Convergence towards an Erdős-Rényi graph structure in network contraction processes

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In a highly influential paper twenty years ago, Barabási and Albert [Science **286**, 509 (1999)] showed that networks undergoing generic growth processes with preferential attachment evolve towards scale-free structures. In any finite system, the growth eventually stalls and is likely to be followed by a phase of network contraction due to node failures, attacks, or epidemics. Using the master equation formulation and computer simulations, we analyze the structural evolution of networks subjected to contraction processes via random, preferential, and propagating node deletions. We show that the contracting networks converge towards an Erdős-Rényi network structure whose mean degree continues to decrease as the contraction proceeds. This is manifested by the convergence of the degree distribution towards a Poisson distribution and the loss of degree-degree correlations.

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I. INTRODUCTION

A central observation in contemporary science is that many of the processes explored take place in complex network architectures [1–3]. Therefore, it is of great importance to analyze the geometries and topologies encountered in complex networks and their temporal evolution. Since the 1960s, mathematical studies of networks were focused on model systems such as the Erdős-Rényi (ER) network, which exhibits a Poisson degree distribution with no degree-degree correlations [4–6]. In an ER network each pair of nodes is connected randomly and independently, with equal probability [7]. In fact, ER networks form a maximum entropy ensemble under the constraint that the mean degree is fixed [8–11]. In the 1990s, the growing availability of data on large biological, social and technological networks revolutionized the field. Motivated by the observation that the World Wide Web [12] and scientific citation networks [13] exhibit power-law degree distributions, Barabási and Albert (BA) introduced a simple model that captures the essential growth dynamics of such networks [14]. A key feature of the BA model is the preferential attachment mechanism, namely the tendency of new nodes to attach preferentially to high degree nodes. Using mean-field equations and computer simulations, it was shown that the combination of growth and preferential attachment leads to the emergence of scale-free networks with power-law degree distributions [14]. This result was later confirmed and generalized using more rigorous formulations based on the master equation [15,16] and using combinatorial methods [17]. It was subsequently found that a large variety of empirical networks exhibit such scale-free structures, which are remarkably different from ER networks [18,19].

Networks are often exposed to node failures, attacks and epidemics, which may halt their growth and lead to their contraction and eventual collapse. Since network growth and contraction are kinetic nonequilibrium processes, they are irreversible, namely the contraction process is not the same as the growth process played backwards in time. This hysteretic behavior is analogous to the response of magnetic systems to an external magnetic field, where the magnetization depends not only on the instantaneous field but also on its history.

One can distinguish between three generic scenarios of network contraction: the *random deletion* scenario that describes inadvertent random failures of nodes [20], the *preferential deletion* scenario that describes intentional attacks [21], which are more likely to focus on high-degree nodes, and the *propagating deletion* scenario that describes cascading failures that spread throughout the network [22–24]. Using the framework of percolation theory, it was shown that in the final stages of the contraction process the network breaks down into disconnected components [20,21,25,26]. However, the evolution of the network structure throughout the contraction phase has not been studied in a systematic way.

In this paper we analyze the structural evolution of networks during the contraction process. To this end we derive a master equation for the time dependence of the degree distribution during network contraction via the random, preferential, and propagating node deletion scenarios. We show that the Poisson distribution with a time dependent mean degree c_t is a solution of the master equation. Moreover, using the relative entropy between the degree distribution $P_t(k)$ of the contracting network at time t and the corresponding Poisson distribution $\pi_t(k)$ with the same mean degree $c_t = \langle K \rangle_t$, we show that the Poisson distribution is an attractive solution for the degree distributions of random networks that contract via these three network contraction scenarios. Thus, the degree distribution $P_t(k)$ converges towards $\pi_t(k)$ during the contraction process. Using computer simulations of contracting networks, we show that if the initial network exhibits degree-degree correlations then these correlations decay during the contraction process. We thus conclude that the contracting networks converge towards an ER structure whose mean degree continues to decrease as the contraction proceeds.

