Determination of energy flux rate in homogeneous ferrohydrodynamic turbulence using two-point statistics

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Under the influence of an external magnetic field **H**, the suspended ferromagnetic particles of a laminar ferrofluid flow try to be oriented along **H** through a relaxation mechanism. Turbulence affects the interaction between the magnetization of each suspended particle and the external field thereby leading to a large relaxation time and hence a slow relaxation process. This can be obtained by replacing viscous drag force with turbulent drag force in Brownian motion. We show that the total energy is an inviscid invariant in turbulent ferrofluids. Using two-point statistics we formulate an exact relation in the inertial zone of incompressible ferrofluid turbulence. This exact relation gives an accurate measure of the energy dissipation rate in a turbulent ferrofluid. We also show that $(\mathbf{u} \times \boldsymbol{\omega})$, $(\mathbf{M} \times \mathbf{H})$, $(\mathbf{M} \cdot \nabla)\mathbf{H}$, and $(\boldsymbol{\omega} \times \mathbf{M})$ play the major role in energy cascading in turbulent ferrofluids.

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I. INTRODUCTION

A ferrofluid is a suspension of solid magnetic particles (magnetite) in dielectric liquids (e.g., water, oil, organic solvent), often called the carrier fluid. The diameter of the suspended particles are of the order of 10 nm and hence each particle contains roughly one single magnetic domain [1]. These magnetic particles are free to move and rotate in the carrier medium. Due to their very small size, the particles interact with each other when subject to a magnetic field. However, because of the Brownian motion, they do not settle under the influence of the external magnetic field. The ferrofluid particles are usually coated with surfactants, which are long-chained molecules having a polar head and a nonpolar tail or vice versa (e.g., oleic acid and tetramethylammonium hydroxide). This coating prevents the particles from (i) immediate agglomeration among themselves by producing steric hinderance and (ii) being extracted from the fluid, when the external field is sufficiently strong. This surfactant partially overcomes the attractive Van der Waals and magnetic forces between the particles with electrostatic repulsion [2]. When an external magnetic field is applied, the magnetic moments of these ferromagnetic particles, which are assumed to be fixed with each particle, immediately try to be oriented along the field direction. When the field is removed, these magnetic moments quickly randomize leading to a zero net magnetization [3,4]. This property, along with the fact that ferrofluids possess magnetic susceptibilities (≈0.5) almost three orders of magnitude larger than the common paramagnetic salt, classify the ferrofluids as superparamagnetic fluids [1]. Owing to their high controllability by external magnetic fields and convective effects in microgravity environments, ferrofluids find a broad range of applications starting from hermetic seals in pumps, rotary seals of computer hard drives to the loudspeakers to extract heat from voice coils [1,5,6]. Ferrofluids are also considered as effective carriers of concentrated medications to specific location in the body [7].

In this article, we mainly investigate the comportment of a ferrofluid flow in the external magnetic field (**H**) gradient when **H** is steady, i.e., time independent. Upon the application of steady **H**, the particle magnetic moments (which are supposed to be rigidly fixed to each particle) start getting oriented along it thereby giving an equilibrium state in a characteristic time τ, called the relaxation time [2]. In the equilibrium state all the particles get settled and oriented along **H** (note that this orientation is achieved only on average due to the presence of thermal fluctuation [8]). In order to achieve a sustaining ferrofluid flow, the process of agglomeration of the particles needs to be slow. As we discuss in Sec. II, this can be achieved through the generation of turbulence in a ferrofluid flow. Turbulence modifies the nature of the viscous drag and increases the relaxation time of particles to settle. In addition, the interaction between the ferrofluid and **H** considerably influences the properties of turbulence. Although a large number of studies [9–12] have been accomplished for the laminar Poiseuille flow of ferrofluids and more particularly the drag reduction in the presence of rotating and oscillating magnetic fields, only a few works have been done to study the turbulence in such fluids. Some of them are dedicated to the study of the effects of pressure drop in the pipe flow of ferrofluids [13,14] whereas some investigate systematically the onset of turbulence in Taylor-Couette ferrofluid flow which takes place at Reynolds numbers at least ten times lower than those of the other fluids [15].

The study of completely developed homogeneous turbulence in ferrofluidic flows was carried out, for the first time, by Schumacher et al. [1]. Starting from the very basic equations of ferrofluids, using the one-point Reynolds decomposition

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method, the authors derived the expression for average dissipation rate of the translational kinetic energy of the fluid, the rotational kinetic energy of the suspended ferrofluid particles, and the internal energy. They also identified the terms which represent the conversion of one type of energy into the other. The analytical expressions were then numerically calculated using direct numerical simulation (DNS). The external field $\mathbf{H}$ was assumed to be steady throughout. The effect of a spatially uniform but oscillating $\mathbf{H}$ on homogeneous ferrofluid turbulence was studied later by the same authors [16] and an increased rate of energy loss was reported. The study also showed a direct dependence of turbulent properties on the oscillation frequency of the external magnetic field and the choice of magnetization equation. The properties of ferrohydrodynamic turbulence in a channel flow under both steady and oscillating magnetic fields were also studied using DNS and the results were found to match satisfactorily with the $k-\epsilon$ model of turbulence adapted to ferrofluids [17].

