Rapid Communications

Surprising variants of Cauchy's formula for mean chord length

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We examine isotropic and anisotropic random walks which begin on the surface of linear (N), square ($N \times N$), or cubic ($N \times N \times N$) lattices and end upon encountering the surface again. The mean length of walks is equal to N and the distribution of lengths n generally scales as $n^{-1.5}$ for large n. Our results are interesting in the context of an old formula due to Cauchy that the mean length of a chord through a convex body of volume V and surface S is proportional to V/S. It has been realized in recent years that Cauchy's formula holds surprisingly even if chords are replaced by irregular insect paths or trajectories of colliding gas molecules. The random walk on a lattice offers a simple and transparent understanding of this result in comparison to other formulations based on Boltzmann's transport equation in continuum.

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Cauchy (1789–1857) [1] derived a number of mathematically rigorous results with far reaching applications in physics. Cauchy's theorem for line integrals of holomorphic functions in the complex plane is a prime example. Another result which has found numerous applications in recent years concerns the mean length of a chord inside a d-dimensional spheroid body. Cauchy's formula states that the average chord length (over an ensemble of straight lines AB joining a randomly chosen pair of points A and B on the inside surface of the body) is proportional to the volume V of the body divided by its surface S. The constant of proportionality η_d depends on the dimension; $\eta_3 = 4$ for a sphere and $\eta_2 = \pi$ for a circle. The simplicity of this result is appealing although not too surprising because the volume and the surface are the only free parameters in the problem and V/S has the dimension of a length. What is surprising is that the result seems to hold even if the straight chord AB is replaced by a random zigzag trajectory of a gas molecule entering V at point A and leaving it at point B (first exit). Even more surprising is the apparent independence of the result from the details of collisions between molecules. The mean chord length plays a key role in several practical problems including neutron scattering with nuclei [2], stereology [3], image analysis [4], and understanding heterogeneous materials [5]. As may be expected, a result with random walks replacing chords would have a much greater applicability. It is observed that Cauchy's formula applies to some biological problems as well pertaining to insect behavior. The average distance traveled by an ant between its entry into a circular domain and the first exit from it is proportional to the radius of the circle [6]. These and other potential applications have inspired several theoretical studies in recent years in generalizing the original Cauchy's formula.

Extant studies assume that the trajectory of a particle between its first entry and exit from V comprises n line segments of lengths $\ell_1, \ell_2, \dots, \ell_n$ oriented randomly with respect to each other. The molecule moves at constant speed in continuum space and suffers $n \geqslant 0$ collisions with other molecules during its stay in V. Analysis based on Feynman's path integral [7] as well as simplified versions of the Boltzmann transport equation [8] leads to similar conclusions. It predicts the average length of the trajectory $\langle L \rangle = \langle \ell_1 + \ell_2 + \cdots + \ell_n \rangle = \eta_d R$, where R is the radius of the bounding d-dimensional sphere for isotropic random walks. If the average length of a segment ℓ_i between two successive collisions $\lambda = \langle \ell_i \rangle = \langle L \rangle / n$ exists in the limit $n \to \infty$, a mean-field-like solution predicts that segment lengths ℓ_i are distributed exponentially according to the probability $P(\ell_i) = \exp(-\ell_i/\lambda)/\lambda$. The result $\langle L \rangle = \eta_d R$ also holds for an arbitrary distribution of ℓ_i if a constraint is imposed between the distribution of the first step ℓ_1 that injects the walker inside V and the distribution of subsequent step lengths [8]. In the present Rapid Communication, we study the problem on an $N \times N$ square lattice. Here each step of the walk is of unit length and the total length of the walk is simply the number of steps n between entry into the lattice and first exit from it. Our main finding is that the key feature of Cauchy's formula holds for isotropic as well as anisotropic random walks on the lattice. The average length of the walk $\langle n \rangle$ scales linearly with N but surprisingly the distribution of walk lengths n in different realizations of the walk follows a power law. In the following we present numerical results as well as theoretical support for them. The results may be easily generalized for a d-dimensional hypercubic lattice for $d > 2$. The results extend Cauchy's formula on lattices but more interestingly provide a simple intuitive understanding of the same based on a one-dimensional random walk with two absorbers.

