Hidden complexity in Life-like rules

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An alternative way to study the rules of life-like cellular automata is presented. The proposed perspective studies some multifractal and informational properties of Boolean functions behind these rules. Results from this approach challenge the traditional argument about the simplicity of Lifelike rules.

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I. INTRODUCTION

The famous cellular automaton called *Life* introduced by J. H. Conway as a zero-player game game was described by M. Gardner in 1970 [1]. Through the years this game has fascinated a wide and diverse audience, which has found in it a motivating example of what a complex system can be. Usually it is pointed out that this automaton is capable of producing global interesting behaviors from local interactions governed by a simple deterministic rule [2–4].

A big part of the study around this automaton has focused on the design of special initial conditions and to run long-time simulations in order to characterize resulting dynamics and patterns [5]. A lot has been learned from the automaton following this way, motivating us to study other automata whose rules are expressed in a similar manner, e.g., Life-like (LL) [6] and Larger-than-Life (LtL) automata [7].

This paper approaches the study of LL rules from a different perspective, which does not appeal to simulate the automaton evolution but to explore the local informational behavior at the core of the rule which is a Boolean function [8,9]. Our findings from this approach suggest LL rules hide interesting phenomena. Exploring the informational behavior of LL rules makes sense because partial differential equations or Boolean delay equations models of complex systems find common ground with cellular automata at microscopic scales [10].

The paper is organized as follows: Section II summarizes some concepts and definitions regarding LL and LtL rules. Section III describes the proposed study approach. Section IV presents our results and findings. Finally, we discuss and draw some conclusions in Sec. V.

II. LL AND LTL CELLULAR AUTOMATA

A cellular automaton is a model of a complex system that contains a large number of identical components with local interactions. The automaton consists of a lattice, where each location has a finite set of possible values. Such values evolve synchronously in discrete steps according to identical rules. The next value of a location is a function of the current and previous values of a neighborhood around it. Thus, cellular automata can be classified as discrete dynamical systems [11].

In some cases, evolution of cellular automata is irreversible and characterized by exhibiting self-organization from arbitrary initial states without structure. In others, this evolution resembles the dynamics of certain chaotic systems that produce patterns with fractal structure [2,12]. From computer science, it has been demonstrated that certain cellular automata are capable of universal computing [13].

Within the universe of cellular automata, the subset called LL is characterized for having properties inspired by *Life* and the subset called LtL extends the properties of the famous automaton. It has been noted that LL and LtL automata are capable of producing an interesting set of behaviors such as spatial oscillators, chaotic dynamics, and production of self-organized spatial patterns that sustain over time and consistent structures that move (i.e., gliders and spaceships) [6,14]. Particularly, simulations of Life show this automaton is characterized by self-organized criticality [3] and its mean field approximation reveals it operates in the border of extinction [5].

According to Ref. [14] an LtL automaton is defined as follows: Consider a bi-dimensional lattice $\mathbf{Z}^2$, where each of its locations has two states, *alive* (1) or *dead* (0). The neighborhood Υ^x of a site x consists of $(2\rho + 1) \times (2\rho + 1)$ surrounding locations, including the site itself. Each step of time k are updated simultaneously according to a deterministic rule Γ.

Let Γ be the rule that governs the cellular automaton. This is $\Gamma : \{0, 1\}^{\mathbf{Z}^2} \rightarrow \{0, 1\}^{\mathbf{Z}^2}$ and $\psi_x(k) \in \{0, 1\}$, the status of the site $x \in \mathbf{Z}^2$ at the instant k, so $\psi(k)$ represents the status of all sites in $\mathbf{Z}^2$ at instant k.

