Rapid Communications

Simple Hamiltonian approach to describe large-amplitude and higher-order parametric resonances

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Parametric resonance is a complex phenomenon that touches many aspects of scientific and technical society, but is still not well understood because of the intensive calculations required to describe the behavior. Thus the importance of developing simple mathematical approaches to describe parametric resonance cannot be overstated. Here a consistent theory of the parametric resonance of a harmonic oscillator under any periodic frequency modulation is constructed. Using a Hamiltonian approach and resonance approximation, simple equations were derived and critical amplitudes for all parametric resonance orders were obtained for any periodic modulation function. The theory agrees with the well-known result for the main resonance and gives correct power dependence on damping. In addition, the theory qualitatively predicts behavior at large modulations. This simplified approach revealed unique features—"safety windows" at large modulation amplitudes where parametric resonance does not occur which were then qualitatively confirmed with numerical simulations. The Hamiltonian approach should serve as a framework that a greater understanding of parametric resonance at large amplitudes and higher orders can be built upon.

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I. INTRODUCTION

The parametric resonance of a harmonic oscillator is a well-known phenomenon that manifests in oscillating systems with periodic modulation of the frequency, whereby when a certain threshold amplitude is reached leads to an exponentially rapid increase in the amplitude of the excited oscillations. This common interdisciplinary effect manifests itself in many completely different areas and environments, such as in mechanics [1–5], electronics [6–8], ion traps [9,10], atomic microscopy [11], DNA [12], Bose-Einstein condensate [13], nanoparticles [14], and many other phenomena. Sinusoidal modulation is usually considered with the Mathieu equation whose mathematical analysis is extremely complex. In the approximate and somewhat heuristic method of analyzing this model (see, e.g., [1,2,5]), small oscillations and small deviations from the resonant frequency are considered using perturbations in the dynamic equation of motion. In this approach each resonance condition should be in a certain level of accuracy and calculated separately, which is very inconvenient and restricts the physical picture. Usually the main resonance at double frequency is considered and the thresholds of other resonances are assumed to be very high and not of interest.

In this Rapid Communication we study periodic frequency modulation of the harmonic oscillator using the classical Hamiltonian formalism. The use of such an approach is logical because Hamiltonian formalism is the natural language for describing nonlocal oscillations of the medium and it has been used to consider the well-understood behavior of first-order parametric resonance of waves [15–18]. Building upon this work, here we demonstrate the power of this technique to simplify and describe parametric resonance under conditions that are typically very computationally challenging. More specifically, this Rapid Communication serves to demonstrate that the Hamiltonian approach significantly simplifies the analysis of parametric resonance conditions at large amplitudes and higher-order resonances. The only assumption of the theory that we use is a validity of the resonance approximation, in which variables are selected in the linear model that oscillate with only one frequency, while the role of the other oscillations is considered negligible. To understand the limitations of the expressions obtained, numerical simulations were conducted and compared with the theory.

II. HARMONIC OSCILLATOR

Without loss of generality, we consider the Hamiltonian of the LC circuit of the form

$$\mathcal{H} = \frac{p^2}{2L} + \frac{L\omega_0^2 q^2}{2}, \tag{1}$$

where $\omega_0 = 1/\sqrt{LC}$ is the resonance frequency, L is the inductance and C is the capacitance. For the generalized coordinate q, describing the charge, and the generalized angular momentum $p = Ldq/dt$, we have the following equations of motion:

$$\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}. \tag{2}$$

Let us introduce complex variables

$$a = \frac{1}{2}\left(\frac{q}{q_0} + i\frac{p}{p_0}\right), \qquad a^* = \frac{1}{2}\left(\frac{q}{q_0} - i\frac{p}{p_0}\right), \tag{3}$$

where $q_0 = \sqrt{\varkappa/2L\omega_0}$ and $p_0 = \sqrt{\varkappa L\omega_0/2}$; $\varkappa$ is a dimensional (energy×time) constant which is introduced here for convenience so that the complex variables are dimensionless

