# Analytic solution for the zero-order postshock profiles when an incident planar shock hits a planar contact surface

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An explicit analytical solution to calculate the profiles after the shock collision with a planar contact surface is presented. The case when a shock is reflected after the incident shock refraction is considered. The goal of this work is to present explicit formulas to obtain the quantities behind the transmitted and reflected shocks valid for arbitrary initial preshock parameters.

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## I. INTRODUCTION

Shock waves have the capability of generating matter with extreme conditions of pressure and temperature with different substances either in gas, liquid or solid phases. Anytime a planar shock wave hits normally the surface that separates two fluids with different thermodynamic properties, a shock is always transmitted into the second fluid and a shock or a rarefaction wave may be reflected inside the first fluid. The type of the reflected wave depends on the initial configuration, as extensively discussed in Refs. [1–4]. This is a basic scenario occurring in a large number of different phenomena to study the behavior of matter at high energy densities [1,5,6], ranging from laboratory created experiments to natural environments. The use of shock waves has been suggested as an important tool to diagnose material properties [7–9] within the domains of high-energy-density physics (HEDP) experiments or within the domain of geophysics and planetary sciences [10–13]. The generation of shock waves is also very important in inertial confinement fusion (ICF) experiments, where typically several shock waves are launched in succession to compress the thermonuclear target with the aim of obtaining fusion energy at the end of the process [14–16].

When a planar or corrugated shock front impinges a rippled interface, the refracted wavefronts become also rippled in shape. The corrugated fronts generate perturbations inside both fluids, in the form of acoustic fluctuations and/or vorticity and entropy perturbations, which promotes the growth of the initial ripple of the contact surface. These scenarios, the classical Richtmyer-Meshkov instability (RMI), are being studied for a long time, either analytically, experimentally and numerically [2–4,17–64]. Besides a wider class of instabilities called RM-like [34], not being caused by the shock-interface refraction, are still driven by exactly the same physical mechanisms as the classical RMI. This class includes instabilities produced by the collision of perturbed fluid layers, instabilities excited in simulating the evolution of an initial discontinuity in a fluid (the perturbed Riemann problem), among others. Independently of the process which trigger the RM-like instability, every model (theoretical or numerical) have to obtain the zero-order postshock profiles as a prior step to develop their calculations.

In 1947, Taub studied the oblique refraction of a plane shock at a contact discontinuity and obtained the postshock quantities in terms of the preshock parameters after solving a polynomial of degree 12 [65]. In this work, we only consider normal refraction and the reflected shock case. The postshock quantities behind the reflected and the transmitted shock fronts are related to the preshock parameters through the continuity of pressure and normal velocity at the contact surface together with the Rankine-Hugoniot equations at the wavefronts [25]. Up to date, no explicit analytical solution for normal refraction has ever been reported in the literature. Nevertheless, approximate solutions in the form of Taylor series expansions in different physical limits as weak/strong incident shock or small and large initial density jump across the interface were provided in Ref. [3].

The aim of this work is to show an exact analytical solution for the aforementioned system of equations for arbitrary preshock configuration, thus providing another tool, besides the numerical calculation of the zero-order postshock profiles. The explicit formulas shown in the work can be easily evaluated and/or included in numerical codes. Additionally, since it is exact, the solution can be used to benchmark any other numerical approach.

We structure the work in the following manner: in Sec. II, we present the basic equations for the zero-order quantities behind the shocks as a function of the preshock parameters and the transmitted and reflected shock Mach numbers. The system of equations formed by the continuity of pressure and normal velocity at the interface is written, and a polynomial equation of degree 6 in the reflected shock Mach number is obtained. In Sec. III, the polynomial equation is reduced to an equivalent polynomial of degree 4, whose roots can be calculated analytically. The four roots are presented in Sec. IV, where the selection of the physical solution is discussed. A summary is given in Sec. V.

## II. BASIC EQUATIONS

### A. Downstream quantities across the different wave-fronts

We consider a shock wave traveling from right to left inside fluid "$b$" along the $\hat{x}$ axis with speed $-D_i\,\hat{x}$ as measured in the

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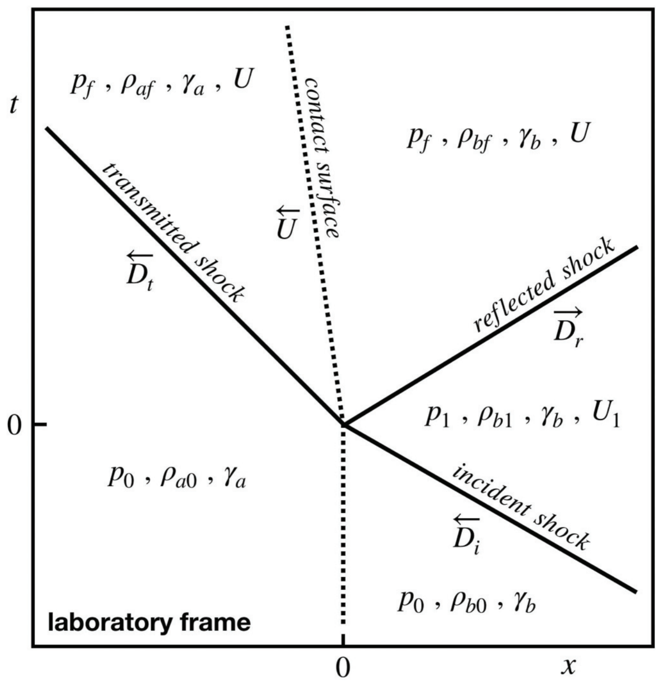


FIG. 1. A space-time diagram of the five regimes of the problem. Each regime is characterized by its pressure, density, isentropic exponent, and velocity. The solid lines are shock fronts and the dotted line is the contact surface between fluids "$a$" and "$b$." The velocity of each front is indicated next to its line.

laboratory frame (see Fig. 1). The fluid in front of the shock front is an ideal gas with preshock density $\rho_{b0}$ and sound speed $c_{b0} = \sqrt{\gamma_b p_0/\rho_{b0}}$, where $\gamma_b$ is the isentropic exponent and $p_0$ is the initial fluid pressure. The shock is driven by pressure $p_1$ downstream and the fluid acquires, after compression, a velocity $-U_1\hat{x}$ in the laboratory frame. The density is increased to the value $\rho_{b1}$ and the downstream sound speed is $c_{b1}$. All the quantities behind the incident shock can be obtained through the Rankine-Hugoniot equations [5,6] and are shown below. The shock strength can be characterized in several ways, either by defining the relative change in pressure across the front, or with the Mach numbers associated to the front. We define the relative change of pressure for the incident shock, as [66]

$$z_i = \frac{p_1 - p_0}{p_0}. \tag{1}$$

We assume that all these quantities are time independent. The upstream Mach number $M_i$ is defined as [3]

$$1 < M_i = \frac{D_i}{c_{b0}} = \sqrt{1 + \frac{\gamma_b + 1}{2\gamma_b} z_i}, \tag{2}$$

An important quantity is the velocity of the compressed fluid in the laboratory frame, which can be also calculated either in terms of $z_i$ or $M_i$:

$$\frac{U_1}{c_{b0}} = \frac{z_i}{\gamma_b\sqrt{1 + \frac{\gamma_b+1}{2\gamma_b} z_i}} = \frac{2}{\gamma_b + 1}\frac{M_i^2 - 1}{M_i}. \tag{3}$$