From these two definitions we can say that $\psi^{\Lambda}(k) = \Gamma^k(\Lambda) = \Lambda(t)$, which means that starting from the configuration $\psi(0) = \Lambda$, the set of occupied sites is reached after k iterations of the Γ rule. Particularly in the case of an LtL rule,

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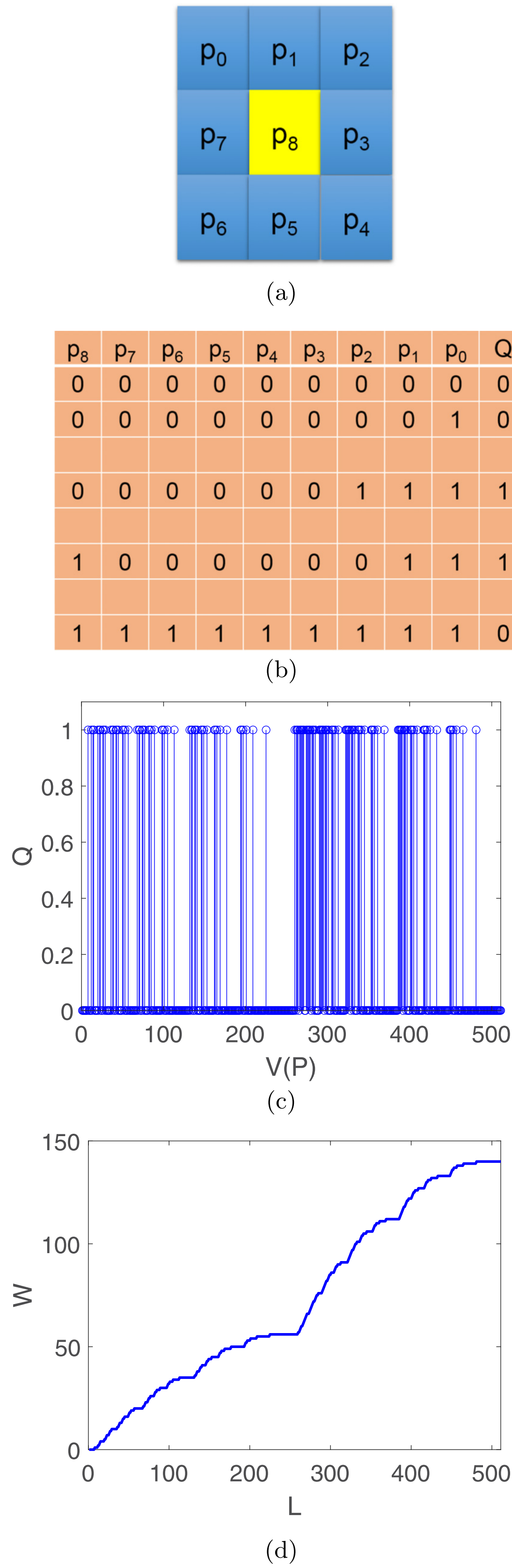

FIG. 1. An alternative study of an LL rule (*B3S23*). (a) Arrangement of the Moore neighborhood as a binary tuple. (b) Truth table of the rule. (c) Equivalent Boolean map obtained from natural binary encoding. (d) Cumulative function of the map.

	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	Integer
Birth encoding	0	0	0	1	0	0	0	0	0	32
Survive encoding	0	0	1	1	0	0	0	0	0	96

FIG. 2. Encoding used to characterize some properties of the LL rule universe.

it is expressed as:

$$\psi_x(k+1)=\begin{cases}1 & \text{if } \psi_x(k)=0 \text{ and } |\Upsilon^x \bigcap \psi(k)| \in [\beta_1,\beta_2] \\ & \text{or} \\ & \text{if } \psi_x(k)=1 \text{ and } |\Upsilon^x \bigcap \psi(k)| \in [\delta_1,\delta_2] \\ 0 & \text{in other case}\end{cases}, \quad (1)$$

where β_1, β_2, δ_1, and δ_2 are natural numbers.

The above can be expressed in words such as:

(i) **Born**: A site that is dead at instant k will live at instant $k+1$ if and only if the number of live sites in its neighborhood at instant k is in the closed interval $[\beta_1, \beta_2]$.

(ii) **Survive**: A site that is alive at instant k will still alive at instant $k+1$ if and only if the number of live sites in its neighborhood (including the same) at instant k is in the closed interval $[\delta_1, \delta_2]$.

(iii) **Die**: A site that is dead at time k and does not live at instant $k+1$ will still be dead at instant $k+1$. A site that is alive at instant k and does not live at instant $k+1$ will die at instant $k+1$.