Figure 1 illustrates a computer generated isotropic random walk on a small 10×10 lattice. In this particular realization the walker takes a total of $n = 68$ steps through 27 lattice points. Starting at the entry point (6,1) on the edge of the lattice, she takes the first step to (6,2) and randomly walks on for a total of 68 steps until she reaches the boundary

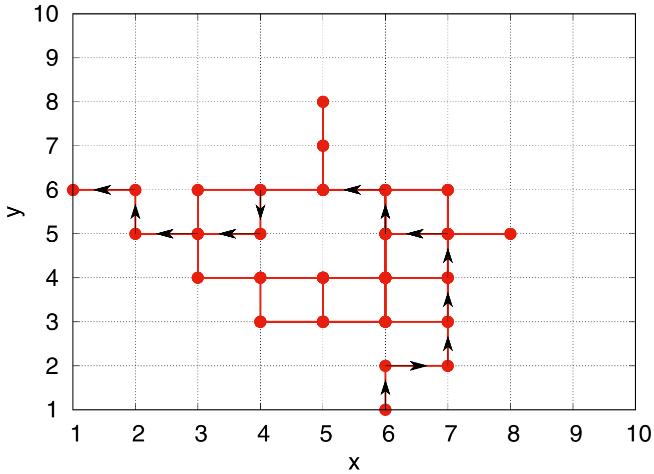

FIG. 1. An isotropic random walk on a 10×10 lattice that starts and ends on the boundary.

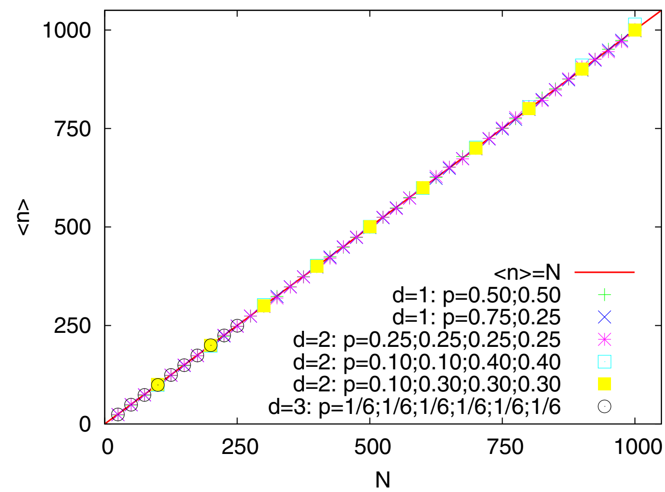

FIG. 2. Average length $\langle n \rangle$ of a random walk starting and ending on a surface of a d-dimensional hypercubic lattice of linear size N. The values of p are the probabilities of walking in $2d$ different directions. Thus the figure shows three isotropic and three anisotropic walks. The data for two anisotropic walks is rescaled (see text) so that all cases collapse on a line $\langle n \rangle = N$.

again at (1,6) and terminates the walk. The first and last steps are necessarily perpendicular to the boundary but other steps occur with equal probability in any of the four directions. The directions of the first 8 and the last 5 of the 68 steps are shown by arrows. The remaining steps are left unmarked because these are traversed back and forth several times making overlapping loops of various lengths. This is not an artifact of the small system size but rather a general feature of restricted (surface to surface) random walks. The walks tend to be loopy and localized in crowded neighborhoods which are separated from each other by longer and less loopy paths. On account of the number of loops that a walker may make in a localized region, the size of the localized region is not a true indicator of the length of walk through it. This feature endows the system with a scale invariant property. It gives rise to a power-law distribution for key quantities. In our simulations on hypercubic lattices of N^d sites we focus on two quantities: (i) $\langle n \rangle$, the mean length of the walk, and (ii) $P(n; \{p\}; N)$, the distribution of lengths. Here $\{p\}$ is a set of $2d$ probabilities for moving in different directions on the lattice. For example, an isotropic walk on a square lattice is denoted by $p = 0.25; 0.25; 0.25; 0.25$. It serves to distinguish between symmetric and asymmetric walks. Our main findings are as follows: (i) $\langle n \rangle$ scales linearly with N for symmetric and asymmetric walks, and (ii) $P(n; \{p\}; N)$ shows power laws for symmetric and a few asymmetric cases as well. The result for $\langle n \rangle$ is in conformity with the general idea of Cauchy's formula. The power-law distribution is in contrast to extant studies in continuum where a mean-field solution produces an exponential distribution.