We therefore define the downstream Mach number $\beta_i$ in terms of the relative shock velocity with respect to the compressed fluid:

$$0 < \beta_i = \frac{D_i - U_1}{c_{b1}} = \sqrt{\frac{2\gamma_b + (\gamma_b - 1)z_i}{2\gamma_b(1 + z_i)}} = \sqrt{\frac{(\gamma_b - 1)M_i^2 + 2}{2\gamma_b M_i^2 - \gamma_b + 1}} < 1. \tag{4}$$

The density compression ratio across the incident shock is

$$R_i = \frac{\rho_{b1}}{\rho_{b0}} = \frac{D_i}{D_i - U_1} = \frac{2\gamma_b + (\gamma_b + 1)z_i}{2\gamma_b + (\gamma_b - 1)z_i} = \frac{(\gamma_b + 1)M_i^2}{(\gamma_b - 1)M_i^2 + 2}, \tag{5}$$

the pressure ratio is

$$\frac{p_1}{p_0} = 1 + z_i = \frac{2\gamma_b M_i^2 - \gamma_b + 1}{\gamma_b + 1}, \tag{6}$$

and the ratio of sound speeds is

$$\frac{c_{b1}}{c_{b0}} = \sqrt{\frac{1 + z_i}{R_i}} = \frac{\sqrt{(2\gamma_b M_i^2 - \gamma_b + 1)[(\gamma_b - 1)M_i^2 + 2]}}{(\gamma_b + 1)M_i}. \tag{7}$$

We assume that at $x = 0$ there is a planar contact discontinuity that separates fluid "$b$" from a different fluid "$a$." The initial density of fluid "$a$" is $\rho_{a0}$ and its isentropic exponent is $\gamma_a$. We define the preshock density ratio $R_0 = \rho_{a0}/\rho_{b0}$. The initial sound speed of fluid "$a$" is given by: $c_{a0} = \sqrt{\gamma_a p_0/\rho_{a0}}$. Immediately after the incident shock arrives to the contact discontinuity, a transmitted shock propagates inside fluid "$a$." A shock or a rarefaction wave may be reflected backwards inside fluid $b$. The conditions to have one or the other kind of wavefront to be reflected back have been discussed in the literature [3,23]. These conditions involve the fluids compressibilities as well as the preshock density ratio and the shock strength, and are extensively discussed in Refs. [1,2,23]. We can continuously go from a situation where a shock or a rarefaction is reflected by changing the value of $R_0$. If we fix the incident shock Mach number and the isentropic exponents of the gases, then there exists a value of the initial density ratio for which a shock is always reflected, if the preshock density ratio stays above that value. Otherwise, we would have a rarefaction wave traveling backwards. We indicate this particular value of the preshock density ratio with the symbol: $R_0^{tt}$. The superindex $tt$ is the acronym of *total transmission*, and it refers to the situation where no wave front is reflected inside fluid "$b$." The expression of $R_0^{tt}$ as a function of the other parameters is

$$R_0^{tt} = \frac{\gamma_b(\gamma_b + 1)M_1^2}{\gamma_a - \gamma_b + \gamma_b(\gamma_a + 1)M_i^2}. \tag{8}$$

We will only concentrate here on the cases in which a shock is always reflected after the incident wavefront hits the contact surface. For fluids with equal isentropic exponents, $R_0^{tt} = 1$, that is, the incident shock comes from the less denser fluid. For fluids with different isentropic exponents, $R_0^{tt}$ can be smaller or larger than unity. In Fig. 1 we show the transmitted and reflected shocks after the incident front has been refracted at the contact surface. The transmitted shock moves to the left with velocity $-D_t\,\hat{x}$ in the laboratory frame. The sound speed of the compressed fluid "$a$" is $c_{af}$. The reflected shock moves to the right with velocity $+D_r\,\hat{x}$. The fluid in between both fronts moves to the left with velocity $-U\,\hat{x}$. The pressure in between both shocks is $p_f$. The relative pressure jump across the transmitted shock is $z_t = (p_f - p_0)/p_0$ and across the reflected shock is $z_r = (p_f - p_1)/p_1$. The transmitted and reflected shock upstream Mach numbers are given by

$$1 < M_t = \frac{D_t}{c_{a0}} = \sqrt{1 + \frac{\gamma_a + 1}{2\gamma_a} z_t}, \tag{9}$$

$$1 < M_r = \frac{D_r + U_1}{c_{b1}} = \sqrt{1 + \frac{\gamma_b + 1}{2\gamma_b} z_r}. \tag{10}$$

The fluids velocity at both sides of the contact surface can be written as

$$\frac{U}{c_{a0}} = \frac{z_t}{\gamma_a\sqrt{1 + \frac{\gamma_a+1}{2\gamma_a} z_t}} = \frac{2}{\gamma_a + 1}\frac{M_t^2 - 1}{M_t}. \tag{11}$$

The quantities across the transmitted front are related by

$$R_t = \frac{\rho_{af}}{\rho_{a0}} = \frac{D_t}{D_t - U} = \frac{2\gamma_a + (\gamma_a + 1) z_t}{2\gamma_a + (\gamma_a - 1) z_t} = \frac{(\gamma_a + 1) M_t^2}{(\gamma_a - 1) M_t^2 + 2}, \tag{12}$$

$$\frac{p_f}{p_0} = 1 + z_t = \frac{2\gamma_a M_t^2 - \gamma_a + 1}{\gamma_a + 1}, \tag{13}$$

$$\frac{c_{af}}{c_{a0}} = \sqrt{\frac{1 + z_t}{R_t}} = \frac{\sqrt{(2\gamma_a M_t^2 - \gamma_a + 1)[(\gamma_a - 1) M_t^2 + 2]}}{(\gamma_a + 1) M_t}. \tag{14}$$

The downstream transmitted shock Mach number, $\beta_t$, is

$$0 < \beta_t = \frac{D_t - U}{c_{af}} = \sqrt{\frac{2\gamma_a + (\gamma_a - 1) z_t}{2\gamma_a (1 + z_t)}} = \sqrt{\frac{(\gamma_a - 1) M_t^2 + 2}{2\gamma_a M_t^2 - \gamma_a + 1}} < 1. \tag{15}$$

The mass density of fluid "$b$" behind the reflected shock is indicated by $\rho_{bf}$. The final sound speed of fluid "$b$" is $c_{bf}$. The quantities at both sides of the reflected shock are related by

$$R_r = \frac{\rho_{bf}}{\rho_{b1}} = \frac{D_r + U_1}{D_r + U} = \frac{2\gamma_b + (\gamma_b + 1) z_r}{2\gamma_b + (\gamma_b - 1) z_r} = \frac{(\gamma_b + 1) M_r^2}{(\gamma_b - 1) M_r^2 + 2}, \tag{16}$$

$$\frac{p_f}{p_1} = 1 + z_r = \frac{2\gamma_b M_r^2 - \gamma_b + 1}{\gamma_b + 1}, \tag{17}$$

$$\frac{c_{bf}}{c_{b1}} = \sqrt{\frac{1 + z_r}{R_r}} = \frac{\sqrt{(2\gamma_b M_r^2 - \gamma_b + 1)[(\gamma_b - 1) M_r^2 + 2]}}{(\gamma_b + 1) M_r}. \tag{18}$$