The Γ rule can be formally expressed as a tuple $\Gamma = \{\rho, \beta_1, \beta_2, \delta_1, \delta_2\}$. For example, the rule of *Life* can be expressed in this format as $\Gamma_{life} = \{1, 3, 3, 3, 4\}$, since the same position is included where the rule is evaluated in the case of surviving.

Now regarding a LL automaton [7], the neighborhood Υ^x is restricted to the traditional Moore neighborhood with $\rho = 1$, not including the site x. Therefore the Γ rule is expressed as:

$$\psi_x(k+1)=\begin{cases}1 & \text{if } \psi_x(k)=0 \text{ and} \\ & |\Upsilon^x \bigcap \psi(k)| \in \{\beta_1,\ldots,\beta_Q\} \\ & \text{or} \\ & \text{if } \psi_x(k)=1 \text{ and} \\ 0 & |\Upsilon^x \bigcap \psi(k)| \in \{\delta_1,\ldots,\delta_P\} \\ & \text{in other case}\end{cases}, \quad (2)$$

where $0 \leqslant Q \leqslant 8$, $0 \leqslant P \leqslant 8$, $\beta_j \in [0, 8]$, and $\delta_k \in [0, 8]$.

The LL rules can be expressed as an alphanumeric string of the form "*Bxxx*...*Syyy*...." The letter *B* indicates the set of possibilities of active neighbors that give life to a site $\{\beta_1, \beta 2, \ldots\}$, which is expressed by the string *xxx*.... The letter *S* indicates the set of possibilities of active neighbors that allow a site to survive $\{\delta_1, \delta 2, \ldots\}$, expressed by the string *yyy*...; for example, in the case of *Life* this string is expressed as *B3S23*.

III. AN ALTERNATIVE APPROACH TO STUDY LL RULES

The traditional study of an LL rule is carried out by observing the dynamics of the automaton evolving from a given initial condition. Initial conditions are modified and different patterns might emerge during simulation [15]. In some cases when different rules exhibit similar patterns and behaviors,

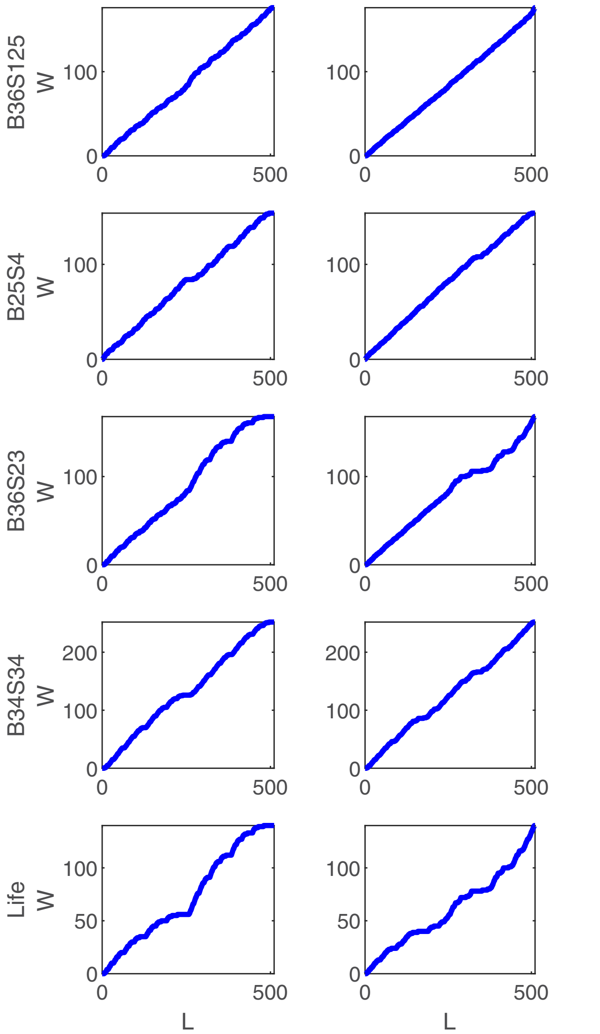

FIG. 3. Cumulative functions of some LL rules obtained from natural binary encoding (left column) and Gray encoding (right column).

taxonomies are proposed to group these rules [6,7,13]. This way has allowed to discover several interesting properties about LL automata regarding their complexity and behavior near critical states [5].

