Localization behavior induced by asymmetric disorder for the one-dimensional Anderson model

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The behavior of the Lyapunov exponent under a small asymmetric disorder distribution is investigated for the one-dimensional Anderson model in the vicinity of the band center and for finite in-band energies. The special effect that could be found in systems with an asymmetric disorder distribution is shown to be small through a perturbation calculation. We obtain a quadratic formula for the Lyapunov exponent and show the enhancement of localization close to the band center induced by asymmetric disorder distribution. We find zero correction for an asymmetric disorder distribution with finite in-band energies. This quantitative behavior of the Lyapunov exponent explains why various asymmetric factors could be neglected in weakly disordered real systems. It also shows in what situation the asymmetric property of the disorder distribution should be considered during study of the localization behavior with a higher accuracy.

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I. INTRODUCTION

Since the ground-breaking work of Anderson [1], the theory of particles moving in one-dimensional random potentials has been well understood. In contrast to ordered systems, the one-dimensional wave functions are exponentially localized, no matter how small the disorder strength is, which implies the distinctive properties of disordered systems. The phenomena were not only rigorously proved by analytical approaches [2] but also observed in ultracold atomic systems [3] and photonic systems [4]. The one-dimensional wave function is characterized by the inverse localization length, or the Lyapunov exponent. The Lyapunov exponent has been extensively studied for various models depending on the disorder strength with or without correlation [5].

The one-dimensional Anderson model is not an exactly solvable model. In the weak-disorder limit, employing the regular perturbation theory, Thouless [6] obtained the leading-order formula for the Lyapunov exponent. Soon, Kappus and Wegner [7] found the anomalous behavior arising in the vicinity of the band center, which deviated from the Thouless formula. The anomaly effect was revealed by Derrida and Gardner [8]. They proposed a perturbation method which could overcome the divergence encountered in the standard perturbation theory and demonstrated that similar anomalies exist in the neighborhood of $E = 2\cos\alpha\pi$ for any rational number α. A recent work [9] pointed out that the band center and the band edge are the only points where the Thouless formula fails. Tessieri and Izrailev [10] obtained the anomaly using the Hamiltonian map method. New insight for the single-parameter scaling theory of disordered systems was provided for the band center [11].

Most of the earlier studies focused on the Anderson model with symmetric disorder distributions, such as Gaussian distribution [12], Lévy type [13], or white noises with a zero mean and a finite variance. And there are hardly any experimental reports on effects for an asymmetric disorder distribution. However, disorder distribution is not perfectly symmetric in practice, for example, the ion doping potentials in layers of semiconductor devices and laser speckle potentials [14]. Therefore, after the study of a symmetric disorder distribution, the next interesting investigation should explain the absence of asymmetric effects in experiments and the acceptance of the symmetric model in real disordered systems. In this paper, we explore the behavior of the Lyapunov exponent in weak asymmetric disorder potential. Using a parametrization method based on a transfer matrix, we analyze how the Lyapunov exponent is modified in the presence of a small third moment of a disorder distribution in comparison to the disorder strength. We show that at the band center there is a small, but nonvanishing, contribution to the Lyapunov exponent from the asymmetric property of the disorder distribution, and at finite in-band energies the contribution from the asymmetric property of disorder distribution is zero in the weak-disorder limit.

The article is organized as follows. In Sec. II, the problem is formulated using the parametrization method proposed in our previous work [15]. The integral equation for stationary distribution and the equation to obtain the Lyapunov exponent are presented. In Sec. III, the corresponding partial differential equation is derived in the weak-disorder limit for the band center with a nonzero third moment of a disorder distribution. A perturbation solution is obtained from the differential equation. In Sec. IV, we calculate the series expansion of the Lyapunov exponent up to the second order in energy and the third moment of a disorder distribution at the band center. In Sec. V, we resolve the integral equation at finite in-band

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energies and obtain the Lyapunov exponent with the third moment of a disorder distribution. A short summary then closes this study.