The fluid velocity jump across the reflected shock is given by

$$\frac{U_1 - U}{c_{b1}} = \frac{z_r}{\gamma_b\sqrt{1 + \frac{\gamma_b+1}{2\gamma_b} z_r}} = \frac{2}{\gamma_b + 1}\frac{M_r^2 - 1}{M_r}. \tag{19}$$

The downstream reflected shock Mach number $\beta_r$ is given by

$$0 < \beta_r = \frac{D_r + U}{c_{bf}} = \sqrt{\frac{2\gamma_b + (\gamma_b - 1) z_r}{2\gamma_b (1 + z_r)}} = \sqrt{\frac{(\gamma_b - 1) M_r^2 + 2}{2\gamma_b M_r^2 - \gamma_b + 1}} < 1. \tag{20}$$

The initial conditions given by the four parameters: $\gamma_a$, $\gamma_b$, $R_0$, and the incident shock strength, represented either by $z_i$ or $M_i$ determine without ambiguities the final state of each fluid, after shock refraction. The key point is to determine $z_t$ and $z_r$ (equivalently, $M_t$ and $M_r$) as functions of the four preshock parameters. Once the reflected and transmitted Mach numbers are obtained, the postshock quantities behind each wavefront are easily calculated using the equations above. The procedure is to ask for continuity of pressure and normal velocity at the contact surface, after shock refraction. It is easy to see that pressure continuity at the material surface can be put in terms of the quantities $z_i$, $z_r$, and $z_t$ in the form

$$z_t = z_i + (1 + z_i) z_r, \tag{21}$$

or in terms of the upstream shock Mach numbers,

$$M_t^2 = 1 + \frac{\gamma_b(\gamma_a + 1)}{\gamma_a(\gamma_b + 1)}(M_i^2 - 1) + \frac{\gamma_b}{\gamma_a}\left[\frac{\gamma_a + 1}{\gamma_b + 1} + \frac{2\gamma_b(\gamma_a + 1)}{(\gamma_b + 1)^2}(M_i^2 - 1)\right](M_r^2 - 1). \tag{22}$$

The continuity of the normal velocity can be written as

$$U = U_1 - (U_1 - U), \tag{23}$$

and in dimensionless form

$$\frac{U}{c_{a0}} = \frac{c_{b0}}{c_{a0}}\frac{U_1}{c_{b0}} - \frac{c_{b0}}{c_{a0}}\frac{c_{b1}}{c_{b0}}\frac{U_1 - U}{c_{b1}}. \tag{24}$$

The last equation can be rewritten, using the relationships derived before, in terms of $z_i$, $z_r$, and $z_t$, as

$$\frac{z_t}{\gamma_a\sqrt{1 + \frac{\gamma_a+1}{2\gamma_a} z_t}} = \sqrt{\frac{\gamma_b R_0}{\gamma_a}}\left[\frac{z_i}{\gamma_b\sqrt{1 + \frac{\gamma_b+1}{2\gamma_b} z_i}} + \sqrt{\frac{1 + z_i}{R_i}}\frac{z_r}{\gamma_b\sqrt{1 + \frac{\gamma_b+1}{2\gamma_b} z_r}}\right]. \tag{25}$$

We can also rewrite the last equation in terms of $M_i$, $M_r$, and $M_t$:

$$\frac{M_t^2-1}{(\gamma_a+1)M_t}=\sqrt{\frac{\gamma_b R_0}{\gamma_a}}\left[\frac{M_i^2-1}{(\gamma_b+1)M_i}+\frac{\sqrt{\left(2\gamma_b M_i^2-\gamma_b+1\right)\left[(\gamma_b-1)M_i^2+2\right]}}{(\gamma_b+1)M_i}\frac{M_r^2-1}{M_r}\right]. \tag{26}$$

To solve for the postshock quantities, we must solve Eqs. (21) and (25) or the equivalent system formed by Eqs. (22) and (26). The evident difficulty with the quantities $z_r$ and $z_t$ is that the velocity continuity gives us an algebraic equation in the unknowns that can not be solved analytically and only a numerical solution seems feasible. However, by considering the Mach numbers ($M_r$ and $M_t$) instead of the relative pressure jumps, we easily arrive to a sixth-degree polynomial equation in $M_r$. The details of how to solve this equation are shown next.

### B. Polynomial equation for $M_r$

If we substitute Eq. (22) into (26) and operate algebraically, then we arrive to the following polynomial equation:

$$a_0+a_1M_r+a_2M_r^2+a_3M_r^3+a_4M_r^4+a_5M_r^5+a_6M_r^6=0, \tag{27}$$

where the coefficients $a_i$ ($0\leqslant i\leqslant 6$) are given by

$$a_0=-\frac{4\left(2\gamma_b M_i^2-\gamma_b+1\right)\left[(\gamma_b-1)M_i^2+2\right]\left[-\gamma_a+\gamma_b(2-\gamma_a\gamma_b)+(\gamma_a+1)(\gamma_b-1)\gamma_b M_i^2\right]}{\gamma_a(\gamma_b+1)^6M_i^2}, \tag{28}$$

$$a_1=-\frac{8\left(M_i^2-1\right)\left[-\gamma_a+\gamma_b(2-\gamma_a\gamma_b)+(\gamma_a+1)(\gamma_b-1)\gamma_b M_i^2\right]}{\gamma_a(\gamma_b+1)^5M_i^2}\sqrt{\left(2\gamma_b M_i^2-\gamma_b+1\right)\left[(\gamma_b-1)M_i^2+2\right]}, \tag{29}$$

$$\begin{aligned}a_2=&\frac{1}{\gamma_a(\gamma_b+1)^6R_0M_i^2}\{\left(-12\gamma_a+32(\gamma_a+1)\gamma_b-8(8+3\gamma_a)\gamma_b^2+32\gamma_a\gamma_b^3+4\gamma_a\gamma_b^4\right)R_0\\&+\left[-16\gamma_b(\gamma_b+1)^2+\left(-16(5\gamma_a+1)\gamma_b+16(5\gamma_a+13)\gamma_b^2-64(2\gamma_a+1)\gamma_b^3\right)R_0\right]M_i^2\\&+\left[-16\gamma_b(\gamma_b+1)(\gamma_b-1)^2+\left(4\gamma_a+8(3\gamma_a-1)-8(11\gamma_a+4)\gamma_b^2+152(\gamma_a+1)\gamma_b^3-4(4+7\gamma_a)\gamma_b^4\right)R_0\right]M_i^4\\&+\left[-4\left(\gamma_b^2-1\right)^2\gamma_b^2+\left(4(\gamma_b-1)^2\gamma_b+28(\gamma_b-1)^2\gamma_b^2+4\gamma_a(\gamma_b-1)^2(7\gamma_b+1)\gamma_b\right)R_0\right]M_i^6\},\end{aligned} \tag{30}$$

$$a_3=\frac{8\left(M_i^2-1\right)\left[-\gamma_a+3\gamma_b+\gamma_a\gamma_b-\gamma_b^2-2\gamma_a\gamma_b^2+(\gamma_a+1)(3\gamma_b-1)\gamma_b M_i^2\right]}{\gamma_a(\gamma_b+1)^5M_i^2}\sqrt{\left(2\gamma_b M_i^2-\gamma_b+1\right)\left[(\gamma_b-1)M_i^2+2\right]}, \tag{31}$$

