From power law to Anderson localization in nonlinear Schrödinger equation with nonlinear randomness

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We study the propagation of coherent waves in a nonlinearly induced random potential and find regimes of self-organized criticality and other regimes where the nonlinear equivalent of Anderson localization prevails. The regime of self-organized criticality leads to power-law decay of transport [Y. Sharabi *et al.*, Phys. Rev. Lett. **121**, 233901 (2018)], whereas the second regime exhibits exponential decay.

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I. INTRODUCTION

In this work, we consider a problem of polynomial to exponential localization in the one-dimensional (1D) nonlinear Schrödinger equation (NLSE) in a random potential. A special property of the system is that randomness is incorporated into nonlinearity in the form $\beta\eta(x)|\psi(x)|^2$, where $\psi = \psi(x)$ is a wave function, $\eta(x)$ is a random field, and β is a nonlinearity parameter. This formulation of randomness and nonlinearity differs essentially from the NLSE in random potential, which reads

$$i\partial_t \Psi = -\partial_x^2 \Psi + \eta(x)\Psi + \beta|\Psi|^2\Psi. \tag{1}$$

The model, (1), has been extensively studied, and a variety of results have been observed over the years. In particular, a stationary counterpart of Eq. (1), when

$$\Psi(x,t) = e^{-i\omega t}\psi(x), \tag{2}$$

relates to the Anderson localization [1,2] of the stationary states of the NLSE. Another important task is wave propagation in nonlinear media [3–11], where the problems of spreading of wave packets and transmission are not simply related [6,7,12,13], in contrast with the linear case. This problem is relevant for experiments in nonlinear optics, for example, disordered photonic lattices [14,15], where Anderson localization was found in the presence of nonlinear effects. This long-lasting task is far from being completely solved, and many fundamental problems are still open in both dynamical and stationary cases, like the Berry phase and the semiclassical, also known as the adiabatical, approximation in the NLSE [16].

Here, we study the propagation of coherent waves in a random potential that is induced (nonlinearly) by the wave itself. The problem was first proposed in the context of nonlinear optics [17], but it is in fact a universal problem, relevant to any coherent wave system, for example, cold atoms in the Gross-Pitaevskii regime [18], and more. The underlying model is fundamentally different from the NLSE, (1), because the random potential is strictly nonlinear, with a mean value around 0. Technically, this situation corresponds to the study of stationary solutions of the 1D NLSE in a random potential, where the latter relates now to the nonlinear part of the NLSE, and this situation relates to numerical observation of the power-law decay of diffusive waves [17] in 1D dielectric disordered, nonlinear media. The model in task reads [17]

$$\partial_x^2\psi + \omega\psi + \beta\eta(x)|\psi|^2\psi = 0, \tag{3}$$

where ω is the energy of the real stationary solution ψ. Here the variables are chosen in dimensionless units and the Planck constant is $\hbar = 1$. Statistical properties of the random potential $\eta(x)$ are specified when necessary in the text. To admit the difference between the NLSE, (1) [with solution (2)], and Eq. (3), we call the latter the random nonlinear Schrödinger equation (RNLSE).

When $\omega = k^2$, where k is a wave number[1] and β/k^2 is the Kerr coefficient, Eq. (3) is the Helmholtz equation, which corresponds to the experimental setup in Ref. [17], where the power-law decay of the intensity of the wave has been observed numerically. Therefore, polynomial decay of the wave function is anticipated for the solution of Eq. (3).

The paper is organized as follows. In Sec. II some heuristic arguments based on the random walk theory are presented to explain the experimental setup. Section III is devoted to the estimation of the transmission coefficient based on the RNLSE under the condition of a nonzero constant probability current. A completely original approach to Anderson localization is developed in Sec. IV, and its numerical verification is presented in Sec. V. A summary of the results and conclusion are presented in Sec. VI.

II. HEURISTIC ARGUMENTS OF EXPERIMENTAL SETUP IN REF. [17]

Returning to the experiment on wave diffusion, the power-law decay of wave transmission has been explain by heuristic

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[1]Without restriction of the generality, the refractive index of the medium is taken to be 1.

arguments as follows. Due to the Kerr effect, the transmitting characteristic is a function of the intensity of the wave $\mathcal{T} = \mathcal{T}(I) = \sigma I$. Therefore, the Boltzman equation, which describes the intensity of the diffusive wave, reads [19,20] $dI/dx = -\mathcal{T}(I)I = -\sigma I^2$, with the boundary condition $I(x=0) = I_0$. This equation defines the power-law decay of the propagating wave amplitude: $I(L) = I_0/(1 + \mathcal{T}(I_0)L)$. It is a simplified scheme of a more sophisticated toy model suggested in Ref. [17].

Random walk approximation

The kinetic theory of diffusive light in slab geometry can also be considered in the framework of random walk theory based on the universality of the probability of escape from a half-space [21,22]. This phenomenology is completely relevant for the multicollision dynamics for transmission through a finite slab, considering diffusive waves as a Brownian particle. In this approach, the transmission probability is determined by the first passage of a Brownian particle at $x = L$.

