Comment on "Slow passage through resonance"

Claire Bourquard [*] and Nicolas Noiray [†]
CAPS Laboratory, Department of Mechanical and Process Engineering, ETH Zürich, Switzerland

(Received 29 March 2019; published 7 October 2019)

In a recent numerical and analytical study, Park *et al.* [Phys. Rev. E **84**, 056604 (2011)] presented the statement that a linearly damped harmonic oscillator subject to a linear frequency chirp $\omega_f(t) = \omega_0 + \epsilon t$ experiences "an early onset of resonance, setting in when the ramped forcing frequency is midway between its initial value ω_0 and the natural frequency ω_n for resonance in the unforced problem," i.e., the resonance occurs when the instantaneous frequency ω_{inst} approaches $(\omega_0 + \omega_n)/2$. This statement is not valid because the instantaneous frequency of the forcing function actually grows twice as fast as stated in the paper, and the resonance actually occurs for instantaneous forcing frequency ω_{inst} approximately equal to the natural frequency ω_n. We also highlight the difference between the critical frequency ramp rate for resonance amplitude measurement and the critical frequency ramp rate for resonance frequency and damping measurements.

DOI: 10.1103/PhysRevE.100.047001

Park *et al.* [1] studied the slow passage through resonance of a damped harmonic oscillator, with the "forcing frequency ω_f as a slowly varying parameter in time, that is, $\omega_f(t) = \omega_0 + \epsilon t$, where ω_0 is an initial frequency and ϵ is a ramp rate." They derived interesting numerical results on the maximum amplitude reached and the influence of the frequency ramp rate ϵ and the damping coefficient γ of the linear harmonic oscillator on this maximum amplitude. Those results were complemented by analytical derivations, and the authors drew practical conclusions on the critical ramp rate beyond which the response amplitude is lower than the one attained in the static-parameter model. However, what is presented as the main finding of the study, namely that the resonance occurs midway between the starting frequency of the linear forcing chirp and the resonance frequency of the undamped oscillator ω_n, is not valid because ω_f is not the instantaneous forcing frequency. The differential equation in Ref. [1] is

$$\ddot{x} + \gamma\dot{x} + x = \sin(\omega_f t) \quad \text{with} \quad \omega_f = \omega_0 + \epsilon t, \tag{1}$$

with the natural resonance frequency $\omega_n = 1$. It is reported in Ref. [1] that the instantaneous forcing frequency is ω_f directly, whereas it actually is the derivative of the instantaneous phase $\phi(t)$, which gives:

$$\omega_{\text{inst}}(t) = \frac{d\phi}{dt} = \frac{d}{dt}[(\omega_0 + \epsilon t)t] = \omega_0 + 2\epsilon t. \tag{2}$$

The actual instantaneous frequency then grows twice as fast as the statement made in Ref. [1], i.e., the actual ramp rate used is 2ϵ and not ϵ. Highlighting the difference between ω_f and ω_{inst} is crucial since it is not the first occurrence of this mistake in the literature (see, for example, the book from Evan-Iwanowski [2], pages 78 and 86 therein).

First, it is important to mention that this problem of "slow passage through resonance," for which the frequency of the harmonic forcing is ramped, is restricted to stable linear oscillators, i.e., with γ being a real positive constant. This problem differs from the "slow passage through Hopf bifurcations," for which one ramps at a finite rate the damping coefficient from a positive to a negative value (e.g., Refs. [3,4]).

In what follows, we clarify the problem by computing numerical solutions of the following ordinary differential equation:

$$\frac{d^2x}{d\tau^2} + 2\nu\frac{dx}{d\tau} + \omega_n^2 x = F\sin\left[\left(\omega_i + \frac{\beta}{2}\tau\right)\tau\right], \tag{3}$$

with ν the damping rate, ω_n the undamped natural angular frequency, F the amplitude of the forcing, ω_i the initial frequency of the chirp, and β the frequency ramp rate. Using the nondimensional time $t = \omega_n\tau$, this equation becomes

$$\ddot{x} + \gamma\dot{x} + x = \sin(\omega_f t) \quad \text{with} \quad \omega_f = \omega_0 + \frac{\epsilon}{2}t, \tag{4}$$

and with $\gamma = 2\nu/\omega_n$, $\omega_0 = \omega_i/\omega_n$, and $\epsilon = \beta/\omega_n^2$. Note that we also assume $F/\omega_n^2 = 1$. In contrast with Eq. (1) for which the instantaneous forcing frequency is $\omega_0 + 2\epsilon t$, one now obtains:

$$\omega_{\text{inst}}(t) = \frac{d}{dt}\left[\left(\omega_0 + \frac{\epsilon}{2}t\right)t\right] = \omega_0 + \epsilon t. \tag{5}$$

