

Multiparameter universality and directional nonuniversality of exact anisotropic critical correlation functions of the two-dimensional Ising universality class

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We prove the validity of multiparameter universality for the exact critical bulk correlation functions of the anisotropic square-lattice and triangular-lattice Ising models on the basis of the exact scaling structure of the correlation function of the two-dimensional anisotropic scalar φ^4 model with four nonuniversal parameters. The correlation functions exhibit a directional nonuniversality due to principal axes whose orientation depends on microscopic details. We determine the exact anisotropy matrices governing the bulk and finite-size critical behavior of the φ^4 and Ising models. We also prove the validity of multiparameter universality for an exact critical bulk amplitude relation.

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The concept of bulk universality classes plays a fundamental role in the theory of critical phenomena [1–3]. They are characterized by the spatial dimension d and the symmetry of the ordered state which we assume here to be $O(n)$ symmetric with an n -component order parameter. Within a given (d, n) universality class, critical exponents and bulk scaling functions are independent of microscopic details, such as the couplings of (short-range) interactions or the lattice structure. It was asserted that, once the universal quantities of a universality class are given, two-scale-factor universality [2–5] implies that the asymptotic [small $t = (T - T_c)/T_c$] critical behavior of any particular system of this universality class is known completely provided that only two nonuniversal amplitudes are specified. It has been shown [6–9], however, that it is necessary to distinguish subclasses of isotropic and weakly anisotropic systems within a universality class and that two-scale-factor universality is not valid for the subclass of weakly anisotropic systems. In the latter systems there exists no unique bulk correlation-length amplitude but rather d independent nonuniversal amplitudes in the d principal directions. This has a significant effect on the anisotropic bulk order-parameter correlation function $G(\mathbf{x}, t)$ but a clear classification of its universality properties was not developed [8].

Recently [10] the notion of *multiparameter universality*, originally introduced for critical amplitude relations [8], was formulated for the scaling structure of $G(\mathbf{x}, t)$ within the anisotropic φ^4 theory where $G(\mathbf{x}, t)$ depends on up to $d(d + 1)/2 + 1$ independent nonuniversal parameters in d dimensions, i.e., up to four or seven parameters in two or three dimensions, respectively. It was hypothesized that multiparameter universality of $G(\mathbf{x}, t)$ is valid not only for “soft-spin” φ^4 models but also for all weakly anisotropic systems within a given universality class including fixed-length spin models such as Ising ($n = 1$), XY ($n = 2$), and Heisenberg ($n = 3$) models. No general proof was given for this hypothesis, except for an analytic verification for a special example within the two-dimensional anisotropic Ising model at $T = T_c$ [11].

A unique opportunity for a significant test of the validity of multiparameter universality is provided by an analysis of the exact results for the bulk correlation function of the anisotropic “square-lattice” and “triangular-lattice” Ising models [12,13] in the asymptotic scaling region near T_c . Such an analysis is made possible by deriving the exact scaling structure of $G(\mathbf{x}, t)$ of the general anisotropic two-dimensional scalar φ^4 lattice model which belongs to the same universality class as the $d = 2$ Ising model [14]. We introduce angular-dependent correlation lengths which permits us to determine the principal axes via an extremum criterion and to derive the exact anisotropy matrices. This leads to a proof of multiparameter universality for the Ising models with three or four nonuniversal parameters for the square-lattice or triangular-lattice model, respectively. However, the correlation functions exhibit a directional nonuniversality due to the principal axes whose orientation depends on microscopic details. This dependence is different for Ising and φ^4 models. Our results are expected to make an impact on scaling theories for $G(\mathbf{x}, t)$ of real anisotropic systems such as magnetic materials [15], superconductors [16], alloys [17], and solids with structural phase transitions [18], where angular-dependent correlation functions are measurable quantities. Multiparameter universality is relevant also for finite-size effects, e.g., the critical Casimir force [10].

It is necessary to first reformulate the standard scaling form of $G_{\pm}^{\text{iso}}(\mathbf{x}, t)$ for isotropic systems. In the limit of large $|\mathbf{x}|$ and large $\xi_{\pm}^{\text{iso}}(t) = \xi_{0\pm}^{\text{iso}}|t|^{-\nu}$ at fixed $|\mathbf{x}|/\xi_{\pm}^{\text{iso}} \geq 0$ the scaling form for $n = 1$ and $2 \leq d < 4$ reads [3,19]

$$G_{\pm}^{\text{iso}}(\mathbf{x}, t) = D_1^{\text{iso}} |\mathbf{x}|^{-d+2-\eta} \Phi_{\pm}(|\mathbf{x}|/\xi_{\pm}^{\text{iso}}), \quad (1)$$

with the universal scaling function $\Phi_{\pm}(y)$ above (+) and below (−) T_c and the nonuniversal amplitudes $D_1^{\text{iso}}, \xi_{0\pm}^{\text{iso}}$, where $\xi_{0+}^{\text{iso}}/\xi_{0-}^{\text{iso}}$ is universal but D_1^{iso} still contains a universal part. We employ the “true” (exponential) correlation lengths ξ_{\pm}^{iso} which are defined by the exponential decay $\sim \exp(-|\mathbf{x}|/\xi_{\pm}^{\text{iso}})$ of Φ_{\pm} for large $|\mathbf{x}|/\xi_{\pm}^{\text{iso}}$ and which are universally related to

the second-moment correlation lengths [3]. The exact sum rule [8] $\chi_{\pm}^{\text{iso}}(t) = \int d^d \mathbf{x} G_{\pm}^{\text{iso}}(\mathbf{x}, t)$ yields the susceptibility

$$\chi_{\pm}^{\text{iso}}(t) = D_1^{\text{iso}} [\xi_{\pm}^{\text{iso}}(t)]^{2-\eta} \tilde{\Phi}_{\pm} = \Gamma_{\pm}^{\text{iso}} |t|^{-\gamma}, \quad (2)$$

$$\tilde{\Phi}_{\pm} = 2\pi^{d/2} \Gamma(d/2)^{-1} \int_0^{\infty} ds s^{1-\eta} \Phi_{\pm}(s), \quad (3)$$

with $2 - \eta = \gamma/\nu$ and the universal quantities $\tilde{\Phi}_+$, $\tilde{\Phi}_-$, and $\Gamma_+^{\text{iso}}/\Gamma_-^{\text{iso}}$. This implies $D_1^{\text{iso}} = \Gamma_+^{\text{iso}} (\xi_{0+}^{\text{iso}})^{-2+\eta}/\tilde{\Phi}_+$, thus G_{\pm}^{iso} can be uniquely divided into universal and nonuniversal parts