Figure 1 depicts just a single walk for illustrative purposes. In simulations on larger systems presented below, each point on the boundary (except corners on square and cubic lattices) was assigned an equal probability of being chosen as the starting point of the walk. The direction of the first step was restricted to the nearest neighbor of the starting point lying inside the boundary. The corner sites were excluded because they did not have a nearest neighbor inside the boundary. No restriction was placed on the direction of the walk after the first step. Results were obtained by (a) averaging over a large number of randomly selected starting points on the boundary, and (b) averaging over all points on the boundary as starting points. As may be expected, results in the two cases are indistinguishable from each other on the scale of the figures. The numerical procedure outlined above is a natural realization on lattices of the two conditions for the validity of Cauchy's formula [8]: (i) the starting position should be uniformly distributed over the boundary, and (ii) the starting direction should satisfy isotropic incident flux condition.

Figure 2 shows the result for $\langle n \rangle$ on an N^d lattice for $10 \leqslant N \leqslant 10^3$, $d \leqslant 3$, and different cases of symmetric as well as asymmetric walks. We find $\langle n \rangle$ to be proportional to N in each case within numerical errors. The constant of proportionality is unity for symmetric walks but different from unity for asymmetric walks for reasons that are simple to understand and explained in the following. After scaling by appropriate weight factors, the data for all cases collapse on a single line as shown in Fig. 2. For a symmetric walk, the walker moves to any of its nearest neighbors with equal probability. In an asymmetric walk, the probabilities to go to different neighbors are different. On a square lattice, we consider two cases of asymmetry for exit from site (i, j). Case I: the probability to go to $(i-1, j)$ or $(i+1, j)$ is equal to 0.10 but the probability to go to $(i, j-1)$ or $(i, j+1)$ is equal to 0.40. Case II: the probability to go to $(i-1, j)$ is equal to 0.10, and to $(i+1, j)$, $(i, j-1)$, or $(i, j+1)$ is equal to 0.30 in each direction. In other words, the asymmetry in case I is between left-right (x axis) and up-down (y axis) steps. In case II there is an asymmetry along the x axis in addition to asymmetry between the two axes. The probabilities to go to different neighbors add up to unity ensuring the walker does move to a new site in each step. The proportionality of $\langle n \rangle$ to N and the constants of proportionality can be understood by focusing on the one-dimensional case. We shall return to it shortly after presenting the results for the distribution of walk lengths.

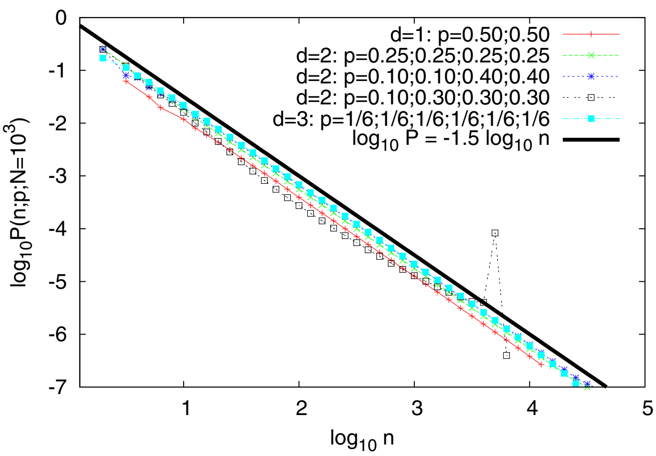

FIG. 3. Probability of a walk comprising n steps shows a power-law behavior except for one case shown in open black squares (see text).