Despite these extensive studies, to our knowledge, no exact relation has been derived for homogeneous ferrofluid turbulence using two-point statistics. Exact relations are crucial as they can directly express the average energy dissipation rate in terms of two-point fluctuations which corresponds to the quantities as a function of the length scale. The first exact relation for incompressible hydrodynamic turbulence was derived by Kolmogorov [18]. Later such exact relations were derived for incompressible magnetohydrodynamics [19,20] and also for various types of compressible turbulence [21–23]. Very recently Banerjee and Galtier [24,25] have proposed an alternative method of derivation of the exact relations. This method is found to have several advantages and the final exact relation becomes more compact than the previous ones. This method was successfully generalized for compressible turbulence of both neutral and plasma fluids [26].

Following [25,26], in this paper, we derive an exact relation corresponding to the total energy conservation in the so-called inertial zone of incompressible ferrofluid turbulence. The paper is organized as follows. The governing equations of dynamics are presented in Sec. II. In Sec. III, the total energy conservation is shown, and Sec. IV contains the detailed derivation of an exact relation using two-point statistics. Finally in Sec. V, we summarize the results and conclude.

II. BASIC EQUATIONS OF DYNAMICS

The basic equations of dynamics for ferrofluids consist of the linear momentum equation of the fluid, the internal angular momentum equation of the suspended ferromagnetic particles, and the magnetization equation. The governing equation for linear momentum evolution is derived from the theory of structured continua [1,3] and can be written as in the following:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p+\mu\nabla^2\mathbf{u}+\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}-\zeta\nabla\times(\boldsymbol{\Omega}-2\boldsymbol{\omega})+\mathbf{f}_u, \tag{1}$$

where ρ is the density of ferrofluid, $\mathbf{u}$ the fluid velocity, p the dynamic pressure, μ the dynamic viscosity, ζ the vortex viscosity, $\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}$ the magnetic body force, μ_0 the free space permeability, $\mathbf{M}$ the magnetization vector, $\mathbf{H}$ the steady external magnetic field, $\nabla\times\mathbf{u}=\boldsymbol{\Omega}$ the vorticity of the fluid, $\boldsymbol{\omega}$ the local ferrofluid rotation rate, and $\mathbf{f}_u$ the stationary large scale force. The evolution equation of the internal angular momentum, which is the total angular momentum minus the moment of the total linear momentum [3], is given by

$$\rho I\left[\frac{\partial \boldsymbol{\omega}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}\right]=\eta\nabla^2\boldsymbol{\omega}+\mu_0(\mathbf{M}\times\mathbf{H})+2\zeta(\boldsymbol{\Omega}-2\boldsymbol{\omega}), \tag{2}$$

where I is the moment of inertia of ferrofluid particle, η is the spin viscosity, and $\mu_0(\mathbf{M}\times\mathbf{H})$ represents the magnetic body couple force. In addition to these two equations, we assume incompressibility which gives $\nabla\cdot\mathbf{u}=0$. Again Maxwell's equations for ferrofluid with no current are written as

$$\nabla\times\mathbf{H}=0,\quad \nabla\cdot\mathbf{B}=0; \tag{3}$$

and the relation between $\mathbf{M}$, $\mathbf{H}$, and $\mathbf{B}$ is

$$\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}). \tag{4}$$

Now in order to close the system of equations, one needs to know the evolution equation for $\mathbf{M}$, often called the magnetization equation. Including the effect of the magnetic body couple force in Debye theory [27], for the first time, Shliomis derived the a magnetization equation for ferrofluids [4], which can be written as

$$\frac{\partial \mathbf{M}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{M}=\boldsymbol{\omega}\times\mathbf{M}-\frac{1}{\tau}(\mathbf{M}-\mathbf{M}_0), \tag{5}$$

where τ is the relaxation time, $\boldsymbol{\omega}\times\mathbf{M}$ represents the rate of change in magnetization due to rotation of magnetic particles, and $\frac{1}{\tau}(\mathbf{M}-\mathbf{M}_0)$ represents the change in magnetization towards an equilibrium magnetization ($\mathbf{M}_0$) via relaxation. This equilibrium magnetization is given by

$$\mathbf{M}_0=M_sL(\xi)\frac{\mathbf{H}}{H}, \tag{6}$$

where

$$L(\xi)=\coth(\xi)-\frac{1}{\xi}\quad\text{and}\quad\xi=\frac{\mu_0 mH}{k_BT}, \tag{7}$$

and M_s and m are the magnitudes of saturation magnetization and magnetic moment of a single particle. Several other magnetization equations were derived later using the Fokker-Planck equation [8], irreversible thermodynamics [28,29], and also using general principles without considering the angular momentum of the suspended ferrofluid particles [30]. All these models are different from each other (see [31,32]), and it can be interesting to study their effects in ferrofluid turbulence. However, in this paper, as a first step, we use Eq. (5).