$$G_{\pm}^{\text{iso}}(\mathbf{x}, t) = \Gamma_{\pm}^{\text{iso}} (\xi_{0\pm}^{\text{iso}})^{-2+\eta} |\mathbf{x}|^{-d+2-\eta} \Psi_{\pm}(|\mathbf{x}|/\xi_{\pm}^{\text{iso}}), \quad (4)$$

$$\Psi_+(y) = \Phi_+(y)/\tilde{\Phi}_+, \quad \Psi_-(y) = \Phi_-(y)/\tilde{\Phi}_+, \quad (5)$$

with two nonuniversal amplitudes Γ_+^{iso} and ξ_{0+}^{iso} and the universal scaling function $\Psi_{\pm}(y)$. At T_c it is related to the universal constants [2,8,20] \tilde{Q}_3 and Q_3 by

$$\Psi_+(0) = \Psi_-(0) = \tilde{Q}_3 = \frac{2^{d-2+\eta} \Gamma[(d-2+\eta)/2]}{(4\pi)^{d/2} \Gamma[(2-\eta)/2]} Q_3. \quad (6)$$

For $d = 2$, Γ_+^{iso} and ξ_{0+}^{iso} are related to the amplitude B^{iso} of the order parameter $\mathcal{M}^{\text{iso}} = B^{\text{iso}} |t|^{1/8}$ and to the specific-heat amplitude A^{iso} through [3]

$$(B^{\text{iso}})^2 (\Gamma_+^{\text{iso}})^{-1} (\xi_{0+}^{\text{iso}})^2 = Q_c \quad (7)$$

and $A^{\text{iso}} (\xi_{0+}^{\text{iso}})^2 = (R_{\xi}^+)^2$ where Q_c and $R_{\xi}^+ = (2\pi)^{-1/2}$ are universal constants according to Eqs. (2.50), (3.49), and (6.31) of [2]. We present the exact value of Q_c in Eq. (76) below. Our only assumption is the validity of two-scale-factor universality for isotropic systems which implies that Ψ_{\pm} , Q_c , R_{ξ}^+ are the same for isotropic Ising and φ^4 models with $\gamma = 2 - \eta = 7/4$, $\nu = 1$, and $\xi_{0+}^{\text{iso}}/\xi_{0-}^{\text{iso}} = 2$.

We first consider the anisotropic scalar φ^4 model on \tilde{N} lattice points $\mathbf{x}_i \equiv (x_{i1}, x_{i2})$ of a square lattice with lattice spacing \tilde{a} and finite-range interactions $K_{i,j}$. The Hamiltonian divided by $k_B T$ and the bulk correlation function are defined by [8]

$$H = \tilde{a}^2 \left[\sum_{i=1}^{\tilde{N}} \left(\frac{r_0}{2} \varphi_i^2 + u_0 \varphi_i^4 \right) + \sum_{i,j=1}^{\tilde{N}} \frac{K_{i,j}}{2} (\varphi_i - \varphi_j)^2 \right], \quad (8)$$

$$G_{\pm}(\mathbf{x}_i - \mathbf{x}_j, t) = \lim_{\tilde{N} \rightarrow \infty} [\langle \varphi_i \varphi_j \rangle - \mathcal{M}^2], \quad (9)$$

where $\mathcal{M}^2 = \lim_{|\mathbf{x}_i - \mathbf{x}_j| \rightarrow \infty} \langle \varphi_i \varphi_j \rangle$. The large-distance anisotropy is described by the anisotropy matrix

$$\mathbf{A} = (A_{\alpha\beta}) = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad (10)$$

$$A_{\alpha\beta} = \lim_{\tilde{N} \rightarrow \infty} \tilde{N}^{-1} \sum_{i,j=1}^{\tilde{N}} (x_{i\alpha} - x_{j\alpha})(x_{i\beta} - x_{j\beta}) K_{i,j}, \quad (11)$$

where weak anisotropy requires $\det \mathbf{A} > 0$, $a > 0$, $b > 0$ which ensures unchanged critical exponents [6]. It has been shown recently [10] that $G_{\pm}(\mathbf{x}, t)$ has the asymptotic scaling form

$$G_{\pm}(\mathbf{x}, t) = \frac{\Gamma_+(\xi_{0+})^{-7/4}}{[\mathbf{x} \cdot \bar{\mathbf{A}}^{-1} \mathbf{x}]^{1/8}} \Psi_{\pm} \left(\frac{[\mathbf{x} \cdot \bar{\mathbf{A}}^{-1} \mathbf{x}]^{1/2}}{\bar{\xi}_{\pm}(t)} \right) \quad (12)$$

with $\bar{\mathbf{A}} = \mathbf{A}/(\det \mathbf{A})^{1/2}$ where Ψ_{\pm} is the same scaling function as that in Eq. (4) for isotropic systems ($\bar{\mathbf{A}} = \mathbf{1}$). We have obtained (12) from Eqs. (5.61) and (5.32) of [10] by employing the sum rule for the susceptibility of the anisotropic system $\chi_{\pm}(t) = \int d^2 \mathbf{x} G_{\pm}(\mathbf{x}, t) = \Gamma_{\pm} |t|^{-7/4}$ which yields the nonuniversal constant $D_1 = \Gamma_+(\xi_{0+})^{-7/4}/\tilde{\Phi}_+$ [10]. Here $\bar{\xi}_{\pm}(t)$ is the geometric mean

$$\bar{\xi}_{\pm}(t) = \bar{\xi}_{0\pm} |t|^{-1}, \quad \bar{\xi}_{0\pm} = [\xi_{0\pm}^{(1)} \xi_{0\pm}^{(2)}]^{1/2} \quad (13)$$