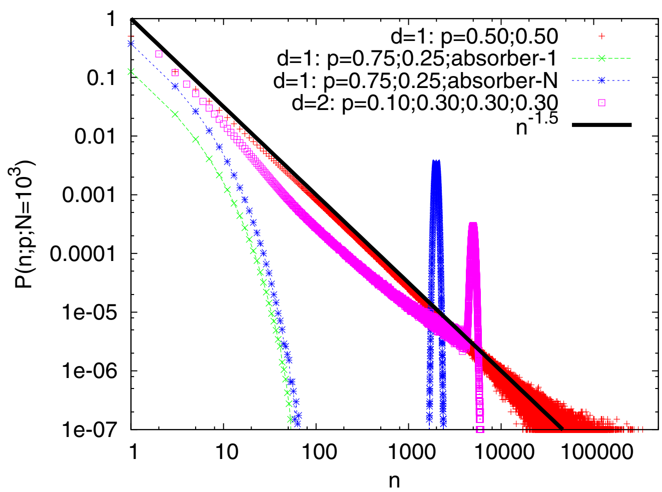

FIG. 4. Similar plot as in Fig. 3 but without logarithmic binning of data and for selected cases. In the $1d$ asymmetric case the probabilities of reaching the two absorbers are plotted separately. It explains the shape of the anisotropic walk in $2d$ (see text).

The distributions $P(n;\{p\};N)$ for fixed $\{p\}$ and different N are qualitatively similar. The range of n increases with increasing N but the scatter in the data reduces when larger number of realizations of the walk are used to calculate the distribution. We also note that $P(n;\{p\};N)$ remains finite for $n \gg N$. Thus even for $N \leqslant 10^3$, simulations generate huge data files, particularly so for symmetric walks. Figure 3 is drawn using reduced data based on logarithmic binning of the raw data into a reasonable number of bins that preserve the trends of the raw data. The raw data for Fig. 3 was obtained for $N = 10^3$ and more than 10^6 realizations of the walk. The important features of Fig. 3 are the following. For symmetric walks, $P(n;\{p\};N)$ shows a power-law decrease with increasing n over several decades. Within numerical errors, $P(n;\{p\};N) \approx n^{-1.5}$ for large n. For an asymmetric walk belonging to case I, the distribution of walk lengths is qualitatively similar to the one for the symmetric walk. However, it is different for the asymmetric case II. In this case we do not see a clear power law although the departure from the power law does not look very pronounced on the scale of Fig. 3. We can understand case II more clearly by referring to Fig. 4, which shows the results for symmetric and asymmetric walks on a $1d$ lattice of length N.

In one dimension, the problem can be solved analytically [9,10]. On a linear lattice $1, 2, \dots, m, \dots N-1, N$ with p the probability of moving towards N, $q = 1 - p$ the probability of moving towards 1, sites 1 and N being absorbers, the average number of steps S_m required to start from m and get absorbed at 1 is given by

$$S_m = \frac{m}{q-p} - \frac{N}{q-p}\left[\frac{1-r^m}{1-r^N}\right] \qquad (\text{if } p > q; r = q/p);$$
$$S_m = m(N-m) \qquad (\text{if } p = q = 1/2). \tag{1}$$

The probability that it takes exactly n steps to get absorbed into attractor 1 is given by

$$U(1,n) = \frac{1}{N} 2^n p^{(n-1)/2} q^{(n+1)/2} \sum_{\nu=1}^{N-1} \cos^{n-1}\frac{\nu\pi}{N} \sin\frac{\nu\pi}{N} \sin\frac{\nu\pi m}{N}. \tag{2}$$