A random walk of a particle in random media, starting at $x(t=0) = 0$, after n identically distributed steps $\Delta x(t_j)$, related to $n-1$ collisions, finishes at a random position $x(t_n) = \sum_{j=0}^{n-1} \Delta x(t_j)$, with mean squared displacement $\langle (x(t_n))^2 \rangle \sim \sigma^2 t_n$. Note that the free pass variance between collisions $\sigma^2 = \langle (x(t_n))^2 \rangle$ is a well-defined value from the experimental setup. Therefore, the mean transition time reads $t_L \sim (L/\sigma)^2$ for $\sigma \ll L < \infty$ [21]. It has been shown in Ref. [22] that the probability of reaching the boundary L, which corresponds to transmission and is determined as the superposition of all first-boundary passages, reads[2]

$$\mathcal{T}_L \sim \mathrm{Prob}(t > t_L) \sim \frac{1}{\sqrt{\pi t_L}} \sim (L/\sigma)^{-1}. \tag{4}$$

Although these heuristic arguments on wave diffusion, based on either the Boltzman equation or random walks, provide physically reasonable and relevant explanations, these approaches are far from analytical rigor, related to Eq. (3). Another fundamental question is about the localization length, or Lyapunov exponents of the stationary solution of the RNLSE, (3). As has been shown for the NLSE, (1), in Ref. [9], the nonlinearity parameter β does not contribute to the Lyapunov exponents of the linear counterpart. It is also well known that in the linear case for a random system of finite length L, the transmission coefficient decays exponentially with L, including the linear part of the NLSE, (1) [5,9]. However, a specific feature of the RNLSE, (3), is that for $\beta = 0$, the medium is transparent and the transmission coefficient does not decay at all. Therefore, our next consideration of the transmission is in the framework of Eq. (3).

[2]Following the Sparre Andersen theorem [23], one obtains that the first-passage probability of escaping from a half infinite line for any symmetrical random walk reads $P_\infty(t) \sim t^{-3/2}$. Therefore, to find a particle outside the boundary L after the mean transit time is $\mathrm{Prob}(t > t_L) = \int_{t_L}^{\infty} P_\infty(t) \sim t_L^{-1/2}$ for an asymptotically large L.

III. THE RNLSE AS A HELMHOLTZ EQUATION: DEVILLARD-SOULLIARD APPROACH

In this section, we investigate the initial-value problem, where the wave is launched from $x = 0$ with some initial amplitude and induces nonlinear changes in the potential as it propagates. In this regime, we find that the wave follows self-organized criticality, as it exhibits power-law decay while propagating into the structure.

Let us consider 1D wave propagation in the slab geometry, which is described by Eq. (3),

$$\partial_x^2 \psi + \omega\psi + \beta\eta(x)|\psi|^2\psi = 0. \tag{5}$$

The boundary conditions for the random potential are $\eta(x) = 0$ for $x < 0$ and $x > L$. Therefore the incident and reflected (with coefficient $\mathcal{R}$) waves on the left and outgoing (with transmission coefficient $\mathcal{T}$) wave on the right read

$$\psi(x) = e^{ikx} + \mathcal{R}e^{-ikx}, \quad x < 0, \tag{6a}$$

$$\psi(x) = \mathcal{T}e^{ikx}, \quad x > L. \tag{6b}$$

For the fixed output condition, the conservation condition for the current reads

$$J(x) = [\psi^* \partial\psi - \psi\partial\psi^*]/2i = |\mathcal{T}|^2 = 1 - |\mathcal{R}|^2. \tag{7}$$

In this section, we follow Devillard and Souillard's consideration in Ref. [5]. Namely, we follow their improved (theoretically and numerically) Theorem (3), which states that *for any J the transmission $\mathcal{T}$ cannot tend toward 0 faster than L^{-1} as $L \to \infty$*. This theorem has been proved for the NLSE, (1), and we prove it here for the RNLSE, (3), or (5) following the method in Ref. [5], modified for the present model, (5). In this sense, this extension of the Devillard-Souillard estimation of the wave transmission in the framework of the RNLSE, (3), can be considered as a corollary of Theorem (3) in Ref. [5]. Note also that the present case is simpler, and the result immediately follows from the theorem conditions.

We make the partition $L = N\Delta x \equiv \sum \Delta$ with constant randomness $\eta(x_n) = \eta_n$ at each step $x_n \in (n\Delta x\,,\, n\Delta x + \Delta x)$. Therefore, for every interval Δx_n with constant value η_n there is a Hamiltonian/energy H_n, which produces, by the Hamiltonian form of Eq. (5) for each step n,

$$H_n = |\partial_x\psi|^2 + \omega|\psi|^2 + \beta\eta_n|\psi|^4/2. \tag{8}$$

Since ψ and $\partial_x\psi$ are continuous at edges of the steps [5], we have

$$H_{n+1} - H_n = \beta(\eta_n - \eta_{n+1})|\psi|^4/2. \tag{9}$$

Taking $\Delta\eta = \eta_{\max} - \eta_{\min}$ as the maximum fluctuation, we obtain from Eq. (9)