An LL rule is capable of producing diverse dynamics over an automaton depending on initial conditions [2,7]. Besides, it is well known that some rules produce more complex dynamics than others [5,6,16]. However, LL rules are static maps in their nature themselves. Thus the approach that will be presented is focused on studying the properties of these functions.

This work aims to understand the behavior of a rule itself considered as a simple algorithm [17] or a Boolean map [16]. Our approach investigates the nature of these rules, not in terms of the complexity footprint induced in the automaton dynamics but in terms of possible nuances of complexity that these rules might contain. Thus we take distance from the algorithmic understanding of the rule to rather focus on its logical-mathematical structure as a Boolean function [8]. The proposed method is illustrated over the $B3S23$ rule as follows:

(i) Start arranging the traditional Moore neighborhood, shown in Fig. 1(a), as a nine element binary tuple $P = p_8 p_7 ... p_0$. The central site of the neighborhood is chosen as the most significant position p_8. The remaining positions can be arranged in the tuple arbitrarily without loss of generality. This is possible since LL rules are outer-totallistic, and thus they are independent from the particular spatial arrangement of neighbors [5].

(ii) Consider this tuple as the input argument of a logic expression in the disjunctive normal form $Q = \Gamma(P)$. Apply the LL rule over each case of the tuple and construct the truth table with 2^9 cases that describe the entire Γ function, as shown in Fig. 1(b) for some cases of the $B3S23$ rule.

(iii) Order the cases according to an encoding $V(P)$ and represent Q as an integer function of this order. Figure 1(c) presents the result when natural binary encoding is used.

(iv) Finally, compute the cumulative function of Q as

$$W_L = \sum_{j=0}^{L-1} Q_j \tag{3}$$

for $L = 1 \cdots 2^9$, as shown in Fig. 1(d).

IV. RESULTS AND FINDINGS

Two ways are considered to study the behavior of LL rules from the alternative approach proposed in this work.

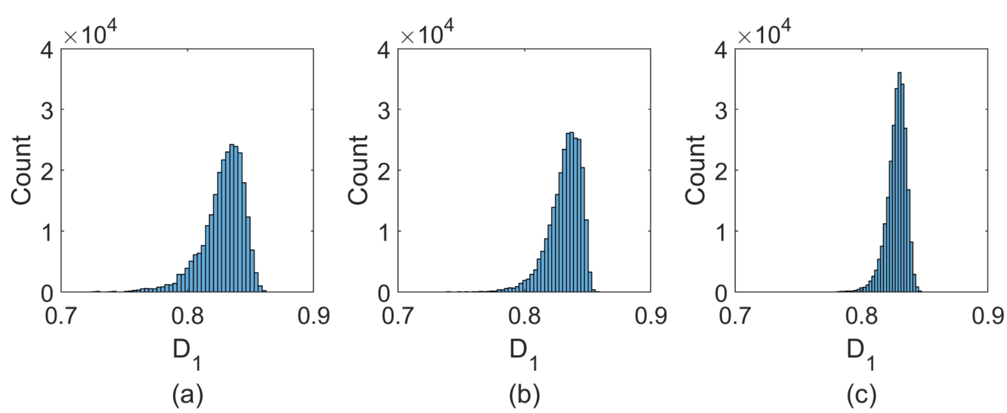

FIG. 4. Histograms of D_1 over the LL universe for (a) natural binary encoding, (b) Gray encoding, and (c) random encoding.

TABLE I. Distribution moments for D_1 and D_2 over the LL universe considering three encodings.

		D_1		D_2	
Coding	$\tilde{H}$	Mean	Std	Mean	Std
Gray	1.0	0.8314	0.0136	0.7	0.0083
Natural	1.97	0.8277	0.0172	0.7	0.0116
Random	4.51	0.8272	0.0082	0.7	0.001

TABLE II. Information dimension D_1 obtained from cumulative functions and cumulative dynamics of automata for some LL rules.

