Anomalous diffusion under stochastic resettings: A general approach

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We present a general formulation of the resetting problem which is valid for any distribution of resetting intervals and arbitrary underlying processes. We show that in such a general case, a stationary distribution may exist even if the reset-free process is not stationary, as well as a significant decreasing in the mean first-passage time. We apply the general formalism to anomalous diffusion processes which allow simple and explicit expressions for Poissonian resetting events.

I. INTRODUCTION

There has been in recent years a great deal of work on a special type of composite random process known as stochastic resetting, which basically consists of superposition of a given underlying random process in line with the so-called resetting events that bring the process into a fixed position x_r at random instants of time. The combined processes may have two remarkable properties since, on one hand, it may become stationary and, hence, in some way stabilized even if the underlying process is not. On the other hand, and most importantly, resettings may considerably diminish the first-passage time to any preassigned value x_c, which has a great potential of applications in many fields, especially in searching processes of all kinds, such as protein identification in DNA [1–3], animal foraging [4,5], and data mining [6–8], just to name a few. In recent years we have seen a large amount of literature on the subject of which we cite just a small sample [9–16].

To our knowledge resetting mechanisms have been amply studied when the underlying process is the Brownian motion, with some generalizations within continuous-time random walks [12], Lévy flights [13], some bounded diffusion processes [17,18], and very recently to telegraphic processes [19]. Even though in most cases the resetting mechanism is governed by Poisson processes, the fact that the two key characteristics mentioned above appear for different kinds of underlying processes suggests the universal character of the resetting mechanism to stabilize the process and reduce the first-passage time, as it has been explained from diverse points of view in Refs. [14,15,20–26].

In the present paper we will insist on the universal character of the resetting mechanism by presenting a formulation of the problem for any distribution of the resetting events and underlying processes and apply the formalism for obtaining explicit expressions for anomalous diffusions under resettings. To our knowledge the study of the effect of resettings on anomalous diffusion processes has been scarcely studied.

A recent attempt in this direction is the work of Ref. [27], which is, however, limited to the spectral analysis of fractional Brownian motion with stochastic resettings. Another very recent and independent approach, which is along the lines presented here, deals with subdiffusion under resettings [28] (see also Ref. [26] for interesting asymptotic results about the same problem).

Anomalous diffusion shows up in the transport of particles through extremely disordered media (random media, fractal structures, and the like [29,30]). The most distinctive characteristic is that the mean-square displacement follows an asymptotic law of the form

$$\langle x^2(t) \rangle \sim t^{\alpha}, \quad (t \to \infty, \ \alpha > 0),$$

showing subdiffusion when $0 < \alpha < 1$ and superdiffusion if $\alpha > 1$. The concept of anomalous transport has been the object of very intense research during the past two decades and it extends to many areas of physical research and not only to transport phenomena (there are countless of excellent and complete reports on the subject of which we cite a very few [31–36]).

We will also address the resetting problem on anomalous diffusion processes and explore the consequences of a quite general resetting mechanism on the anomalous transport of the underlying process. There is no unique mathematical approach to anomalous diffusion processes (see Ref. [37] for a recent report focused on subdiffusive processes). Among them one of the most used approaches for anomalous diffusion is that based on the continuous-time random walk (CTRW) of Montroll and Weiss [32,38,39]. This is the procedure implicit in our development.

To this end, and in tune with Ref. [26], we will consider a broad framework in which an underlying process is brought to a fixed location at random times, drawn from an arbitrary distribution. Once we obtain general expressions for the different statistics of interest, we will specifically suppose that the reset-free process is the time-fractional Brownian motion on the line, whose probability density function (PDF) obeys the time-fractional diffusion equation:

$$\frac{\partial^{\alpha} p}{\partial t^{\alpha}} = D \frac{\partial^2 p}{\partial x^2} \tag{1}$$

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$(0 < \alpha < 2)$, where $\partial^\alpha / \partial t^\alpha$ is the fractional Caputo derivative to be formally defined in Sec. IV. Since we are interested in finding and analyzing closed formulas not only in asymptotic results [26], we will further assume that the inter-reset times are Poissonian, which is not required within our setup but approaches our results to those reported in Ref. [28] for subdiffusive processes.

The paper is organized as follows. In Sec. II we set the general formalism and obtain overall expressions for the propagator of the combined process and show the possible existence of a stationary distribution even if the reset-free process is not stationary. In Sec. III we present the general approach to the first-passage problem and obtain general expressions for the mean first-passage time showing a considerable decrease in the value of this statistic when the reset-free process has an infinite time. In Sec. IV we apply the formalism to time-fractional diffusion processes and some concluding remarks are in Sec. V.