$$|H_{n+1} - H_n| \leqslant \beta\Delta\eta|\psi|^4/2 \leqslant \frac{\beta\Delta\eta}{2\omega^2} H_n^2 = AH_n^2, \tag{10}$$

where $A = \frac{\beta\Delta\eta}{2\omega^2}$ and the second inequality is valid for the positive random potential $\eta(x) \geqslant 0$. This inequality yields a decay of the energy H_n with n not faster than $1/n$ in the limiting case of $n \gg 1$. We also have from Eq. (10) that the

maximum decay[3] of the energy reads

$$H_n = \frac{H_0}{1 + AH_0 n}. \tag{11}$$

From the partition it follows that $\mathcal{T} = |\psi_N|^2/|\psi_0|^2$, and taking into account Eqs. (10) and (11), the transmission coefficient reads

$$\mathcal{T} = \frac{|\psi_N|^2}{|\psi_0|^2} \sim \frac{H_N}{H_0} \geqslant \frac{1}{1 + bL}, \tag{12}$$

where $AH_0 = b\Delta x$ and $L = N\Delta x$. Therefore, the transmission $\mathcal{T}$ cannot tend toward 0 faster than L^{-1} as L tends to ∞. This behavior is also supported by numerical investigations of the scattering problem in Eqs. (5) and (6), reported in Ref. [17].

IV. FOKKER-PLANCK EQUATION

In this section, we consider 1D localization of stationary solutions of the RNLSE, (3), in a random δ-correlated potential $\eta(x)$ with a Gaussian distribution (white noise), of zero mean and variance $2D$: namely, $\langle \eta(x)\eta(x')\rangle = 2D\delta(x - x')$. Following Refs. [9] and [10], we study Anderson localization of stationary solutions with energies ω. In this case, the wave functions are real, $\psi(x) = \psi^*(x) = \phi(x)$.

We specifically calculate $\langle \phi^2(x)\rangle$ of solutions of Eq. (3) that are found for a certain ω, with given boundary conditions at some point, for example, $\phi(x = 0)$ and $\phi'(x = 0)$, where the prime means the derivative with respect to x. This is done with the help of the analogy with the Langevin equation [9,25,26]. In particular, we calculate the growth rate of the second moments $\langle \phi^2(x)\rangle$ and $\langle (\phi'(x))^2\rangle$. Therefore, considering coordinate x as the formal time on the half-axis with the definition $\sqrt{\omega}x \equiv \tau \in [0\,, \infty)$, Eq. (3) reduces to the classical Langevin equation

$$\ddot{\phi} + \phi = \beta_\omega \eta(\tau)\phi^3, \quad \beta_\omega = \frac{\beta}{\omega}. \tag{13}$$

Here $\eta(\tau)$ is considered the δ-correlated Gaussian noise[4] $\langle \eta(\tau)\eta(\tau')\rangle = \delta(\tau - \tau')$.

The dynamical process in the presence of the Gaussian δ-correlated noise is described by the distribution function $P = P(u, v, \tau) = \langle \delta(\phi(\tau) - u)\delta(\dot{\phi}(\tau) - v)\rangle$ that satisfies the Fokker-Planck equation [9]

$$\partial_\tau P = -v\partial_u P + u\partial_v P + \beta_\omega^2 u^6 \partial_v^2 P. \tag{14}$$

The Fokker-Planck equation produces an infinite chain of equations for the averages

$$r_{m,n} = \langle u^m v^n\rangle = \int dv du u^m v^n P(u, v, \tau),$$

where $m + n = 2 + 4l$ with $l, m, n = 0, 1, 2, \ldots$. This yields the system of equations

$$\begin{aligned} \dot{r}_{2,0} &= 2r_{1,1}, \\ \dot{r}_{1,1} &= -r_{2,0} + r_{0,2}, \\ \dot{r}_{0,2} &= -2r_{1,1} + 2\beta_\omega^2 r_{6,0}, \\ \dot{r}_{6,0} &= 6r_{5,1}. \\ &\vdots \end{aligned} \tag{15}$$

In vector notation, it reads

$$\dot{\mathbf{R}} = \hat{M}\mathbf{R}\,. \tag{16}$$

Here $\hat{M}$ is a dynamical matrix, which is defined from Eq. (15) later in the text.

A. First order of the perturbation theory

The chain of equations in system (15) is truncated at the term $r_{6,0} = \langle u^6\rangle$, taking $\dot{r}_{6,0} = 0$, and $2\beta_\omega^2 r_{6,0} = h$ is a small constant value.[5]

It should be admitted that the rest of this infinite chain of linear equations, determined by matrix $\hat{M}$, contributes only to the term $r_{6,0}(\tau)$. However, in the perturbation approach, when $h = 0$, the material is transparent. Therefore, neglecting h terms in the rest of the matrix $\hat{M}$, we obtain it in the Jordan block form, where every block matrix $M[(3 + 4l) \times (3 + 4l)]$ has the same structure, determined by operator $-v\partial_u + u\partial_v$. Its integral curves are circles in the (u, v) phase space [27]. Correspondingly, every block matrix has imaginary eigenvalues, and $2\beta_\omega^2 r_{6,0} \leqslant h = \text{const}$.