II. PARAMETRIZATION METHOD

The Anderson model is known as the simplest one to describe the behavior of electrons moving through a disorder lattice, whose Schrödinger equation can be expressed as

$$\psi_{i-1} + \psi_{i+1} = (E - \epsilon_i)\psi_i, \tag{1}$$

where ψ_i is the wave function on the ith site, E is the eigenenergy. ϵ_i are random on-site energies, which are uncorrelated to each other and share the same normalized distribution $p_\epsilon(\epsilon)$. The hopping energy has been set as a unit, so E and ϵ_i are dimensionless here. In this paper, we do not address any exact form of disorder distribution but assume that

$$\langle\epsilon\rangle = 0, \quad \langle\epsilon^2\rangle = \sigma^2, \quad \langle\epsilon^3\rangle = m_3\sigma^3, \tag{2}$$

where σ^2 is the variance of disorder distribution and m_3 is a constant satisfying $m_3\sigma \ll 1$. We investigate the effect of the $\langle\epsilon^3\rangle$ term, the asymmetric property for the disorder distribution $p_\epsilon(\epsilon)$.

In the transfer matrix method, we can rewrite the Schrödinger equation by two component vectors $\Psi_i = (\psi_i \quad \psi_{i-1})^T$,

$$\Psi_{L+1} = \begin{pmatrix} E-\epsilon_L & -1 \\ 1 & 0 \end{pmatrix}\Psi_L = \boldsymbol{T}_L \cdots T_1\Psi_1, \tag{3}$$

where $\boldsymbol{T}_i$ is the transfer matrix. In the parametrization method proposed in our previous work [15], an orthogonal matrix $\boldsymbol{U}(\theta_L)$ is used to diagonalize the product of transfer matrices $\boldsymbol{M}_L = \boldsymbol{T}_i\boldsymbol{T}_{i-1}\cdots\boldsymbol{T}_1$. We obtain the recursion relation of the parameter θ_L in the thermodynamic limit:

$$\tan\theta_{L+1} = \frac{1}{E - \epsilon_{L+1} - \tan\theta_L}. \tag{4}$$

We focus on the large-L limit, $L \to \infty$, and do not restrict to any sequence realization of ϵ_i. The recursive relation gives the exact integral equation for the stationary distribution function $p(\theta)$ of variable θ,

$$p(\theta)\sin^2\theta = \int_{-\infty}^{\infty} p_\epsilon(\epsilon)p(\theta')\cos^2\theta' d\epsilon, \tag{5}$$

where $\tan\theta' = E - \epsilon - \frac{1}{\tan\theta}$. As we can seen from the integral equation, the disorder distribution $p_\epsilon(\epsilon)$ and the energy E totally determine the properties of system.

With the normalized solution $p(\theta)$ of the integral equation, the inverse localization length, or the Lyapunov exponent can be expressed as

$$\begin{aligned}\gamma &= -\int_{-\pi/2}^{\pi/2} p(\theta)\ln|\tan\theta| d\theta \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta \int_{-\infty}^{\infty} d\epsilon \quad p(\theta)p_\epsilon(\epsilon) \\ &\quad \times \ln[1 + (E-\epsilon)^2\cos^2\theta - (E-\epsilon)\sin 2\theta].\end{aligned} \tag{6}$$

The first expression used by Ishii [2] for the Lyapunov exponent is equivalent to the second one. We see that the second expression is more practical to obtain the expansion in small values of σ^2 and $m_3\sigma^3$ for $|E| \ll 1$ in the band center region, while the first expression is better to calculate the expansion of the Lyapunov exponent at finite in-band energies.

III. EXPANSION NEAR THE BAND CENTER

The behavior of the Lyapunov exponent near the band center is dominated by the ratio of energy to disorder strength. In the case near the band center and with a weak-disorder limit with small asymmetry, $|E| \ll 1$, $\sigma^2 \ll 1$, and $m_3\sigma \ll 1$. In this section, we utilize the small value expansion method to obtain the differential equation for the stationary distribution $p(\theta)$ and then derive perturbation terms for the Lyapunov exponent in the next section.