of the principal correlation lengths $\xi_{\pm}^{(\alpha)}(t) = \xi_{0\pm}^{(\alpha)} |t|^{-1}$ where the principal axes are defined by the eigenvectors $\mathbf{e}^{(\alpha)}$ determined by $\mathbf{A} \mathbf{e}^{(\alpha)} = \lambda_{\alpha} \mathbf{e}^{(\alpha)}$, $\alpha = 1, 2$. The eigenvalues $\lambda_{\alpha} > 0$ determine the amplitudes $\xi_{0\pm}^{(\alpha)} = \lambda_{\alpha}^{1/2} \xi'_{0\pm}$ with $\xi_{0+}^{(\alpha)}/\xi_{0-}^{(\alpha)} = \xi'_{0+}/\xi'_{0-} = 2$ where $\xi'_{0\pm}$ is the correlation length of the isotropic system obtained after a shear transformation that consists of a rotation and a rescaling in the $\mathbf{e}^{(\alpha)}$ directions [6–8,10]. The amplitudes $\xi_{0+}^{(\alpha)}$ are independent of the amplitude B of the order parameter $\mathcal{M} = B |t|^{1/8}$ of the anisotropic model. From the shear transformations [7,8,10] $(\xi'_{0+})^2 = (\det \mathbf{A})^{-1/2} (\xi_{0+})^2$, $(B')^2 = (\det \mathbf{A})^{1/2} B^2$, $A' = (\det \mathbf{A})^{1/2} A$, and $\Gamma'_+ = \Gamma_+$ we find the relations for the anisotropic system

$$B^2 \Gamma_+^{-1} (\xi_{0+})^2 = Q_c \quad (14)$$

and $A(\xi_{0+})^2 = (R_{\xi}^+)^2$ where Q_c and R_{ξ}^+ are the same as in the isotropic case. Thus the susceptibility amplitude Γ_+ is determined by *three* independent nonuniversal parameters $\xi_{0+}^{(1)}$, $\xi_{0+}^{(2)}$, B whereas the specific-heat amplitude A is determined by two parameters $\xi_{0+}^{(1)}$, $\xi_{0+}^{(2)}$, and B^2 can be expressed as $B^2 = A \Gamma_+ Q_c / (R_{\xi}^+)^2$. The individual lengths $\xi_{0+}^{(\alpha)}$ cannot be determined from A , B , and Γ_+ .

Contours of constant correlations are ellipses determined by $\mathbf{x} \cdot \bar{\mathbf{A}}^{-1} \mathbf{x} = \text{const}$ whose excentricity and orientation are characterized by

$$q = (\lambda_1/\lambda_2)^{1/2} = \xi_{0\pm}^{(1)}/\xi_{0\pm}^{(2)} \quad (15)$$

and by the angle Ω determining the principal axes, i.e.,

$$\mathbf{e}^{(1)} = \begin{pmatrix} \cos \Omega \\ \sin \Omega \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} -\sin \Omega \\ \cos \Omega \end{pmatrix}. \quad (16)$$

For $a \neq b$ we define

$$\lambda_1 = \frac{a+b}{2} + \frac{a-b}{2} w, \quad \lambda_2 = \frac{a+b}{2} - \frac{a-b}{2} w, \quad (17)$$

$$w = [1 + 4c^2/(a-b)^2]^{1/2} \geq 1, \quad (18)$$

$$\tan \Omega = [b - a + (a-b)w]/(2c), \quad (19)$$

and for $a = b$, $c \neq 0$

$$\lambda_1 = a + c, \quad \lambda_2 = a - c, \quad \Omega = \pi/4. \quad (20)$$

From $\bar{\mathbf{A}} = \mathbf{U}^{-1} \bar{\lambda} \mathbf{U}$ with the rotation and rescaling matrices $\mathbf{U} = \begin{pmatrix} c_{\Omega} & s_{\Omega} \\ -s_{\Omega} & c_{\Omega} \end{pmatrix}$ and $\bar{\lambda} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ we obtain

$$\bar{\mathbf{A}}(q, \Omega) = \begin{pmatrix} q c_{\Omega}^2 + q^{-1} s_{\Omega}^2 & (q - q^{-1}) c_{\Omega} s_{\Omega} \\ (q - q^{-1}) c_{\Omega} s_{\Omega} & q s_{\Omega}^2 + q^{-1} c_{\Omega}^2 \end{pmatrix} \quad (21)$$

with the abbreviations $c_{\Omega} \equiv \cos \Omega$, $s_{\Omega} \equiv \sin \Omega$. Using polar coordinates $\mathbf{x} = (x_1, x_2) = (r \cos \theta, r \sin \theta)$ (Fig. 1) we

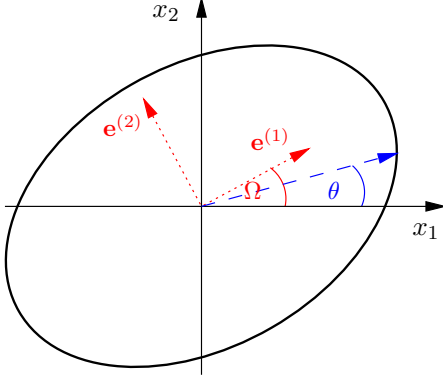


FIG. 1. Elliptical contour of constant correlations of $G_{\pm}(\mathbf{x}, t)$, (23)–(25). Dotted arrows $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$: principal directions for $a > b > 0, c > 0$ with $0 < \Omega < \pi/4$. Dashed arrow: vector \mathbf{x} in the direction θ .

define the angular-dependent correlation length $\xi_{\pm}(t, \theta, q, \Omega)$ by

$$[\mathbf{x} \cdot (\bar{\mathbf{A}}^{-1}\mathbf{x})]^{1/2}/\bar{\xi}_{\pm}(t) = r/\xi_{\pm}(t, \theta, q, \Omega) \quad (22)$$

which yields the exact reformulation of (12)

$$G_{\pm}(\mathbf{x}, t) = \frac{\Gamma_{+}(\bar{\xi}_{0+})^{-7/4}}{[rf(\theta, q, \Omega)]^{1/4}} \Psi_{\pm}\left(\frac{r}{\xi_{\pm}(t, \theta, q, \Omega)}\right) \quad (23)$$

where the directional dependence is described by

$$\xi_{\pm}(t, \theta, q, \Omega) = \bar{\xi}_{\pm}(t)/f(\theta, q, \Omega), \quad (24)$$

$$\begin{aligned} f(\theta, q, \Omega) &= [(q \sin^2 \Omega + q^{-1} \cos^2 \Omega) \cos^2 \theta \\ &\quad + (q \cos^2 \Omega + q^{-1} \sin^2 \Omega) \sin^2 \theta \\ &\quad + (q^{-1} - q) \cos \Omega \sin \Omega \sin(2\theta)]^{1/2} \\ &= [q \sin^2(\theta - \Omega) + q^{-1} \cos^2(\theta - \Omega)]^{1/2}. \end{aligned} \quad (25)$$