The key points are that the average length of the walk scales linearly with N, and the probability that a symmetric walk is completed in n steps scales as $n^{-1.5}$ for $n \to \infty$. Simulations show that these features hold in two and three dimensions as well. The reason is as follows. Consider isotropic walks on a square lattice. A walker may start from a randomly selected site on an edge, say site $(1, m)$ on the top edge (first row). The first step brings her straight down to the second row at point $(2, m)$. Thereafter she can move in any direction with equal probability taking one step at a time until she hits one of the edges. We may imagine that each step is decided by two tosses of a coin, the first toss deciding if she would move along a column or a row, and the second deciding the direction. Now consider a modified walk where she decides to ignore the first toss and stays on the column m. She uses the second toss to move up or down on column m with equal probability. This modified walk is a one-dimensional walk with rescaled time. The average number of steps needed to hit either the top or the bottom edge is equal to $N - 1$, i.e., the average length of the walk scales linearly with N. We can also imagine a walk confined to the first row moving left or right with equal probability but skipping steps in the vertical direction. The average length of the walk before it hits the left or the right edge of the square is equal to $m(N - m)$, again scaling linearly with N. On a d-dimensional hypercubic lattice we may consider components of the walk along each dimension independently and so the qualitative features of the composite walk are the same as for each component. This explains why the average length of the walk scales linearly with N, and the distribution of lengths as $n^{-1.5}$ irrespective of the dimensionality of the lattice.

Similar reasoning may be used to understand asymmetric walks. Figure 4 shows the distribution of lengths for three cases: (i) $1d$ asymmetric; (ii) $1d$ symmetric (for comparison); and (iii) $2d$ asymmetric. The idea behind Fig. 4 is to use the $1d$ example to understand the scaling of data for asymmetric walks in Fig. 2 which make it collapse on a common curve for symmetric walks, and also to understand the shape of one $2d$ curve in Fig. 3 which does not follow a clear power law. At the risk of some repetition, the data for the $1d$ curves in Fig. 4 is obtained as follows. On the lattice $1, 2, \ldots, N-1, N$, start from site 2 or site $(N-1)$ with equal probability. Take a step towards N with probability p or towards site 1 with probability $q = 1 - p$. Count the number of steps n to reach site 1 or site N, whichever occurs first. Average the distribution of n over 10^7 realizations of the walk. For $p = 0.50$ both absorbers are reached equally frequently and the probability of reaching any one of them scales as $n^{-1.5}$ for large n. In higher dimensions as well, the power-law distribution is obtained whenever the two absorbers on opposite sides of a coordinate axis have equal probability of encounter.

For the $1d$ asymmetric case, the two absorbers are not reached with equal probability. Figure 4 shows that the probabilities of hitting absorber 1 or absorber N at the end of n steps are in the ratio 0.25:0.75. The probability of hitting N is larger because the walk is biased towards it. To calculate the average length $\langle n\rangle$ we have to use different weight factors for the two absorbers. This is relatively easy in $1d$ because there are only two absorbers but gets tedious in higher dimensions. A simplification is provided by the fact that the weight factors of different absorbers do not depend on the length n. Thus we may not count the frequency of different absorbers separately but simply rescale the cumulative number of instances where the walk hits an absorber after n steps. In the two cases of asymmetric walk $\{d = 2; p = 0.10; 0.10; 0.40; 0.40\}$ and $\{d = 2; p = 0.10; 0.30; 0.30; 0.30\}$, the cumulative frequencies need to be weighted by 0.89 and 1.20, respectively, to make $\langle n\rangle$ vs N data collapse on a single line as shown in Fig. 2.

The probability to hit the absorber N comprises two disjointed parts: a part that starts at 0.375 at $n = 1$ and decreases rapidly for higher odd values of n, and another part with a peak around $n \approx 2000$. The exponentially decreasing part corresponds to a walk starting at $N - 1$ and getting absorbed at N after $n = 1, 2, 3 \ldots$ steps. The simulation data agrees with the analytic solution mentioned above but the peak is a finite size effect. There are two parameters of interest, n and N. The analytic result applies to $n \leqslant N$. However, in numerical simulations we may have $n \gg N$, and a substantial number of walks terminate at N for $n \approx 2N$. This accounts for walks that start from site 2 and continue towards site N; for $p = 0.75$, $q = 0.25$, it takes nearly $2N$ steps to cover the distance to N. As the asymmetry decreases, the following effects set in: (i) rapidly decreasing branch tends to a slower power-law decrease and extends to higher n; (ii) the peak diminishes as well and may not remain disjoint from the other part of the curve.