From Eq. (5), considering the precondition $\sigma^2 \to 0$ and $E \to 0$, we can easily verify that $p(\theta) = p(\theta - \frac{\pi}{2})$. We cannot directly expand Eq. (5). This difficulty can be avoided by iterating the recursive integral equation twice:

$$p(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\epsilon_1 d\epsilon_2 \frac{\cos^2\theta_2 p_\epsilon(\epsilon_1)p_\epsilon(\epsilon_2)}{\sin^2\theta\tan^2\theta_1} p(\theta_2), \tag{7}$$

where $\tan\theta_1 = E - \epsilon_1 - \frac{1}{\tan\theta}$, $\tan\theta_2 = E - \epsilon_2 - \frac{1}{\tan\theta_1}$. The advantage gained by iterating twice is that θ_2 differs from θ in a small magnitude when $\sigma^2 \to 0$, $E \to 0$. Hence $p(\theta_2)$ can be written in terms of $p(\theta)$ and its derivatives.

The right-hand side of the above integral equation is expressible as a series in powers of E, ϵ_1, and ϵ_2. Keeping terms up to E and $\langle\epsilon^3\rangle$, and neglecting all the other higher-order terms in the limit of $\sigma^2 \to 0$, $E \to 0$, we get the following differential equation:

$$\begin{aligned}p(\theta) &= p(\theta) + Ep'(\theta) \\ &\quad + \sigma^2\left[\frac{(3+\cos 4\theta)p'(\theta)}{8} - \frac{(\sin 4\theta)p(\theta)}{4}\right]' - m_3\sigma^3 \\ &\quad \times \left\{\left[\frac{(3\cos 4\theta + 5)p'(\theta)}{48}\right]' - \frac{(3\cos 4\theta + 1)p(\theta)}{12}\right\}',\end{aligned} \tag{8}$$

where $p'(\theta)$ means $\frac{dp(\theta)}{d\theta}$, and the primes over the groups of expressions are the derivative with respect to θ. This equation possesses the effect for the third moment of the disorder distribution, which gives asymmetric features.

In the weak-disorder limit, we define $x = E/\sigma^2$, $y = m_3\sigma = m_3\sigma^3/\sigma^2$ as small parameters. Then we look for a series solution of the differential equation (8) for the stationary distribution $p(\theta)$,

$$\begin{aligned}p(\theta) &= p_0(\theta)[1 + xp_x(\theta) + yp_y(\theta) \\ &\quad + x^2p_{xx}(\theta) + xyp_{xy}(\theta) + y^2p_{yy}(\theta) + \cdots].\end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (8), we obtain the differential equation for each order of x and y. $p(\theta)$ should meet requirements for normalization and periodicity condition,

$$\begin{aligned}&\int_{-\pi/2}^{\pi/2} p_0(\theta)d\theta = 1, \quad \int_{-\pi/2}^{\pi/2} p_0(\theta)p_{xn,ym}(\theta)d\theta = 0, \\ &p_{xn,ym}(\theta) = p_{xn,ym}(\theta + \pi),\end{aligned} \tag{10}$$

where $p_{xn,ym}$ represents the perturbation term of order x^ny^m.

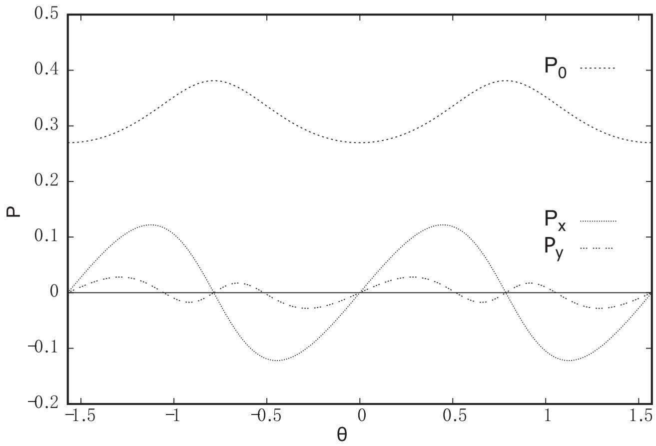

FIG. 1. $p_0(\theta)$, $p_x(\theta)$, and $p_y(\theta)$, the zero- and the first-order expansion terms for the stationary distribution in Eq. (9).