Consequently, after the truncation, Eq. (16) becomes an inhomogeneous equation with $l = 0$, which reads

$$\dot{\mathbf{R}} = \hat{M}\mathbf{R} + h\mathbf{V}, \tag{17}$$

where

$$\mathbf{R} = \begin{pmatrix} r_{2,0} \\ r_{1,1} \\ r_{0,2} \end{pmatrix}, \quad \hat{M} = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{18}$$

We take the "initial" condition $\mathbf{R}_0^T = (0, 0, 1)$. Performing the Laplace transformation $\tilde{\mathbf{R}}(s) = \mathcal{L}[\mathbf{R}(\tau)](s)$, we arrive at the equation for the Laplace image

$$\tilde{\mathbf{R}}(s) = \frac{1}{s(s - \hat{M})}(\mathbf{R}_0 s + h\mathbf{V}). \tag{19}$$

The inverse Laplace transformation yields the solution

$$\mathbf{R}(\tau) = e^{\hat{M}\tau}\mathbf{R}_0 + h\frac{e^{\hat{M}\tau} - 1}{\hat{M}}\mathbf{V}. \tag{20}$$

[3] From Eq. (10), we have equality $H_{n+1} - H_n = -AH_n^2$. Denoting $z_n = AH_n$, we obtain the iteration $z_{n+1} = z_n(1 - z_n)$, which maps the unit interval $[0\,, 1]$ onto itself. There is a fixed point defined by $z^* = z^*(1 - z^*)$, and the iterations converge to $z^* = 0$ [24]. Therefore, the energy H_n decays to 0 at the maximal rate according Eq. (11) with the initial condition at the incident point at $x = 0$.

[4] In this notation, the correlator is scaled over the variance $2D$.

[5] In the numerical experiment [17], fluctuations of the refractive index $\beta \sim 10^{-3}$. However it cannot be neglected, since it is incorporated into the highest derivative in the Fokker-Planck equation, (14).

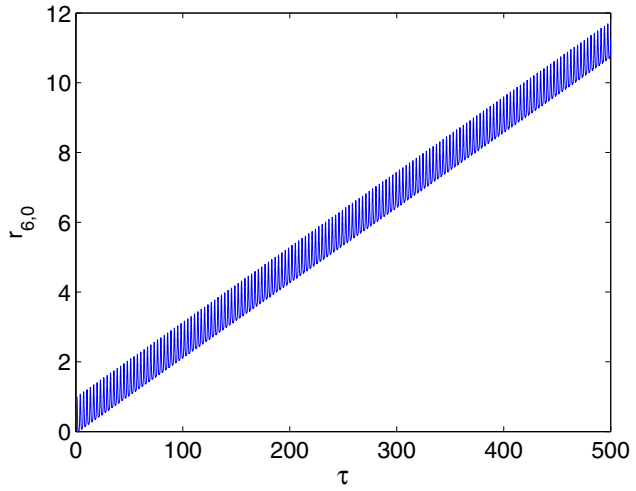

FIG. 1. Dynamics of the sixth moment $r_{6,0}$ vs time τ as a result of the numerical solution of Eq. (17) for $l = 1$ and $\beta_\omega^2 = 10^{-3}$ (MATLAB, ode45).

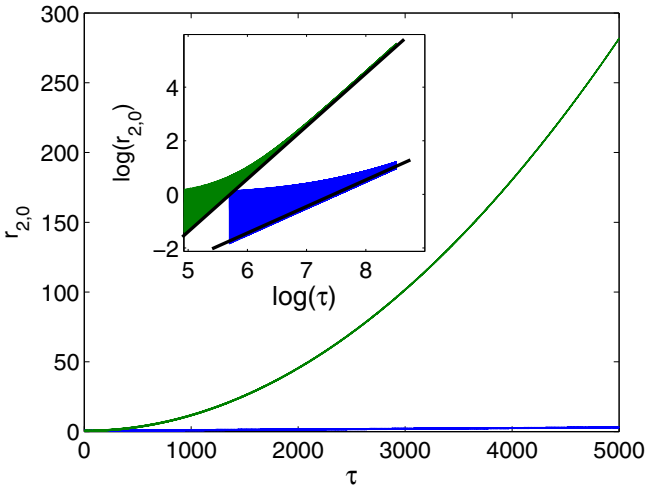

FIG. 2. Dynamics of the second moments obtained numerically for $\beta_\omega^2 = 10^{-3}$ (MATLAB, ode45). The upper curve corresponds to the second order of the perturbation and the lower curve corresponds to the first order of the perturbation.