II. GENERAL FRAMEWORK

Let $X(t)$ be a random process on the line which, starting at $X(\mathcal{T}_0 = 0) = x_r$, undergoes resettings to this initial position x_r at times $\mathcal{T}_1, \mathcal{T}_2, \ldots$, from where it continues afresh. Resettings occur at random times and we denote by $\psi_r(\tau)$ the PDF of the time interval between two consecutive resettings, $\tau_n \equiv \mathcal{T}_n - \mathcal{T}_{n-1}, n = 1, 2, \ldots$,

$$\psi_r(\tau)d\tau = \text{Prob}\{\tau < \tau_n \leqslant \tau + d\tau\}. \quad (2)$$

In what follows we will assume that $\psi_r(\tau)$ has finite moments which in particular implies a finite mean time between consecutive resettings,

$$\langle \tau_n \rangle = \int_0^\infty \tau \psi_r(\tau)d\tau < \infty. \quad (3)$$

In other words, the mean frequency of resetting (also called resetting rate) defined as

$$r = \frac{1}{\langle \tau_n \rangle} > 0 \quad (4)$$

is nonvanishing.[1] The probability $\Psi_r(\tau)$ that no resettings occur for a time interval greater than τ is given by

$$\Psi_r(\tau) = \int_\tau^\infty \psi_r(\tau')d\tau'. \quad (5)$$

Before proceeding further let us note that if we keep a general resetting density $\psi_r(\tau)$, then the combined process with resettings is not Markovian, even if the evolution between resets is a Markov process. Specifically, this implies that the propagator p_r of the combined process is not simply a function of the present state of the system, $X(t_0) = x_0$, but

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[1]This assumption is needed for obtaining a stationary distribution [see Eq. (16)]. We, therefore, deal with a resetting mechanism which is not governed by a power law (see, for instance, Ref. [40] for a very recent generalization along these lines). Even so, most of the general expressions to be derived will remain valid in this case.

also a function of the last resetting event previous to t_0:

$$p_r(x, t|x_0, t_0; x_r, t_r)dx$$
$$= \text{Prob}\{x < X(t) \leqslant x + dx | X(t_0) = x_0; X(t_r) = x_r\},$$

where t_r is the time of the last reset before t_0,

$$t_r \equiv \max\{\mathcal{T}_n | \mathcal{T}_n \leqslant t_0\}.$$

The only exception to this rule is when t_0 *exactly coincides* with a reset, $t_0 = t_r$, since then

$$p_r(x, t|x_0, t_0; x_r, t_r) = p_r(x, t|x_0, t_0) \quad (6)$$

and, hence, $x_0 = x_r$. However, for the sake of clarity in the upcoming development we will keep x_0 as the position associated to t_0, and eventually set $x_0 = x_r$ when required.

To obtain an expression for the complete propagator p_r, we will first address the case when t_0 coincides with a reset time t_r and briefly postpone the analysis of the general case.[2] Note that when $t_0 = t_r$ the propagator $p_r(x, t|x_0, t_0)$ obeys the following renewal equation in terms of the reset-free propagator $p_0(x, t|x_0, t_0)$:

$$p_r(x, t|x_0, t_0) = \Psi_r(t - t_0)p_0(x, t|x_0, t_0)$$
$$+ \int_{t_0}^t \psi_r(t' - t_0)p_r(x, t|x_r, t')dt', \quad (7)$$

where the first term on the right-hand side accounts for the probability density when no reset event has occurred up to time t. The second term represents the probability density that the first resetting (ulterior to the one at t_0 and bringing again the process to $x_r = x_0$) occurred during any intermediate time $t_0 < t' \leqslant t$.

In what follows we will assume that the underlying process is not only Markovian but also time homogeneous, which implies that the propagator only depends on time differences, $p_0(x, t|x_0, t_0) = p_0(x, t - t_0|x_0)$. This leads to the conclusion that the same property holds for the combined process when t_0 coincides with a reset, that is, $p_r(x, t|x_0, t_0) = p_r(x, t - t_0|x_0)$, and Eq. (7) can be written as

$$p_r(x, t|x_0) = \Psi_r(t)p_0(x, t|x_0)$$
$$+ \int_0^t \psi_r(t')p_r(x, t - t'|x_r)dt'. \quad (8)$$

After taking the Laplace transform,

$$\hat{p}_r(x, s|x_0) = \int_0^\infty e^{-st}p_r(x, t|x_0)dt,$$

the integral Eq. (8) reduces to the following algebraic equation:

$$\hat{p}_r(x, s|x_0) = \mathcal{L}\{\Psi_r(t)p_0(x, t|x_0)\} + \hat{\psi}_r(s)\hat{p}_r(x, s|x_r),$$

and, since $x_0 = x_r$, we get [26]

$$\hat{p}_r(x, s|x_r) = \frac{1}{1 - \hat{\psi}_r(s)}\mathcal{L}\{\Psi_r(t)p_0(x, t|x_r)\}. \quad (9)$$

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[2]We will show below that in the case of Poissonian resettings for which $\psi_r(\tau) = re^{-r\tau}$, as in Ref. [19], it is not necessary to consider such a distinction.

In all these expressions, $\mathcal{L}\{\cdot\}$ represents the Laplace transform with respect to the time variable, and $\mathcal{L}^{-1}\{\cdot\}$ its inverse.