III. INVISCID INVARIANCE OF TOTAL ENERGY IN TURBULENT FERROFLUIDS

A turbulent state is characterized by the scale invariant flux rate of an inviscid invariant of the flow. An exact relation of turbulence relates that flux rate to the statistical moments of two-point increments of various flow and relevant electromagnetic fields, often called the structure functions. In this section we discuss, in detail, the conservation of total energy in a turbulent ferrofluidic flow in the limit of infinitely large Reynolds numbers where the viscosity is negligibly small. For the current system, the total energy has three components: the translational kinetic energy of the fluid, the rotational kinetic energy of the suspended particles, and the internal energy due to the work done by the ferromagnetic particles under the influence of the external field. The evolution equation for translational kinetic energy can be obtained from Eq. (1) as

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)+\nabla\cdot\left(\frac{u^2}{2}\mathbf{u}\right)=-\nabla\cdot(p\mathbf{u})+\nu\mathbf{u}\cdot\nabla^2\mathbf{u}+2\zeta\mathbf{u}\cdot(\nabla\times\boldsymbol{\omega})-\zeta\mathbf{u}\cdot(\nabla\times\boldsymbol{\Omega})+\mu_0\mathbf{u}\cdot(\mathbf{M}\cdot\nabla)\mathbf{H}+\mathbf{u}\cdot\mathbf{f}_u, \tag{8}$$

and similarly the evolution equation for rotational kinetic energy is obtained from (2) as

$$I\frac{\partial}{\partial t}\left(\frac{\omega^2}{2}\right)+\nabla\cdot\left(I\frac{\omega^2}{2}\mathbf{u}\right)=\eta\boldsymbol{\omega}\cdot\nabla^2\boldsymbol{\omega}+2\zeta\boldsymbol{\omega}\cdot(\boldsymbol{\Omega}-2\boldsymbol{\omega})+\mu_0\boldsymbol{\omega}\cdot(\mathbf{M}\times\mathbf{H}), \tag{9}$$

where we have used the relation $\boldsymbol{\omega}(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}=\mathbf{u}\cdot\nabla(\omega^2/2)$. Integrating Eqs. (8) and (9) over the entire volume and adding, we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int\frac{u^2}{2}+I\frac{\boldsymbol{\omega}^2}{2}\right)dV &= -\int\nabla\cdot\left(\frac{u^2}{2}+I\frac{\boldsymbol{\omega}^2}{2}+p\right)\mathbf{u}\,dV\\ &\quad-\zeta\int[\mathbf{u}\cdot\nabla\times(\boldsymbol{\Omega}-2\boldsymbol{\omega})-2\boldsymbol{\omega}\cdot(\boldsymbol{\Omega}-2\boldsymbol{\omega})]dV\\ &\quad+\int\left[\nu\mathbf{u}\cdot\nabla^2\mathbf{u}+\eta\boldsymbol{\omega}\cdot\nabla^2\boldsymbol{\omega}\right]dV\\ &\quad+\mu_0\int[\mathbf{u}\cdot(\mathbf{M}\cdot\nabla)\mathbf{H}+\boldsymbol{\omega}\cdot(\mathbf{M}\times\mathbf{H})]dV. \end{aligned}\tag{10}$$

For inviscid limit $\mu \to 0$, Schumacher *et al.* (2003) [14] estimated that $\zeta/\mu=0.5$ and $\eta=2\times10^{-15}$ kg m s^{-1}, so one can reasonably set $\zeta=0$ and $\eta=0$. By using the Gauss divergence equation (10), we get

$$\frac{\partial}{\partial t}\left(\int\frac{u^2}{2}+I\frac{\boldsymbol{\omega}^2}{2}\right)dV=\mu_0\int\mathbf{u}\cdot(\mathbf{M}\cdot\nabla)\mathbf{H}dV+\mu_0\int\boldsymbol{\omega}\cdot(\mathbf{M}\times\mathbf{H})dV. \tag{11}$$

Since the external magnetic field is steady, e.g., time independent, we have $\partial\mathbf{H}/\partial t=0$. The rate of work done of the ferrofluid particles due to their orientation towards the state of minimum potential energy will therefore be [2]