Let us consider the series of the exponential. To this end we use the Cayley-Hamilton theorem.[6] The characteristic equation of matrix $\hat{M}$ is $\lambda^3 + 4\lambda = 0$. Therefore, $\hat{M}^3 = -4\hat{M}$, and correspondingly $\hat{M}^4 = -4\hat{M}^2$. These equalities determine the exponential in Eq. (20) as follows:

$$e^{\tau\hat{M}} = 1 + \frac{1}{4}\left[1 - \frac{1}{4}\cos(2\tau)\right]\hat{M}^2 + \frac{1}{2}\sin(2\tau)\hat{M}, \tag{21a}$$

$$\frac{e^{\tau\hat{M}} - 1}{\hat{M}} = \tau + \frac{1}{8}[2\tau - \sin(2\tau)]\hat{M}^2 + \frac{1}{4\tau}[1 - \cos(2\tau)]\hat{M}. \tag{21b}$$

Applying these expressions to the solution, (20), we eventually obtain that $r_{2,0}$ and $r_{0,2}$ increase linearly for large values of τ. Therefore the second moments of the wave function and its derivative grow linearly with the coordinate x:

$$\langle\phi^2(x)\rangle \sim 1 + \beta_\omega^2 \omega^{\frac{1}{2}} x. \tag{22}$$

An important part of the analysis is the perturbation approach, which is valid only when the nonlinear randomness/(random nonlinearity) is small. The situation becomes completely unclear for the strong nonlinearity. Therefore. we take into account all the orders of the perturbation theory.

B. Iteration procedure for higher orders of the perturbation theory

Now we consider Eq. (17) for the next Jordanian block with $l = 1$, which reads $\dot{\mathbf{R}}_1 = \hat{M}_1\mathbf{R}_1 + h\mathbf{V}_1$. It describes moments $r_{m,n}$ with $m + n = 6$. Therefore, $\mathbf{R}_1^T = (r_{6,0}, r_{6,1}, \ldots, r_{0,6})$ and $V_1^T = (0\,,0\,,2\,,6\,,12\,,20\,,30)$. We are interested in the dynamics of $r_{6,0}$, which also grows linearly with τ. Its numerical solution is shown in Fig. 1. Therefore, the second order of the perturbation, shown in Fig. 2, yields quadratic growth of the moment $r_{2,0}(\tau) \sim \tau^2$.

Since the structures of the Jordanian blocks are just the same for all $l = 0, 1, \ldots$, then each lth block has its solution in the form of Eq. (20), which reads

$$\mathbf{R}_l(\tau) = e^{\hat{M}_l\tau}\mathbf{R}_0 + h\frac{e^{\hat{M}_l\tau} - 1}{\hat{M}_l}\mathbf{V}_l. \tag{23}$$

This solution corresponds to the linear growth of the moment $r_{2+4l,0}(\tau) \sim \beta_\omega^2\tau$. Therefore in the lth order of the perturbation theory, after l integrations, we obtain $r_{6,0}(\tau)$ in Eq. (17) as

$$h \sim \sum_{j=0}^{l} \frac{(\beta_\omega^2\tau)^j}{j!}, \tag{24}$$

where constants in the primitives at every integration are taken to be 1, to obtain the truncated series of an exponential function. Continuing this iteration procedure *ad infinitum*, we eventually obtain the second moment in the exponential form as follows:

$$\langle\phi^2(x)\rangle = r_{2,0}(\tau) \simeq e^{\beta_\omega^2\tau} = \exp(\beta^2\omega^{-3/2}x). \tag{25}$$

C. Anderson localization of the wave function

The rate of this exponential growth of the second moment is determined by the so-called generalized Lyapunov exponent [32]

$$2\gamma = \lim_{x\to\infty} \frac{\ln\langle\phi^2(x)\rangle}{x} = \frac{\omega^{3/2}}{\beta^2}. \tag{26}$$

This exponential growth is a strong indication of the exponential, Anderson, localization of the stationary states $\psi_\omega(x)$ with localization length $\xi = 1/\gamma$. Note that it is different from the usually studied self-averaging quantity

$$\gamma_s = \frac{1}{2}\frac{d}{dx}\langle\ln\phi^2(x)\rangle = \frac{1}{2}\lim_{x\to\infty}\frac{\ln\phi^2(x)}{x}, \tag{27}$$

[6] According to the Cayley-Hamilton theorem, every complex square matrix satisfies its own characteristic equation; see, for example, [28].

which determines a genuine localization length.[7] In the NLSEs (1) and (2), the Lyapunov exponent γ is independent of β and coincides with the linear limit [9]. However, here the limit with $\beta = 0$ does not exist, and as reported above, β cannot be a parameter for a perturbation consideration. We return to this value in Sec. V, while here, following [9] and [10], we explore *à la* Borland arguments: since the distribution of the random potentials is translationally invariant, it is independent of the choice of the initial point as $x = 0$. As in the linear case, starting from a specific initial condition, $\phi(x)$ will typically grow. For specific values of ω at some point this function will start to decay, so that a normalized eigenfunction is found. Borland's arguments [29,30] are rigorous for the linear case [31]. In heuristic form, applied to the nonlinear case, the envelope of the wave function will grow exponentially if we start either from the right or from the left. The value of ω results from the matching condition, so that an eigenfunction has some maximum and decays in both directions as required by the normalization condition. The exponential decay is an asymptotic property, while the matching is determined by the potential in the vicinity of the maximum. This observation is crucial for the validity of this approach and enables us to determine the exponential decay rate of states from the solution of the initial value problem in the form of both the NLSE, (1), and the RNLSE, (3).