Taking the inverse Laplace transform of Eq. (9) we obtain the following expression for the propagator of the combined process in the form of a convolution integral:

$$p_r(x,t|x_r) = \int_0^t F_r(t-t')\Psi_r(t')p_0(x,t'|x_r)dt', \qquad (10)$$

where

$$F_r(t) = \mathcal{L}^{-1}\left\{\frac{1}{1-\hat{\psi}_r(s)}\right\}. \qquad (11)$$

Let us next consider the more general case in which t_0 does not coincide with a reset time, that is, $t_0 > t_r$ and x_0 can take arbitrary values. The complete propagator, $p_r(x,t|x_0,t_0;x_r,t_r)$, is now given by

$$p_r(x,t|x_0,t_0;x_r,t_r) = \frac{\Psi_r(t-t_r)}{\Psi_r(t_0-t_r)}p_0(x,t-t_0|x_0)$$
$$+ \int_{t_0}^t \frac{\psi_r(t'-t_r)}{\Psi_r(t_0-t_r)}p_r(x,t-t'|x_r)dt', \qquad (12)$$

where $p_r(x,t-t'|x_r)$ is given by Eq. (10). Equation (12) is the generalization of Eq. (7) in which the probability of having a reset at time t after $t_0 = t_r$ has been replaced by the conditional probability of having a reset at time t knowing that no reset has occurred yet at t_0 ($t_r < t_0 \leqslant t$), that is,

$$\psi_r(t-t_0) \longrightarrow \frac{\psi_r(t-t_r)}{\int_{t_0}^\infty \psi_r(t-t_r)dt} = \frac{\psi_r(t-t_r)}{\Psi_r(t_0-t_r)}. \qquad (13)$$

Note also that in general $p_r(x,t|x_0,t_0;x_r,t_r)$ is a function of two time intervals $t-t_r$ and t_0-t_r, except for Poissonian reset times. Indeed, if the random instants of time when resettings occur are a Poissonian set of events, then the density $\psi_r(\tau)$ and the probability $\Psi_r(\tau)$ are given by

$$\psi_r(\tau) = re^{-r\tau}, \quad \Psi_r(\tau) = e^{-r\tau}, \qquad (14)$$

where r is the mean rate of resettings. In such a case,

$$\frac{\psi_r(t-t_r)}{\Psi_r(t_0-t_r)} = re^{-r(t-t_0)}, \quad \frac{\Psi_r(t-t_r)}{\Psi_r(t_0-t_r)} = e^{-r(t-t_0)},$$

and from Eq. (12) we see that $p_r(x,t|x_0,t_0;x_r,t_r)$ is no longer a function of t_r, and the process becomes Markovian.[3]

The stationary distribution

We know that the stationary distribution is defined as the long-time limit of the propagator:

$$p_r^{(st)}(x) \equiv \lim_{t\to\infty} p_r(x,t|x_0,t_0;x_r,t_r).$$

From Eq. (12) and recalling the definition of $\Psi_r(t)$ we have

$$p_r^{(st)}(x) = \lim_{t\to\infty} p_r(x,t|x_r),$$

which, using a well-known property of the Laplace transform, can be written as

$$p_r^{(st)}(x) = \lim_{s\to 0}[s\hat{p}_r(x,s|x_r)],$$

and from Eq. (9) we conclude that

$$p_r^{(st)}(x) = \lim_{s\to 0}\left[\frac{s}{1-\hat{\psi}_r(s)}\int_0^\infty e^{-st}\Psi_r(t)p_0(x,t|x_r)dt\right]. \qquad (15)$$

Note that we are dealing with a time-interval PDF $\psi_r(t)$ having finite moments which implies that the Laplace transform $\hat{\psi}_r(s)$ can be expanded as

$$\hat{\psi}_r(s) = 1 - \frac{s}{r} + O(s^2),$$

where $r = \langle \tau_n \rangle^{-1}$ is the mean rate of resettings. Substituting into Eq. (15) yields

$$p_r^{(st)}(x) = r\int_0^\infty \Psi_r(t)p_0(x,t|x_r)dt. \qquad (16)$$

Let us recall that a random process is stationary if there exist a nonnull stationary density $p_r^{(st)}(x)$ defined as the long-time limit of the complete propagator $p_r(x,t|x_0,t_0;x_r,t_r)$, which is independent of x_0. Note that if $p_r^{(st)}(x) = 0$ the process in the long-time limit cannot be found at some state x. In other words, the existence of $p_r^{(st)}(x) \neq 0$ stabilizes the process around some equilibrium point which in our case is the resetting position x_r.

In the present case of a combined process with stochastic resettings, the stationary distribution in the most general case is given by Eq. (16). Let us note that, attending the positive character of both $\Psi_r(t)$ and $p_0(x,t|x_0)$, the integral in Eq. (16) will not vanish in general, even if

$$p_0^{(st)}(x) = \lim_{t\to\infty} p_0(x,t|x_0) = 0.$$

That is to say, the combined process is always stationary regardless the stationary character of the underlying, reset-free, process. This proves, in a rather general manner, the stabilizing effect of resettings which otherwise is a rather intuitive effect.