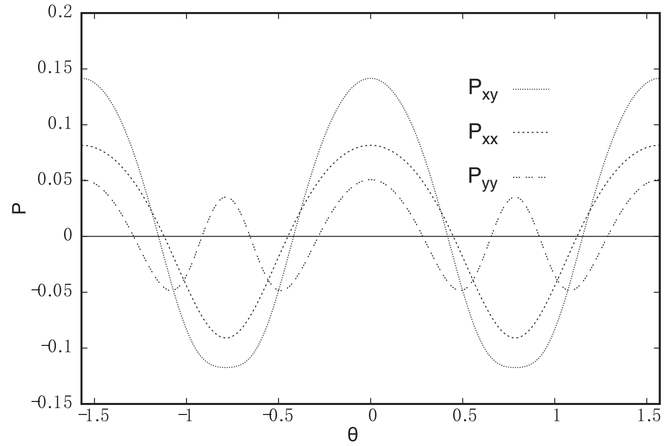

FIG. 2. $p_{xy}(\theta)$, $p_{xx}(\theta)$, and $p_{yy}(\theta)$, the second-order expansion terms for the stationary distribution in Eq. (9).

The differential equation for the first term $p_0(\theta)$ corresponding to the limit $\sigma^2 \to 0$, $E = 0$ without asymmetry is

$$\frac{3+\cos 4\theta}{8}p_0''(\theta) - \frac{3\sin 4\theta}{4}p_0'(\theta) - (\cos 4\theta)p_0(\theta) = 0. \quad (11)$$

The normalized solution to this equation is

$$p_0(\theta) = \frac{1}{K\left(\frac{1}{\sqrt{2}}\right)\sqrt{3+\cos 4\theta}}, \quad (12)$$

where $K(\frac{1}{\sqrt{2}})$ is the complete elliptic integral of the first kind, $K(k) = \int_0^{\pi/2} d\phi/\sqrt{1-k^2\sin^2\phi}$. Equation (11) is consistent with the results of Refs. [16] and [17] obtained by the quasidegenerate perturbation theory and the Hamiltonian map method, respectively.

First, let us consider the leading perturbation terms. It is clear that p_x stands for the energy perturbation term with a zero asymmetric property, and p_y is induced by the third moment of the disorder distribution when energy equals zero strictly. We calculate them from the differential equation up to their order, respectively:

$$p_x(\theta) = \int_0^\theta \left[2\sqrt{2}\pi\, p_0(\phi) - \frac{8}{3+\cos 4\phi}\right]d\phi, \quad (13)$$

$$p_y(\theta) = \int_0^\theta \left[\frac{\pi\, p_0(\phi)}{4\sqrt{2}} - \frac{8\sin^2 4\phi}{(3+\cos 4\phi)^3}\right]d\phi. \quad (14)$$

Both $p_x(\theta)$ and $p_y(\theta)$ can be written out in elliptic integral functions. We plot the zero-order stationary distribution $p_0(\theta)$ and the first-order corrections $p_x(\theta)$ and $p_y(\theta)$ in Fig. 1. It shows that the magnitude of $p_y(\theta)$ is smaller than the magnitude of $p_x(\theta)$.

Repeating the same procedure as above, perturbation terms of the second order are given by

$$p_{xx}(\theta) = C_1 - \int_0^\theta \frac{8}{3+\cos 4\phi}p_x(\phi)d\phi, \quad (15)$$

$$p_{xy}(\theta) = C_2 + \frac{3\cos 4\theta + 5}{6(3+\cos 4\theta)}p_x'(\theta) - \int_0^\theta \left[\frac{8(\sin^2 4\phi)p_x(\phi)}{(3+\cos 4\phi)^3} + \frac{8p_y(\phi)}{3+\cos 4\phi}\right]d\phi, \quad (16)$$

$$p_{yy}(\theta) = C_3 + \frac{3\cos 4\theta + 5}{6(3+\cos 4\theta)}p_y'(\theta) - \int_0^\theta \frac{8(\sin^2 4\phi)p_y(\phi)}{(3+\cos 4\phi)^3}d\phi, \quad (17)$$