V. NUMERICAL PERTURBATION THEORY

In this section we develop an analytical approach based on the numerical evaluation of a small parameter, which is necessary for the perturbation theory. We rewrite Eq. (3) in the form

$$-\hbar_{\rm eff}^2 \partial_x^2 \psi + \beta\eta(x)|\psi|^2\psi = \omega\psi, \tag{28}$$

where we have introduced a dimensionless Planck constant $\hbar_{\rm eff}$, which should be small enough for correct numerical evaluation of Anderson localization.

Parameter β cannot be 0: $\beta \neq 0$ since it does not fulfill the boundary condition $\psi(x = \pm\infty) = 0$ for the equation $\partial_x^2\psi + \omega\psi = 0$, if $\beta = 0$. Therefore a perturbation theory over β does not exist even for small $\beta \ll 1$. Therefore we keep $\beta = 1$.

Let us perform an identity transformation by adding and subtracting the linear random term $\beta\eta(x)\psi(x)$ in Eq. (28). We have

$$\begin{aligned}\hbar_{\rm eff}^2\partial_x^2\psi + \omega\psi - \beta\eta(x)\psi(x) &\equiv \omega\psi - \hat{H}_0\psi \\ &= -\beta\eta(x)\psi(x) + \beta\eta(x)\psi^3,\end{aligned} \tag{29}$$

where $\hat{H}_0$ is the Anderson Hamiltonian. Therefore, the left-hand side of Eq. (29) corresponds to the eigenvalue problem of Anderson localization

$$\hat{H}_0\varphi_n(x) = \big[-\hbar_{\rm eff}^2\partial_x^2 + \beta\eta(x)\big]\varphi_n(x) = \sigma_n\varphi_n(x), \tag{30}$$

where the eigenvalues are functions of β, namely, $\sigma_n = \sigma_n(\beta)$ and

$$\int_{-\infty}^{\infty} \varphi_m(x)\varphi_n(x)dx = \delta_{m,n}. \tag{31}$$

The Hamiltonian $\hat{H}_0$ is Hermitian, and $\{\varphi_n(x)\}$ is a complete set of orthogonal functions. Therefore, the stationary states can be expanded over the Anderson modes

$$\psi(x) \equiv \psi_\omega(x) = \sum_n a_n(\omega)\varphi_n(x). \tag{32}$$

Substituting expansion (32) into Eq. (29), we obtain

$$\begin{aligned}\omega\sum_n a_n\varphi_n(x) - \sum_n \sigma_n\varphi_n(x) &= -\beta\eta(x)\sum_n a_n\varphi_n(x) \\ &\quad + \beta\sum_{n_1,n_2,n_3} a_{n_1}a_{n_2}a_{n_3}\varphi_{n_1}(x)\varphi_{n_2}(x)\varphi_{n_3}(x),\end{aligned} \tag{33}$$

where $a_n \equiv a_n(\omega)$.

A. Algebraic equation and small parameter

Multiplying Eq. (33) by $\varphi_m(x)$ and integrating with respect to x, and taking into account Eq. (31), we arrive at the algebraic equation

$$a_m(\omega - \sigma_m) = -\beta\sum_n A_{m,n}a_n + \beta\sum_{n_1,n_2,n_3} B^m_{n_1,n_2,n_3}a_{n_1}a_{n_2}a_{n_3}, \tag{34}$$

where the overlapping integrals are

$$A_{m,n} = \int \eta(x)\varphi_m(x)\varphi_n(x)dx, \tag{35a}$$

$$B^m_{n_1,n_2,n_3} = \int \eta(x)\varphi_{n_1}(x)\varphi_m(x)\varphi_{n_2}(x)\varphi_{n_3}(x)dx. \tag{35b}$$

The overlapping integrals in Eqs. (35) are estimated numerically, and the results are presented in Figs. 3 and 4. These integrals are definitely less than 1, and we take them as the first order of the approximation by a small parameter[8] ε: $\varepsilon < A_{m,n} \sim B^m_{n_1,n_2,n_3} < 1$.

[7] In the linear case, with a Gaussian noise, these values are related [32], $\gamma = \gamma_s + a$, where a is due to the width of the Gaussian process.