For Poissonian resettings the expression given by Eq. (16) is simpler and more explicit. In this case, $\Psi_r(t) = e^{-rt}$ and Eq. (16) reduces to

$$p_r^{(st)}(x) = r\hat{p}_0(x,r|x_r), \qquad (17)$$

where $\hat{p}_0(x,s|x_r)$ is the Laplace transform of the reset-free propagator. In this case the possible stationarity of the combined process depends on the existence of the Laplace transform of the reset-free propagator.

III. THE FIRST-PASSAGE PROBLEM

We next address within the general framework described above the first-passage problem for a random process with stochastic resettings. This has been the object of intense research because resettings may significantly reduce the mean-first passage time, a fact with many practical applications and that, in particular, optimizes any search process based on the combined process.

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[3]Obviously, $p_r(x,t|x_0,t_0)$ still depends on x_r.

The characterization of the first-passage problem under resettings has been mostly addressed for the Brownian motion (i.e., unbounded diffusion process) under Poissonian resettings even though there have been recent works stressing the universal character of resetting processes [14,15,24,25]. In this section we present another view on this universal character.

Let us focus on the first-passage problem to some preassigned value x_c, also called critical value or threshold. The problem is characterized by the survival probability (SP) to threshold x_c, $S_r(t|x_0, t_0; x_r, t_r)$, which is the probability that the process, being in x_0 at t_0, does not reach x_c during the interval $[t_0, t]$. As before t_r denotes the last reset time prior to t_0, $t_r \leqslant t_0$. For diffusion processes with Poissonian resettings [9] the survival probability obeys an inhomogeneous Fokker-Planck equation with initial and boundary conditions, respectively, given by $S_r(t_0|x_0, t_0) = 1$ and $S_r(t|x_c, t_0) = 0$, independent of t_r.[4]

As in the previous section with the analysis of the propagator and the stationary distribution, we can also obtain a general equation for the survival probability for any distribution of resetting events and any class of underlying process. Such an equation is an integral equation based on the renewal principle and a version of it has been recently used by Pal, Kundu, and Evans [11] in their study of diffusions with time-dependent resetting rates.

Let us denote by $S_0(t|x_0, t_0)$ the survival probability to threshold x_c for the underlying (i.e., reset-free) process, while $S_r(t|x_0, t_0; x_r, t_r)$ denotes the SP to x_c for the combined process with resettings. Once again, we begin by assuming that at time t_0 a resetting event has occurred which implies that $t_r = t_0$ and $x_r = x_0$. In this case, the integral equation for the SP of the combined process $S_r(t|x_0, t_0)$ is

$$S_r(t|x_0, t_0) = \Psi_r(t - t_0)S_0(t|x_0, t_0)$$
$$+ \int_{t_0}^{t} \psi_r(t' - t_0)S_0(t'|x_0, t_0)S_r(t|x_r, t')dt', \quad (18)$$

where the first term on the right-hand side gives the probability that neither a reset has occurred at time t nor any hitting to threshold x_c between t_0 and t. The second term represents the probability that the first resetting (after the one at t_0) to position x_r[5] occurred at some instant of time t' with no hitting to x_c between t_0 and t' and no hitting either from t' and t, all of this integrated over any intermediate time $t' \in [t_0, t]$.

Since we are assuming time homogeneity we have that $S_0(t|x_0, t_0) = S_0(t - t_0|x_0)$ and similarly for $S_r(t|x_0, t_0)$. This allows us to set $t_0 = 0$ and rewrite Eq. (18) in the simpler form:

$$S_r(t|x_0) = \Psi_r(t)S_0(t|x_0)$$
$$+ \int_0^t \psi_r(t')S_0(t'|x_0)S_r(t - t'|x_r)dt'. \quad (19)$$

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[4]For telegraphic processes with Poissonian resettings $S_r(t|x_0, t_0)$ obeys an inhomogeneous telegrapher's equation [19].

[5]In fact $x_r = x_0$ although we keep x_0 and x_r for the clearness of the exposition.

We can easily solve this integral equation in the Laplace space. To this end we first define the auxiliary quantities

$$H(t|x_0) \equiv \Psi_r(t)S_0(t|x_0), \quad h(t|x_0) \equiv \psi_r(t)S_0(t|x_0), \quad (20)$$

which allow us to write Eq. (19) as

$$S_r(t|x_0) = H(t|x_0) + \int_0^t h(t'|x_0)S_r(t - t'|x_r)dt'. \quad (21)$$

The Laplace transform,

$$\hat{S}_r(s|x_0) = \int_0^\infty e^{-st}S_r(t|x_0)dt, \quad (22)$$

turns Eq. (21) into the simple algebraic equation

$$\hat{S}_r(s|x_0) = \hat{H}(s|x_0) + \hat{h}(s|x_0)\hat{S}_r(s|x_r),$$

whose solution reads (recall that here $x_0 = x_r$)

$$\hat{S}_r(s|x_r) = \frac{\hat{H}(s|x_r)}{1 - \hat{h}(s|x_r)}, \quad (23)$$

where $\hat{H}(s|x_r)$ and $\hat{h}(s|x_r)$ are the Laplace transform of the functions $H(t|x_r)$ and $h(t|x_r)$ defined in Eq. (20).

