{\dot{I}}{2}[\eta_1\eta_4 - \eta_2\eta_3]C_2 + e^{-(I/2)[\eta_1\eta_4-\eta_2\eta_3]}\frac{\partial\widetilde{C}_2}{\partial t}. \tag{A9}$$

But,

$$\frac{\partial\widetilde{C}_1}{\partial t} = i\langle v_x\rangle\eta_1\widetilde{C}_1 + i\langle\dot{v}_x\rangle\eta_3\widetilde{C}_1 - \frac{1}{2}[\dot{F}\eta_1^2 + 2\dot{H}\eta_1\eta_3 + \dot{G}\eta_3^2]\widetilde{C}_1, \tag{A10}$$

$$\frac{\partial\widetilde{C}_2}{\partial t} = i\langle v_y\rangle\eta_2\widetilde{C}_2 + i\langle\dot{v}_y\rangle\eta_4\widetilde{C}_2 - \frac{1}{2}[\dot{F}\eta_2^2 + 2\dot{H}\eta_2\eta_4 + \dot{G}\eta_4^2]\widetilde{C}_2. \tag{A11}$$

If we define $\hat{C}_1$ and $\hat{C}_2$ as

$$\hat{C}_1 \equiv e^{-(I/2)[\eta_1\eta_4-\eta_2\eta_3]}\frac{\partial\widetilde{C}_1}{\partial t} = i\langle v_x\rangle\eta_1 C_1 + i\langle\dot{v}_x\rangle\eta_3 C_1 - \frac{1}{2}[\dot{F}\eta_1^2 + 2\dot{H}\eta_1\eta_3 + \dot{G}\eta_3^2]C_1, \tag{A12}$$

$$\hat{C}_2 \equiv e^{-(I/2)[\eta_1\eta_4-\eta_2\eta_3]}\frac{\partial\widetilde{C}_2}{\partial t} = i\langle v_y\rangle\eta_2 C_2 + i\langle\dot{v}_y\rangle\eta_4 C_2 - \frac{1}{2}[\dot{F}\eta_2^2 + 2\dot{H}\eta_2\eta_4 + \dot{G}\eta_4^2]C_2, \tag{A13}$$

then

$$C_2\frac{\partial C_1}{\partial t} = -\frac{\dot{I}}{2}[\eta_1\eta_4 - \eta_2\eta_3]C_1C_2 + \hat{C}_1C_2, \tag{A14}$$

$$C_1\frac{\partial C_2}{\partial t} = -\frac{\dot{I}}{2}[\eta_1\eta_4 - \eta_2\eta_3]C_1C_2 + C_1\hat{C}_2. \tag{A15}$$

Also, from Eqs. (A4) and (A5), we get

$$\frac{1}{\widetilde{C}_1}\frac{\partial\widetilde{C}_1}{\partial\eta_1} = i\langle x\rangle - (F\eta_1 + H\eta_3), \tag{A16}$$

$$\frac{1}{\widetilde{C}_1}\frac{\partial\widetilde{C}_1}{\partial\eta_3} = i\langle v_x\rangle - (H\eta_1 + G\eta_3), \tag{A17}$$

$$\frac{1}{\widetilde{C}_2}\frac{\partial\widetilde{C}_2}{\partial\eta_2} = i\langle y\rangle - (F\eta_2 + H\eta_4), \tag{A18}$$

$$\frac{1}{\widetilde{C}_2}\frac{\partial\widetilde{C}_2}{\partial\eta_4} = i\langle v_y\rangle - (H\eta_2 + G\eta_4). \tag{A19}$$

If we now define the functions of time,

$$\mathcal{A} \equiv \mathcal{A}(t) = \chi_0(t) + \Omega^2\omega^2\chi_2(t), \tag{A20}$$

$$\mathcal{B} \equiv \mathcal{B}(t) = \Omega\omega^2 H_1(t), \tag{A21}$$

$$\mathcal{C} \equiv \mathcal{C}(t) = H_0(t) - \Omega^2\mathcal{H}_2(t), \tag{A22}$$

$$\mathcal{D} \equiv \mathcal{D}(t) = \Omega\mathcal{H}_1(t), \tag{A23}$$

we can write the mean values of Sec. III as

$$\langle x(t)\rangle = \mathcal{A}(t)x_0 - \mathcal{B}(t)y_0 + \mathcal{C}(t)v_{x0} + \mathcal{D}(t)v_{y0}, \tag{A24}$$