where C_1, C_2, and C_3 are integral constants. These constants are calculated by normalizing the correction terms of the stationary distribution to zero numerically: $C_1 = 0.0816$, $C_2 = 0.00953$, and $C_3 = 0.000966$. We plot the second-order terms $p_{xy}(\theta)$, $p_{xx}(\theta)$, and $p_{yy}(\theta)$ in Fig. 2. It shows that the magnitude of the $p_{yy}(\theta)$ correction is smaller than the magnitude of $p_{xx}(\theta)$. After obtaining all the perturbation terms up to the second order for the stationary distribution, we calculate the Lyapunov exponent up to the order of σ^2 next.

IV. LYAPUNOV EXPONENT NEAR THE BAND CENTER

The first expression in Eq. (6) requires the σ^2-order term of the stationary distribution $p(\theta)$ in order to give the σ^2-order term for γ. Alternatively, we use the second expression for the Lyapunov exponent in Eq. (6) for the band center, by which we only need the terms of zero order of σ^2 for the stationary distribution $p(\theta)$, and these terms have been obtained in the previous section. We expand the logarithmic function in the second expression for the Lyapunov exponent in Eq. (6), and integrate ϵ out up to $\langle\epsilon^3\rangle$ terms:

$$\gamma = \frac{\sigma^2}{2}\int_{-\pi/2}^{\pi/2} d\theta\, p(\theta)\left[\frac{1+2\cos 2\theta + \cos 4\theta}{4} - x\sin 2\theta - y(\sin 2\theta)\frac{2+3\cos 2\theta+\cos 4\theta}{6}\right]. \quad (18)$$

After integrating ϵ out, we find that the second expression for the Lyapunov exponent in Eq. (6) gives zero coefficient for the zero order of σ^2 when $E = 0$. It is the convenient way to evaluate γ for the band center region.

We write γ in power expansion of x and y:

$$\gamma = \gamma_0 + x\gamma_x + y\gamma_y + x^2\gamma_{xx} + xy\gamma_{xy} + y^2\gamma_{yy} + \cdots. \quad (19)$$

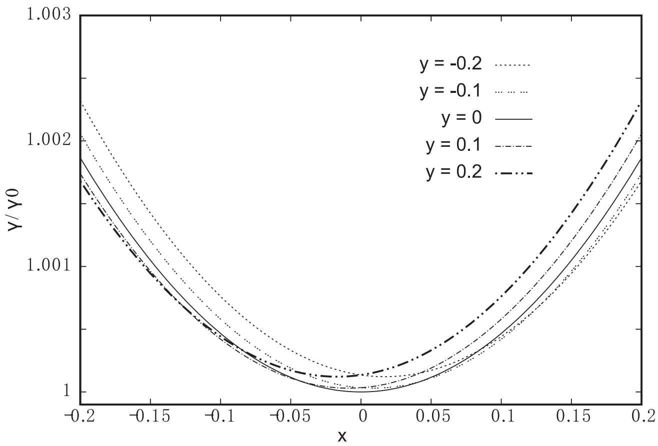

FIG. 3. Dependence of Lyapunov exponent on $x = E/\sigma^2$. The lines are for several small values of the third moment $y = \langle\epsilon^3\rangle/\sigma^2$. The location of minimum points of each curve is at $x = -0.086y$, with a lifted Lyapunov exponent as the absolute magnitude of y increases.

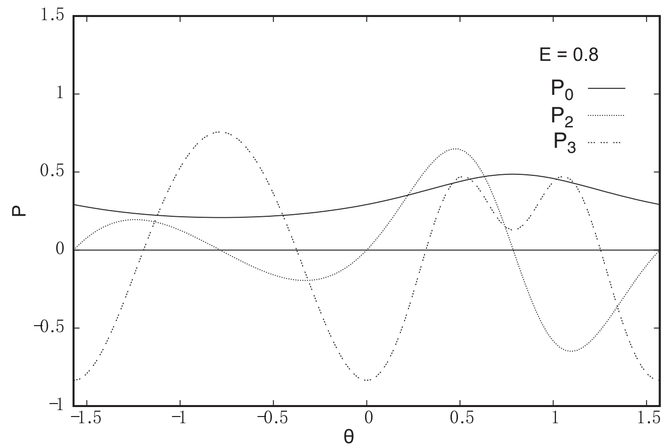

FIG. 4. The expansion of the stationary distribution at a finite energy $E = 0.8$. The lines $p_0(\theta)$, $p_2(\theta)$, and $p_3(\theta)$ are defined in Eq. (22).