The paper is organized as follows. In Sec. II we describe the three network contraction scenarios considered in this paper and discuss related examples of contraction processes in empirical networks. In Sec. III we present the master equation for these three network contraction scenarios. In Sec. IV we show that a Poisson distribution with a time dependent mean degree is a stationary solution of the master equation. In Sec. V we use direct integration of the master equation in conjunction with computer simulations to examine the convergence of the degree distribution of a contracting network towards a Poisson distribution. In Sec. VI we use the relative entropy S_t between the degree distribution $P_t(k)$ of the contracting network at time t and the corresponding Poisson distribution $\pi_t(k)$ with the same mean degree $c_t = \langle K \rangle_t$ to quantify the rate of convergence of $P_t(k)$ towards $\pi_t(k)$. In Sec. VII we use computer simulations to evaluate the decay rate of the degree-degree correlation function Γ_t during the contraction process. The results are discussed in Sec. VIII and summarized in Sec. IX. In Appendix A we present a detailed derivation of the master equation for the three network contraction scenarios. In Appendix B we present an exact solution for the time dependent degree distribution $P_t(k)$ of a network contracting via the random node deletion scenario.

II. NETWORK CONTRACTION PROCESSES

We consider network contraction processes in which at each time step a single node is deleted together with its links. The size of the network at time t is thus $N_t = N - t$, where $N_0 = N$ is the size of the initial network. Consider a node of degree k, whose neighbors are of degrees k'_r, $r = 1, 2, \dots, k$. Upon deletion of such node the degrees of its neighbors are reduced to $k'_r - 1$, $r = 1, 2, \dots, k$. The node deleted at each time step is selected randomly. However, the probability of a specific node to be deleted in the next time step may depend on its degree as well as on other properties, according to the specific network contraction scenario. Here we focus on three generic scenarios of network contraction: the random node deletion scenario that describes the random, inadvertent failure of nodes; the preferential node deletion scenario that describes intentional attacks that are more likely to focus on highly connected nodes; and the propagating node deletion scenario that describes cascading failures that spread throughout the network.

In the random deletion scenario, at each time step a random node is selected for deletion. In this scenario each one of the nodes in the network at time t has the same probability to be selected for deletion, regardless of its degree or any other properties. Since at time t there are N_t nodes in the network, the probability of each one of them to be selected for deletion is $1/N_t$. This scenario may describe a situation in which random nodes in a communication network become dysfunctional independently of each other due to technical failures or random attacks [20,25].

In the preferential deletion scenario the probability of a node to be targeted for deletion at a given time step is proportional to its degree. This means that the probability of a given node of degree k to be deleted at time t is $k/[N_t \langle K \rangle_t]$. This is equivalent to picking a random edge in the network and randomly selecting for deletion one of the two nodes at its ends. This scenario may describe attacks in which high degree nodes are more likely to be targeted [21].

In the propagating deletion scenario at each time step the node to be deleted is randomly selected among the neighbors of the node deleted in the previous time step. If the node deleted in the previous time step does not have any yet-undeleted neighbor, we pick a random node, randomly select one of its neighbors for deletion, and continue the process from there. This scenario may describe cascading failures in which the failure of a node increases the load on its neighbors and causing their subsequent failure. Such situations may occur in power grids and transportation networks [27,28]. Another mechanism of cascading failures was identified in social networks in which a user who leaves the network may encourage some of his/her friends to leave the network too, possibly for joining a competing network [29,30].

III. THE MASTER EQUATION

Consider an ensemble of networks of size N_0 at time $t = 0$, with degree distribution $P_0(k)$ and mean degree $\langle K \rangle_0$, which are exposed to network contraction via node deletion. Below we derive a master equation that describes the time evolution of the degree distribution $P_t(k)$ throughout the contraction process. The master equation consists of a set of coupled first-order differential equations of the form [31,32]

$$\frac{d}{dt}\vec{P}_t = M\vec{P}_t, \quad (1)$$

where $\vec{P}_t$ is a vector whose elements are the probabilities $P_t(k)$, $k = 0, 1, 2, \dots$, and M is the transition matrix.