$$\langle v_x(t)\rangle = \dot{\mathcal{A}}(t)x_0 - \dot{\mathcal{B}}(t)y_0 + \dot{\mathcal{C}}(t)v_{x0} + \dot{\mathcal{D}}(t)v_{y0}, \tag{A25}$$

$$\langle \dot{v}_x(t)\rangle = \ddot{\mathcal{A}}(t)x_0 - \ddot{\mathcal{B}}(t)y_0 + \ddot{\mathcal{C}}(t)v_{x0} + \ddot{\mathcal{D}}(t)v_{y0}, \tag{A26}$$

$$\langle y(t)\rangle = \mathcal{A}(t)y_0 + \mathcal{B}(t)x_0 + \mathcal{C}(t)v_{y0} - \mathcal{D}(t)v_{x0}, \tag{A27}$$

$$\langle v_y(t)\rangle = \dot{\mathcal{A}}(t)y_0 + \dot{\mathcal{B}}(t)x_0 + \dot{\mathcal{C}}(t)v_{y0} - \dot{\mathcal{D}}(t)v_{x0}, \tag{A28}$$

$$\langle \dot{v}_y(t)\rangle = \ddot{\mathcal{A}}(t)y_0 + \ddot{\mathcal{B}}(t)x_0 + \ddot{\mathcal{C}}(t)v_{y0} - \ddot{\mathcal{D}}(t)v_{x0}. \tag{A29}$$

Upon substitution of $\langle x\rangle$, $\langle v_x\rangle$, $\langle y\rangle$, and $\langle v_y\rangle$ into Eqs. (A16)–(A19), we obtain a system of four equations for the initial conditions ix_0, iy_0, iv_{x0}, and iv_{y0}, and the corresponding solution can be written as

$$ix_0 = \frac{1}{\Delta_m}[\widehat{A}\hat{a}_x(t) + \widehat{C}\hat{c}_x(t) - \widehat{B}\hat{b}_x(t) - \widehat{D}\hat{d}_x(t)], \tag{A30}$$

$$iy_0 = \frac{1}{\Delta_m}[-\widehat{A}\hat{a}_y(t) + \widehat{C}\hat{c}_y(t) - \widehat{B}\hat{b}_y(t) - \widehat{D}\hat{d}_y(t)], \tag{A31}$$

$$iv_{x_0} = \frac{1}{\Delta_m}[\widehat{A}\hat{a}_{v_x}(t) - \widehat{C}\hat{c}_{v_x}(t) + \widehat{B}\hat{b}_{v_x}(t) - \widehat{D}\hat{d}_{v_x}(t)], \tag{A32}$$

$$iv_{y_0} = \frac{1}{\Delta_m}[\widehat{A}\hat{a}_{v_y}(t) + \widehat{C}\hat{c}_{v_y}(t) + \widehat{B}\hat{b}_{v_y}(t) + \widehat{D}\hat{d}_{v_y}(t)], \tag{A33}$$

where

$$\begin{aligned}\widehat{A} &= \frac{1}{\widetilde{C}_1}\frac{\partial\widetilde{C}_1}{\partial\eta_1} + (F\eta_1 + H\eta_3),\\ \widehat{B} &= \frac{1}{\widetilde{C}_1}\frac{\partial\widetilde{C}_1}{\partial\eta_3} + (H\eta_1 + G\eta_3),\\ \widehat{C} &= \frac{1}{\widetilde{C}_2}\frac{\partial\widetilde{C}_2}{\partial\eta_2} + (F\eta_2 + H\eta_4),\\ \widehat{D} &= \frac{1}{\widetilde{C}_2}\frac{\partial\widetilde{C}_2}{\partial\eta_4} + (H\eta_2 + G\eta_4),\end{aligned} \tag{A34}$$

with the determinant

$$\begin{aligned}\Delta_m = &(\mathcal{A}^2 + \mathcal{B}^2)(\dot{\mathcal{C}}^2 + \dot{\mathcal{D}}^2) + (\mathcal{C}^2 + \mathcal{D}^2)(\dot{\mathcal{A}}^2 + \dot{\mathcal{B}}^2)\\ &- 2(\mathcal{A}\mathcal{C} - \mathcal{B}\mathcal{D})(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}})\\ &- 2(\mathcal{A}\mathcal{D} + \mathcal{B}\mathcal{C})(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\end{aligned} \tag{A35}$$