Inserting into Eq. (18) the stationary distribution $p_0(\theta)$ for $E = 0$ and $\langle\epsilon^3\rangle = 0$, we have

$$\gamma_0 = \left[\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]^2 \sigma^2, \tag{20}$$

which was obtained in Refs. [8,16,17]. From the symmetry of $p_x(\theta)$, $p_y(\theta)$, one immediately knows that the terms corresponding to order x and y in γ equal zero. Substituting the expansions of $p(\theta)$ into Eq. (18) and summing up all terms in each order of x^2, xy, and y^2, we obtain the power series for γ numerically,

$$\gamma = \gamma_0 + \sigma^2(0.00533x^2 + 0.000912xy + 0.000387y^2). \tag{21}$$

The coefficients before y terms are much smaller than the one before x^2. We plot γ as the function of x and y in Fig. 3. It shows that γ is affected little by the small asymmetric shape of the disorder distribution $p_\epsilon(\epsilon)$.

The terms corresponding to the first order of x and y vanish in γ; therefore, the Lyapunov exponent is given by a quadratic form in x and y. The absence of the first-order terms can be understood from the Hamiltonian of system (1). Let $\{\psi_i\}$ be the wave function for a realization $\{\epsilon_i\}$ of $p_\epsilon(\epsilon)$. Then $\{(-1)^i\psi_i\}$ is a wave function for the Hamiltonian with energy $-E$ and disorder realization $\{-\epsilon_i\}$, and we have $\tan\theta' = E - \epsilon - 1/\tan\theta \;\to\; \tan(-\theta') = (-E) - (-\epsilon) - 1/\tan(-\theta)$. At the order of x, no asymmetric property is needed and we can take $p_\epsilon(-\epsilon) = p_\epsilon(\epsilon)$. Consequently, it results in the relation $p(\theta, E) = p(-\theta, -E)$. By $\gamma = -\int_{-\pi/2}^{\pi/2} p(\theta)\ln|\tan\theta| d\theta$, an even function of E, we reach $\gamma_x = 0$. Energy can be set as zero when studying γ_y. When the asymmetry is included, we have $p_\epsilon(\epsilon) \neq p_\epsilon(-\epsilon)$. However, in the same way, $\{(-1)^i\psi_i\}$ is the wave function for disorder distribution $p_\epsilon(-\epsilon)$, and so $p(\theta, m_3\sigma^3) = p(-\theta, -m_3\sigma^3)$. Therefore, γ is an even function of y, and we get $\gamma_y = 0$.

As plotted in Fig. 3, when the third moment increases, the position of the minimum point of the Lyapunov exponent curve is gradually moving away from the band center. The moving direction of the minimum position is opposite to the sign of the third moment. It is evident that the shift comes from the combined effect of nonzero energy and asymmetry. The term γ_{yy} is responsible for adjustment of height. Figure 3 shows that the minimum value of the γ curve tends to increase along with the absolute value of the third moment. It means that, given a zero mean and a finite variance, a larger third moment will lead to a more localized system.

V. LYAPUNOV EXPONENT FOR NONZERO ENERGIES

In this section we study the asymmetric correction for γ at a finite in-band energy. We show this correction is zero at the order of σ^2. For a continuous distribution with a zero mean, a standard deviation σ^2, and a third moment $\langle\epsilon^3\rangle = y\sigma^2$, setting other higher-order moments as zero, we have the stationary distribution for θ expanded up to the order of σ^2,

$$p(\theta) = p_0(\theta)\{1 + \sigma^2[p_2(\theta) + yp_3(\theta)] + \cdots\}. \tag{22}$$