At each time step during the contraction process a single node is deleted from the network. In addition to the primary effect of the loss of the deleted node, the network sustains a secondary effect as the neighbors of the deleted node lose one link each. An intrinsic property of the secondary effect is that it is of a preferential nature, namely the likelihood of a node to be a neighbor of the deleted node is proportional to its degree. The time dependent degree distribution is given by

$$P_t(k) = \frac{N_t(k)}{N_t}, \quad (2)$$

where $N_t(k)$ is the number of nodes of degree k at time t. The mean degree of the contracting network at time t is given by

$$\langle K \rangle_t = \sum_{k=0}^{\infty} k P_t(k), \quad (3)$$

while the second moment of the degree distribution is given by

$$\langle K^2 \rangle_t = \sum_{k=0}^{\infty} k^2 P_t(k). \quad (4)$$

Here we analyze three generic scenarios of network contraction: the random deletion scenario, in which a randomly selected node is deleted at each time step; the preferential deletion scenario, in which the probability of a node to be targeted for deletion is proportional to its degree; and the propagating deletion scenario, in which at each time step we delete a random neighbor of the last deleted node. To demonstrate the derivation of the master equation, we consider below the relatively simple case of random node deletion (for a detailed derivation of the master equation for all three network contraction scenarios see Appendix A). The time dependence of $N_t(k)$ depends on the primary effect, given by the probability that the node selected for deletion is of degree k, as well as on the secondary effect of node deletion on neighboring nodes of degrees k and $k+1$. In random node deletion the probability that the node selected for deletion at time t is of degree k is given by $N_t(k)/N_t$. Thus, the rate in which $N_t(k)$ decreases due to the primary effect of the deletion of nodes of degree k is given by

$$R_t(k \to \varnothing) = \frac{N_t(k)}{N_t}, \tag{5}$$

where $\varnothing$ represents the empty set. If the node deleted at time t is of degree k', it affects k' adjacent nodes, which lose one link each. The probability of each one of these k' nodes to be of degree k is given by $kN_t(k)/[N_t\langle K\rangle_t]$. We denote by $W_t(k \to k-1)$ the expectation value of the number of nodes of degree k that lose a link at time t and are reduced to degree $k-1$. Summing up over all possible values of k', we find that the secondary effect of random node deletion on nodes of degree k is

$$W_t(k \to k-1) = \frac{kN_t(k)}{N_t}. \tag{6}$$

Similarly, the secondary effect on nodes of degree $k+1$ is

$$W_t(k+1 \to k) = \frac{(k+1)N_t(k+1)}{N_t}. \tag{7}$$

The time evolution of $N_t(k)$ can be expressed in terms of the forward difference

$$\Delta_t N_t(k) = N_{t+1}(k) - N_t(k). \tag{8}$$

Combining the primary and the secondary effects on the time dependence of $N_t(k)$, we obtain

$$\Delta_t N_t(k) = -R_t(k \to \varnothing) + [W_t(k+1 \to k) - W_t(k \to k-1)]. \tag{9}$$

Inserting the expressions for $R_t(k \to \varnothing)$, $W_t(k \to k-1)$, and $W_t(k+1 \to k)$ from Eqs. (5), (6) and (7), respectively, we obtain

$$\Delta_t N_t(k) = \frac{(k+1)[N_t(k+1) - N_t(k)]}{N_t}. \tag{10}$$

Since nodes are discrete entities the process of node deletion is intrinsically discrete. Therefore, the replacement of the forward difference $\Delta_t N_t(k)$ by a time derivative of the form $dN_t(k)/dt$ involves an approximation. In fact, it is closely related to the approximation made in numerical integration of differential equations using the Euler method [33]. In the Euler method the time derivative df_t/dt is replaced by $(f_{t+h} - f_t)/h$, where h is a suitably chosen time step. In our case $h = 1$. Below we evaluate the error associated with this approximation. To this end we use a series expansion of the form

$$\Delta_t N_t(k) = \frac{d}{dt}N_t(k) + \frac{1}{2}\frac{d^2}{dt^2}N_t(k) + \cdots. \tag{11}$$

Typical degree distributions, which are not too narrow, satisfy $N_t(k) \ll N_t$ for any value of k. For such distributions the second time derivative satisfies

$$\frac{1}{2}\frac{d^2}{dt^2}N_t(k) \sim O\left(\frac{1}{N_t^2}\right), \tag{12}$$

and quickly vanishes for sufficiently large networks. This means that the replacement of the forward difference by a time derivative has little effect on the results. Thus, the difference equation (10) can be replaced by the differential equation

$$\frac{d}{dt}N_t(k) = \frac{(k+1)[N_t(k+1) - N_t(k)]}{N_t} + O\left(\frac{1}{N_t^2}\right). \tag{13}$$

In a more rigorous approach one could define a reduced time $\theta = t/N$ and a density $n(\theta, k) = N_t(k)/N$, as done in Refs. [34–38]. Using this approach, one can show that the random variable $N_{t=\theta N}(k)/N$ concentrates, in the large N limit, around the deterministic density $n(\theta, k)$ which is the solution of the corresponding differential equation.