Rule	Automaton	Gray	Natural	Random
B36S125	0.7620	0.8340	0.8165	0.8229
B25S4	0.7633	0.8252	0.8193	0.8200
B36S23	0.7621	0.8058	0.8248	0.8202
B34S34	0.7620	0.8073	0.8272	0.8276
B3S23 (Life)	0.7615	0.7849	0.8085	0.8155

The former attempts to characterize the self-similarity of cumulative functions $W(Q)$, whereas the latter measures some informational features of the discrete map Q.

The LL rule universe is explored by using a rule-expression format adapted from Ref. [5], as shown in Fig. 2. In this example, the rule $B3S23$ is expressed as the pair of integers $(B_e, S_e) = (32, 96)$. According to the string format of LL rules, each part of the string (i.e., *Bxxx*.. and *Syyy*..) might have the combination of nine integer literals [0,8]. Thus 2^9 cases per part are produced, which sets the size of the LL universe in 2^{18} rules. Strings filled with zeros are ignored so that $2^{10} - 1$ rules are not considered in the analysis.

A. Multifractal properties

The cumulative functions W_L of some representative LL rules [7] are presented in Fig. 3 for two encodings: natural binary and Gray [18]. In the natural binary encoding, each case of P produces an integer according to $V(P) = \sum_{i=0}^{8} 2^i p_i$. The order induced by this encoding corresponds to the natural order of integers. On the other hand, Gray encoding orders codes by guarantying that two successive cases of P only differ in one position.

The shape of cumulative functions obtained by means of these two encodings does not go unnoticed, since at first glance it appears the functions would have self-similar behavior. In addition, it also draws attention the resemblance of these functions with *Devil's staircases*, which are known to have fractal properties [19]. Devil's staircases appear in various natural processes and can be explained from the theory of nonlinear dynamic systems [20]. However, these functions can also be generated by means of mathematical maps of the type $[0, 1] \to [0, 1]$ [21].

The apparent self-similarity observed in cumulative functions is tested by means of a multifractal analysis [19] over the LL universe. The cumulative function of each rule is generated by means of binary natural, Gray, and random encodings. The multifractal analysis of each cumulative function is summarized in its information D_1 and correlation D_2 dimensions. Regarding the random encoding, dimensions are averaged over 33 independent random permutations of the natural binary code.

Results of the multifractal analysis are presented in Fig. 4. An interesting similarity between the histograms of D_1 over the LL universe for the three encodings can be noted. This particularity is also reflected in the closeness of the first distribution moment, as presented in Table I. This result suggests that the multifractal properties of cumulative functions would be preserved over three different representations. The attribute to differentiate the encodings is the average hamming distance between successive codes $\tilde{H}$, which exhibits a significant increment from one encoding to the other.

In addition, dimensions of the five rules considered in Fig. 3 were compared with dimensions obtained from cumulative dynamics in the corresponding cellular automata. The cumulative dynamics is the accumulation of values produced in each location of the automaton during M iterations. In this case, 30×30 automata were configured so that 900 cumulative dynamics were analyzed. Dimensions are averaged over the locations of 66 independent automata with random initialization and $M = 512$ iterations.

Results are presented in Table II and Table III. It can be noted that multifractal attributes of the cumulative dynamics do not overpass those detected in cumulative functions no matter the encoding used to construct them. Life reported the smallest values in D_1 for all encodings as well as in the cumulative dynamics of the automaton. In addition, these values are the closest ones among explored rules.

B. Informational properties

The Boolean map Q [Fig. 1(c)] is also studied in terms of its structure and some informational properties inspired by Ref. [22]. Four indices are proposed to support this study:

(i) Structural burden (S_B): Structural burden is the number of nine-variable logic terms required (without optimization) to implement the rule as a sum of products (disjunctive normal form). This is the number of ones N_o found in the map Q. S is the normalization of this number computed as:

$$S_B = \frac{N_o}{2^9}. \tag{4}$$

(ii) Information (I): Information is the Shannon entropy of a binary string. It represents how much variety is found in the string, computed as:

$$I = -p(0)\log_2 p(0) - p(1)\log_2 p(1), \tag{5}$$

TABLE III. Correlation dimension D_2 obtained from cumulative functions and cumulative dynamics of automata for some LL rules.