Following the procedure we did for the zero-energy region in previous sections, we insert the above $p(\theta)$ into Eq. (5), then solve the equation up to the order of $\langle\epsilon^3\rangle$. We obtain

$$p_0(\theta) = \frac{\sqrt{4 - E^2}}{\pi(2 - E\sin 2\theta)}, \tag{23}$$

$$p_2(\theta) = \frac{\sin 4\theta}{E(2 - E\sin 2\theta)^2}, \tag{24}$$

$$p_3(\theta) = \frac{1}{E^3(E^2-1)}\left[\frac{2(E^2-16)}{E\sin 2\theta - 2} - \frac{4(2E^4 - 13E^2 + 24)}{(E^2-4)^2} + \frac{8(E^2-7)}{(E\sin 2\theta - 2)^2} + \frac{8(E^2-4)}{(E\sin 2\theta - 2)^3}\right], \tag{25}$$

where the distribution $p_0(\theta)$ has been normalized to 1. Integrations over $[-\pi/2, \pi/2]$ for $p_0(\theta)p_2(\theta)$ and $p_0(\theta)p_3(\theta)$ are zero. We illustrate the obtained expansion in Fig. 4 for $E = 0.8$.

The inverse of the localization length is obtained by the first expression for γ in Eq. (6). The only nonzero result of the involved integrations is

$$-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_0(\theta)p_2(\theta)\log|\tan\theta|\,d\theta = \frac{1}{2(4-E^2)}. \quad (26)$$

We obtain $\gamma = \frac{\sigma^2}{2(4-E^2)}$, the same result found in previous studies [6]. At a finite in-band energy the localization length depends on σ^2 only. It is not affected by the small asymmetric property of disorder distribution in the weak-disorder limit. Therefore, we should not expect much experimental phenomena from the asymmetric property of a disorder distribution in the weak disorder. We found no experimental reports on this aspect either. This conclusion is in agreement with one-parameter scaling theory, which fails only for the zero-energy region [11].

A refinement for the degenerate point $E = 1$ [8,9,16,18] is worth noting. There is no further correction to $\gamma = \frac{\sigma^2}{6}$ at the order of σ^2. However, $p_2(\theta)$ and $p_3(\theta)$ can be fixed only by higher-order equations accurate for σ^4 with $\langle\epsilon^4\rangle$ terms. The solutions for $p_2(\theta)$ and $p_3(\theta)$ presented in Eqs. (24) and (25) are not correct for the $E = 1$ point.

Finally we should consider the boundary between the two distinct behaviors in the weak-disorder limit. The small asymmetric disorder has a positive effect on the Lyapunov exponent close to the band center while it has no effect away from it. The energy scale, beyond which the effect of small asymmetric disorder vanishes, is $E \sim \sigma^2$. The crossover region cannot be quantified through perturbation series in powers of σ^2/E or E/σ^2 from either side. We could examine briefly the crossover region by numerical calculation.

To compute $p(\theta)$ numerically from Eq. (5), we consider terms up to the third moment $\langle\epsilon^3\rangle$ and neglect all the other higher-order moments in Eq. (5),

$$p(\theta)\sin^2\theta = \cos^2\theta_0\bigg(p(\theta_0) + \frac{\sigma^2}{2}(\cos^2\theta_0(\cos^2\theta_0 p(\theta_0))')' - \frac{y\sigma^2}{6}(\cos^2\theta_0(\cos^2\theta_0(\cos^2\theta_0 p(\theta_0))')')'\bigg), \quad (27)$$

where $\tan\theta_0 = E - 1/\tan\theta$, $y = \langle\epsilon^3\rangle/\sigma^2$, and the primes over the groups of expressions are derivatives with respect to θ_0. We compute $\gamma(E, y)$ through the numerical solution $p(\theta)$ of the above differential equation for $\sigma = 10^{-4}$.