The derivation of the master equation is completed by taking the time derivative of Eq. (2), which is given by

$$\frac{d}{dt}P_t(k) = \frac{1}{N_t}\frac{d}{dt}N_t(k) - \frac{N_t(k)}{N_t^2}\frac{d}{dt}N_t. \tag{14}$$

Inserting the time derivative of $N_t(k)$ from Eq. (13), and using the fact that $dN_t/dt = -1$, we obtain the master equation for the random deletion scenario, which is given by

$$\frac{d}{dt}P_t(k) = \frac{1}{N_t}[(k+1)P_t(k+1) - kP_t(k)]. \tag{15}$$

The derivation of the master equations for the preferential deletion and the propagating deletion scenarios can be performed along similar lines. The detailed derivations of the master equations for all three scenarios appear in Appendix A. Interestingly, the resulting master equations for these three network contraction scenarios can be written in a unified manner, in the form

$$\frac{d}{dt}P_t(k) = \frac{A_t}{N_t}[(k+1)P_t(k+1) - kP_t(k)] - \frac{B_t(k)}{N_t}P_t(k), \tag{16}$$

where the coefficients are given by

$$A_t = \begin{cases} 1, & \text{random deletion,} \\ \frac{\langle K^2\rangle_t}{\langle K\rangle_t^2}, & \text{preferential deletion,} \\ \frac{\langle K^2\rangle_t - 2\langle K\rangle_t}{\langle K\rangle_t^2}, & \text{propagating deletion} \end{cases} \tag{17}$$

and

$$B_t(k) = \begin{cases} 0, & \text{random deletion,} \\ \frac{k - \langle K\rangle_t}{\langle K\rangle_t}, & \text{preferential deletion,} \\ \frac{k - \langle K\rangle_t}{\langle K\rangle_t}, & \text{propagating deletion.} \end{cases} \tag{18}$$

The master equation consists of a set of coupled ordinary differential equations for $P_t(k)$, $k = 0, 1, 2, \ldots, k_{\max}$. In order to calculate the time evolution of the degree distribution $P_t(k)$ during the contraction process, one solves the master equation using direct numerical integration, starting from the initial network that consists of N_0 nodes whose degree distribution is $P_0(k)$. For any finite network the degree distribution is bounded from above by an upper bound denoted by $k_{\max}$, which satisfies the condition $k_{\max} \leqslant N_0 - 1$. Since the contraction process can only delete edges from the remaining nodes and cannot increase the degree of any node, the upper cutoff $k_{\max}$ is maintained throughout the contraction process.

Expressing the master equation in terms of the transition rate matrix formulation of Eq. (1), it is found that the matrix M is an upper bidiagonal matrix, whose diagonal elements are given by

$$M_{k,k} = -\frac{kA_t + B_t(k)}{N_t}, \tag{19}$$

the off-diagonal elements are given by

$$M_{k,k+1} = \frac{(k+1)A_t}{N_t}, \tag{20}$$

and $M_{k,k'} = 0$ for $k' < k$ and $k' > k+1$.

The rate coefficients on the right-hand side of the master equation (16) include a combination of explicit and implicit time dependence. The overall factor of $1/N_t$ is the only components that exhibits an explicit time dependence, while the moments $\langle K \rangle_t$ and $\langle K^2 \rangle_t$ depend implicitly on the time via the instantaneous degree distribution $P_t(k)$. Since their coefficients are time dependent they need to be updated throughout the numerical integration of Eq. (16). In particular, the instantaneous network size N_t should be updated at each time step. The time derivatives of the moments $\langle K \rangle_t$ and $\langle K^2 \rangle_t$ scale with the network size like $1/N_t$. Therefore, they may be considered as slow variables and updated once every several time steps during the integration of the master equation.