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[19]. Thus, the Hamiltonian (1) takes the form

$$\mathcal{H}/\varkappa = \omega_0 a^* a. \tag{4}$$

The equations of motion become classical analogs of the Heisenberg equations

$$i\frac{da}{dt} = [a, \mathcal{H}/\varkappa]_c, \quad i\frac{da^*}{dt} = [a^*, \mathcal{H}/\varkappa]_c, \tag{5}$$

where

$$[\mathcal{A}, \mathcal{B}]_c = \frac{\partial \mathcal{A}}{\partial a}\frac{\partial \mathcal{B}}{\partial a^*} - \frac{\partial \mathcal{B}}{\partial a}\frac{\partial \mathcal{A}}{\partial a^*} \tag{6}$$

is the classical analog of operator commutator.

Usually, the *LC* circuit equation contains a term describing ohmic losses:

$$\frac{d^2q}{dt^2} + 2\eta\frac{dq}{dt} + \omega_0^2 q = 0. \tag{7}$$

Here $\eta = R/L$; R is the resistence in the circuit. The above losses can be accounted for in the system (5) if we supplement these equations with the damping η:

$$\begin{aligned} i\left(\frac{da}{dt} + \eta a\right) &= \frac{\partial \mathcal{H}/\varkappa}{\partial a^*}, \\ i\left(\frac{da^*}{dt} + \eta a^*\right) &= -\frac{\partial \mathcal{H}/\varkappa}{\partial a}. \end{aligned} \tag{8}$$

Here we neglect the nonsecular damping terms ηa^* and ηa, respectively. Since the second equation is simply complex conjugate to the first one, it is not necessary to write it down. Note that the method of complex variables for oscillatory processes that we use has long been known in the literature, e.g., [6,15,20].

III. TIME-DEPENDENT PARAMETER

Without loss of generality, we can consider a parametric change in the frequency of the form

$$\omega_0^2 \Rightarrow \omega_0^2[1 + f(t)] = \omega_0^2 + \omega_0^2 f(t). \tag{9}$$

Then the Hamiltonian (1) becomes

$$\mathcal{H} = \frac{p^2}{2L} + \frac{L\omega_0^2 q^2}{2} + \frac{L\omega_0^2 f(t) q^2}{2}. \tag{10}$$

Using complex variables (3), from (10) we get

$$\mathcal{H}/\varkappa = \omega_0\left[1 + \frac{f(t)}{2}\right]a^* a + \frac{\omega_0}{4} f(t)(a^2 + a^{*2}). \tag{11}$$

Then Eq. (8) acquires the following form:

$$i\left(\frac{da}{dt} + \eta a\right) = \omega_0\left[1 + \frac{f(t)}{2}\right]a + \frac{\omega_0}{2} f(t) a^*. \tag{12}$$

Here as expected, dimensional constant $\varkappa$ dropped out of the equations.

One can apply a change of variable

$$a = b\exp\left[-i\frac{\omega_0}{2}\int_0^t f(t')dt'\right] \tag{13}$$

and obtain from Eq. (12)

$$i\left(\frac{db}{dt} + \eta b\right) = \omega_0 b + \frac{\omega_0}{2} F(t) b^*, \tag{14}$$

where

$$F(t) = f(t)\exp\left[i\omega_0\int_0^t f(t')dt'\right]. \tag{15}$$

Let us now consider a periodic time dependence $f(t) = f(t + T)$, where T is a period. Then the function (15) can be expanded in the Fourier series and Eq. (14) becomes

$$i\left(\frac{db}{dt} + \eta b\right) = \omega_0 b + \frac{\omega_0}{2}\sum_{n=-\infty}^{\infty} F_n \exp(in\omega t) b^*, \tag{16}$$

where $\omega = 2\pi/T$ and

$$F_n = F_{-n}^* = \frac{1}{T}\int_0^T F(t)\exp(-in\omega t)dt. \tag{17}$$

IV. RESONANCE $\omega_0 \simeq n\omega/2$ APPROXIMATION

The fast-oscillating dependence on time in Eq. (16) can be partly eliminated by transforming b and b^* to so-called, slow variables c and c^* as follows:

$$b = ce^{-in\omega t/2}, \qquad b^* = c^* e^{in\omega t/2}. \tag{18}$$

As a result one obtains

$$\begin{aligned} i\frac{dc}{dt} &= \left(\omega_0 - \frac{n\omega}{2} - i\eta\right)c + \frac{\omega_0}{2}[F_n^* + \mathcal{F}_n^*(t)]c^*, \\ i\frac{dc^*}{dt} &= -\left(\omega_0 - \frac{n\omega}{2} + i\eta\right)c^* - \frac{\omega_0}{2}[F_n + \mathcal{F}_n(t)]c, \end{aligned} \tag{19}$$

where

$$\mathcal{F}_n(t) = \sum_{\nu\neq n} F_\nu \exp[i(\nu - n)\omega t].$$

It is easy to see that in Eq. (19) the only term with an explicit time dependence is $\mathcal{F}_n(t)$. If $|\omega_0 - n\omega/2| \ll \omega$, then the function $\mathcal{F}_n(t)$ consists of rapidly oscillating terms compared to the slow dynamics of the rest of the system. The resonance approximation consists in the assumption that we can neglect the role of the rapidly oscillating terms (their average is assumed to be zero) and consider slow dynamics of c and c^* variables which describe the behavior of the system in the vicinity of the resonance $\omega_0 \simeq n\omega/2$. Thus from Eq. (19) one obtains

$$i\frac{d}{dt}\begin{pmatrix} c \\ c^* \end{pmatrix} = \begin{pmatrix} \omega_0 - \frac{n\omega}{2} - i\eta & \frac{\omega_0 F_n^*}{2} \\ -\frac{\omega_0 F_n}{2} & -\omega_0 + \frac{n\omega}{2} - i\eta \end{pmatrix}\begin{pmatrix} c \\ c^* \end{pmatrix}. \tag{20}$$

Now we are looking for an instability of the form

$$c = c_1 e^{\lambda t}, \qquad c^* = c_1^* e^{\lambda t}. \tag{21}$$

From Eq. (20) follows

$$\det\begin{pmatrix} \omega_0 - \frac{n\omega}{2} - i(\eta + \lambda) & \frac{\omega_0 F_n^*}{2} \\ -\frac{\omega_0 F_n}{2} & -\omega_0 + \frac{n\omega}{2} - i(\eta + \lambda) \end{pmatrix} = 0. \tag{22}$$

As the result one gets

$$\lambda_\pm = -\eta \pm \sqrt{\left(\frac{\omega_0}{2}|F_n|\right)^2 - \left(\omega_0 - \frac{n\omega}{2}\right)^2}. \tag{23}$$

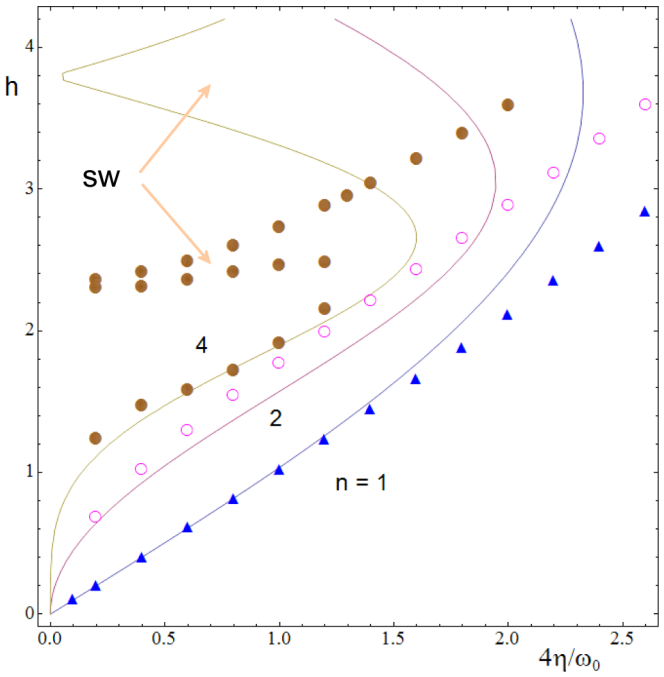

FIG. 1. Solid lines represent the critical amplitudes for various n versus relative damping obtained from Eq. (29). Points represent the results of numeric simulation of Eq. (33). The up arrow indicates the theoretical area of theoretical safety window, and the down arrow indicates the result of the numerical simulation.