$$\begin{aligned}\frac{d}{dt}\int W dV &= \frac{\partial}{\partial t}\int W dV=-\int\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}dV=-\mu_0\int\mathbf{H}\cdot\frac{\partial\mathbf{M}}{\partial t}dV\\ &= -\mu_0\int\mathbf{H}\cdot\left(-(\mathbf{u}\cdot\nabla)\mathbf{M}+\boldsymbol{\omega}\times\mathbf{M}-\frac{1}{\tau}(\mathbf{M}-\mathbf{M}_0)\right)dV\\ &= \mu_0\int\nabla\cdot[\mathbf{M}(\mathbf{u}\cdot\mathbf{H})]dV-\mu_0\int\mathbf{u}\cdot(\mathbf{M}\cdot\nabla)\mathbf{H}dV\\ &\quad-\mu_0\int\nabla\cdot[(\mathbf{u}\times\mathbf{M})\times\mathbf{H}]dV\\ &\quad-\mu_0\int\mathbf{H}\cdot(\boldsymbol{\omega}\times\mathbf{M})dV+\frac{\mu_0}{\tau}\int[\mathbf{H}\cdot(\mathbf{M}-\mathbf{M}_0)]dV\\ &= -\mu_0\int\mathbf{u}\cdot(\mathbf{M}\cdot\nabla)\mathbf{H}dV-\mu_0\int\mathbf{H}\cdot(\boldsymbol{\omega}\times\mathbf{M})dV\\ &\quad+\frac{\mu_0}{\tau}\int[\mathbf{H}\cdot(\mathbf{M}-\mathbf{M}_0)]dV, \end{aligned}\tag{12}$$

where the relation $\mathbf{H}\cdot(\mathbf{u}\cdot\nabla)\mathbf{M}=\nabla\cdot[\mathbf{M}(\mathbf{u}\cdot\mathbf{H})]-\mathbf{u}\cdot(\mathbf{M}\cdot\nabla)\mathbf{H}-\nabla\cdot[(\mathbf{u}\times\mathbf{M})\times\mathbf{H}]$ is used. The evolution equation for the total energy, in the inviscid limit, is thus given by

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int\frac{u^2}{2}+I\frac{\omega^2}{2}\right)dV-\frac{\partial}{\partial t}\int W dV &= \frac{\partial}{\partial t}\left(\int\frac{u^2}{2}+I\frac{\omega^2}{2}-\mu_0\mathbf{H}\cdot\mathbf{M}\right)dV\\ &= -\frac{\mu_0}{\tau}\int\mathbf{H}\cdot(\mathbf{M}-\mathbf{M}_0)dV. \end{aligned}\tag{13}$$

In general $\mathbf{M}\neq\mathbf{M_0}$. Therefore, for the total energy to be an inviscid invariant, we should have $\tau\to\infty$ when $\mu\to 0$. The magnetization of a colloidal ferrofluid particle can relax through particle rotation (Brownian relaxation) or by the rotation of the magnetization vector due to thermal fluctuation inside the particle (Néel relaxation) [2]. In the current study, we only consider a dilute solution of ferrofluid particles of size greater than 10 nm and assume that the particle magnetic moment is not varying much with thermal fluctuation. As a result, the Néel mechanism is practically blocked and the effective relaxation time is determined by the Brownian relaxation time τ_B [1,4,33]. For a laminar flow where the drag force is mainly coming from viscous drag $F^v=6\pi\mu r u$ [34], with r being the radius of the particle, the Brownian relaxation time becomes

$$\tau_B^v=3\frac{V_p\mu}{k_BT}\approx\tau, \tag{14}$$

where V_p is the hydrodynamic volume of the magnetic particle and k_BT is the thermal energy. In the limit $\mu\to 0$, hence we have $\tau\to 0$ which, in turn leads to a state of near equilibrium with $\mathbf{M}\to\mathbf{M_0}$, thereby rendering the right-hand side of Eq. (13) a nonzero finite value. This indicates a finite energy leakage of the ferrofluid flow in the laminar limit. Surprisingly previous studies of turbulent ferrofluids [1,14] used viscous Brownian time τ_B^v for their study. This problem is addressed in the current work by simply replacing the viscous drag by turbulent drag force which is given by [34,35]

$$F^t=\tfrac{1}{2}\rho u^2C_dA, \tag{15}$$

where C_d is the drag coefficient which is more or less constant at high Reynolds number and A is the cross-sectional area. The Einstein equation for Brownian relaxation time is then modified as

$$\tau_B^t=\frac{\rho\Delta^2uC_dA}{4k_BT}. \tag{16}$$

Here relaxation time ($\tau \approx \tau_B^t$) is directly proportional to the velocity, which is very high for very large Reynolds number. So in turbulent regime, τ becomes very large and $\frac{1}{\tau}$ becomes negligibly small. As a result the relaxation process becomes very slow and hence $\mathbf{M}$ will be tending to $\mathbf{M}_0$ slowly. This renders ($\mathbf{M} - \mathbf{M_0}$) a moderate finite value and makes the ratio $\frac{\mathbf{M}-\mathbf{M_0}}{\tau}$ negligibly small. Then the right-hand term of Eq. (13) becomes negligibly small and the resulting magnetization equation becomes:

$$\frac{\partial \mathbf{M}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{M} = \boldsymbol{\omega}\times\mathbf{M} \tag{17}$$

and guaranteeing the inviscid invariance of the total energy in a turbulent regime as

$$\frac{\partial}{\partial t}\int\left(\frac{u^2}{2} + I\frac{\omega^2}{2} - \mu_0\mathbf{H}\cdot\mathbf{M}\right)dV = 0. \tag{18}$$