[8] Some heuristic arguments for supporting this statement are in order. If the localization length is large, the segment of integration is large enough to apply the ergodic theorem for the Markov process, which yields $\int \eta(x)dx \approx \langle\eta(x)\rangle = 0$. Therefore, the integrals can be considered as small parameters ε. Moreover, the larger the segment of integration is, the smaller ε is, and we can account for only diagonals and a few nearest off-diagonals in matrices A and B. In the opposite case, when the localization length is small, then overlapping of Anderson modes is small, and only diagonal and nearest-neighbor matrix elements should be taken into account. The off-diagonal elements are small values, $\varepsilon \ll 1$, however, for the diagonal elements the integrals are just less than 1 and these terms contribute to the energy ω. Integrals $B^m_{n_1,n_2,n_3}$ are four-tensors, however, in the first order of ε, only terms with $n_1 = n_2 = n_3 = n$ are accounted for.

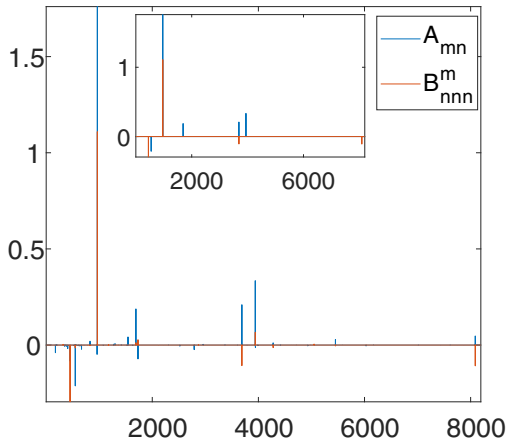

FIG. 3. Overlapping integrals $A_{m,n}$ and $B^m_{n,n,n}$ for the random mode $m = 965$ with $\sigma_m = -2.8822$. The Anderson modes are the result of numerical calculation of Eq. (30) in a chain with zero boundary condition, number of cites $N = 8192$, and $\hbar_{\rm eff} = 1.5915 \times 10^{-8}$. Inset: Overlapping integrals, which contribute to the spectrum for $n = m$ and stationary states $\psi_\omega(l)$ for $n \neq m$ in the first order of perturbation, when the absolute values of the overlapping integrals are larger then $\varepsilon = 0.1$.

B. Perturbation theory

After taking into account only nearest neighbors of the mth mode, algebraic equation (34) reads

$$a_m(\omega - \sigma_m) = -\beta A_{m,m} a_m + \beta B^m_{m,m,m} a_m^3 - \sum_n{}' \left[\beta A_{m,n} a_n - \beta B^m_{n,n,n} a_n^3\right], \quad (36)$$

where the prime means $n \neq m$. Taking into account that the overlapping integrals in Eqs. (35) are small, we cast the solution for a_n in the form of the decomposition/expansion over ε up to the first order of ε:

$$a_m = a_m^{(0)} + \varepsilon a_m^{(1)}, \quad m = 1, \ldots, N. \quad (37)$$

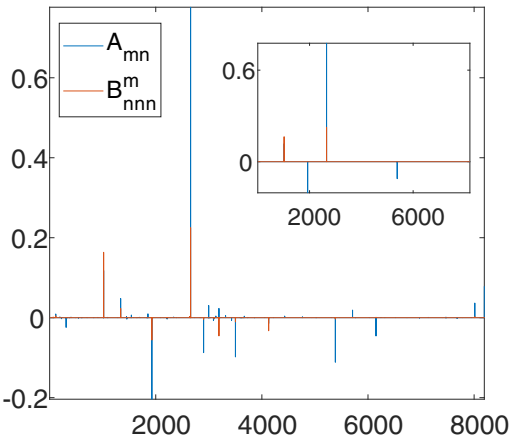

FIG. 4. The same as Fig. 3 for the random mode $m = 2657$ with $\sigma_m = -1.6727$.

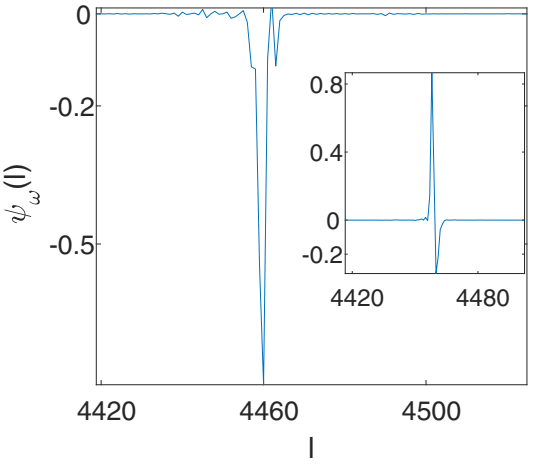

FIG. 5. The normalized stationary state $\psi_{\omega_m}(l)$ constructed near the Anderson mode (inset) $\varphi_m(l)$ for $m = 965$ with $\sigma_m = -2.8822$.