As in the previous section, we can recover the general solution for the SP $S_r(t|x_0, t_0; x_r, t_r)$, when $t_r \leqslant t_0$ and x_0 is any point, by using the rule Eq. (13) in Eq. (18):

$$S_r(t|x_0, t_0; x_r, t_r) = \frac{\Psi_r(t - t_r)}{\Psi_r(t_0 - t_r)}S_0(t - t_0|x_0)$$
$$+ \int_{t_0}^{t} \frac{\psi_r(t' - t_r)}{\Psi_r(t_0 - t_r)}S_0(t' - t_0|x_0)$$
$$\times S_r(t - t'|x_r)dt', \quad (24)$$

where $S_r(t - t'|x_r)$ appearing on the right-hand side is given by Laplace inverting Eq. (23). Let us, however, note that $S_r(t|x_0, t_0; x_r, t_r)$ thus defined does not guarantee that the process has never reached x_c before t_0 and, in particular, during the interval $[t_r, t_0]$. Indeed, recall that t_r is the time of the last reset before t_0, in consequence the evolution of the process between t_r and t_0 could have perfectly crossed threshold x_c and go afterwards to x_0 at time $t_0 > t_r$. To avoid that eventuality we must perform the additional replacement

$$S_0(t - t_0|x_0) \longrightarrow S_0(t - t_0|x_0)$$
$$\times \left[1 - \frac{p_0(2x_c - x_0, t_0 - t_r|x_r)}{p_0(x_0, t_0 - t_r|x_r)}\right]$$

in Eq. (24), where we have applied the reflection principle which is valid for underlying processes that are symmetric and space homogeneous [41].

In what follows we will, therefore, restrict our attention to the situation in which $t_0 = t_r$ and thus $x_0 = x_r$. In such a case and taking into account time homogeneity we can set $t_0 = 0$ and the SP is given by the Laplace inversion of Eq. (23). Knowing $S_r(t|x_0)$, the mean first-passage time (MFPT) to threshold x_c from x_0 in the presence of random resettings is given by the integral

$$T_r(x_0) = \int_0^\infty S_r(t|x_0)dt,$$

which in terms of the Laplace transform of the SP is simply given by

$$T_r(x_0) = \hat{S}_r(s = 0|x_0). \tag{25}$$

From Eqs. (20), (23), and (25) we see that

$$T_r(x_0) = \frac{\int_0^\infty \Psi_r(t) S_0(t|x_0) dt}{1 - \int_0^\infty \psi_r(t) S_0(t|x_0) dt}. \tag{26}$$

Therefore, the MFPT will be finite as long as the integrals in the right-hand side of Eq. (26) exist even if the MFPT for the reset-free process,

$$T_0(x_0) = \int_0^\infty S_0(t|x_0) dt,$$

is infinite. Since the existence of the integrals in Eq. (26) is ensured for a wide class of sojourn densities $\psi_r(t)$ that have finite moments, we see that resettings may considerably reduce the MFPT with the subsequent increase of efficiency.

A key property of Poissonian resettings on diffusion processes is the nonmonotonous behavior of the MFPT as a function of the resetting rate r. In other words, the MFPT attains a minimum for a particular value of r. Let us now argue that this characteristic is kept for a wide class of resetting mechanisms and underlying processes. Indeed, as seen in Sec. II, any resetting mechanism governed by a PDF $\psi_r(t)$ with finite and nonzero mean, has a finite resetting rate r defined in Eq. (4). Note that as $r \to 0$ the complete process approaches the reset-free process, so that $T_r(x_0) \to T_0(x_0)$. Let us also assume that, as in Brownian motion, $T_0(x_0) = \infty$. Therefore,

$$\lim_{r \to 0} T_r(x_0) = \infty. \tag{27}$$

However, as $r \to \infty$ the average time between consecutive resettings tends to 0 which means that the system becomes circumscribed to a shrinking neighborhood around $x_0 = x_r$ if the underlying process is continuous enough. Let us further assume that r is the only scale parameter of $\psi_r(t)$, that is,

$$\psi_r(t) = r f(rt), \tag{28}$$

for a certain nonnegative function $f(u)$ which, due to the normalization condition of $\psi_r(t)$ and Eqs. (3) and (4), satisfies

$$\int_0^\infty f(u) du = \int_0^\infty u f(u) du = 1. \tag{29}$$

Therefore, the scaling Eq. (28) allows us to express Eq. (26) as

$$T_r(x_0) = \frac{\int_0^\infty du f(u) \int_0^{u/r} S_0(t|x_0) dt}{1 - \int_0^\infty f(u) S_0(u/r|x_0) du}. \tag{30}$$