IV. DERIVATION OF EXACT RELATION

It is important to note that $\mu_0(\mathbf{H}\cdot\mathbf{M})$ is not the work done by the magnetic field. Since here the magnetic field $\mathbf{H}$ is time independent, the time rate of the magnetic energy density is equal (as shown above) to the time rate of $\mu_0(\mathbf{H}\cdot\mathbf{M})$. Because in the derivation of the exact relation, one is particularly interested in the time rate of the two-point correlation functions of the inviscid invariants, the two-point symmetric correlation function of energy [25] for the current case can be written as

$$\begin{aligned}\mathcal{R} &= \frac{R_E + R_E'}{2}\\ &= \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{u}' + I\boldsymbol{\omega}\cdot\boldsymbol{\omega}' - \mu_0(\mathbf{H}\cdot\mathbf{M}' + \mathbf{H}'\cdot\mathbf{M})\rangle,\end{aligned} \tag{19}$$

where unprimed and primed quantities represent the variables at the point $\mathbf{x}$ and $\mathbf{x}'(\equiv \mathbf{x}+\mathbf{r})$ respectively. The evolution equation of the correlation function by using the basic equations of dynamics [25] is

$$\begin{aligned}\frac{\partial \mathcal{R}}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial t}\langle \mathbf{u}\cdot\mathbf{u}' + I\boldsymbol{\omega}\cdot\boldsymbol{\omega}' - \mu_0(\mathbf{H}\cdot\mathbf{M}' + \mathbf{H}'\cdot\mathbf{M})\rangle\\ &= \frac{1}{2}\langle(\mathbf{u}\times\boldsymbol{\Omega})\cdot\mathbf{u}' + (\mathbf{u}'\times\boldsymbol{\Omega}')\cdot\mathbf{u} - I(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}\cdot\boldsymbol{\omega}' - I(\mathbf{u}'\cdot\nabla')\boldsymbol{\omega}'\cdot\boldsymbol{\omega}\rangle\\ &\quad + \frac{1}{2}\mu_0\langle(\mathbf{M}\times\mathbf{H})\cdot\boldsymbol{\omega}' + (\mathbf{M}'\times\mathbf{H}')\cdot\boldsymbol{\omega} - \mathbf{H}'\cdot(\boldsymbol{\omega}\times\mathbf{M}) - \mathbf{H}\cdot(\boldsymbol{\omega}'\times\mathbf{M}') + \mathbf{u}\cdot(\mathbf{M}'\cdot\nabla')\mathbf{H}'\rangle\\ &\quad + \frac{1}{2}\mu_0\langle\mathbf{u}'\cdot(\mathbf{M}\cdot\nabla)\mathbf{H} + \mathbf{H}'\cdot(\mathbf{u}\cdot\nabla)\mathbf{M} + \mathbf{H}\cdot(\mathbf{u}'\cdot\nabla')\mathbf{M}'\rangle + D_u + D_\zeta + D_w + F_u,\end{aligned} \tag{20}$$

where D_u, D_w represent kinematic viscous dissipation, D_ζ represents vortex dissipation, and F_u represents the forcing contribution, and are given below:

$$D_u = \frac{\nu}{2}\langle \mathbf{u}\cdot\nabla'^2\mathbf{u}' + \mathbf{u}'\cdot\nabla^2\mathbf{u}\rangle, \tag{21}$$

$$D_\zeta = -\frac{\zeta}{2}\langle \mathbf{u}'\cdot\nabla\times\boldsymbol{\Omega}_r + \mathbf{u}\cdot\nabla'\times\boldsymbol{\Omega}_r' - 2(\boldsymbol{\omega}'\cdot\boldsymbol{\Omega}_r + \boldsymbol{\omega}\cdot\boldsymbol{\Omega}_r')\rangle, \tag{22}$$

$$D_w = \frac{\eta}{2}\langle \boldsymbol{\omega}'\cdot\nabla^2\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla'^2\boldsymbol{\omega}'\rangle, \tag{23}$$

$$F_u = \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{f}_u' + \mathbf{u}'\cdot\mathbf{f}_u\rangle, \tag{24}$$

where $\boldsymbol{\Omega}_r = \boldsymbol{\Omega} - 2\boldsymbol{\omega}$. For incompressible and statistical homogeneous systems we can show the relations $\langle\nabla\cdot(\frac{u^2}{2}\mathbf{u}')\rangle = \langle\nabla'\cdot(\frac{u^2}{2}\mathbf{u}')\rangle = 0$ and $\langle\mathbf{u}'\cdot\nabla p\rangle = \langle\nabla\cdot(p\mathbf{u}')\rangle = \langle\nabla'\cdot(p\mathbf{u}')\rangle = 0$. Using statistical homogeneity, one can prove that