and

$$\begin{aligned}
\hat{a}_x(t) &= \mathcal{A}(\dot{\mathcal{C}}^2 + \dot{\mathcal{D}}^2) - \mathcal{C}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{D}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{c}_x(t) &= \mathcal{B}(\dot{\mathcal{C}}^2 + \dot{\mathcal{D}}^2) + \mathcal{D}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{C}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{b}_x(t) &= \mathcal{B}(\mathcal{C}\dot{\mathcal{D}} - \dot{\mathcal{C}}\mathcal{D}) + \mathcal{C}(\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C}) + \mathcal{D}(\mathcal{A}\dot{\mathcal{D}} - \dot{\mathcal{A}}\mathcal{D}),\\
\hat{d}_x(t) &= \mathcal{B}(\mathcal{C}\dot{\mathcal{C}} + \mathcal{D}\dot{\mathcal{D}}) - \mathcal{C}(\mathcal{A}\dot{\mathcal{D}} + \mathcal{C}\dot{\mathcal{B}}) + \mathcal{D}(\mathcal{A}\dot{\mathcal{C}} - \mathcal{D}\dot{\mathcal{B}}),
\end{aligned} \tag{A36}$$

$$\begin{aligned}
\hat{a}_y(t) &= \mathcal{B}(\dot{\mathcal{C}}^2 + \dot{\mathcal{D}}^2) + \mathcal{D}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{C}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{c}_y(t) &= \mathcal{A}(\dot{\mathcal{C}}^2 + \dot{\mathcal{D}}^2) - \mathcal{C}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{D}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{b}_y(t) &= \mathcal{A}(\mathcal{C}\dot{\mathcal{D}} - \dot{\mathcal{C}}\mathcal{D}) - \mathcal{C}(\mathcal{B}\dot{\mathcal{C}} - \dot{\mathcal{B}}\mathcal{C}) - \mathcal{D}(\mathcal{B}\dot{\mathcal{D}} - \dot{\mathcal{B}}\mathcal{D}),\\
\hat{d}_y(t) &= \mathcal{A}(\mathcal{C}\dot{\mathcal{C}} + \mathcal{D}\dot{\mathcal{D}}) + \mathcal{C}(\mathcal{B}\dot{\mathcal{D}} - \mathcal{C}\dot{\mathcal{A}}) - \mathcal{D}(\mathcal{B}\dot{\mathcal{C}} + \mathcal{D}\dot{\mathcal{A}}),
\end{aligned} \tag{A37}$$

$$\begin{aligned}
\hat{a}_{v_x}(t) &= \mathcal{C}(\dot{\mathcal{A}}^2 + \dot{\mathcal{B}}^2) - \mathcal{A}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{B}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{c}_{v_x}(t) &= \mathcal{D}(\dot{\mathcal{A}}^2 + \dot{\mathcal{B}}^2) + \mathcal{B}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{A}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{b}_{v_x}(t) &= \mathcal{A}(\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{A}}\mathcal{C}) + \mathcal{B}(\mathcal{B}\dot{\mathcal{C}} - \dot{\mathcal{B}}\mathcal{C}) - \mathcal{D}(\mathcal{A}\dot{\mathcal{B}} - \dot{\mathcal{A}}\mathcal{B}),\\
\hat{d}_{v_x}(t) &= \mathcal{A}(\mathcal{A}\dot{\mathcal{D}} + \mathcal{C}\dot{\mathcal{B}}) + \mathcal{B}(\mathcal{B}\dot{\mathcal{D}} - \mathcal{C}\dot{\mathcal{A}}) - \mathcal{D}(\mathcal{A}\dot{\mathcal{A}} + \mathcal{B}\dot{\mathcal{B}}),
\end{aligned} \tag{A38}$$