Taking the limit $r \to \infty$, applying L'Hôpital's rule and bearing in mind the initial condition $S_0(0|x_0) = 1$ and Eq. (29), we get

$$\lim_{r \to \infty} T_r(x_0) = -\frac{1}{\partial_t S_0(t|x_0)|_{t=0}}, \tag{31}$$

whenever $x_c \neq x_0$. Hence, as long as $S_0(t|x_0)$ satisfies the condition

$$\partial_t S_0(t|x_0)|_{t=0} = 0, \tag{32}$$

the MFPT also becomes infinite as $r \to \infty$:

$$\lim_{r \to \infty} T_r(x_0) = \infty. \tag{33}$$

Since under mild conditions $T_r(x_0)$ is a continuous function of r, from Eqs. (27) and (33) we see that the MFPT must attain a minimum value for r somewhere in between 0 and ∞. This is a complementary discussion to that of Refs. [14,15,20–25] on the generality of resettings.

For Poissonian resettings, the above expressions are simpler and more explicit. Indeed, substituting the exponential forms given in Eq. (14) into the definitions Eq. (20) of the auxiliary quantities $h(t|x_0)$ and $H(t|x_0)$ and taking the Laplace transform we get

$$\hat{H}(s|x_0) = \hat{S}_0(r + s|x_0), \quad \hat{h}(s|x_0) = r\hat{S}_0(r + s|x_0),$$

which finally results in [9]

$$\hat{S}_r(s|x_0) = \frac{\hat{S}_0(r + s|x_0)}{1 - r\hat{S}_0(r + s|x_0)} \tag{34}$$

and

$$T_r(x_0) = \frac{\hat{S}_0(r|x_0)}{1 - r\hat{S}_0(r|x_0)}. \tag{35}$$

This last equation showing that for Poissonian resettings the MFPT of the combined process is finite regardless the exact nature of the reset-free process which is a direct consequence of the fact that the Laplace transform of any survival probability always exists.[6]

IV. ANOMALOUS DIFFUSION AND RESETTINGS

We will next address the resetting problem when the underlying is a time-fractional diffusive process or fractional Brownian motion.

A. Stationary distribution

In this case the propagator of the reset-free process, $p_0(x, t|x_0)$, obeys the time-fractional diffusion-wave equation [19,39,42]

$$\frac{\partial^\alpha p_0(x, t|x_0)}{\partial t^\alpha} = D \frac{\partial^2 p_0(x, t|x_0)}{\partial x^2}, \tag{36}$$

$(0 < \alpha < 2)$[7] with the initial condition

$$p_0(x, t = 0|x_0) = \delta(x - x_0). \tag{37}$$

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[6]Indeed, the survival probability (like any probability) is always less or equal to one, $S_0(t|x_0) \leqslant 1$, and hence $\hat{S}_0(s|x_0) \leqslant 1/s$ is finite (we thank an anonymous referee for recalling us this elementary result).

[7]In the literature the most frequently studied case of time-fractional diffusion equation corresponds to the subdiffusive case where $0 < \alpha < 1$. In our opinion the main reason for such a restriction on the values α can take lies in the fact that this subdiffusive case can be easily derived from microscopic models based on the continuous time random walk (see, for instance, Ref. [44] for a short review). However, the superdiffusive case where $1 < \alpha < 2$, which is also included in the time-fractional diffusion-wave Eq. (36), can be derived from microscopic models based on the fractional persistent random walk (see Ref. [39] for details).

When $1 < \alpha < 2$ the initial condition Eq. (37) has to be supplemented with a second initial condition, which usually is

$$\left.\frac{\partial p_0(x, t|x_0)}{\partial t}\right|_{t=0} = 0. \tag{38}$$

The operator $\partial^\alpha / \partial t^\alpha$ is the fractional Caputo derivative defined as

$$\frac{\partial^\alpha \phi(t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\phi^{(n)}(t')dt'}{(t-t')^{1+\alpha-n}}, & n-1 < \alpha < n, \\ \phi^{(n)}(t), & \alpha = n \end{cases} \tag{39}$$

$(n = 1, 2, 3, \dots)$. Using this definition the Laplace transform of the Caputo derivative is found to be [43,44]

$$\mathcal{L}\left\{\frac{\partial^\alpha \phi(t)}{\partial t^\alpha}\right\} = s^\alpha \hat{\phi}(s) - s^{\alpha-1}\phi(0) - \sum_{j=1}^{n-1} s^{\alpha-1-j}\phi^{(j)}(0) \tag{40}$$

$(n = 1, 2, 3, \dots; n-1 < \alpha < n)$, where $\hat{\phi}(s) = \mathcal{L}\{\phi(t)\}$. Note that when $0 < \alpha < 1$ we have $n = 1$ and this transform reduces to

$$\mathcal{L}\left\{\frac{\partial^\alpha \phi(t)}{\partial t^\alpha}\right\} = s^\alpha \hat{\phi}(s) - s^{\alpha-1}\phi(0). \tag{41}$$

We also observe that if $1 < \alpha < 2$ but $\phi'(0) = 0$, the Laplace transform for the Caputo derivative is also given by Eq. (41).