The effect from small asymmetric disorders is too small to display in a figure for $\gamma(E, y)$ with different values of E and y. We adopt $\delta\gamma(E, y)$ to display this effect in Figs. 5 and 6:

$$\delta\gamma(E, y) = \frac{[\gamma(E, y) - \gamma(E, 0)]}{\sigma^2/(1 - E^2/4)}. \quad (28)$$

In Fig. 5 we demonstrate the Lyapunov exponent close to the band center. It shows that γ changes along with E and y as Eq. (21) near the band center, and deviates from Eq. (21) when $E > 0.1\sigma^2$. In Fig. 6 we demonstrate the Lyapunov exponent out of the band center. It shows that γ changes along with E and y with a $\frac{y}{E/\sigma^2}\frac{\sigma^2}{1-E^2/4}$ term when E is big. This kind of σ^4 term is not exactly zero in the perturbation series for finite E in the weak-disorder limit. γ deviates from this finite E behavior when $E/\sigma^2 < 10$. As in this brief numerical examination, the energy of the crossover region is on the order of $E \sim \sigma^2$.

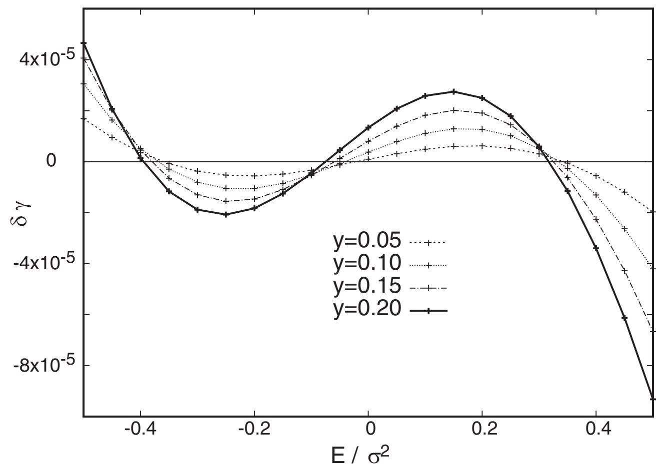

FIG. 5. Dependence of the Lyapunov exponent on asymmetric disorder in the band center region. $\delta\gamma$ is given in Eq. (28).

VI. CONCLUSION

This study has been mainly concerned with the consequence of an asymmetry disorder distribution in the one-dimensional Anderson model. We analyzed the Lyapunov exponent of the one-dimensional Anderson model with diagonal disorder. The third moment of a disorder distribution was used to represent asymmetric property. A small third moment as a perturbation term was included in a differential equation for the model. A small third moment shifted the curve of the Lyapunov exponent versus energy a little upward, and lifted its minimum value around the band center. We found that the small asymmetric property has no effect on the Lyapunov exponent at finite in-band energies.

These results provide some understanding on the absence of the effect of a small asymmetric disorder distribution in real systems in experimental observations. They also provide a quantitative estimation on the required precision of an

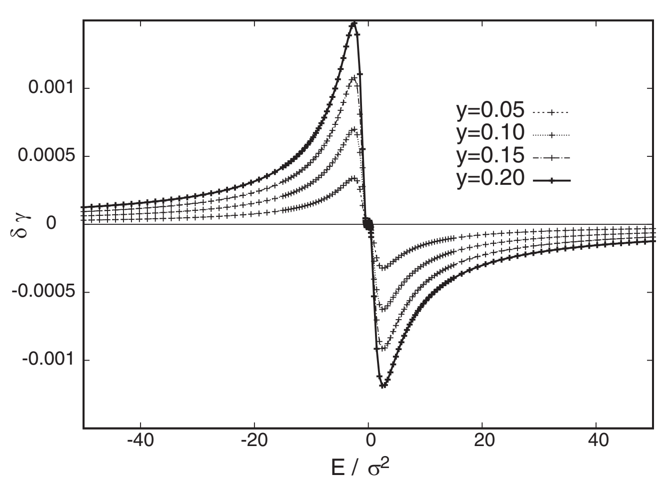

FIG. 6. Dependence of the Lyapunov exponent on asymmetric disorder out of the band center region. $\delta\gamma$ is given in Eq. (28).

experiment when the effect of a small asymmetric disorder distribution is targeted. To approach the effect of disorder strength as small but not in the weak-disorder limit, we expect the problem to be much more formidable since the fourth moment of the disorder distribution will come into play.

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