$$\begin{aligned}a_4=&\frac{4\left(2\gamma_b M_i^2-\gamma_b+1\right)}{\gamma_a(\gamma_b+1)^6R_0M_i^2}\{\left(2\gamma_a+(\gamma_b-1)(\gamma_b+7)\gamma_b+\left(\gamma_b^2+8\gamma_b-3\right)\gamma_a\right)R_0\\&+\left[4(\gamma_b+1)^2\gamma_b+\left(-\gamma_a+(3\gamma_a+4)\gamma_b-(19\gamma_a+20)\gamma_b^2+\gamma_a\gamma_b^3\right)R_0\right]M_i^2\\&+\left[2(\gamma_b-1)(\gamma_b+1)^2\gamma_b-4(\gamma_a+1)(\gamma_b-2)\gamma_b^2R_0\right]M_i^4\},\end{aligned} \tag{32}$$

$$a_5=-\frac{8(\gamma_a+1)\gamma_b\left(M_i^2-1\right)}{\gamma_a(\gamma_b+1)^5M_i^2}\left(2\gamma_b M_i^2-\gamma_b+1\right)^{3/2}\sqrt{(\gamma_b-1)M_i^2+2}, \tag{33}$$

$$a_6=\frac{4(\gamma_a+1)\gamma_b\left(2\gamma_b M_i^2-\gamma_b+1\right)^2}{\gamma_a(\gamma_b+1)^5}\left[\frac{\gamma_b+1}{\gamma_a+1}\frac{1}{R_0}-\frac{(\gamma_b-1)M_i^2+2}{(\gamma_b+1)M_i^2}\right]. \tag{34}$$

It is noted that the last factor between brackets in $a_6$ above, is exactly equal to

$$\frac{\gamma_b+1}{\gamma_a+1}\frac{1}{R_0}-\frac{1}{R_i}, \tag{35}$$

where $R_i$ is the density compression ratio across the incident shock [see Eq. (5)]. This fact will be useful later on, when discussing the behavior of the roots of the polynomial as a function of the defining parameters. In fact, for given values of $\gamma_a$, $\gamma_b$, and $R_0$ there could be situations in which, at a specific incident shock Mach number, the value of $a_6$ becomes 0 and therefore, the polynomial defining $M_r$ changes to a polynomial of fifth degree.

## III. SOLUTIONS OF EQ. (27)

### A. Numerical solution

At first, we try a numerical solution of Eqs. (21) [or (22)] and (25) [or (26)] for a specific choice of the preshock parameters using the Mathematica software [67]. We choose $\gamma_a=5/3$, $\gamma_b=7/5$, $R_0=3$, and $M_i=3$. For this case, the incident shock relative pressure jump is $z_i=28/3$. We obtain the following values for the relative pressure jumps and/or shock Mach numbers:

$$z_r = 0.636971, \quad z_t = 15.91537,$$
$$M_r = 1.24337, \quad M_t = 3.705711. \tag{36}$$

The roots of the polynomial given by Eq. (27) can also be numerically calculated with the same software and we obtain the following set of six numbers:

$$\begin{aligned} M_{r1} &= -21.0031, \\ M_{r2} &= -0.475191, \\ M_{r3} &= -0.475191, \\ M_{r4} &= 0.350483, \\ M_{r5} &= -0.376014, \\ M_{r6} &= 1.24337, \end{aligned} \tag{37}$$

where we recognize that the only physical solution is given by $M_r = 1.24337$. There are five spurious solutions that do not represent realizable values for the reflected shock Mach number. The question is: Would it be possible to obtain the six roots of Eq. (27) in analytical form? We know that if the polynomial is of degree 4 or less, this is always possible. For the equation we have at hand, the answer is uncertain. There is, however, an intriguing aspect of the set of roots displayed above: There is a double root. Let us change the value of the preshock density ratio and repeat the previous procedure. Let us choose $R_0 = 5$ with the other parameters held fixed. We obtain a new set of relative pressure jumps and refracted Mach numbers ($z_i$ and $M_i$ are, however, the same as before):

$$z_r = 0.95919, \quad z_t = 19.24494,$$
$$M_r = 1.34987, \quad M_t = 4.04919. \tag{38}$$

And the six roots of Eq. (27) for this new situation are

$$\begin{aligned} M_{r1} &= 10.2733, \\ M_{r2} &= -0.475191, \\ M_{r3} &= -0.475191, \\ M_{r4} &= 0.350075, \\ M_{r5} &= -0.364392, \\ M_{r6} &= 1.34987. \end{aligned} \tag{39}$$

We see that the new situation shows the same double roots as before. In fact, it will be verified that if we keep $\gamma_b$ and $M_i$ fixed, the double roots $M_{r2}$ and $M_{r3}$ are the same for the whole set of permissible $\gamma_a$ and $R_0$ values. If this double root could be found analytically, then we could always extract it from the original polynomial given by Eq. (27) and obtain a fourth-degree polynomial equation in $M_r$ which can always be solved analytically [68]. That this is indeed the case is shown in the following subsection.

### B. Calculation of the double root of Eq. (27)

Let us indicate with $\sigma_0$ the double root of Eq. (27). From elementary algebra, we know that it is also a root of the derivative (with respect to $M_r$) of the original Eq. (27). Therefore, we can always write a system of two equations like

$$a_0 + a_1\sigma_0 + a_2\sigma_0^2 + a_3\sigma_0^3 + a_4\sigma_0^4 + a_5\sigma_0^5 + a_6\sigma_0^6 = 0, \tag{40}$$

$$a_1 + 2a_2\sigma_0 + 3a_3\sigma_0^2 + 4a_4\sigma_0^3 + 5a_5\sigma_0^4 + 6a_6\sigma_0^5 = 0. \tag{41}$$

If we combine both equations in the form: $6\times$ Eq. (40) – $\sigma_0\times$ Eq. (41), then we can substract the term proportional to $\sigma_0^6$ and obtain a new equation which is of degree 5 in $\sigma_0$:

$$6a_0 + 5a_1\sigma_0 + 4a_2\sigma_0^2 + 3a_3\sigma_0^3 + 2a_4\sigma_0^4 + a_5\sigma_0^5 = 0. \tag{42}$$

We can proceed along the same line of reasoning, by adequately combining Eq. (41) with Eq. (42) to eliminate the term proportional to $\sigma_0^5$ and remain with a fourth-order polynomial in $\sigma_0$. Proceeding further, at each step we combine the previous equations to obtain a new equation that is a polynomial in $\sigma_0$ but whose degree is decreased in one unit with respect to the degree of the previous expression. We will finally arrive to a first-order linear equation in $\sigma_0$ which gives us the double root $\sigma_0$ as a function of the original coefficients $a_i$. The intermediate calculations can be easily followed with a software like Mathematica. After careful simplification of the final result, we obtain the surprisingly simple expression

$$\sigma_0 = -\beta_i = -\sqrt{\frac{(\gamma_b - 1)M_i^2 + 2}{2\gamma_b M_i^2 - \gamma_b + 1}}, \tag{43}$$

which coincides with minus the incident downstream shock Mach number. We confirm our prediction, that $\sigma_0$ is only a function of the isentropic coefficient of the "incident" fluid $\gamma_b$ and of the incident shock strength $M_i$. With Eq. (43) we can immediately reproduce the double roots of the cases studied before.