$$\begin{aligned}
\hat{a}_{v_y}(t) &= \mathcal{D}(\dot{\mathcal{A}}^2 + \dot{\mathcal{B}}^2) + \mathcal{B}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{A}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{c}_{v_y}(t) &= \mathcal{C}(\dot{\mathcal{A}}^2 + \dot{\mathcal{B}}^2) - \mathcal{A}(\dot{\mathcal{A}}\dot{\mathcal{C}} - \dot{\mathcal{B}}\dot{\mathcal{D}}) - \mathcal{B}(\dot{\mathcal{A}}\dot{\mathcal{D}} + \dot{\mathcal{B}}\dot{\mathcal{C}}),\\
\hat{b}_{v_y}(t) &= \mathcal{A}(\mathcal{A}\dot{\mathcal{D}} - \dot{\mathcal{A}}\mathcal{D}) + \mathcal{B}(\mathcal{B}\dot{\mathcal{D}} - \dot{\mathcal{B}}\mathcal{D}) + \mathcal{C}(\mathcal{A}\dot{\mathcal{B}} - \dot{\mathcal{A}}\mathcal{B}),\\
\hat{d}_{v_y}(t) &= \mathcal{A}(\mathcal{A}\dot{\mathcal{C}} - \dot{\mathcal{B}}\mathcal{D}) + \mathcal{B}(\mathcal{B}\dot{\mathcal{C}} + \mathcal{D}\dot{\mathcal{A}}) - \mathcal{C}(\mathcal{A}\dot{\mathcal{A}} + \mathcal{B}\dot{\mathcal{B}}).
\end{aligned} \tag{A39}$$

It can be checked that $\hat{a}_x(t) = \hat{c}_y(t)$, $\hat{a}_y(t) = \hat{c}_x(t)$, $\hat{b}_x(t) = \hat{d}_y(t)$, $\hat{d}_x(t) = -\hat{b}_y(t)$, $\hat{a}_{v_x}(t) = \hat{c}_{v_y}(t)$, $\hat{a}_{v_y}(t) = \hat{c}_{v_x}(t)$, $\hat{b}_{v_x}(t) = \hat{d}_{v_y}(t)$, and $\hat{b}_{v_y}(t) = \hat{d}_{v_x}(t)$.

After a very long and careful algebra, we arrive at the generalized phase-space Fokker-Planck equation for a harmonic oscillator in a magnetic field, which we write as

$$\begin{aligned}
&\frac{\partial P}{\partial t} + \left(v_x\frac{\partial P}{\partial x} + v_y\frac{\partial P}{\partial y}\right) + \mathcal{Q}_1(t)\left(x\frac{\partial P}{\partial v_x} + y\frac{\partial P}{\partial v_y}\right) - \mathcal{P}_2(t)I\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right)\\
&= \mathcal{P}_1(t)\left(x\frac{\partial P}{\partial y} - y\frac{\partial P}{\partial x}\right) + \mathcal{P}_2(t)\left(v_x\frac{\partial P}{\partial y} - v_y\frac{\partial P}{\partial x}\right) - \mathcal{R}_1(t)\left(x\frac{\partial P}{\partial v_y} - y\frac{\partial P}{\partial v_x}\right) - \mathcal{R}_3(t)\left(v_x\frac{\partial P}{\partial v_y} - v_y\frac{\partial P}{\partial v_x}\right)\\
&\quad - \mathcal{Q}_3(t)\left(\frac{\partial v_x P}{\partial v_x} + \frac{\partial v_y P}{\partial v_y}\right) - \mathcal{S}_1(t)\left(\frac{\partial^2 P}{\partial v_x^2} + \frac{\partial^2 P}{\partial v_y^2}\right) - \mathcal{S}_2(t)\left(\frac{\partial^2 P}{\partial x\partial v_x} + \frac{\partial^2 P}{\partial y\partial v_y}\right) + \mathcal{S}_3(t)\left(\frac{\partial^2 P}{\partial x\partial v_y} - \frac{\partial^2 P}{\partial y\partial v_x}\right),
\end{aligned} \tag{A40}$$

where the time-dependent coefficients are given by

$$\mathcal{S}_1(t) = \mathcal{Q}_1(t)H - \mathcal{Q}_2(t)I + \mathcal{Q}_3(t)G - \frac{1}{2}\dot{G}, \tag{A41}$$

$$\mathcal{S}_2(t) = G + [\mathcal{Q}_4(t) - \mathcal{P}_1(t)]I + \mathcal{Q}_1(t)F + \mathcal{Q}_3(t)H - \dot{H}, \tag{A42}$$

$$\begin{aligned}
\mathcal{S}_3(t) = {}& \mathcal{Q}_2(t)F - \mathcal{P}_2(t)G + [\mathcal{Q}_4(t) - \mathcal{P}_1(t)]H\\
&- \mathcal{Q}_3(t)I + \dot{I},
\end{aligned} \tag{A43}$$