In the limit $c \rightarrow 0$ at fixed $a - b \neq 0$ a “rectangular” anisotropy is obtained with $\lambda_1 = a, \lambda_2 = b, q = (a/b)^{1/2} = \xi_{0\pm}^{(1)}/\xi_{0\pm}^{(2)}, \Omega = 0$, and

$$f(\theta, q, 0) \equiv f_{\text{rec}}(\theta, q) = (q^{-1} \cos^2 \theta + q \sin^2 \theta)^{1/2}. \quad (26)$$

For $q \neq 1$ the requirement $\partial \xi_{\pm}(t, \theta, q, \Omega)/\partial \theta = 0$ yields $\sin[2(\theta - \Omega)] = 0$ implying that ξ_{\pm} has extrema at $\theta^{(1)} = \Omega$ and $\theta^{(2)} = \Omega + \pi/2$ defining the two principal directions.

In contrast to $G_{\pm}^{\text{iso}}(\mathbf{x}, t)$, $G_{\pm}(\mathbf{x}, t)$ depends on four independent nonuniversal parameters $\Gamma_{+}, \bar{\xi}_{0+}, q, \Omega$ which violates two-scale-factor universality. Unlike q , the angle $\Omega(a, b, c)$ cannot be parametrized in terms of $\xi_{0\pm}^{(\alpha)}$ but depends on the lattice structure and the microscopic couplings $K_{i,j}$ through a, b, c . Thus $\bar{\mathbf{A}}$ depends not only on bulk correlation lengths through q but also on other microscopic details through Ω . The parametrization of (15)–(26) is valid in the unrestricted range $0 < q < \infty$ above, at, and below T_c [21]. The same matrix $\bar{\mathbf{A}}$ also enters the finite-size critical behavior [10]. These results derived for a φ^4 model on a square lattice remain valid more generally for a φ^4 model with couplings $K_{i,j}$ on two-dimensional Bravais lattices [8].

The hypothesis of multiparameter universality [10] predicts that the critical correlation functions of all anisotropic

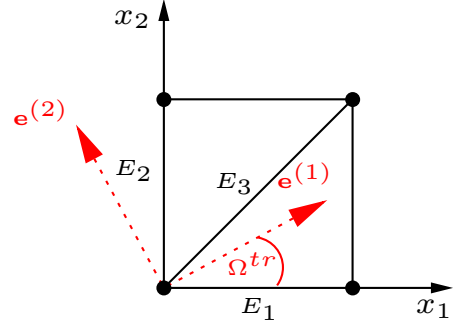


FIG. 2. Lattice points of the “triangular-lattice” Ising model (27) on a square lattice with couplings E_1, E_2, E_3 . Dotted arrows $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$: principal directions for $E_1 > E_2 > 0, E_3 > 0$ with $0 < \Omega^{\text{tr}} < \pi/4$.

Ising models with short-range interactions can be expressed in the same form as (23)–(25) with the same universal functions Ψ_{\pm} and f and the same critical exponents, but with up to four different nonuniversal parameters. We shall show that this is indeed valid for Ising models with the Hamiltonian [12,13,22]

$$H^{\text{Is}} = \sum_{j,k} [-E_1 \sigma_{j,k} \sigma_{j,k+1} - E_2 \sigma_{j,k} \sigma_{j+1,k} - E_3 \sigma_{j,k} \sigma_{j+1,k+1}], \quad (27)$$

where $\sigma_{j,k} = \pm 1$ are spin variables on a square lattice (with the lattice spacing $\tilde{a} = 1$) with horizontal, vertical, and diagonal couplings $E_1 > 0, E_2 > 0, E_3$ (Fig. 2). The exact correlation function $\langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm}$ at vanishing external field was calculated for $E_3 = 0$ in Ref. [12] and for positive and negative $E_3 \neq 0$ in Ref. [13], resulting in the scaling form

$$\langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm} = R^{-1/4} \mathcal{F}_{\pm}(R/\xi_{\pm}^{\text{Is}}, E_1, E_2, E_3) \quad (28)$$

with a nonuniversal scaling function \mathcal{F}_{\pm} , a distance $R(E_1, E_2, E_3)$, and a correlation length $\xi_{\pm}^{\text{Is}} = \xi_{0\pm}^{\text{Is}}(E_1, E_2, E_3)|t|^{-1}$ with $\xi_{+}^{\text{Is}}/\xi_{-}^{\text{Is}} = 2$. The exact amplitude $C_{0\pm}(E_1, E_2, E_3)$ of the susceptibility was also calculated. So far the universality properties of (28) have not been analyzed in the literature [2,3,12,13,23], and the universal part of the function \mathcal{F}_{\pm} has not been identified. In particular, the principal axes and principal correlation lengths of the triangular-lattice model ($E_3 \neq 0$) are as yet unknown, and only a conjecture exists for the correlation lengths in the direction of the bonds [24]. For comparison with (9) below T_c we need to consider the *subtracted* correlation function

$$\langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm}^{\text{sub}} = \langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm} - (\mathcal{M}^{\text{Is}})^2, \quad (29)$$

where $\mathcal{M}^{\text{Is}}(E_1, E_2, E_3)$ is the spontaneous magnetization, with $\mathcal{M}^{\text{Is}} = 0$ for $T \geq T_c$. We shall analyze three cases.

We start from the isotropic case $E_1 = E_2 = E > 0, E_3 = 0$, where $R(E, E, 0) = (M^2 + N^2)^{1/2} \equiv R$, and [12]

$$\xi_{0+}^{\text{Is}}(E, E, 0) = (4\beta_c E)^{-1} = [2 \ln(1 + 2^{1/2})]^{-1} \equiv \xi_{0+}^{\text{Is}}, \quad (30)$$

$$\mathcal{M}^{\text{Is}}(E, E, 0) = 2^{5/16} [\ln(1 + 2^{1/2})]^{1/8} |t|^{1/8} \equiv B^{\text{Is}} |t|^{1/8}, \quad (31)$$

$$C_{0+}(E, E, 0) = 2^{19/8} \pi [\xi_{0+}^{\text{Is}}(E, E, 0)]^{7/4} p_+ \equiv C_{0+}^{\text{Is}}, \quad (32)$$