Substituting Eq. (37) into Eq. (36) and collecting terms with corresponding orders of ε, we have for zero order

$$a_m^{(0)}(\omega - \sigma_m) = 0, \quad (38)$$

which yields $a_m^{(0)}(\omega) = 1$ for $\omega = \sigma_m$ and $a_m^{(0)}(\omega) = 0$ for $\omega \neq \sigma_m$ and $m = 1, \ldots, N$. Formally, we define it as the Kronecker delta:

$$a_m^{(0)} \equiv a_m^{(0)}(\omega) = \delta(\omega - \sigma_m), \quad m = 1, 2, \ldots, N. \quad (39)$$

For the first order of ε, we have

$$a_m^{(1)}(\omega - \sigma_m) = -\beta \sum_n{}' \left[A_{m,n} a_n^{(0)} - \beta B^m_{n,n,n} \left(a_n^{(0)}\right)^3\right] \quad (40)$$

This equation has a solution only for $\omega = \sigma_n$ with $n \neq m$, which yields

$$a_m^{(1)}(\omega = \sigma_n) = \frac{\beta}{\sigma_m - \sigma_n}, \quad (41)$$

while the spectrum reads $\omega = \omega_n = \sigma_m - \beta(A_{m,m} - B^m_{m,m,m})$. Here we have used that $a_m^{(0)}(\omega = \sigma_n) = 0$.

Eventually, we obtain that the stationary states $\psi_\omega(x)$ are localized only for the discrete spectrum $\omega \in \text{spec}(\sigma_n + B^m_{n,n,n} - A_{m,n})$, $n = 1, 2, \ldots$. For example,

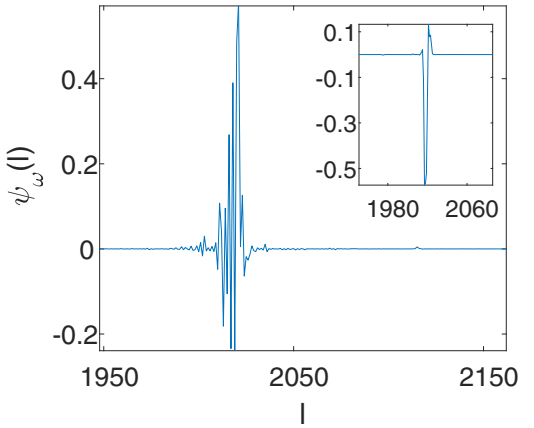

FIG. 6. The same as Fig. 5 for $m = 2657$ and $\sigma_m = -1.6727$.

for $\omega = \sigma_n$, the stationary state $\psi_{\sigma_n}(x)$ contains all a_n's with $\omega = \omega_n$, which are $a_n^{(0)}(\sigma_n)$ and $a_n^{(1)}(\sigma_n')$. Therefore, the stationary state reads

$$\psi_{\omega_n}(x) = \varphi_{\sigma_n}(x) + \beta \sum_{k \neq n} \frac{\varphi_{\sigma_k}(x)}{\sigma_n - \sigma_k}\left(A_{n,k} - B^n_{k,k,k}\right), \qquad (42)$$

with eigenenergy

$$\omega_n = \sigma_n - \beta\left(A_{n,n} - B^n_{n,n,n}\right). \qquad (43)$$

Results of numerical evaluation of two random stationary modes are presented in Figs. 5 and 6.

VI. CONCLUSION

We have considered two related problems and obtained two results. The first is the power-law decay of the transmission coefficient as a function of the slab length L. This result is obtained in both heuristic and microscopic (in the framework of the Helmholtz-RNLSE) approaches. We present these alternative ways to explain the experimental setup [17] of wave propagation in nonlinear media. The second result corresponds to the general microscopic consideration of Anderson localization in the RNLSE, (3). We construct a perturbation theory over the nonlinearity parameter. Constructing an iteration procedure, we are able to take into account all orders of the perturbation theory by means of a resummation procedure over all the orders of the small parameter β_ω^2. As the result of the developed method, we obtain exponential (Anderson) localization of the wave function $\phi(x) \sim e^{-x/\xi}$, estimating the mean squared second-moment growth. It should be stressed that this approach does not determine the localization length ξ, and we estimate it approximately according to the generalized Lyapunov exponent γ: $\xi = 1/\gamma \sim 1/2\sqrt{\omega}\beta_\omega^2 \gg 1$.[9]

We have also suggested numerical verification of Anderson localization of the stationary states $\psi_\omega(x)$ for discrete spectrum ω. To this end we introduce a linear counterpart of the RNLSE in the form of Eqs. (29) and (30). In this case a small parameter ε can be introduced in the form of overlapping integrals of the linear Anderson modes. This makes it possible to construct a perturbation theory which is well controlled numerically.[10]

In conclusion, we admit that the obtained results on the power-law and exponential decays correspond to two different approaches, which are so-called "fixed output" and "fixed input" [7]. The former, considered in Sec. III, leads to the power-law localization, while the latter, considered in Secs. IV and V, leads to the exponential (Anderson) localization. Therefore, there is an essential difference between Eq. (5) and Eq. (13). Namely, in the former case there is a finite constant probability current that corresponds to the fixed output conditions, while in the latter case, the probability current is 0, which is a necessary condition of localization.

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[9] We stress that it is not a genuine localization length, defined in Eq. (27).

[10] It is the first step of the numerical approach. For example, numerical study of the localization length as a function of the spectrum $\xi = \xi(\omega)$ is an important task, which should be considered separately.

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