$$\mathcal{P}_1(t) = \frac{1}{\Delta_m}\big[\dot{\mathcal{A}}\hat{c}_x(t) - \dot{\mathcal{B}}\hat{c}_y(t) - \dot{\mathcal{C}}\hat{c}_{v_x}(t) + \dot{\mathcal{D}}\hat{c}_{v_y}(t)\big],$$

$$\mathcal{P}_2(t) = \frac{1}{\Delta_m}\big[-\dot{\mathcal{A}}\hat{d}_x(t) + \dot{\mathcal{B}}\hat{d}_y(t) - \dot{\mathcal{C}}\hat{d}_{v_x}(t) + \dot{\mathcal{D}}\hat{d}_{v_y}(t)\big],$$

$$\mathcal{P}_3(t) = \frac{1}{\Delta_m}\big[\dot{\mathcal{B}}\hat{a}_x(t) - \dot{\mathcal{A}}\hat{a}_y(t) - \dot{\mathcal{D}}\hat{a}_{v_x}(t) + \dot{\mathcal{C}}\hat{a}_{v_y}(t)\big],$$

$$\mathcal{P}_4(t) = \frac{1}{\Delta_m}\big[-\dot{\mathcal{B}}\hat{b}_x(t) - \dot{\mathcal{A}}\hat{b}_y(t) - \dot{\mathcal{D}}\hat{b}_{v_x}(t) + \dot{\mathcal{C}}\hat{b}_{v_y}(t)\big],$$

$$\mathcal{Q}_1(t) = \frac{1}{\Delta_m}\big[\ddot{\mathcal{A}}\hat{a}_x(t) + \ddot{\mathcal{B}}\hat{a}_y(t) + \ddot{\mathcal{C}}\hat{a}_{v_x}(t) + \ddot{\mathcal{D}}\hat{a}_{v_y}(t)\big],$$

$$\mathcal{Q}_2(t) = \frac{1}{\Delta_m}\big[\ddot{\mathcal{A}}\hat{c}_x(t) - \ddot{\mathcal{B}}\hat{c}_y(t) - \ddot{\mathcal{C}}\hat{c}_{v_x}(t) + \ddot{\mathcal{D}}\hat{c}_{v_y}(t)\big],$$

$$\mathcal{Q}_3(t) = \frac{1}{\Delta_m}\big[-\ddot{\mathcal{A}}\hat{b}_x(t) + \ddot{\mathcal{B}}\hat{b}_y(t) + \ddot{\mathcal{C}}\hat{b}_{v_x}(t) + \ddot{\mathcal{D}}\hat{b}_{v_y}(t)\big],$$

$$\mathcal{Q}_4(t) = \frac{1}{\Delta_m}\big[-\ddot{\mathcal{A}}\hat{d}_x(t) + \ddot{\mathcal{B}}\hat{d}_y(t) - \ddot{\mathcal{C}}\hat{d}_{v_x}(t) + \ddot{\mathcal{D}}\hat{d}_{v_y}(t)\big],$$

$$\mathcal{R}_1(t) = \frac{1}{\Delta_m}\big[\ddot{\mathcal{B}}\hat{a}_x(t) - \ddot{\mathcal{A}}\hat{a}_y(t) - \ddot{\mathcal{D}}\hat{a}_{v_x}(t) + \ddot{\mathcal{C}}\hat{a}_{v_y}(t)\big],$$

$$\mathcal{R}_2(t) = \frac{1}{\Delta_m}\big[\ddot{\mathcal{B}}\hat{c}_x(t) + \ddot{\mathcal{A}}\hat{c}_y(t) + \ddot{\mathcal{D}}\hat{c}_{v_x}(t) + \ddot{\mathcal{C}}\hat{c}_{v_y}(t)\big],$$

$$\mathcal{R}_3(t) = \frac{1}{\Delta_m}\big[-\ddot{\mathcal{B}}\hat{b}_x(t) - \ddot{\mathcal{A}}\hat{b}_y(t) - \ddot{\mathcal{D}}\hat{b}_{v_x}(t) + \ddot{\mathcal{C}}\hat{b}_{v_y}(t)\big],$$
$$\mathcal{R}_4(t) = \frac{1}{\Delta_m}\big[-\ddot{\mathcal{B}}\hat{d}_x(t) - \ddot{\mathcal{A}}\hat{d}_y(t) + \ddot{\mathcal{D}}\hat{d}_{v_x}(t) + \ddot{\mathcal{C}}\hat{d}_{v_y}(t)\big]. \tag{A44}$$