$$\langle(\mathbf{u}\times\boldsymbol{\Omega})\cdot\mathbf{u}' + (\mathbf{u}'\times\boldsymbol{\Omega}')\cdot\mathbf{u}\rangle = -\langle\delta(\mathbf{u}\times\boldsymbol{\Omega})\cdot\delta\mathbf{u}\rangle, \tag{25}$$

$$\begin{aligned}&\langle\mathbf{M}\times\mathbf{H})\cdot\boldsymbol{\omega}' + (\mathbf{M}'\times\mathbf{H}')\cdot\boldsymbol{\omega} - \mathbf{H}'\cdot(\boldsymbol{\omega}\times\mathbf{M}) - \mathbf{H}\cdot(\boldsymbol{\omega}'\times\mathbf{M}')\rangle\\ &\quad = \langle\delta\mathbf{H}\cdot\delta(\boldsymbol{\omega}\times\mathbf{M})\rangle - \langle\delta\boldsymbol{\omega}\cdot\delta(\mathbf{M}\times\mathbf{H})\rangle, \text{ and}\end{aligned} \tag{26}$$

$$\langle-(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}\cdot\boldsymbol{\omega}' - (\mathbf{u}'\cdot\nabla')\boldsymbol{\omega}'\cdot\boldsymbol{\omega}\rangle = \langle\delta\boldsymbol{\omega}\cdot\delta((\mathbf{u}\cdot\nabla)\boldsymbol{\omega})\rangle, \tag{27}$$

where for any field $\boldsymbol{\psi}$, $\delta\boldsymbol{\psi} \equiv \boldsymbol{\psi}' - \boldsymbol{\psi}$, and by the incompressibility condition, we have $\langle\boldsymbol{\omega}\cdot(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}\rangle = \langle\mathbf{u}\cdot\nabla(\frac{\omega^2}{2})\rangle = \langle\nabla\cdot(\frac{\omega^2}{2}\mathbf{u})\rangle = \langle\nabla'\cdot(\frac{\omega^2}{2}\mathbf{u})\rangle = 0$. Again we can show

$$\begin{aligned}\langle\delta\mathbf{u}\cdot\delta((\mathbf{M}\cdot\nabla)\mathbf{H})\rangle &= \langle(\mathbf{u}' - \mathbf{u})\cdot[(\mathbf{M}'\cdot\nabla')\mathbf{H}' - (\mathbf{M}\cdot\nabla)\mathbf{H}]\rangle\\ &= \langle\mathbf{u}'\cdot(\mathbf{M}'\cdot\nabla')\mathbf{H}' - \mathbf{u}\cdot(\mathbf{M}'\cdot\nabla')\mathbf{H}' - \mathbf{u}'\cdot(\mathbf{M}\cdot\nabla)\mathbf{H} + \mathbf{u}\cdot(\mathbf{M}\cdot\nabla)\mathbf{H}\rangle, \text{ and}\end{aligned} \tag{28}$$

$$\begin{aligned}\langle\delta\mathbf{H}\cdot\delta((\mathbf{u}\cdot\nabla)\mathbf{M})\rangle &= \langle(\mathbf{H}' - \mathbf{H})\cdot[(\mathbf{u}'\cdot\nabla')\mathbf{M}' - (\mathbf{u}\cdot\nabla)\mathbf{M}]\rangle\\ &= \langle\mathbf{H}'\cdot(\mathbf{u}'\cdot\nabla')\mathbf{M}' - \mathbf{H}\cdot(\mathbf{u}'\cdot\nabla')\mathbf{M}' - \mathbf{H}'\cdot(\mathbf{u}\cdot\nabla)\mathbf{M} + \mathbf{H}\cdot(\mathbf{u}\cdot\nabla)\mathbf{M}\rangle\end{aligned}$$

$$= \langle \nabla' \cdot [(\mathbf{M}' \cdot \mathbf{H}')\mathbf{u}'] - \mathbf{M}' \cdot (\mathbf{u}' \cdot \nabla')\mathbf{H}' - \mathbf{H} \cdot (\mathbf{u}' \cdot \nabla')\mathbf{M}' - \mathbf{H}' \cdot (\mathbf{u} \cdot \nabla)\mathbf{M}\rangle$$
$$+ \langle \nabla \cdot [(\mathbf{M} \cdot \mathbf{H})\mathbf{u}] - \mathbf{M} \cdot (\mathbf{u} \cdot \nabla)\mathbf{H}\rangle. \qquad (29)$$