For steady harmonic excitation at angular frequency ω, the response amplitude is

$$A(\omega) = \frac{1}{\sqrt{\omega^2\gamma^2 + (1-\omega^2)^2}}. \tag{6}$$

For the chirp excitation of Eq. (4), with instantaneous frequency given by Eq. (5), the responses for three different values of ϵ can be seen in Fig. 1(b): For a very slow frequency ramp rate (green), the shape of the envelope of the response approaches the response amplitude. For increasing frequency ramp rates (blue and red), the resonance is delayed and reaches a lower maximum amplitude. The effect of frequency

[*] clairebo@ethz.ch
[†] noirayn@ethz.ch

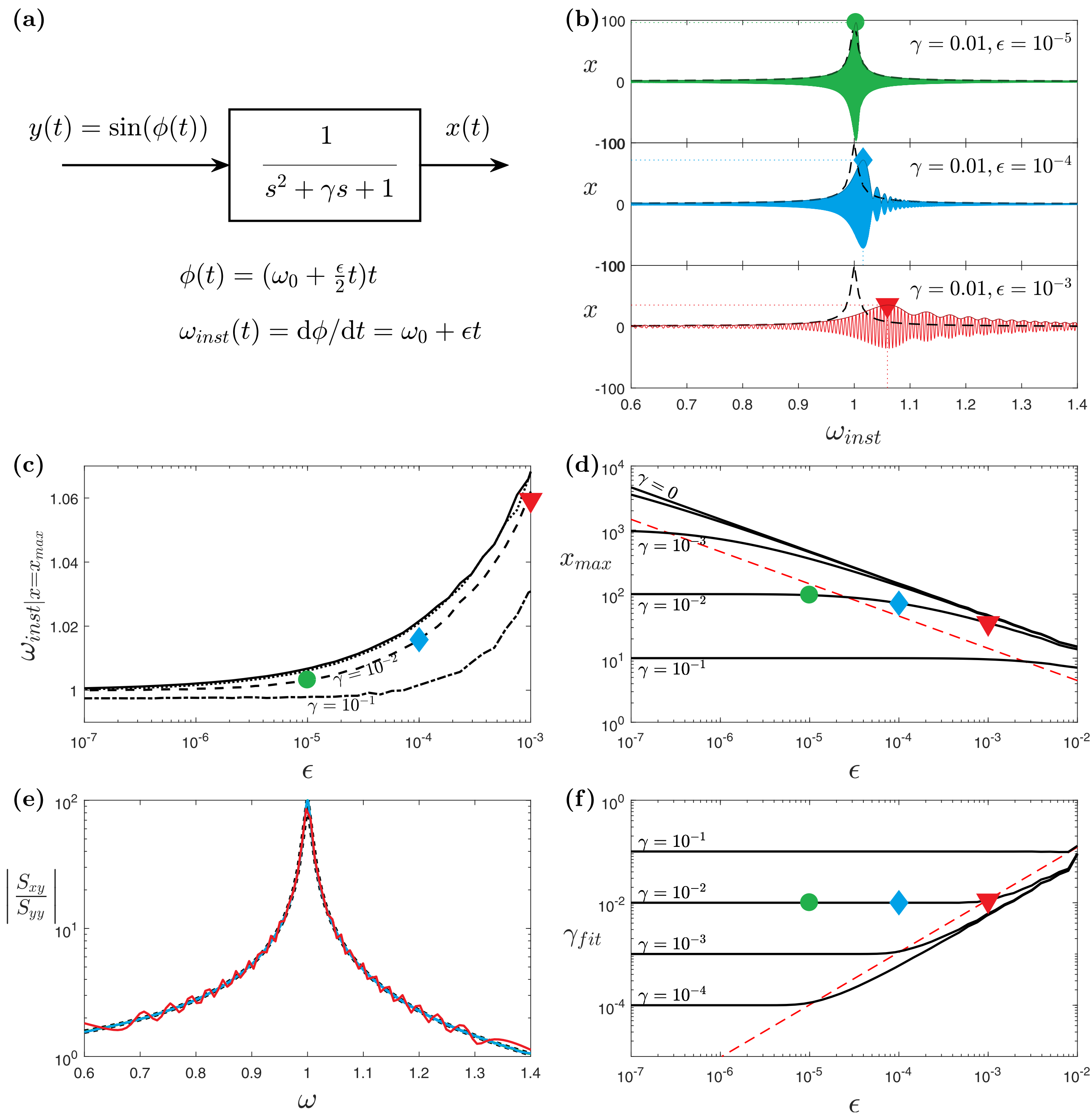

FIG. 1. (a) Transfer function of the studied system with s the Laplace variable. (b) Time traces of $x(t)$ as function of instantaneous angular frequency ω_{inst} for $\gamma = 10^{-2}$ and $\epsilon = 10^{-5}$ (green), 10^{-4} (blue), and 10^{-3} (red). (c) Influence of damping γ and frequency ramp rate ϵ on the instantaneous frequency at which the maximum amplitude is attained (black lines). Superimposed markers correspond to the time traces on top. (d) Influence of damping γ and frequency ramp rate ϵ on the maximum amplitude reached by the oscillator (black lines). Superimposed markers correspond to the time traces on top. Red dashed line: Limit of the maximum ϵ to be used for staying within 10% error on the maximum amplitude, given by $\epsilon_a = \gamma^2/4$. (e) Steady-state transfer function (blacked dashed) line, with almost perfectly superimposed transfer functions from the green, blue, and red sweep signals on top. The transfer function of the red signal (highest ϵ, i.e. fastest frequency sweep) is wavy but still reproduces the general shape accurately. (f) Influence of damping γ and frequency ramp rate ϵ on the damping γ_{fit} obtained from a fit on the transfer function (black lines). Superimposed markers correspond to the time traces on top. Red dashed line: Limit of the maximum ϵ to be used for staying within 10% error on γ_{fit}, given by $\epsilon_d = \gamma/10$.

ramp rate on the instantaneous frequency at which maximum response occurs is displayed in Fig. 1(c) for different oscillator damping: The quasisteady chirp asymptotes exhibit instantaneous frequencies at maximum response that are lower than 1, which corresponds to the expected resonance frequency for harmonic forcing $\omega_r = \sqrt{1 - \gamma^2/2}$. The latter frequency is different from the frequency of the free damped oscillations $\sqrt{1-\gamma^2/4}$. Evaluating Eq. (6) at ω_r indeed gives the approximation given in Eq. (2) in Ref. [1]. Figure 1(c), similarly to Fig. 6 in Ref. [1], also underlines the delay in resonance for fast chirps, where the maximum amplitude is reached at a higher instantaneous frequency than the resonance frequency. This effect is accentuated when the damping γ decreases. The effect of the frequency ramp rate on the maximum reached amplitude for different damping coefficients is presented in Fig. 1(d) (similarly to Fig. 4(a) in Ref. [1]). One can see that for increasing oscillator damping, the critical chirp rate, above which the steady-state resonance amplitude is not attained any more, increases. This critical ramp rate is $\epsilon_a \approx \gamma^2/4$, which is consistent with previous studies [5–7].