From the condition $\lambda = 0$ follows the threshold of parametric resonance

$$\left(\frac{\omega_0}{2}|F_n|\right)^2 = \left(\omega_0 - \frac{n\omega}{2}\right)^2 + \eta^2, \tag{24}$$

which is minimal at the exact resonance condition $\omega_0 = n\omega/2$:

$$|F_n| = 2\eta/\omega_0. \tag{25}$$

Thus, our theory suggests that the parametric resonance should occur in the range

$$-\Delta_n < \frac{2}{n} - \frac{\omega}{\omega_0} < \Delta_n,$$
$$\Delta_n = \frac{1}{n}\sqrt{|F_n|^2 - (2\eta/\omega_0)^2}, \tag{26}$$

provided $\Delta_n \ll 1$. If the latter strong inequality does not hold, then the estimate of the interval where the parametric resonance occurs unlikely can be done rigorously. But the close vicinity ω/ω_0 to $2/n$ may be hoped to be suitable even in this case at least for $\Delta_n \sim 1$ (see Fig. 1).

Formulas (25) and (26) represent a general solution in a resonance approximation for the harmonic oscillator parametric resonance problem in a periodic field.

V. EXAMPLE: SINE FUNCTION

Let us consider $f(t) = h\cos(\omega t)$; then one has $\int_0^t f(t')dt' = (h/\omega)\sin(\omega t)$. Using the following relation [21]

$$\exp(iz\sin\theta) = \sum_{k=-\infty}^{\infty} J_k(z)\exp(ik\theta), \tag{27}$$

we obtain

$$F_n = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} h\cos(\omega t)\exp\left[\frac{ih\omega_0}{\omega}\sin(\omega t) - in\omega t\right]dt$$
$$= \frac{h}{2}\left[J_{n-1}\left(\frac{\omega_0 h}{\omega}\right) + J_{n+1}\left(\frac{\omega_0 h}{\omega}\right)\right]. \tag{28}$$

At the resonance $\omega_0 = n\omega/2$ from Eqs. (25) and (28) we get the equation for critical amplitude $h_c^{(n)}$,

$$h_c^{(n)}\left|J_{n-1}\left(\frac{n}{2}h_c^{(n)}\right) + J_{n+1}\left(\frac{n}{2}h_c^{(n)}\right)\right| = \frac{4\eta}{\omega_0}. \tag{29}$$

Using Bessel function approximation $J_n(x) \simeq (x/2)^n/n!$ at $x \ll 1$, from (29) one obtains at $nh_c^{(n)}/2 \ll 1$:

$$h_c^{(n)} \simeq C_n(2\eta/\omega_0)^{1/n}, \tag{30}$$

where

$$C_n = \left(\frac{4}{n}\right)^{1-1/n}[2(n-1)!]^{1/n}. \tag{31}$$

With the accuracy of notations from this relation we have well-known results [1]: $h_c^{(1)} \simeq 4\eta/\omega_0$ for the main resonance $n = 1$ and dependence $h_c^{(n)} \propto \eta^{1/n}$ for $n \geqslant 1$. For $n = 1$ the formula (26) coincides with the classical results [1]:

$$-\sqrt{\left(\frac{\omega_0 h}{2}\right)^2 - 4\eta^2} < 2\omega_0 - \omega < \sqrt{\left(\frac{\omega_0 h}{2}\right)^2 - 4\eta^2}. \tag{32}$$

Several numerical values of the coefficient C_n from the formula (31) are given in the first row of Table I. We note that the relation of the form (30) was considered in Refs. [9,10]. In Ref. [9], the coefficient C_n was calculated numerically with the accuracy 5% for $n = 1, \ldots, 5$. These coefficients are shown in the second row of Table I. It can be seen that the results of our work and numerical calculation coincide with good accuracy. In Ref. [10], an analytical expression of the form (30) was obtained using the perturbation method in the Mathieu equation with damping. For this case numerical estimates of the coefficient C_n are given in the third row of Table I. We see the agreement of this theory with numerical calculation only for $n = 1$. As n increases, there is a significant discrepancy between theory [10] and numerical calculation [9].