### C. Reduction of Eq. (27) to a fourth-degree polynomial equation

Since we have the value of the double root $\sigma_0$, we can factor the original Eq. (27) into the product of $(M_r - \sigma_0)^2$ and a fourth-order polynomial. We divide the original Eq. (27) by the coefficient $a_6$ to simplify the intermediate algebraic steps and propose a factorization of the form

$$\frac{a_0}{a_6} + \frac{a_1}{a_6}M_r + \frac{a_2}{a_6}M_r^2 + \frac{a_3}{a_6}M_r^3 + \frac{a_4}{a_6}M_r^4 + \frac{a_5}{a_6}M_r^5 + M_r^6 = (M_r - \sigma_0)^2\big(b_0 + b_1M_r + b_2M_r^2 + b_3M_r^3 + M_r^4\big). \tag{44}$$

After some algebra, and equating equal powers of $M_r$ at both sides of the last equation, we can find the coefficients $b_i$ as explicit functions of $\gamma_a$, $\gamma_b$, $R_0$, and $M_i$:

$$b_0 = \frac{[-(\gamma_a+1)(\gamma_b-1)\gamma_b + (\gamma_b+1)(\gamma_a-\gamma_b)]R_0 + (\gamma_a+1)(\gamma_b-1)\gamma_b R_0 M_i^2}{-4(\gamma_a+1)\gamma_b R_0 + [(\gamma_b+1)^2 - (\gamma_a+1)(\gamma_b-1)\gamma_b R_0]M_i^2}, \tag{45}$$

$$b_1 = \frac{4R_0}{\gamma_b(2\gamma_b M_i^2 - \gamma_b + 1)^{3/2}[(\gamma_b - 1)M_i^2 + 2]^{1/2}[-2(\gamma_a + 1) + ((\gamma_b + 1)^2 - (\gamma_a + 1)(\gamma_b - 1))R_0 M_i^2]} \times [4(\gamma_a - 2\gamma_b + \gamma_a\gamma_b^2) + 4(\gamma_b + 1)(\gamma_a - (\gamma_a + 3)\gamma_b + \gamma_a\gamma_b)M_i^2 + (\gamma_b - 1)^2\gamma_b(-6 + \gamma_a(\gamma_b - 4))M_i^4 - (\gamma_a + 1)(\gamma_b - 1)^3\gamma_b M_i^6], \tag{46}$$

$$b_2 = \frac{1}{2\gamma_b(2\gamma_b M_i^2 - \gamma_b + 1)} \frac{\gamma_b}{[2(\gamma_b(\gamma_a + 1)(\gamma_b - 1) - 2\gamma_b(\gamma_a + 1)(\gamma_b + 1))R_0 + 2\gamma_b((\gamma_b + 1)^2 - (\gamma_a + 1)(\gamma_b - 1)R_0 M_i^2)]} \times [-4(2\gamma_a - (\gamma_a + 3)\gamma_b + (\gamma_a + 1)\gamma_b^2)R_0 + 4R_0(2\gamma_b(\gamma_b + 1) + (\gamma_b - 1)(-\gamma_a + (3\gamma_a + 4)\gamma_b))M_i^2 + 4\gamma_b[\gamma_b^2 - 1 - 3((\gamma_a + 1) - 4(\gamma_b - 1) - 4\gamma_a(\gamma_b - 1) + \gamma_b + \gamma_a\gamma_b)R_0]M_i^4], \tag{47}$$

$$b_3 = \frac{-8(\gamma_a + 1)R_0 + 4[(\gamma_b + 1)^2 - 2(\gamma_a + 1)\gamma_b]M_i^2}{2(\gamma_a + 1)(\gamma_b - 1)R_0 + [(\gamma_b + 1)^2 - 2(\gamma_a + 1)\gamma_b R_0]M_i^2}\sqrt{\frac{(\gamma_b - 1)M_i^2 + 2}{2\gamma_b M_i^2 - \gamma_b + 1}}. \tag{48}$$

Therefore, the problem of determining the postshock quantities of the normal shock refraction problem consists in identifying the physically meaningful solution of the reduced equation

$$b_0 + b_1 M_r + b_2 M_r^2 + b_3 M_r^3 + M_r^4 = 0, \tag{49}$$

where the coefficients $b_i$ are given above.

## IV. SOLUTIONS OF EQ. (49)

### A. Analytic solution of Eq. (49)

The analytical form of the four roots of Eq. (49) (which we call from now on: $\sigma_1$, $\sigma_2$, $\sigma_3$, and $\sigma_4$) can be obtained following the strategy explained, for example, in Ref. [68]. However, they can also be readily obtained with the Mathematica software. If written explicitly as functions of the four preshock parameters, then the expressions for $\sigma_i$ would become extremely large and not practical. It is convenient to define them through auxiliary functions. The explicit forms of $\sigma_i$ are

$$\sigma_1 = -\frac{b_3}{4} - \frac{\theta}{2} - \frac{1}{2}\sqrt{-\frac{4}{3}b_2 + \frac{1}{2}b_3^2 - P_5 - \frac{P_4}{4\theta}}, \tag{50}$$

$$\sigma_2 = -\frac{b_3}{4} - \frac{\theta}{2} + \frac{1}{2}\sqrt{-\frac{4}{3}b_2 + \frac{1}{2}b_3^2 - P_5 - \frac{P_4}{4\theta}}, \tag{51}$$

$$\sigma_3 = -\frac{b_3}{4} + \frac{\theta}{2} - \frac{1}{2}\sqrt{-\frac{4}{3}b_2 + \frac{1}{2}b_3^2 - P_5 + \frac{P_4}{4\theta}}, \tag{52}$$

$$\sigma_4 = -\frac{b_3}{4} + \frac{\theta}{2} + \frac{1}{2}\sqrt{-\frac{4}{3}b_2 + \frac{1}{2}b_3^2 - P_5 + \frac{P_4}{4\theta}}, \tag{53}$$

where the necessary auxiliary functions, are given by the following expressions:

$$P_1 = 12b_0 + b_2^2 - 3b_1 b_3, \tag{54}$$

$$P_2 = 27b_1^2 - 72b_0 b_2 + 2b_2^3 - 9b_1 b_2 b_3 + 27b_0 b_3^2, \tag{55}$$

$$P_3 = -4P_1^3 + P_2^2, \tag{56}$$

$$P_4 = -8b_1 + 4b_2 b_3 - b_3^3, \tag{57}$$

$$P_5 = \frac{P_1}{3}\sqrt[3]{\frac{2}{P_2 + \sqrt{P_3}}} + \frac{1}{3}\sqrt[3]{\frac{P_2 + \sqrt{P_3}}{2}}, \tag{58}$$

$$\theta = \sqrt{-\frac{2}{3}b_2 + \frac{1}{4}b_3^2 + P_5}. \tag{59}$$

In principle, we would need all the branches, because as any one of the preshock parameters is varied along its range, we will find particular singularities associated to that parameter that would oblige us to change branch. This certainly happens, for example, when studying the dependence of $M_r$ as a function of $M_i$ at fixed values of $\gamma_a$, $\gamma_b$, and $R_0$ as will be shown with a particular example in the following subsection, where at a particular value $M_i = M_{\text{sing}}$ we must change the branch of the solution.

At a first and naive glance, the explicit calculation of Eqs. (54)–(59) might seem too complicated and superfluous, because $\sigma_i$ $(1 \leqslant i \leqslant 4)$ can always be found numerically. Nevertheless, the analytical expressions Eqs. (50)–(53) are always exact, and hence, can be always used to contrast and/or benchmark any numerical approach to the same problem.