Adding Eqs. (28) and (29), one can get

$$\langle \delta\mathbf{u} \cdot \delta((\mathbf{M} \cdot \nabla)\mathbf{H}) + \delta\mathbf{H} \cdot \delta((\mathbf{u} \cdot \nabla)\mathbf{M})\rangle$$
$$= \langle -\mathbf{u} \cdot (\mathbf{M}' \cdot \nabla')\mathbf{H}' - \mathbf{u}' \cdot (\mathbf{M} \cdot \nabla)\mathbf{H}\rangle$$
$$+ \langle -\mathbf{H} \cdot (\mathbf{u}' \cdot \nabla')\mathbf{M}' - \mathbf{H}' \cdot (\mathbf{u} \cdot \nabla)\mathbf{M} + \mathbf{u}' \cdot (\mathbf{M}' \cdot \nabla')\mathbf{H}' - \mathbf{M}' \cdot (\mathbf{u}' \cdot \nabla')\mathbf{H} + \mathbf{u} \cdot (\mathbf{M} \cdot \nabla)\mathbf{H}\rangle$$
$$+ \langle -\mathbf{M} \cdot (\mathbf{u} \cdot \nabla)\mathbf{H} + \nabla' \cdot [(\mathbf{M}' \cdot \mathbf{H}')\mathbf{u}'] + \nabla \cdot [(\mathbf{M} \cdot \mathbf{H})\mathbf{u}]\rangle$$
$$= \langle -\mathbf{u} \cdot (\mathbf{M}' \cdot \nabla')\mathbf{H}' - \mathbf{u}' \cdot (\mathbf{M} \cdot \nabla)\mathbf{H} - \mathbf{H} \cdot (\mathbf{u}' \cdot \nabla')\mathbf{M}' - \mathbf{H}' \cdot (\mathbf{u} \cdot \nabla)\mathbf{M} + \nabla \cdot [(\mathbf{M} \cdot \mathbf{H})\mathbf{u}]\rangle$$
$$+ \langle \nabla' \cdot [(\mathbf{M}' \cdot \mathbf{H}')\mathbf{u}'] + (\mathbf{M} \times \mathbf{u}) \cdot (\nabla \times \mathbf{H}) + (\mathbf{M}' \times \mathbf{u}') \cdot (\nabla' \times \mathbf{H}')\rangle$$
$$= \langle -\mathbf{u} \cdot (\mathbf{M}' \cdot \nabla')\mathbf{H}' - \mathbf{u}' \cdot (\mathbf{M} \cdot \nabla)\mathbf{H} - \mathbf{H} \cdot (\mathbf{u}' \cdot \nabla')\mathbf{M}' - \mathbf{H}' \cdot (\mathbf{u} \cdot \nabla)\mathbf{M}\rangle, \qquad (30)$$

where we have used the following identities:

$$\nabla \cdot [(\mathbf{M} \cdot \mathbf{H})\mathbf{u}] = \mathbf{M} \cdot (\mathbf{u} \cdot \nabla)\mathbf{H} + \mathbf{H} \cdot (\mathbf{u} \cdot \nabla)\mathbf{M}, \qquad (31)$$

$$\langle \nabla \cdot [(\mathbf{M} \cdot \mathbf{H})\mathbf{u}]\rangle = -\nabla_l \cdot \langle (\mathbf{M} \cdot \mathbf{H})\mathbf{u}\rangle = \langle \nabla' \cdot [(\mathbf{M} \cdot \mathbf{H})\mathbf{u}]\rangle = 0, \quad \text{and} \qquad (32)$$

$$\mathbf{u} \cdot (\mathbf{M} \cdot \nabla)\mathbf{H} - \mathbf{M} \cdot (\mathbf{u} \cdot \nabla)\mathbf{H} = (\mathbf{M} \times \mathbf{u}) \cdot (\nabla \times \mathbf{H}) = 0. \qquad (33)$$

Combining Eqs. (25), (26), (27), and (30) and putting them together in Eq. (20), we get

$$\frac{\partial \mathcal{R}}{\partial t} = \frac{1}{2}\langle -\delta(\mathbf{u} \times \mathbf{\Omega}) \cdot \delta\mathbf{u} + I\delta\boldsymbol{\omega} \cdot \delta((\mathbf{u} \cdot \nabla)\boldsymbol{\omega}) + \mu_0[\delta\mathbf{H} \cdot \delta(\boldsymbol{\omega} \times \mathbf{M}) - \delta\boldsymbol{\omega} \cdot \delta(\mathbf{M} \times \mathbf{H})]\rangle$$
$$- \frac{1}{2}\mu_0\langle \delta\mathbf{u} \cdot \delta((\mathbf{M} \cdot \nabla)\mathbf{H}) + \delta\mathbf{H} \cdot \delta((\mathbf{u} \cdot \nabla)\mathbf{M})\rangle + D_w + D_\zeta + D_u + F_u. \qquad (34)$$