$$p_+ = C_{0+}(E, E, 0)/D = 0.159\,284\,695\,8\dots, \quad (33)$$

with $\beta_c = (k_B T_c)^{-1}$, $\sinh 2\beta_c E = 1$. The constant p_+ is expressed analytically in terms of a Painlevé function of the third kind and its numerical value follows from Eq. (2.52S) of [12]. According to (4) we reformulate (29) as

$$\langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm}^{\text{sub}} = \frac{C_{0+}^{\text{Is}} (\xi_{0+}^{\text{Is}})^{-7/4}}{R^{1/4}} \Psi_{\pm}(R/\xi_{\pm}^{\text{Is}}), \quad (34)$$

$$\Psi_+(y_+) = (2^{19/8} \pi p_+)^{-1} \tilde{F}_+(y_+), \quad (35)$$

$$\Psi_-(y_-) = (2^{19/8} \pi p_+)^{-1} [\tilde{F}_-(y_-/2) - 2^{3/8} (y_-/2)^{1/4}], \quad (36)$$

with $y_{\pm} = R/\xi_{\pm}^{\text{Is}}$, $\xi_{\pm}^{\text{Is}} \equiv \xi_{0\pm}^{\text{Is}}(E, E, 0)|t|^{-1}$ where the functions $\tilde{F}_+(y_+)$ and $\tilde{F}_-(y_-/2)$ are given by the right-hand side of Eq. (2.39) of [12] with $\sinh 2\beta_c E_1 + \sinh 2\beta_c E_2$ replaced by 2 and with the argument θ replaced by $y_+/2$ or $y_-/4$, respectively. The unsubtracted correlation function [12] $\langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm}$ is easily obtained by dropping the last term in Eq. (36) which comes from $-(\mathcal{M}^{\text{Is}})^2$ in Eq. (29). According to two-scale-factor universality the functions Ψ_{\pm} identify the exact universal scaling functions above and below T_c of all systems in the subclass of *isotropic* systems in the $(d=2, \eta=1)$ universality class. In particular, $\Psi_+(0) = \Psi_-(0) = \tilde{Q}_3$ is a universal amplitude. Its exact value is $\tilde{Q}_3 = 0.270\,969\dots$ where we have used Eq. (5.10S) of [12]. This implies $Q_3 = 0.414\,131\dots$

Now we turn to the case of a “rectangular” anisotropy $E_1 \neq E_2, E_3 = 0$ where the condition of criticality is $S_1 S_2 = 1$ with $S_{\alpha} = \sinh 2\beta_c^{\text{rec}} E_{\alpha}$, $\alpha = 1, 2$ [12]. Using $x_1 = N = r \cos \theta$, $x_2 = M = r \sin \theta$ we derive from Eqs. (2.6), (2.8), (2.10), and (2.44) of [12]

$$R(E_1, E_2, 0) = [q^{\text{rec}} M^2 + (q^{\text{rec}})^{-1} N^2]^{1/2} \quad (37)$$

$$= r f_{\text{rec}}(\theta, q^{\text{rec}}), \quad (38)$$

$$q^{\text{rec}} = \xi_{0\pm}^{(1)\text{rec}} / \xi_{0\pm}^{(2)\text{rec}} = (S_1/S_2)^{1/2}, \quad (39)$$

$$\xi_{0+}^{(\alpha)\text{rec}} = S_{\alpha}^{1/2} [2\beta_c^{\text{rec}} E_1/S_1^{1/2} + 2\beta_c^{\text{rec}} E_2/S_2^{1/2}]^{-1} \quad (40)$$

$$= 2\xi_{0-}^{(\alpha)\text{rec}}, \quad (41)$$

$$\mathcal{M}^{\text{Is}}(E_1, E_2, 0) = [q^{\text{rec}} + (q^{\text{rec}})^{-1}]^{1/16} [\bar{\xi}_{-}^{\text{rec}}(t)]^{-1/8} \quad (42)$$

$$\equiv B^{\text{rec}} |t|^{1/8}, \quad (43)$$

$$\bar{\xi}_{\pm}^{\text{rec}}(t) = \bar{\xi}_{0\pm}^{\text{rec}} |t|^{-1}, \quad \bar{\xi}_{0\pm}^{\text{rec}} = [\xi_{0\pm}^{(1)\text{rec}} \xi_{0\pm}^{(2)\text{rec}}]^{1/2}, \quad (44)$$

where $\xi_{0\pm}^{(1)\text{rec}}$ and $\xi_{0\pm}^{(2)\text{rec}}$ are the correlation-length amplitudes in the principal directions (1,0) and (0,1), respectively, corresponding to $\Omega^{\text{rec}} = 0$ and where $f_{\text{rec}}(\theta, q^{\text{rec}})$ is the same function as defined in Eq. (2.6), with q replaced by q^{rec} . From Eq. (2.8) of [12] we derive

$$\xi_{0+}^{\text{Is}}(E_1, E_2, 0) = [2\beta_c^{\text{rec}} (E_1 C_1 + E_2 C_2)]^{-1} \times (S_1 + S_2)^{1/2} = \bar{\xi}_{0+}^{\text{rec}} \quad (45)$$

$$= 2\xi_{0-}^{\text{Is}}(E_1, E_2, 0) = 2\bar{\xi}_{0-}^{\text{rec}}, \quad (46)$$

where $C_{\alpha} = \coth 2\beta_c^{\text{rec}} E_{\alpha}$. This identifies “the correlation length ξ ” in Ref. [12] as $\bar{\xi}_{0+}^{\text{rec}}(t)$ above T_c^{rec} and $2\bar{\xi}_{0-}^{\text{rec}}(t)$ below T_c^{rec} as follows from Eq. (2.31) of [12]. From Eqs. (2.46a) and (2.48) of [12] we obtain the exact amplitude $C_{0+}^{\text{rec}} \equiv C_{0+}(E_1, E_2, 0)$ in the form

$$C_{0+}^{\text{rec}} = 2^{9/4} \pi p_+ (\bar{\xi}_{0+}^{\text{rec}})^{7/4} [q^{\text{rec}} + (q^{\text{rec}})^{-1}]^{1/8}, \quad (47)$$

and from Eq. (2.39) of [12] we obtain

$$\mathcal{F}_{\pm}(y, E_1, E_2, 0) = \{[q^{\text{rec}} + (q^{\text{rec}})^{-1}]/2\}^{1/8} \mathcal{F}_{\pm}(y, E, E, 0). \quad (48)$$