Rule	Automaton	Gray	Natural	Random
B36S125	0.5648	0.7001	0.6994	0.7016
B25S4	0.5525	0.7001	0.7028	0.7023
B36S23	0.5459	0.6866	0.7146	0.7016
B34S34	0.5498	0.6887	0.7050	0.7016
B3S23 (Life)	0.5586	0.6737	0.7073	0.7015

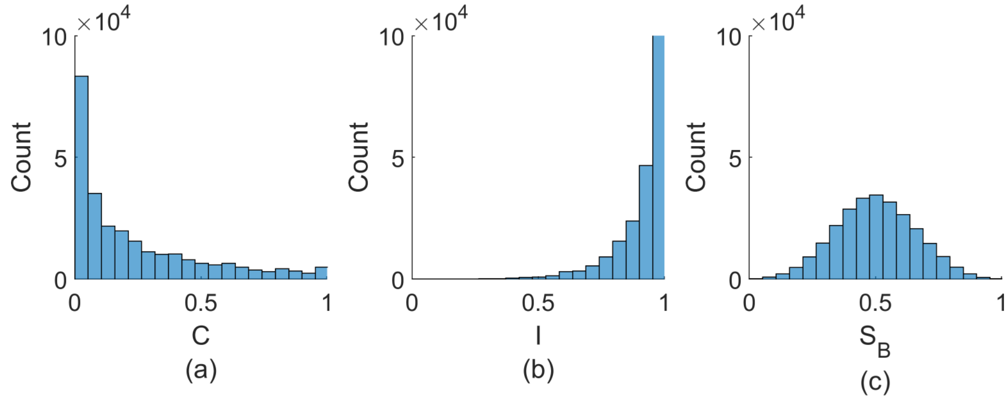

FIG. 5. Histograms of the informational properties over the LL rules universe: (a) complexity, (b) information, and (c) structural burden.

where $p(0)$ and $p(1)$ are the estimated probabilities of finding "0" and "1" in the string, respectively. Here the string is Q ordered by $V(P)$ as depicted in Fig. 1(c).

(iii) Organization (O): Organization represents how much order the string contains. It measures information increase or reduction as:

$$O = 1 - I. \tag{6}$$

(iv) Complexity (C): This is the balance between variety and order computed as:

$$C = kIO = kI(1 - I) \tag{7}$$

with $k = 4$, and C is bounded in [0,1].

Histograms of informational properties are depicted in Fig. 5. The structural burden histogram reveals that rules tending to be equiprobable [i.e., $p(0) = p(1) = 0.5$] are the most frequent ones, which can also be noted as the high number of rules that report information I close to 1. Because of the symmetry in the S_B histogram, the amount of rules with $p(1) = p^*$ is similar to that of rules with $p(0) = p^*$, where p^* is a real value in [0,1]. In addition, it can be inferred that most frequent rules report lower complexities. On the other hand, rules with maximum complexity are rare. Among detailed rules, $B34S34$ is an example of a frequent rule with low complexity, whereas Life is a rule with a medium level of complexity that does not abound.

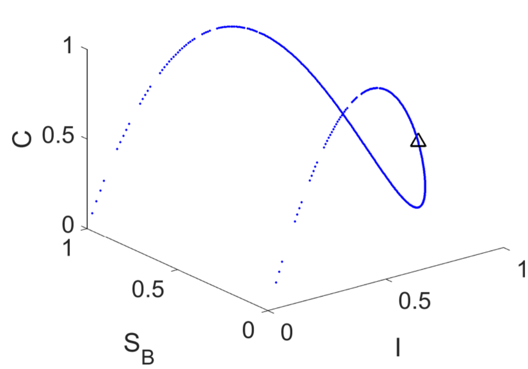

FIG. 6. Characterization of the LL universe in terms of information, structural burden, and complexity. The point corresponding to Life is highlighted.