### B. Dependence of $M_r$ on $M_i$ for fixed values of $\gamma_a$, $\gamma_b$, and $R_0$

#### 1. $M_i \neq M_{\text{sing}}$

We start at first by examining the behavior of the solutions to Eq. (49) for the particular case: $\gamma_a = 5/3$, $\gamma_b = 7/5$, and $R_0 = 3/2$. In Fig. 2(a) we show the four roots in the same plot to identify the branches that are meaningful to this situation. We have found that at $M_i = 5/\sqrt{13}$ there is a change of branch. In fact, for $1 \leqslant M_i \leqslant 5/\sqrt{13}$ the physical solution to Eq. (49) is given by $\sigma_3$. If we increase the Mach number above this particular value, then we are obliged to use the branch $\sigma_4$ to calculate $M_r$ in the interval $M_i \geqslant 5/\sqrt{13}$. The reason for this behavior should be searched in the behavior of the coefficients of the original polynomial equation of degree 6 as a function of $M_i$. With this purpose, we go back to Eq. (27) and examine the dependence of the coefficients $a_i$ when we vary the incident shock Mach number. We soon recognize that the highest power coefficient $a_6$ is zero for a specific value of $M_i > 1$. In fact, looking at the expression given in Eqs. (34) and (35) we see that this happens in all those cases

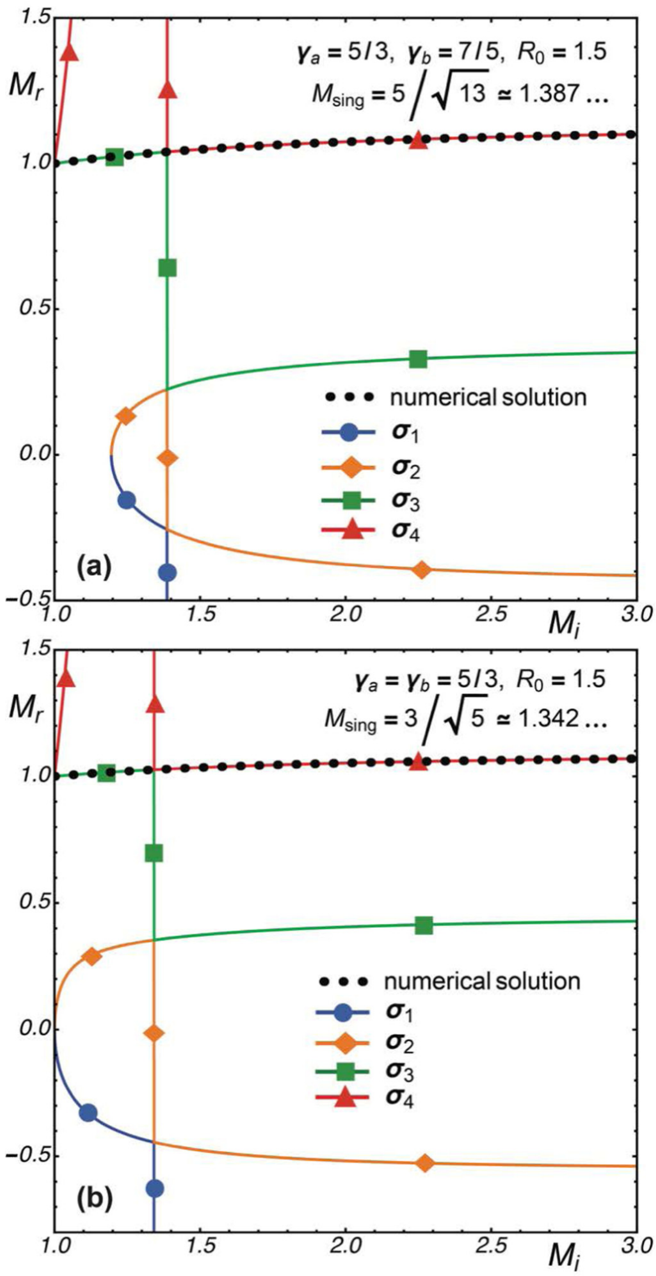


FIG. 2. Dependence of the numerical solution and the four independent roots $\sigma_1$, $\sigma_2$, $\sigma_3$, and $\sigma_4$ of Eq. (49) as a function of $M_i$; (a) for $\gamma_a = 5/3$, $\gamma_b = 7/5$, and $R_0 = 1.5$; (b) for $\gamma_a = \gamma_b = 5/3$ and $R_0 = 1.5$. In both cases, the physical solution for $M_r$ starts from the point $(M_i = 1, M_r = 1)$ and it is given piecewise by $\sigma_3$ (in the interval $1 \leqslant M_i \leqslant M_{\rm sing}$), and $\sigma_4$ (in the interval $M_i \geqslant M_{\rm sing}$). The branches $\sigma_1$ and $\sigma_2$ do not correspond to physically realizable solutions of $M_r$.

in which the density compression at the incident shock front $R_i$ equals the value $(\gamma_a + 1)R_0/(\gamma_b + 1)$. For a diatomic gas with $\gamma_b = 7/5$ this is not difficult to achieve if the shock is strong enough, in particular, for $M_i = 5/\sqrt{13}$. The general expression (arbitrary values of the preshock parameters) for the singularity that obliges us to change from the $\sigma_3$ branch to the $\sigma_4$ branch is given explicitly by calculating the particular value of $M_i$ that makes the coefficient $a_6$ in Eq. (34) equal to zero (equivalently, the coefficients $b_i$ would have a pole at that value of $M_i$). We call it $M_{\rm sing}$ and is given by

$$M_{\rm sing} = \sqrt{\frac{2(\gamma_a + 1)R_0}{(\gamma_b + 1)^2 - (\gamma_a + 1)(\gamma_b - 1)R_0}}. \tag{60}$$

Exactly at $M_i = M_{\rm sing}$, the original Eq. (27) has degree 5. This means that our polynomial in Eq. (49) becomes a polynomial of the third degree. Therefore, a cubic equation must be solved to get the physical solution at the singularity. It is clear that the curve must be continuous and therefore the physical solution of the cubic equation agrees with the right limit of $\sigma_3$ as well with the left limit of $\sigma_4$ when $M_i$ approaches $M_{\rm sing}$ either from the left or the right, respectively. The cubic equation will be presented in the next subsection. Anytime a singularity appears in a solution of this type, it is related to a particular characteristic of the problem at hand. Something special happens that makes that specific choice of parameters unique. To have a clearer picture of this singularity we choose another set of parameters. It is worth to study the case in which both fluids are ideal monatomic gases. We consider the same initial density ratio $R_0 = 3/2$ as before.