Together with (35) and (36) this leads to the exact reformulation of the asymptotic result of [12]

$$\langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm}^{\text{rec,sub}} = \frac{C_{0+}^{\text{rec}} (\bar{\xi}_{0+}^{\text{rec}})^{-7/4}}{[\mathbf{x} \cdot (\bar{\mathbf{A}}^{\text{rec}})^{-1} \mathbf{x}]^{1/2}} \Psi_{\pm} \left(\frac{[\mathbf{x} \cdot (\bar{\mathbf{A}}^{\text{rec}})^{-1} \mathbf{x}]^{1/2}}{\bar{\xi}_{\pm}^{\text{rec}}(t)} \right) \quad (49)$$

$$= \frac{C_{0+}^{\text{rec}} (\bar{\xi}_{0+}^{\text{rec}})^{-7/4}}{[r f(\theta, q^{\text{rec}}, 0)]^{1/4}} \Psi_{\pm} \left(\frac{r}{\xi_{\pm}^{\text{rec}}(t, \theta)} \right), \quad (50)$$

$$\xi_{\pm}^{\text{rec}}(t, \theta) = \bar{\xi}_{\pm}^{\text{rec}}(t)/f(\theta, q^{\text{rec}}, 0), \quad (51)$$

with the angular-dependent correlation lengths $\xi_{\pm}^{\text{rec}}(t, \theta)$ and with $\bar{\mathbf{A}}^{\text{rec}}(q^{\text{rec}}) = \bar{\mathbf{A}}(q^{\text{rec}}, 0)$, in exact agreement with (12) and (21)–(26) for $\Omega^{\text{rec}} = 0$, thus confirming multiparameter universality above, at, and below T_c^{rec} with three nonuniversal parameters $C_{0+}^{\text{rec}}, \bar{\xi}_{0+}^{\text{rec}}, q^{\text{rec}}$. This is valid for both $E_1 \geq E_2$ and $E_1 \leq E_2$ in the unrestricted range $0 < q^{\text{rec}} < \infty$. For $E_1 = E_2$ the isotropic results are recovered.

We proceed to the case of a “triangular” anisotropy [13,22] with $E_3 \neq 0$ and the condition of criticality $\hat{S}_1 \hat{S}_2 + \hat{S}_2 \hat{S}_3 + \hat{S}_3 \hat{S}_1 = 1$ with $\hat{S}_{\alpha} = \sinh 2\beta_c^{\text{tr}} E_{\alpha}$, $\alpha = 1, 2, 3$ [25]. We first determine the angle Ω^{tr} describing the orientation of the principal axes. For $T \approx T_c^{\text{tr}}$ the angular dependence of the distance $R(E_1, E_2, E_3) \equiv R_{\text{tr}}(\theta) = r f_{\text{tr}}(\theta)$ of Eq. (11) of [13] is given by

$$f_{\text{tr}}(\theta) = [(\hat{S}_1 + \hat{S}_3) \sin^2 \theta + (\hat{S}_2 + \hat{S}_3) \cos^2 \theta - \hat{S}_3 \sin 2\theta]^{1/2}. \quad (52)$$

We define the angular-dependent correlation length $\xi_{\pm}^{\text{tr}}(t, \theta)$ by rewriting the scaled variable of Eq. (10) of [13] for $T \approx T_c^{\text{tr}}$ as

$$R^{\text{tr}}(\theta)/\bar{\xi}_{\pm}^{\text{tr}}(t) = r/\xi_{\pm}^{\text{tr}}(t, \theta), \quad (53)$$

$$\xi_{\pm}^{\text{tr}}(t, \theta) = \bar{\xi}_{\pm}^{\text{tr}}(t)/f_{\text{tr}}(\theta). \quad (54)$$

The requirement $\partial \xi_{\pm}^{\text{tr}}(t, \theta)/\partial \theta = 0$ yields $\theta = \Omega^{\text{tr}}$ with

$$\tan 2\Omega^{\text{tr}} = \frac{2\hat{S}_3}{\hat{S}_1 - \hat{S}_2} = \frac{2(1 - \hat{S}_1 \hat{S}_2)}{\hat{S}_1^2 - \hat{S}_2^2} \quad \text{for } E_1 \neq E_2, \quad (55)$$

$$\Omega^{\text{tr}} = \pi/4 \quad \text{for } E_1 = E_2, \quad (56)$$

implying that $\xi_{\pm}^{\text{tr}}(t, \theta)$ has extrema at $\theta^{(1)} = \Omega^{\text{tr}}$ and $\theta^{(2)} = \Omega^{\text{tr}} + \pi/2$ defining the two principal directions. Clearly $\Omega^{\text{tr}}(E_1, E_2, E_3)$ is a nonuniversal quantity that depends on microscopic details and differs from Ω , (19), of the φ^4 model even if the φ^4 and Ising models have the same couplings on the same lattices. This is due to the nonuniversal difference

between a fixed-length spin model and a soft-spin model. From (52)–(56) we determine the ratio $q^{\text{tr}} = \xi_{0\pm}^{(1)\text{tr}}/\xi_{0\pm}^{(2)\text{tr}}$ of the amplitudes of the principal correlation lengths

$$\begin{aligned} q^{\text{tr}}(E_1, E_2, E_3) &= \frac{\xi_{\pm}^{\text{tr}}(t, \Omega^{\text{tr}})}{\xi_{\pm}^{\text{tr}}(t, \Omega^{\text{tr}} + \pi/2)} = \frac{f_{\text{tr}}(\Omega^{\text{tr}} + \pi/2)}{f_{\text{tr}}(\Omega^{\text{tr}})} \\ &= \frac{2 + \hat{S}_1^2 + \hat{S}_2^2 \pm [(\hat{S}_1^2 + \hat{S}_2^2)^2 + 4(1 - 2\hat{S}_1\hat{S}_2)]^{1/2}}{2(\hat{S}_1 + \hat{S}_2)}, \end{aligned} \quad (57)$$

where now the \pm sign in front of the square root term means $E_1 > E_2$ (+) and $E_1 < E_2$ (–), respectively, and

$$q^{\text{tr}}(E, E, E_3) = \frac{f_{\text{tr}}(3\pi/4)}{f_{\text{tr}}(\pi/4)} = \frac{1}{\sinh 2\beta_c^{\text{tr}} E} \quad (58)$$