It also can be checked that $\mathcal{P}_1(t) = -\mathcal{P}_3(t)$, $\mathcal{P}_2(t) = -\mathcal{P}_4(t)$, $\mathcal{Q}_1(t) = \mathcal{R}_2(t)$, $\mathcal{Q}_2(t) = -\mathcal{R}_1(t)$, $\mathcal{Q}_3(t) = \mathcal{R}_4(t)$, and $\mathcal{Q}_4(t) = -\mathcal{R}_3(t)$.

APPENDIX B: GENERALIZED PHASE-SPACE FPE FOR A HARMONIC OSCILLATOR IN A MAGNETIC FIELD AND TIME-DEPENDENT FORCE FIELDS

To achieve the goal, we again begin with the characteristic function, which in this case can be written as $\mathbb{C}(\boldsymbol{\eta}, t) = \mathbb{C}_1(\boldsymbol{\eta}, t)\mathbb{C}_2(\boldsymbol{\eta}, t)$, where

$$\mathbb{C}_1(\boldsymbol{\eta}, t) = e^{iq_x\eta_1 + ip_x\eta_3}C_1(\eta_1, \eta_3), \tag{B1}$$

$$\mathbb{C}_2(\boldsymbol{\eta}, t) = e^{iq_y\eta_2 + ip_y\eta_4}C_2(\eta_2, \eta_4), \tag{B2}$$

with $C_1(\eta_1, \eta_3)$ and $C_2(\eta_2, \eta_4)$ exactly the same as Eqs. (A2) and (A3), respectively, and q_x, q_y, p_x, and p_y come from the solutions given by Eqs. (102), (103), (105), and (106), such that

$$q_x = \int_0^t [H_0(t-t') - \Omega^2\mathcal{H}_2(t-t')]a_x(t')dt' + \Omega\int_0^t \mathcal{H}_1(t-t')a_y(t')dt', \tag{B3}$$

$$q_y = \int_0^t [H_0(t-t') - \Omega^2\mathcal{H}_2(t-t')]a_y(t')dt' - \Omega\int_0^t \mathcal{H}_1(t-t')a_x(t')dt', \tag{B4}$$

$$p_x = \int_0^t [\dot{H}_0(t-t') - \Omega^2\dot{\mathcal{H}}_2(t-t')]a_x(t')dt' + \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')a_y(t')dt', \tag{B5}$$

$$p_y = \int_0^t [\dot{H}_0(t-t') - \Omega^2\dot{\mathcal{H}}_2(t-t')]a_y(t')dt' - \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')a_x(t')dt', \tag{B6}$$

where we note that $p_x = \dot{q}_x$ and $p_y = \dot{q}_y$. Now,

$$\frac{\partial P}{\partial t} = \frac{1}{(2\pi)^4}\int\cdots\int\left(\mathbb{C}_1\frac{\partial\mathbb{C}_2}{\partial t} + \mathbb{C}_2\frac{\partial\mathbb{C}_1}{\partial t}\right)e^{-i\boldsymbol{\eta}\cdot\boldsymbol{\xi}}\,d\boldsymbol{\eta}, \tag{B7}$$

where

$$\mathbb{C}_2\frac{\partial\mathbb{C}_1}{\partial t} = (i\dot{q}_x\eta_1 + i\dot{p}_x\eta_3)\mathbb{C} + e^{iq_x\eta_1 + ip_x\eta_3}\,\mathbb{C}_2\frac{\partial C_1}{\partial t}, \tag{B8}$$

$$\mathbb{C}_1\frac{\partial\mathbb{C}_2}{\partial t} = (i\dot{q}_y\eta_2 + i\dot{p}_y\eta_4)\mathbb{C} + e^{iq_y\eta_1 + ip_y\eta_3}\,\mathbb{C}_1\frac{\partial C_2}{\partial t}. \tag{B9}$$