For the statistical stationary state $\partial_t(R_E + R_E') = 0$ and for the inertial range we can neglect the effect of dissipation and we finally derive the following exact relation:

$$2\varepsilon = \langle \delta(\mathbf{u} \times \mathbf{\Omega}) \cdot \delta\mathbf{u} - I\delta\boldsymbol{\omega} \cdot \delta((\mathbf{u} \cdot \nabla)\boldsymbol{\omega}) + \mu_0[\delta\mathbf{u} \cdot \delta((\mathbf{M} \cdot \nabla)\mathbf{H}) + \delta\mathbf{H} \cdot \delta((\mathbf{u} \cdot \nabla)\mathbf{M})$$
$$- \delta\mathbf{H} \cdot \delta(\boldsymbol{\omega} \times \mathbf{M}) + \delta\boldsymbol{\omega} \cdot \delta(\mathbf{M} \times \mathbf{H})]\rangle, \qquad (35)$$

where $\varepsilon = F_u$, is the mean energy flux rate. Note that the cascading energy is injected through a large scale forcing $\mathbf{f}_u$ which has a larger correlation length than the length scales inside the inertial zone and the correlation length of corresponding velocities. Therefore, within the scale range of the inertial zone, using statistical homogeneity, one can approximately write $\varepsilon(\mathbf{r}) = F_u = \langle \mathbf{u} \cdot \mathbf{f}_u' + \mathbf{u}' \cdot \mathbf{f}_u\rangle/2 \approx \langle \mathbf{u} \cdot \mathbf{f}_u + \mathbf{u}' \cdot \mathbf{f}_u'\rangle/2 = \langle \mathbf{u} \cdot \mathbf{f}_u\rangle = \varepsilon(\mathbf{0})$, which is the one-point energy flux rate (see [36,37]). Equation (35) is the central result of this paper. This expresses the average flux rate of total energy (ε) for completely developed turbulence in an incompressible ferrofluid in terms of two-point increments of different fluid and electromagnetic field variables.

V. DISCUSSION

A. Different limits

In the following, we shall check some known limits using the above exact relation:

(1) For zero external magnetic field, i.e., $\mathbf{H} = 0$, the exact relation (35) reduces to

$$\langle \delta(\mathbf{u} \times \mathbf{\Omega}) \cdot \delta\mathbf{u} - I\delta\boldsymbol{\omega} \cdot \delta((\mathbf{u} \cdot \nabla)\boldsymbol{\omega})\rangle, \qquad (36)$$

which shows that in the absence of a magnetic field, the ferrofluid behaves like a neutral fluid with suspended particles having internal rotational degrees of freedom. Note that this rotational degrees of freedom brings forth internal angular momentum by virtue of the external magnetic field and the fluid vorticity. For the case with $\mathbf{H} = 0$, the only contribution to $\boldsymbol{\omega}$ comes from the fluid vorticity. If in addition, the moment of inertia I is also very small then the term $I\delta\boldsymbol{\omega} \cdot \delta((\mathbf{u} \cdot \nabla)\boldsymbol{\omega})$ is negligible, so Eq. (36) becomes $\varepsilon = \frac{1}{2}\langle \delta(\mathbf{u} \times \mathbf{\Omega}) \cdot \delta\mathbf{u}\rangle$, which is identical to the alternative form of exact relation for incompressible hydrodynamic turbulence as derived by Banerjee and Galtier [25].

(2) For $\mathbf{M} = 0$, we have exactly the same case and Eq.(35) also reduces to Eq. (36).

(3) When magnetic moments of particles align towards the external magnetic field, one can write $\mathbf{M} \propto \mathbf{H}$ or $\mathbf{M} = \alpha\mathbf{H}$, here α is a proportionality variable, then $\mathbf{M} \times \mathbf{H} = 0$. If α is constant, then $\langle \delta\mathbf{u} \cdot \delta((\mathbf{M} \cdot \nabla)\mathbf{H})\rangle = 0$. Hence in Eq. (35) only terms $\delta(\mathbf{u} \times \mathbf{\Omega}) \cdot \delta\mathbf{u}$, $\delta\boldsymbol{\omega} \cdot \delta((\mathbf{u} \cdot \nabla)\boldsymbol{\omega})$, $\delta\mathbf{H} \cdot \delta((\mathbf{u} \cdot \nabla)\mathbf{M})$, and $\delta\mathbf{H} \cdot \delta(\boldsymbol{\omega} \times \mathbf{M})$ survive, of which the last two terms represent the settling of the magnetic particles.

B. Other features and perspective

We see that the $\mathbf{M} \times \mathbf{H}$, $\mathbf{M} \times \boldsymbol{\omega}$, and the Lamb vector $\mathbf{\Omega} \times \mathbf{u}$ play the key roles in energy cascading. For different types of aligned flow states, some or all of them can vanish.

For example, for Beltrami flows where the fluid velocity $\mathbf{u}$ and fluid vorticity $\mathbf{\Omega}$ are collinear, the contribution of $\delta(\mathbf{u} \times \mathbf{\Omega})$ in Eq. (35) goes away and the same will apply for other aligned cases where $\mathbf{M} \parallel \mathbf{H}$, which means that the total magnetization vector is aligned to the external magnetic field and also for $\mathbf{M} \parallel \mathbf{\Omega}$ where the individual particle spin direction is aligned to the total magnetization. It is therefore clear that with each type of alignment, there is a partial suppression of the turbulent contribution to the energy flux rate. Future work can be proposed to derive this type of exact relation for other types of magnetization equations, using a time-varying external magnetic field and adding compressibility to the ferrofluid. This type of relation can be verified using properly designed laboratory experiment or numerical simulations.

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