The joint Fourier-Laplace transform of the free propagator $p_0(x, t|x_0)$ is defined as

$$\hat{\hat{p}}_0(\omega, s|x_0) = \int_{-\infty}^{\infty} e^{-i\omega x}dx \int_0^{\infty} e^{-st} p_0(x, t|x_0)dt,$$

and the use of Eq. (41) leads to the following solution of the initial-value problem Eqs. (36)–(38),

$$\hat{\hat{p}}_0(\omega, s|x_0) = \frac{s^{\alpha-1}e^{-i\omega x_0}}{s^\alpha + D\omega^2}. \tag{42}$$

Recalling the Fourier inversion formula

$$\mathcal{F}^{-1}\left\{\frac{ae^{-i\omega x_0}}{b + c^2\omega^2}\right\} = \frac{a}{2\sqrt{bc}} e^{-|x-x_0|b^{1/2}/c},$$

we get [28]

$$\hat{p}_0(x, s|x_0) = \frac{s^{\alpha/2-1}}{2\sqrt{D}} e^{-|x-x_0|\sqrt{s^\alpha/D}}. \tag{43}$$

The Laplace transform can also be inverted with the result [44]

$$p_0(x, t|x_0) = \frac{1}{2\sqrt{Dt^\alpha}} M_{\alpha/2}\left(\frac{|x-x_0|}{\sqrt{Dt^\alpha}}\right), \tag{44}$$

where $M_{\alpha/2}(\cdot)$ is the Mainardi function defined by the power series [42,45]

$$M_\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(-\beta n + 1 - \beta)}, \quad 0 < \beta < 1. \tag{45}$$

Mainardi's function $M_\beta(z)$ is an entire function of z for $0 < \beta < 1$ [42]. It is a special case of the Wright function [45,46] which is closely related to the rather cumbersome Fox

function, the latter frequently used in the anomalous diffusion literature [32].

The reset-free process is not stationary because as we can easily see from Eq. (44),

$$p_0^{(\text{st})}(x) = \lim_{t \to \infty} p_0(x, t|x_0) = 0.$$

However, as we have seen in Sec. II, the addition of a resetting mechanism turns the process into an stationary one with a nonzero stationary density given by Eq. (16). For Poissonian resettings, cf. Eq. (14), the stationary density is given by Eq. (17), which for the anomalous diffusion process yields the tent-shape density [cf. Eq. (43)]

$$p_r^{(\text{st})}(x) = \frac{r^{\alpha/2}}{2\sqrt{D}} e^{-|x-x_0|\sqrt{r^\alpha/D}}, \tag{46}$$

an expression which has been very recently derived independently in Refs. [26,28].

B. Survival probability

The survival probability to some critical value x_c of the underlying reset-free process, $S_0(t|x_0)$, is the solution to the following equation:

$$\frac{\partial^\alpha S_0(t|x_0)}{\partial t^\alpha} = D\frac{\partial^2 S_0(t|x_0)}{\partial x_0^2} \tag{47}$$

$(0 < \alpha < 2)$, with the initial and boundary conditions

$$S_0(t = 0|x_0) = 1, \quad S_0(t|x_c) = 0. \tag{48}$$

Similar to the case of the propagator discussed above, when $1 < \alpha < 2$ these conditions have to be supplemented with

$$\partial_t S_0(t|x_0)|_{t=0} = 0,$$

which has implications in the behavior of $T_r(x_0)$, as we have seen [cf. Eqs. (31)–(33)].

In the Laplace space, and after using Eq. (41), this problem simply reads

$$D\frac{\partial^2 \hat{S}_0(s|x_0)}{\partial x_0^2} - s^\alpha \hat{S}_0(s|x_0) = -s^{\alpha-1}, \quad \hat{S}_0(s|x_c) = 0. \tag{49}$$

As can be seen by direct substitution, the solution to this problem that is finite as $x_0 \to \pm\infty$ is [28]

$$\hat{S}_0(s|x_0) = \frac{1}{s}[1 - e^{-|x_0-x_c|\sqrt{s^\alpha/D}}]. \tag{50}$$

The Laplace inversion of this expression yields (see Ref. [44] for details)

$$S_0(t|x_0) = 1 - \phi(-\alpha/2, 1, -|x_0 - x_c|/\sqrt{Dt^\alpha}), \tag{51}$$

where $\phi(\rho, \beta, z)$ is the Wright function, which can be defined by the power series [46,47]

$$\phi(\rho, \beta, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(n\rho + \beta)} \tag{52}$$

$(\rho > -1, \beta > 0)$.