In this other case, for equal isentropic exponents, the condition for the singularity reads

$$R_i = R_0. \tag{61}$$

We see that if the incident shock is strong enough as to compress the light fluid such that it raises its density from $\rho_{b0}$ to $\rho_{b1}$ and that this ratio is the same as the preshock density ratio at the contact surface, then this value of incident Mach is singular in the sense that the branch in the solution for $M_r$ must be changed. It is clear that such a situation is quite particular, as if two identical fluids collided at $x = 0$. Even though $\rho_{b1}/\rho_{b0} = R_0$, a shock is always reflected back, because $R_0 > 1$. Substituting $\gamma_a = \gamma_b = 5/3$, we get

$$M_{\rm sing} = \sqrt{\frac{3R_0}{4 - R_0}}. \tag{62}$$

An interesting aspect of the solution is evidenced by Eq. (62) above. If $R_0 < 4$, then $M_{\rm sing} < \infty$ and therefore, for any experimental situation involving this pair of gases, the solution branch must be changed at some incident shock Mach number. If $R_0 > 4$, then the singularity is not real. This is because, for a monatomic gas, the maximum compression ratio achievable at the incident shock is equal to 4. Therefore, if $R_0 > 4$, then there is no incident shock that can fulfill the requirement imposed by Eq. (61). For these other cases, the branch represented by $\sigma_3$ is enough in the whole range $1 < M_i < \infty$. These results are shown in Fig. 2(b).

### 2. $M_i = M_{\rm sing}$

When the incident Mach number equals $M_{\rm sing}$ the equation for $M_r$ is a polynomial of the third degree. It is

$$c_0 + c_1 M_r + c_2 M_r^2 + M_r^3 = 0, \tag{63}$$

where the coefficients $c_i$ are

$$c_0 = -\frac{\{2\gamma_b - \gamma_a[1+\gamma_b^2 - R_0(\gamma_a+1)(\gamma_b-1)]\}\sqrt{1-\gamma_b+R_0(\gamma_a+1)}}{2\sqrt{2}\gamma_b(\gamma_a+1)[1+\gamma_b-R_0(\gamma_a+1)]}, \tag{64}$$

$$c_1 = \frac{2\gamma_b - \gamma_a[1+\gamma_b^2 - R_0(\gamma_a+1)(\gamma_b-1)]}{\gamma_b(\gamma_a+1)[1+\gamma_b-R_0(\gamma_a+1)]}, \tag{65}$$

$$c_2 = \frac{1}{2\sqrt{2}\gamma_b(\gamma_a+1)[1+\gamma_b-R_0(\gamma_a+1)]\sqrt{1-\gamma_b+R_0(\gamma_a+1)}}\big[\gamma_b(-7-2\gamma_b+\gamma_b^2)+\gamma_a(1-3\gamma_b+\gamma_b^3)$$
$$-2R_0(\gamma_a+1)(\gamma_b-1)[\gamma_b+\gamma_a(\gamma_b+1)]+\gamma_b R_0^2(\gamma_a+1)^3\big]. \tag{66}$$

The three solutions of Eq. (63) are

$$\tau_1 = \frac{1}{6}\left[-2c_2 - 2^{4/3}\frac{3c_1-c_2^2}{(Q_1+3\sqrt{3Q_2})^{1/3}} + 2^{2/3}(Q_1+3\sqrt{3Q_2})^{1/3}\right], \tag{67}$$

$$\tau_2 = \frac{1}{12}\left[-4c_2 + \frac{2^{4/3}(1+i\sqrt{3})(3c_1-c_2^2)}{(Q_1+3\sqrt{3Q_2})^{1/3}} + 2^{2/3}(i\sqrt{3}-1)(Q_1+3\sqrt{3Q_2})^{1/3}\right], \tag{68}$$

$$\tau_3 = \frac{1}{12}\left[-4c_2 + \frac{2^{4/3}(1-i\sqrt{3})(3c_1-c_2^2)}{(Q_1+3\sqrt{3Q_2})^{1/3}} - 2^{2/3}(i\sqrt{3}-1)(Q_1+3\sqrt{3Q_2})^{1/3}\right], \tag{69}$$

where the auxiliary functions are given by the expressions

$$Q_1 = -27c_0 + 9c_1c_2 - 2c_2^3,$$
$$Q_2 = 27c_0^2 + 4c_1^3 - 18c_0c_1c_2 - c_1^2c_2^2 + 4c_0c_2^3. \tag{70}$$

For the case under study in Fig. 2(a), $\gamma_a = 5/3$, $\gamma_b = 7/5$, and $R_0 = 1.5$, we have $M_{\text{sing}} = 5/\sqrt{13} \cong 1.38675...$ and the results $\tau_1 = 1.04072...$, $\tau_2 = -0.255629...$, and $\tau_3 = 0.225136...$. The first root $\tau_1$ is the physical solution for this situation. The case shown in Fig. 2(b) shows a similar behavior, the physical solution is given again by $\tau_1 = 1.02636$.

### C. Selection of the root

So far, the problem of the normal refraction of a planar shock wave when a shock is reflected back is analytically solved as we dispose of the four roots [Eqs. (50)–(53)] of Eq. (49). Nevertheless, the solution is not complete if we do not have an unambiguous criterion to select the branch of the solution which corresponds with the physical solution for any choice of the preshock parameters. At least initially, there is no reason to discard any of the four branches. Anyway, the two cases shown in Fig. 2 provide us a hint to follow. In both cases, only two of the branches of the solution are used over the whole the parameter space. From $M_i = 1$ until the singular point $M_{\text{sing}}$, the physical solution is given by $\sigma_3$, and then $\sigma_4$ from that point forward. Besides, in both cases the solution at the singular point (cubic equation) is provided by the first root $\tau_1$. Additionally, we study several different cases trying to cover the whole space of preshock parameters (not shown in the text), and we have confirmed that this behavior is always repeated. The next step is to find analytic proofs of these observations.

It is important to note that $M_{\text{sing}}$ has a unique value, this makes that we only need two branches of the solution to describe the complete parameter space, one in the interval $1 < M_i < M_{\text{sing}}$, and another one when $M_i > M_{\text{sing}}$. When $M_i = M_{\text{sing}}$ the solution is provided by one of the roots of the cubic Eq. (63). Since the physical solution must be continuous, it is clear that the left limit of the branch valid up to the singular point must be equal to the correct root of the cubic equation and to the right limit of the branch used beyond this point.

Therefore, our first task is to determine which branch is valid in the first interval. As we said, the equation system [Eqs. (22) and (26)] has not analytic solution for arbitrary preshock parameters, but it does in some physical limits. In fact, solutions for $M_r$ in the form of a Taylor series expansion was provided in Ref. [3] in the limits of weak incident shock ($M_i - 1 \ll 1$), strong incident shock ($M_i \gg 1$), and large density ratio ($R_0 \gg 1$) and low density ratio ($R_0 - R_0^{tt} \ll 1$). For weak shocks, the solution is

$$M_r)_{M_i-1\ll 1} = 1 + \frac{\sqrt{R_0\gamma_a} - \sqrt{gb}}{\sqrt{R_0\gamma_a} + \sqrt{gb}}(M_i - 1) + O[(M_i-1)^2]. \tag{71}$$

It is clear that weak shock limit of the branch valid when $M_1 - 1 \ll 1$ must match with Eq. (71). In fact, same power coefficients in the small parameter $M_i - 1$ must coincide term to term. If we take the weak shock limit to obtain the Taylor expansions of the four branches of the solution, then we see that only $\sigma_3$ and $\sigma_4$ has a constant term equal to 1. Actually, these branches have the following weak shock expansions:

$$\sigma_3)_{M_i-1\ll 1} = 1 + \left(1 - 2\sqrt{\frac{\gamma_a\gamma_b R_0}{(\gamma_b - \gamma_a R_0)^2}} + \frac{2\gamma_b}{\gamma_a R_0 - \gamma_b}\right)$$
$$\times\ (M_i - 1) + O[(M_i-1)^2], \tag{72}$$

$$\sigma_4)_{M_i-1\ll 1} = 1 + \left(1 + 2\sqrt{\frac{\gamma_a\gamma_b R_0}{(\gamma_b - \gamma_a R_0)^2}} + \frac{2\gamma_b}{\gamma_a R_0 - \gamma_b}\right)$$
$$\times (M_i - 1) + O[(M_i-1)^2]. \tag{73}$$