for $E_1 = E_2 = E > 0$. From (55)–(58) we derive

$$q^{\text{tr}} \cos^2 \Omega^{\text{tr}} + q^{\text{tr}-1} \sin^2 \Omega^{\text{tr}} = \hat{S}_1 + \hat{S}_3, \quad (59)$$

$$q^{\text{tr}} \sin^2 \Omega^{\text{tr}} + q^{\text{tr}-1} \cos^2 \Omega^{\text{tr}} = \hat{S}_2 + \hat{S}_3, \quad (60)$$

$$(q^{\text{tr}-1} - q^{\text{tr}}) \cos \Omega^{\text{tr}} \sin \Omega^{\text{tr}} = -\hat{S}_3, \quad (61)$$

for both $E_1 \geq E_2$ and $E_1 \leq E_2$. Together with (52) these equations prove the validity of the identification $f_{\text{tr}}(\theta) = f(\theta, q^{\text{tr}}, \Omega^{\text{tr}})$ in the unrestricted range [26] $0 < q^{\text{tr}} < \infty$ above, at, and below T_c^{tr} where f is indeed the same function (25) as derived within the φ^4 theory. This completes the determination of the angular dependence of the anisotropy matrix $\bar{\mathbf{A}}^{\text{tr}} \equiv \bar{\mathbf{A}}(q^{\text{tr}}, \Omega^{\text{tr}})$ for the triangular-lattice Ising model (27) where $\bar{\mathbf{A}}$ is the same matrix as in Eq. (21) for the φ^4 model, in exact agreement with multiparameter universality. Our results for rectangular anisotropy are recovered from (52)–(61) in the limit $E_3 \rightarrow 0$.

We mention two earlier conjectures. (i) From (52) and (54) we derive

$$\kappa_1 = \frac{\xi_{0\pm}^{(\text{diag})\text{tr}}}{2^{1/2} \xi_{0\pm}^{(\text{hor})\text{tr}}} = \frac{f_{\text{tr}}(0)}{2^{1/2} f_{\text{tr}}(\pi/4)} = \frac{\cosh 2\beta_c^{\text{tr}} E_2}{\hat{S}_1 + \hat{S}_2}, \quad (62)$$

$$\kappa_2 = \frac{\xi_{0\pm}^{(\text{diag})\text{tr}}}{2^{1/2} \xi_{0\pm}^{(\text{vert})\text{tr}}} = \frac{f_{\text{tr}}(\pi/2)}{2^{1/2} f_{\text{tr}}(\pi/4)} = \frac{\cosh 2\beta_c^{\text{tr}} E_1}{\hat{S}_1 + \hat{S}_2}, \quad (63)$$

where $\xi_{0\pm}^{(\text{diag})\text{tr}}$, $\xi_{0\pm}^{(\text{hor})\text{tr}}$, and $\xi_{0\pm}^{(\text{vert})\text{tr}}$ denote the correlation lengths in the (1,1), (1,0), and (0,1) directions and the factor $2^{1/2}$ accounts for the diagonal lattice spacing. This confirms the conjecture in Eq. (2.6) of [24]. (ii) The ratio (58) used in Sec. V C of [10] was based on the conjecture in Eq. (A22) of [24] and is derived here directly from the exact result (52).

In the remaining analysis of the triangular case we confine ourselves to $E_1 = E_2 = E_3 = E > 0$ where

$$\Omega^{\text{tr}}(E, E, E) = \pi/4, \quad (64)$$

$$q^{\text{tr}}(E, E, E) = 1/\sinh 2\beta_c^{\text{tr}} E = 3^{1/2}, \quad (65)$$

$$f_{\text{tr}}(\theta) = f(\theta, 3^{1/2}, \pi/4) = 3^{-1/4}(2 - \sin 2\theta)^{1/2}. \quad (66)$$

By expanding Eqs. (2) and (10) of [13] around T_c^{tr} to leading order in $|t| = |T - T_c^{\text{tr}}|/T_c^{\text{tr}}$ we determine the magnetization,

the mean correlation lengths $\bar{\xi}_{\pm}^{\text{tr}}(t) = \bar{\xi}_{0\pm}^{\text{tr}}|t|^{-1}$, and the principal correlation lengths $\xi_{0+}^{(\alpha)\text{tr}}$ as

$$\mathcal{M}^{\text{Is}}(E, E, E) = (4 \ln 3)^{1/8} |t|^{1/8} \equiv B^{\text{tr}} |t|^{1/8}, \quad (67)$$

$$\bar{\xi}_{0+}^{\text{tr}} = [\xi_{0+}^{(1)\text{tr}} \xi_{0+}^{(2)\text{tr}}]^{1/2} = 3^{-3/4} 2^{1/2} / \ln 3 \quad (68)$$

$$= 3^{-1/4} \xi_{0+}^{(1)\text{tr}} = 3^{1/4} \xi_{0+}^{(2)\text{tr}} = 2\bar{\xi}_{0-}^{\text{tr}}. \quad (69)$$

From Eqs. (12) and (14)–(16) of [13] we obtain

$$C_{0+}(E, E, E) = 2^{21/8} 3^{-3/16} \pi p_+ (\bar{\xi}_{0+}^{\text{tr}})^{7/4} \equiv C_{0+}^{\text{tr}}, \quad (70)$$

$$\mathcal{F}_{\pm}(y, E, E, E) = 2^{1/4} 3^{-3/16} \mathcal{F}_{\pm}(y, E, E, 0). \quad (71)$$

Together with (35) and (36) this leads to the exact reformulation of the asymptotic result of [13]

$$\begin{aligned} \langle \sigma_{0,0} \sigma_{M,N} \rangle_{\pm}^{\text{tr,sub}} &= \frac{C_{0+}^{\text{tr}} (\bar{\xi}_{0+}^{\text{tr}})^{-7/4}}{[\mathbf{x} \cdot (\bar{\mathbf{A}}^{\text{tr}})^{-1} \mathbf{x}]^{1/8}} \Psi_{\pm} \left(\frac{[\mathbf{x} \cdot (\bar{\mathbf{A}}^{\text{tr}})^{-1} \mathbf{x}]^{1/2}}{\bar{\xi}_{\pm}^{\text{tr}}(t)} \right) \\ &= \frac{C_{0+}^{\text{tr}} (\bar{\xi}_{0+}^{\text{tr}})^{-7/4}}{[r f(\theta, q^{\text{tr}}, \Omega^{\text{tr}})]^{1/4}} \Psi_{\pm} \left(\frac{r}{\xi_{\pm}^{\text{tr}}(t, \theta, \Omega^{\text{tr}})} \right), \end{aligned} \quad (72)$$