TABLE I. Threshold coefficient C_n.

n	1	2	3	4	5
Our result, Eq. (31)	2	2	1.923	1.861	1.814
Numeric calculation [9]	2	2.03	1.83	1.73	1.73
From Mathieu equation [10]	2	1.144	1.165	1.030	0.945

Solid lines in Fig. 1 demonstrate several solutions of Eq. (29). Consider the curve corresponding to the fourth order of parametric resonance. It is clear that the threshold of resonance monotonously increases only to some value of attenuation. Then this function turns in the other direction and

continues to grow to a certain point with a small attenuation. Then there is another turn to the right, then (beyond the framework of Fig. 1) another turn to the left, and so on. All the orders of resonances show the same qualitative behavior; only the vertical scale changes. The largest scale has the main $n = 1$ resonance and with increasing order of resonance the scale decreases. An "oscillating" behavior of critical fields leads to "safety windows," areas where even for large amplitudes of modulation, parametric resonance does not occur.

To determine the range of applicability of the solutions obtained, we carried out a numerical simulation of the parametric resonance threshold using the following equation:

$$\frac{d^2q}{d\tau^2} + \left(\frac{2\eta}{\omega_0}\right)\frac{dq}{d\tau} + \left[1 + h\cos\left(\frac{2\tau}{n}\right)\right]q = 0, \qquad (33)$$

where $\tau = \omega_0 t$. Numerical calculation was carried out independently by two different computer programs. In *Mathematica*, the function NDSOLVE was used and in MATLAB, the subroutine ode23 was used. Calculation results obtained independently coincided. Points in Fig. 1 demonstrate the results of simulations. It can be seen that the theory and numeric results agree well enough for the critical field up to $h \simeq 1$ and do not have large differences up to $h \simeq 2$. It is also easy to see that on the fourth-order resonance curve there is a bend to the left, and then to the right, which lead to the appearance of a safety window shown by the down arrow in Fig. 1. The up arrow shows the corresponding qualitative theoretical prediction in a resonance approximation. Thus the simple theory qualitatively captures this result. The safety windows for first and second orders of resonances were found at large amplitudes $h > 5$, but we did not study them in detail.

It should be noted that the resonance approximation and the corresponding transition to slow variables [as in Eq. (18)] in the vicinity of the resonance is a standard procedure in the theory of oscillations. It has been used in the theory of parametric resonance of waves [15–18], in the theory of magnetic resonance (transition to rotating frame) [22], and in quantum optics [23]. The applicability of the resonance approximation in the theory of oscillations was justified in the development of the asymptotic Krylov-Bogolyubov-Mitropolsky methods [24]. There is a regular procedure for finding corrections to the resonance solution with respect to a small parameter $h \ll 1$ of the modulating function $h\cos(2\tau/n)$. As we see from the results obtained above, the resonance approximation leads to satisfactory agreement with the numerical calculation far above the perturbation approach, up to $h \simeq 2$ and for even greater h shows the qualitative behavior of the system. From these facts follows the need to develop a regular method for calculating critical amplitudes in the region $h \gtrsim 1$.

VI. CONCLUSION

The Hamiltonian theory of parametric resonance was constructed for the general case of any periodic frequency modulation of a harmonic oscillator. The calculations were carried out without using perturbation theory methods. We used the resonance approximation within which the general formulas (24)–(26) were obtained for all orders ($n = 1, 2, 3, \ldots$) of parametric resonance. At large modulation levels, the theory qualitatively predicts safety windows, areas in which parametric resonance does not manifest (see Fig. 1). Numerical simulations supported the validity of the formulas obtained for high modulation amplitudes $h \sim 1$. The work presented in this Rapid Communication stands as strong support that exploring parametric resonance with the Hamiltonian approach can be a revealing alternative to the Mathieu equation under conditions that are often considered prohibitively complex.

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