The Laplace transform, $\hat{S}_r(s|x_0)$, of the survival probability to threshold x_c for the complete process with resettings is

given by Eq. (23), after substituting for $S_0(t|x_0)$ given in Eq. (51) and evaluating the Laplace transforms $\hat{h}(s|x_0)$ and $\hat{H}(s|x_0)$. The resulting expression is virtually impossible to handle for any resetting density $\psi_r(t)$. However, this can be easily done for Poissonian resettings. In such a case, substituting Eq. (50) into Eq. (34) yields

$$\hat{S}_r(s|x_0) = \frac{1 - e^{-(r+s)^{\alpha/2}|x_0-x_c|/\sqrt{D}}}{s + re^{-(r+s)^{\alpha/2}|x_0-x_c|/\sqrt{D}}}. \qquad (53)$$

C. Mean first-passage time

We will first check that the MFPT for the reset-free process is infinite. In effect, expanding Eq. (50) for small values of s we find

$$\hat{S}_0(s|x_0) = \frac{|x_0-x_c|}{\sqrt{D}} s^{\alpha/2-1} + O(s^{\alpha-1}).$$

Hence, for $0 < \alpha < 2$,

$$T_0(x_0) = \lim_{s\to 0} \hat{S}_0(s|x_0) = \infty.$$

For the combined process with resetting events governed by any switch density $\psi_r(t)$, the MFPT $T_r(x_0)$ is given by the general expression Eq. (26) after substituting $S_0(t|x_0)$ by Eq. (51). For Poissonian resettings we can write a simple and explicit expression for $T_r(x_0)$ since in this case taking the limit $s \to 0$ in Eq. (53) yields [28]

$$T_r(x_0) = \frac{1}{r}[e^{|x_0-x_c|r^{\alpha/2}/\sqrt{D}} - 1]. \qquad (54)$$

From this expression we clearly see how $T_r(x_0)$ approaches $T_0(x_0) = \infty$ as $r \to 0$. Indeed,

$$T_r(x_0) = \frac{|x_0-x_c|}{\sqrt{D}} r^{\alpha/2-1}[1 + O(r^{\alpha/2})],$$

which diverges as $r \to 0$ for $0 < \alpha < 2$.

As we have shown above, the MFPT in this case is a nonmonotonous function of rate r presenting a minimum for some rate located between 0 and ∞.[8] For Poissonian resettings we can be more specific and obtain the minimum rate. In effect, in such a case the MFPT is given by Eq. (54) from which we can obtain an explicit expression for the derivative $\partial_r T_r(x_0)$ and the minimum rate r_m will be the solution to the equation $\partial_r T_r(x_0) = 0$. This elementary procedure leads to the transcendental equation[9]

$$1 - \frac{\alpha}{2}\xi = e^{-\xi}, \qquad (55)$$

for the variable ξ defined as

$$\xi \equiv |x_0-x_c|r_m^{\alpha/2}/\sqrt{D}. \qquad (56)$$

Having obtained ξ from the numerical solution of Eq. (55) for a given value of exponent α, the minimum rate is thus given

by [28]

$$r_m = \left(\frac{\xi\sqrt{D}}{|x_0-x_c|}\right)^{2/\alpha}, \qquad (57)$$

a result that we can replace back in Eq. (54) to obtain

$$T_\xi(x_0) = \frac{\alpha\xi^{1-2/\alpha}}{2-\alpha\xi} \cdot \frac{|x_0-x_c|^{2/\alpha}}{D^{1/\alpha}}. \qquad (58)$$

Note how the prefactor in this expression is a numeric quantity that only depends on α.

V. CONCLUDING REMARKS

We have analyzed the effects of general resetting mechanisms on anomalous diffusion processes, specifically on the time-fractional Brownian motion. Although our primary purpose has been focusing on anomalous diffusion as underlying process, we have addressed the problem from a general point of view, assuming that both resettings and underlying processes are described in a general fashion with an arbitrary resetting density, $\psi_r(\tau)$, and an unspecified propagator, $p_0(x, t|x_0)$, for reset-free processes.

From this general analysis we have shown that, under rather general assumptions—basically the existence of a finite first moment for $\psi_r(\tau)$—resettings first stabilize the underlying process, in the sense that a nonstationary process becomes stationary under resettings. Second, resettings may greatly reduce the mean first-passage time to some critical value and presents a minimum value for some critical rate to be determined after the details of the whole process are known. This constitutes a complementary view of the universal character of resettings that has been recently brought forward in the literature [14,15,20–25].

We have finally performed a thorough study of the subject when the underlying process is a time fractional diffusion with exponent $\alpha \in (0, 2)$, covering subdiffusion when $0 < \alpha < 1$ and superdiffusion when $1 < \alpha < 2$. The reset-free process possesses no stationary distribution and the mean first-passage time is infinite. We have shown that for Poissonian resettings the stationary distribution has the form of a tent-shape density (i.e., Laplace distribution) given in Eq. (46). We have finally obtained explicit expressions for the survival probability—in Laplace space [Eq. (53)]—and the mean first-passage time [Eq. (54)]. We have shown that this is a nonmonotonous function of the resetting rate r and have obtained the minimum rate.

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[8]A similar approach to this issue has been very recently and independently derived in Ref. [28].

[9]In the particular case of ordinary diffusion, $\alpha = 1$, Eq. (55) has been recently obtained in Ref. [15] with the result $\xi = 1.594\dots$.

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