Informational characterization of the LL universe is presented in Fig. 6. It can be notice that the LL universe does not span uniformly over the feature space (C, I, S_B). This particularity is given by the dependency among these variables. By looking at Eq. (4), the structural burden is equivalent to the estimated probability of finding a "1" in the string Q, used to compute I in Eq. (5). Also note the resulting map described by Eq. (7), where $S_B \rightarrow I \rightarrow C$, is not entirely sampled by the LL universe. Attributes of rules considered in Fig. 3 are presented in Table IV. Life rule reports the highest complexity among detailed rules; however, it does not reach the maximum value. In fact, there are other rules that exhibit higher complexity in the sense of Eq. (7).

Maximum complexity is achieved at two points of structural burden that produce the same information. It would be interesting to characterize which and how many rules accomplish with this condition. Note that informational properties C, I, S_B depend statistically on the output Q of rules. Order induced in the output string does not influence these results so that informational properties of Q are not dependent of encoding.

Figure 7 depicts a representation of the rule encoding as a Cartesian product of pairs (B_e, S_e). Over this product, rules that have the same properties of those detailed in Fig. 3 are highlighted. It can be observed that these rules cluster symmetrically in two opposing regions. If one region is similar to a simple rectangle, then the other one seems like the intersection of two orthogonal rectangles. Clusters share common structure composed of near rules that form diagonal triplets, duets, and single points. As a particular case, Life is a single point that shares its informational properties with other 160 rules.

TABLE IV. Informational properties of some LL rules.

Rule	S_B	I	O	C
B36S125	0.3438	0.9284	0.0716	0.2660
B25S4	0.3007	0.8822	0.1178	0.4156
B36S23	0.3281	0.9130	0.0870	0.3177
B34S34	0.4921	0.9998	0.0002	0.0007
B3S23 (Life)	0.2734	0.8464	0.1536	0.5202

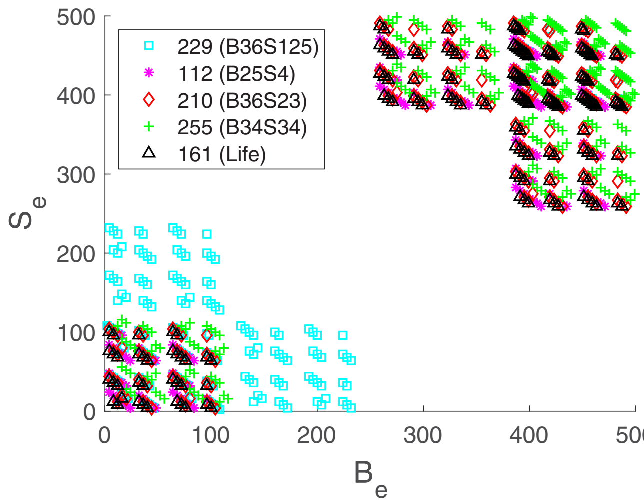

FIG. 7. Clusters of rules that share the same informational properties of rules detailed in Fig. 3.

V. DISCUSSION

To validate whether the cumulative functions obtained from LL rules are devil staircases is a problem that will be left open in this work. A formal response to this concern would indicate that LL rules, which are expressed and understood in a simple way, actually hide an interesting footprint of complexity that can be noted through the multifractal-informational analysis. If the hypothesis of devil's staircases is tested true, then it would be a case of these functions emerging from algorithms such as LL rules.

The experimental evidence presented in this work pointed out that multifractal properties were preserved over three different encodings of the Moore neighborhood used to compute cumulative functions. Thus self-similarity of these functions would not be an attribute of a particular representation. To show whether the self-similarity is preserved over the entire LL universe independent from the enconding is a key question, perhaps similar to the problem of testing whether a theory works independently from the selection of a particular coordinate system. Regarding the implications of partial results concerning automata evolution, it is suggested that multifractal properties of cumulative functions may act as superior limits of the properties of the average cumulative dynamics.

Informational properties of equivalent Boolean maps of LL rules are independent from encoding. The approach followed in this work would be limiting the complexity panorama regarding the local behavior of the rules. However, it pointed out that the feature space composed by (C, I, S_B) is characterized by a nonuniform distribution of these properties over the LL universe. Besides, it suggested that Life exhibits interesting local informational properties that are shared with other rules; however, there exist rules that can produce higher local complexity with smaller structural burden.

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