A combination of the above effects explains the $2d$ asymmetric case $\{p = 0.10; 0.30; 0.30; 0.30\}$ in Fig. 4 if we consider the following points: (i) asymmetry is along the x axis, (ii) there is a peak at $n \approx 5000$, (iii) 10% of 5000 steps are taken towards column 1 and 30% towards column N; (iv) $5000 \times (0.30 - 0.10) = 1000 = N$; (v) thus a walk starting next to first column reaches the last column after 5000 steps and is terminated; (vi) $P(n; \{p\}; N)$ is rather close to a power law because the probability to move along three directions is each equal to 0.30 so there is a good deal of isotropy in the system.

In conclusion, we have studied restricted random walks on bounded d-dimensional hypercubic lattices of size N^d. The walks start and end on the surface of the lattice. The starting points are uniformly distributed over the surface, and the first step of each walk is perpendicular to the surface and directed inwards. These conditions are the lattice analog of uniform and isotropic incident flux associated with the Cauchy formula for random walks in continuum space bounded by an ellipsoid surface [8]. As in the continuum case, the average length of the walk is proportional to the volume divided by the surface of the lattice. The constant of proportionality depends on the geometry of the problem. On the hypercubic lattice the average length is equal to N, i.e., the closest distance between the face of the hypercube where the walk starts and the face opposite to it. Of course, not all walks end on the opposite face and even those that may end on it have a distribution of lengths. We find the lengths of the walk have a power-law distribution with the exponent $-3/2$ within numerical errors. The geometry of the hypercube and the condition of uniform and isotropic incident flux conspire to pair a shorter walk with a longer walk nearly in a detailed balance manner. This leads to an average value of the length equal to N as mentioned above. The lengths have a power-law distribution if the random walk is symmetric, i.e., if it has an equal probability of taking a step towards any one of the $2d$ faces of the cube. The power-law distribution also holds in cases when the walk is symmetric between each pair of opposite faces, but not necessarily symmetric between different pairs. The power law is lost if the symmetry between opposite faces is lost.

The results presented here may have a broader significance on two accounts. Firstly, random walks are used to model a large number of statistical physics problems. They are primary models of diffusion, which is a basic mechanism for the evolution of statistical systems. Diffusion is used not only to understand a system's evolution from a nonequilibrium to an equilibrium state but also to understand equilibrium and steady states. Thus different variants of random walks may be useful in understanding the richness of diffusion phenomena including anomalous diffusion [11]. Recent studies of diffusion with stochastic resetting [12] are also interesting in this context. Stochastic resetting effectively reduces the domain of an otherwise perennial random walk and has a marked effect on its likelihood to hit a specified trap and get terminated. The random walks studied here are confined to a bounded space and this too has a strong effect on their properties. It may be interesting to explore if there are any general principles governing the effect of a bounded domain on the properties of random walks inside it. However, this is somewhat outside the scope of the present study.

Secondly, and not entirely unrelated to the point made above, our study highlights the effect of the surface on the dynamics of the system bounded by it. Often statistical models in the thermodynamic limit ignore surface effects and focus on the deep interior of the volume. This also includes several boundary value problems outside the realm of statistical physics. The role of surface is merely to fix the boundary conditions for the differential equations to be solved. The possibility of some physical quantities depending only on the ratio of volume to surface but independent of the dynamics inside volume may offer new insights and therefore needs further exploration. Even in cases where surfaces play a more direct role in the theoretical understanding, Cauchy's formula may provide a different viewpoint. For example, some reflection shows that it can be used to obtain an alternate understanding of gas laws without the stringent assumptions of an ideal gas. We hope this Rapid Communication is a small step in this direction.

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