If one is interested in the actual oscillation amplitude (during ramp-up of an engine to nominal condition for example [8]), then this criterion holds and one indeed needs very slow frequency ramp rates. However, for identifying the natural frequency ω_n and the damping γ of a linear oscillator, one should not look at the transient amplitude response but at the frequency content of the response to the linear chirp by computing from data the transfer function:

$$H(\omega) = \frac{S_{xy}(\omega)}{S_{yy}(\omega)}, \tag{7}$$

where S_{xy} is the cross-spectral density of the response $x(t)$ and of the chirp excitation $y(t) = \sin[\phi(t)] = \sin[(\omega_0 + \epsilon t/2)t]$ and S_{yy} is the power spectral density of $y(t)$. The transfer functions corresponding to the three examples of Fig. 1(b) are displayed in Fig. 1(e), where all three are almost perfectly superimposed onto the transfer function for steady harmonic excitation. Only the transfer function of the red signal (highest ϵ, i.e. fastest frequency sweep) is wavy but still reproduces the general shape accurately. In order to obtain the resonance frequency and damping of the oscillator, one can then simply fit a second-order transfer function to these data. The effect of frequency ramp rate and system damping on the identified damping γ_{fit} is shown in Fig. 1(f). One can see that the critical chirp rate for this measure to be accurate is higher than the critical ramp rate needed to attain the maximum response amplitude. The critical frequency ramp rate for an accurate damping measurement is $\epsilon_d \approx \gamma/10$. The difference in these critical ramp rates is quite explicit when one looks closer at the red triangles in Figs. 1(c), 1(d), and 1(f): For $\epsilon = 10^{-3}$, one has a maximum amplitude and corresponding instantaneous frequency that significantly differ from the corresponding steady harmonic forcing values. In contrast, the frequency response is very similar in terms of peak frequency, amplitude, and width, thus the fitted damping is very close to the actual one.

In conclusion, we clarified the fact that there is no early onset of resonance when linear harmonic oscillators are submitted to linear chirps, as it was stated by Park *et al.* [1]. In fact, it is a substantial delay which one observes in the case of the linear chirp rate exceeding a critical value.

[1] Y. Park, Y. Do, and J. M. Lopez, Phys. Rev. E **84**, 056604 (2011).

[2] R. Evan-Iwanowski, *Resonance Oscillations in Mechanical Systems* (North-Holland, Amsterdam, 1976).

[3] S. M. Baer, T. Erneux, and J. Rinzel, SIAM J. Appl. Math. **49**, 55 (1989).

[4] N. Berglund, Nonlinearity **13**, 225 (2000).

[5] I. Collinge and J. Ockendon, SIAM J. Appl. Math. **37**, 350 (1979).

[6] L. Sung and K. Stevens, J. Sound Vib. **71**, 497 (1980).

[7] P. Torvik, J. Sound Vib. **330**, 6088 (2011).

[8] F. M. Lewis, Trans. ASME **54**, 253 (1932).