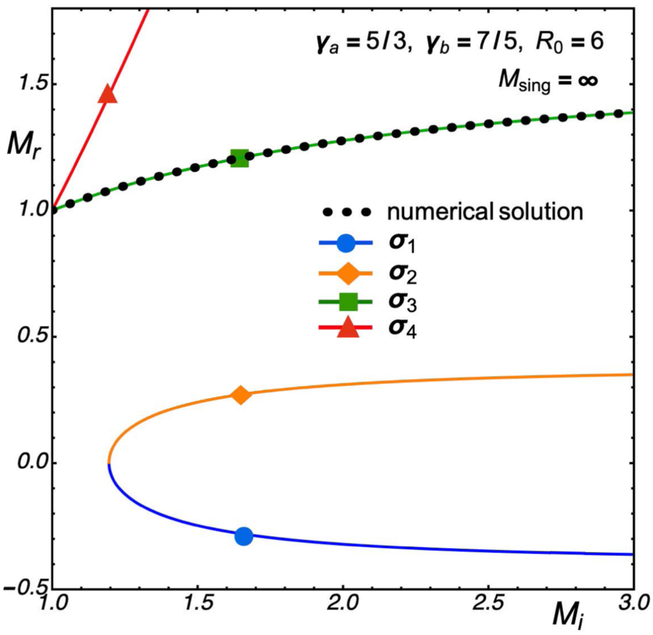


FIG. 3. Dependence of the numerical solution and the four independent roots $\sigma_1$, $\sigma_2$, $\sigma_3$, and $\sigma_4$ of Eq. (49) as a function of $M_i$ for $\gamma_a = 5/3$, $\gamma_b = 7/5$, and $R_0 = 6$. In this particular case, the interval defined by Eq. (74) is $0.9 < R_0 < 5.4$, and, therefore, the physical solution for every $M_i$ is given by $\sigma_3$. The branches $\sigma_1$, $\sigma_2$ and $\sigma_4$ do not correspond to physically realizable solutions of $M_r$.

We see that the linear coefficient in $M_i - 1$ of both expansions depends on the sign of $\gamma_b - \gamma_a R_0$. If $\rho_{a0}\gamma_a > \rho_{b0}\gamma_b$, then the expression inside the square is negative and then $\sqrt{(\gamma_b - \gamma_a R_0)^2} = -\gamma_b + \gamma_a R_0$. In this case, the fist term of $\sigma_3$ coincides with the first term of Eq. (71). However, if $\rho_{a0}\gamma_a > \rho_{b0}\gamma_b$, then $\sqrt{(\gamma_b - \gamma_a R_0)^2} = \gamma_b - \gamma_a R_0$, and the fist term of Eq. (71) is equal to the one provided by $\sigma_4$. So, at first sight, the solution can started with $\sigma_3$ or $\sigma_4$ depending on the preshock parameter choice. Besides, there is another extra condition for weak shocks. It is that the initial density ratio across the interface $R_0$ must be greater than some certain value to get a shock reflected after the incident shock refraction. This particular value is $R_0^{tt}$ written in Eq. (8). If we substitute $M_i = 1$ in Eq. (8), then we obtain that $\rho_{a0}\gamma_a > \rho_{b0}\gamma_b$ if we want a shock reflected back in the first fluid. As a consequence, the solution always starts with $\sigma_3$.

It is important to note that $M_{sing}$ depends on $\gamma_a$, $\gamma_b$ and $R_0$ and it must be greater than unity. There are some combinations of preshock parameters for which $M_{sing} < 1$ or it goes to infinity. In those cases, we have no singular point and $\sigma_3$ is the physical solution for any value of $M_i$. If we apply these restrictions to $M_{sing}$, then we obtain an interval for $R_0$ as a function of the isentropic exponents of the gases:

$$\frac{\gamma_b + 1}{\gamma_a + 1} < R_0 < \frac{(\gamma_b + 1)^2}{(\gamma_a + 1)(\gamma_b - 1)}. \tag{74}$$

When $R_0 < \gamma_b + 1/\gamma_a + 1$, we have $M_{sing} < 1$. For $R_0 \geqslant (\gamma_b + 1)^2/[(\gamma_a + 1)(\gamma_b - 1)]$, $M_{sing} \to \infty$. In Fig. 3, we show the same combination of gases as in Fig. 2(a) but the parameter $R_0$ has been chosen outside of the interval defined in Eq. (74). In this case, $\sigma_3$ is enough to describe the physical solution for any value of $M_i$.

Next, we calculate the left limit of $\sigma_3$ at $M_i = M_{sing}$. We obtain a Taylor series expansion where the small parameter is $M_{sing} - M_i \ll 1$, and we compare the constant term of the expansion with the three roots of the cubic equation. We see that the first root $\tau_1$ coincides with the constant term for every choice of the preshock parameters. Explicit expressions are not shown in the text because of its complexity and length. A similar procedure has been used to determine the branch valid in the interval $M_i < M_{sing}$. We take the right limit of the remaining branches, and obtain a Taylor series expansions in powers of $M_i - M_{sing}$. We compare their constant terms with $\tau_1$, and realize that only $\sigma_4$ matches. The explicit formulas are not shown here for the same reasons explained before.

## V. SUMMARY

We have presented an analytic solution to obtain the zero-order profiles after a planar shock hits a planar contact surface between two fluids valid for arbitrary preshock configurations for the cases in which a shock is always reflected. Using Rankine-Hugoniot conditions at the shock fronts, the quantities behind the shocks can be obtained as analytic formulas which are functions of the preshock parameters and the reflected and transmitted shock Mach numbers. Nevertheless, both Mach numbers must be calculated from the equation system formed by the continuity of the pressure and normal velocity at the interface [Eqs. (22) and (26), respectively]. The key to find an analytic solution lays in obtaining a polynomial equation of degree 6 for the reflected shock Mach number [Eq. (27)]. After the elimination of a double spurious root, it becomes a quartic equation [Eq. (49)] which can be analytically solved. We have shown explicit formulas for the four roots [Eqs. (50)–(53)] of the mentioned quartic equation as function of the initial preshock parameters. In principle, none of the branches of the mathematical solution can be dismissed, and hence, a criterion to select the root which corresponds with the physical solution has been also provided. Studying any conceivable preshock configuration, we have observed and demonstrated that only one or two branches of the solution are needed to cover the whole preshock parameter space. It depends on the possibility to find a singularity in Eq. (49). In the interval $1 < M_i < M_{sing}$ the physical solution is given by $\sigma_3$ [Eq. (52)], and for $M_i > M_{sing}$ the solution is given by $\sigma_4$ [Eq. (53)]. At the singular point $M_{sing}$, the equation for $M_r$ is a polynomial of third degree [Eq. (63)], and the solution is provided by its first root [Eq. (67)]. If, for an initial preshock configuration, $M_{sing}$ is not feasible value, then the branch $\sigma_3$ provided the physical solution for all the range.

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