If we define $\hat{q}_x = q_x\eta_1 + p_x\eta_3$, $\hat{q}_y = q_y\eta_2 + p_y\eta_4$, and use Eqs. (B1) and (B2), we get

$$\mathbb{C}_2\frac{\partial\mathbb{C}_1}{\partial t} = (i\dot{q}_x\eta_1 + i\dot{p}_x\eta_3)\mathbb{C} + e^{i\hat{q}_x\eta_1 + i\hat{p}_x\eta_3}\,C_2\frac{\partial C_1}{\partial t}, \tag{B10}$$

$$\mathbb{C}_1\frac{\partial\mathbb{C}_2}{\partial t} = (i\dot{q}_y\eta_2 + i\dot{p}_y\eta_4)\mathbb{C} + e^{i\hat{q}_y\eta_1 + i\hat{p}_y\eta_3}\,C_1\frac{\partial C_2}{\partial t}. \tag{B11}$$

However, according to Eqs. (A14) and (A15), we have

$$e^{i\hat{q}_x + i\hat{q}_y}C_2\frac{\partial C_1}{\partial t} = -\frac{\dot{I}}{2}[\eta_1\eta_4 - \eta_2\eta_3]\mathbb{C} + e^{i\hat{q}_x + i\hat{q}_y}\hat{C}_1C_2, \tag{B12}$$

$$e^{i\hat{q}_x + i\hat{q}_y}C_1\frac{\partial C_2}{\partial t} = -\frac{\dot{I}}{2}[\eta_1\eta_4 - \eta_2\eta_3]\mathbb{C} + e^{i\hat{q}_x + i\hat{q}_y}\hat{C}_2C_1. \tag{B13}$$

Following similar algebraic steps given in Appendix A and after a very long algebra, it can be concluded that the non-Markovian phase-space Fokker-Planck equation for a harmonic oscillator in a constant magnetic field and time-dependent force fields can be written as

$$\begin{aligned}
&\frac{\partial P}{\partial t} + \left(\dot{p}_x\frac{\partial P}{\partial v_x} + \dot{p}_y\frac{\partial P}{\partial v_y}\right) + \mathcal{P}_1(t)\left(q_x\frac{\partial P}{\partial y} - q_y\frac{\partial P}{\partial x}\right)\\
&\quad - \mathcal{Q}_1(t)\left(q_x\frac{\partial P}{\partial x} + q_y\frac{\partial P}{\partial y}\right) + \mathcal{P}_2(t)\left(p_x\frac{\partial P}{\partial y} - p_y\frac{\partial P}{\partial x}\right)\\
&\quad + \mathcal{Q}_2(t)\left(q_x\frac{\partial P}{\partial v_y} - q_y\frac{\partial P}{\partial v_x}\right) - \mathcal{Q}_3(t)\left(p_x\frac{\partial P}{\partial v_x} + p_y\frac{\partial P}{\partial v_y}\right)\\
&\quad + \mathcal{Q}_4(t)\left(p_x\frac{\partial P}{\partial v_y} - p_y\frac{\partial P}{\partial v_x}\right) + \left(v_x\frac{\partial P}{\partial x} + v_y\frac{\partial P}{\partial y}\right)\\
&\quad + \mathcal{Q}_1(t)\left(x\frac{\partial P}{\partial v_x} + y\frac{\partial P}{\partial v_y}\right) + \mathcal{P}_2(t)I\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right)\\
&= \mathcal{P}_1(t)\left(x\frac{\partial P}{\partial y} - y\frac{\partial P}{\partial x}\right) + \mathcal{P}_2(t)\left(v_x\frac{\partial P}{\partial y} - v_y\frac{\partial P}{\partial x}\right)\\
&\quad - \mathcal{R}_1(t)\left(x\frac{\partial P}{\partial v_y} - y\frac{\partial P}{\partial v_x}\right) - \mathcal{R}_3(t)\left(v_x\frac{\partial P}{\partial v_y} - v_y\frac{\partial P}{\partial v_x}\right)\\
&\quad - \mathcal{Q}_3(t)\left(\frac{\partial v_xP}{\partial v_x} + \frac{\partial v_yP}{\partial v_y}\right) - \mathcal{S}_1(t)\left(\frac{\partial^2 P}{\partial v_x^2} + \frac{\partial^2 P}{\partial v_y^2}\right)\\
&\quad - \mathcal{S}_2(t)\left(\frac{\partial^2 P}{\partial x\partial v_x} + \frac{\partial^2 P}{\partial y\partial v_y}\right) + \mathcal{S}_3(t)\left(\frac{\partial^2 P}{\partial x\partial v_y} - \frac{\partial^2 P}{\partial y\partial v_x}\right).
\end{aligned} \tag{B14}$$

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