$$\xi_{\pm}^{\text{tr}}(t, \theta, \Omega^{\text{tr}}) = \bar{\xi}_{\pm}^{\text{tr}}(t) / f(\theta, q^{\text{tr}}, \Omega^{\text{tr}}), \quad (74)$$

with the same universal functions Ψ_+ , Ψ_- , and f as in Eqs. (23)–(25), (35), and (36) and the same matrix $\bar{\mathbf{A}}^{\text{tr}} \equiv \bar{\mathbf{A}}(q^{\text{tr}}, \Omega^{\text{tr}})$ as in Eq. (21), with the four nonuniversal parameters C_{0+}^{tr} , $\bar{\xi}_{0+}^{\text{tr}}$, q^{tr} , and Ω^{tr} given in Eqs. (70), (68), (65), (64), respectively, thus proving the validity of multiparameter universality for the triangular-lattice Ising model above, at, and below T_c^{tr} , and disproving two-scale-factor universality. Our hypothesis of multiparameter universality predicts the structure of (72)–(74) to be valid also in the general case $E_1 \neq E_2$, $E_3 \neq 0$.

In order to complete our analysis we show that the universal amplitude relations (7) and (14) derived for the φ^4 model remain valid also for the Ising model. We first employ (30)–(33) for the *isotropic* Ising model to derive

$$(B^{\text{Is}})^2 (C_{0+}^{\text{Is}})^{-1} (\xi_{0+}^{\text{Is}})^2 = (4\pi p_+)^{-1}, \quad (75)$$

in structural agreement with (7). Thus our analysis identifies the exact universal constant Q_c for $d = 2$, $n = 1$ as

$$Q_c = (4\pi p_+)^{-1} = 0.499\,592\,701\dots \quad (76)$$

This can be confirmed by means of a different derivation from Eqs. (6.29) and (6.31) of [2] which determines $Q_c = (R_{\xi}^+)^2 / R_C$. From the rectangular and triangular results (42)–(47) and (67)–(70), respectively, we derive

$$(B^{\text{rec}})^2 (C_{0+}^{\text{rec}})^{-1} (\bar{\xi}_{0+}^{\text{rec}})^2 = (4\pi p_+)^{-1} = Q_c, \quad (77)$$

$$(B^{\text{tr}})^2 (C_{0+}^{\text{tr}})^{-1} (\bar{\xi}_{0+}^{\text{tr}})^2 = (4\pi p_+)^{-1} = Q_c, \quad (78)$$

which agrees with (14) for the anisotropic φ^4 model. Thus both the anisotropic Ising and φ^4 models have universal amplitude relations with the same universal constant Q_c as for the isotropic models, in agreement with the hypothesis of multiparameter universality. In the anisotropic cases three independent nonuniversal parameters are involved in Eqs. (77) and (78) for the same reasons as given in the context of (14).

Multiparameter universality for other critical bulk amplitude relations within φ^4 theory in d dimensions follows from Sec. III of [8], e.g., Eqs. (3.32)–(3.36). In particular, multiparameter universality is predicted, for general n , for anisotropic systems at T_c in the presence of an ordering field h with the amplitude Γ_c of the susceptibility and the principal correlation lengths $\xi_c^{(\alpha)}$ according to Eq. (3.35) of [8] for each α , with a universal constant $Q_2(d, n)$ that is the same as for the corresponding relation [20] of isotropic systems at T_c in the same (d, n) universality class. A verification of such relations within anisotropic fixed-length spin models would be interesting.

To summarize, we have determined the exact anisotropy matrix $\bar{\mathbf{A}}$ for anisotropic φ^4 and Ising models [12,13] and have confirmed the validity of multiparameter universality for the exact bulk order-parameter correlation functions of these models above, at, and below T_c , thereby answering the long-standing question [12] as to the universality properties of the Ising models. It is reassuring that the leading scaling part of the detailed expressions for $\langle\sigma_{0,0}\sigma_{M,N}\rangle_{\pm}$ presented in [12,13] can be condensed into the same compact universal forms (50) and (73) as the exact result (23) for the anisotropic φ^4 model, with three universal functions Ψ_+ , Ψ_- , and f . We have also found agreement with multiparameter universality for the exact critical bulk amplitude relations (14), (77), and (78) with three independent nonuniversal parameters. These results support the validity of multiparameter universality for the large class of weakly anisotropic systems within the (d, n) universality classes which is of relevance for studying the correlation functions in real anisotropic systems [15–18]. The significance of multiparameter universality for finite-size effects, e.g., on the critical Casimir force and the specific heat,

has been pointed out in Ref. [10]. In all cases the universal critical exponents are not changed by weak anisotropy [6,8], unlike the case of strong anisotropy [28]. Nonuniversality enters $\langle\sigma_{0,0}\sigma_{M,N}\rangle_{\pm}$ through the anisotropy matrix $\bar{\mathbf{A}}$, the mean correlation length, and the susceptibility amplitude in the prefactor. $\bar{\mathbf{A}}$ is temperature independent and is applicable above, at, and below T_c in bulk and confined systems [10]. As an appropriate parametrization of $\bar{\mathbf{A}}$ we have employed the ratio q of the principal correlation lengths and the angle Ω determining the principal directions. Both parameters are nonuniversal microscopic quantities. While for φ^4 models Ω is known explicitly according to (11) and (19), this is not generally the case for Ising models. We agree with the assertion [29] that, apart from the Ising models [12,13] analyzed in this Rapid Communication, the principal directions “generically depend in an unknown way on the anisotropic interactions.” Since the principal directions enter the angular dependence of correlation functions in a crucial way the unknown dependence of Ω on microscopic details introduces a significant nonuniversality into the correlation functions of weakly anisotropic systems, in contrast to isotropic systems of the same universality class. This underscores the necessity of distinguishing subclasses of isotropic and anisotropic systems within a given (d, n) universality class. The latter are less universal than the former and require significantly more nonuniversal input in order to achieve quantitative predictions. This statement applies also to finite-size effects in anisotropic systems where up to $d(d+1)/2 + 1$ nonuniversal parameters enter the finite-size scaling form of the free energy density [8,10]. This sheds new light on the general belief that the critical behavior of systems with short-range interactions is largely independent of microscopic details [1–3].

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