Since the only explicit time dependence of the rate coefficients on the right-hand side of Eq. (16) is via the overall factor of $1/N_t$, one can multiply both sides of the equation by N_t. The time derivative on the left-hand side of Eq. (16) can then be expressed in terms of $d\tau = dt/N_t$. This implies that the time dependence of $P_t(k)$ is expressed in terms of the ratio N_t/N_0, or more specifically in terms of $\tau = \ln(N_t/N_0)$. This means that the initial network size is essentially an extensive parameter while the time is measured in terms of the fraction of the network that remains. This conclusion is of great practical importance because it means that for any given degree distribution it is sufficient to perform the simulation of network collapse for one size of the initial network.

The first term on the right-hand side of Eq. (16) is referred to as the trickle-down term [39]. This term represents the step by step downwards flow of probability from high to low degrees. The coefficient A_t of the trickle-down term depends on the network contraction scenario. In random deletion $A_t = 1$, because the probability of a node to be selected for deletion does not depend on its degree. In preferential deletion A_t is proportional to $\langle K^2 \rangle_t$ because the probability of a node to be deleted is proportional to its degree k while the magnitude of the secondary effect is also proportional to k.

The second term on the right-hand side of Eq. (16) is referred to as the redistribution term. This term vanishes in the random deletion scenario. However, in the preferential and propagating deletion scenarios the redistribution term is negative for $k > \langle K \rangle_t$ and positive for $k < \langle K \rangle_t$. Thus the redistribution term decreases the probabilities $P_t(k)$ for values of k that are above the mean degree and increases them for values of k that are below the mean degree. Moreover, in absolute value the size of the redistribution term is proportional to $|k - \langle K \rangle_t|$, which means that nodes of degrees that are much higher or much lower than $\langle K \rangle_t$ are most strongly affected by this term.

In general, the master equation accounts for the time evolution of the degree distribution over an ensemble of networks of the same initial size N_0 and degree distribution $P_0(k)$, which are exposed to the same network contraction scenario. A fundamental question in this context is to what extent the solution of a deterministic differential equation describes the results of single instances of the stochastic process in systems of finite size. In the context of network contraction processes, a single instance of the stochastic process at time t is described by $N_t(k)$, $k = 0, 1, \ldots$. The corresponding results of the master equation are given by $\langle N_t(k) \rangle = N_t P_t(k)$, $k = 0, 1, \ldots$. Using the theory of stochastic processes it was shown that under very general conditions the results of single instances, given by $N_t(k)$, are narrowly distributed around $\langle N_t(k) \rangle$, thus the master equation provides a good description of the corresponding stochastic process [34–38].

IV. THE POISSON SOLUTION

Consider an ER network of N_t nodes with mean degree $c_t = \langle K \rangle_t$. Its degree distribution follows a Poisson distribution of the form

$$\pi_t(k) = \frac{e^{-c_t} c_t^k}{k!}. \tag{21}$$

The second moment of the degree distribution satisfies $\langle K^2 \rangle_t = c_t(c_t + 1)$. To examine the contraction process of ER networks we start from an initial network of N_0 nodes whose degree distribution follows a Poisson distribution $\pi_0(k)$ with mean degree c_0. Inserting $\pi_t(k)$ into the master equation (16), we find that the time derivative on the left-hand side is given by

$$\frac{d}{dt}\pi_t(k) = -\frac{dc_t}{dt}\left(1 - \frac{k}{c_t}\right)\pi_t(k), \tag{22}$$

On the other hand, inserting $\pi_t(k)$ on the right-hand side of Eq. (16), we obtain

$$\frac{d}{dt}\pi_t(k) = \frac{A_t}{N_t}(c_t - k)\pi_t(k) - \frac{B_t(k)}{N_t}\pi_t(k), \tag{23}$$

where

$$A_t = \begin{cases} 1, & \text{random deletion,} \\ \frac{c_t+1}{c_t}, & \text{preferential deletion,} \\ \frac{c_t-1}{c_t}, & \text{propagating deletion} \end{cases} \tag{24}$$

and

$$B_t(k) = \begin{cases} 0, & \text{random deletion,} \\ \frac{k-c_t}{c_t}, & \text{preferential deletion,} \\ \frac{k-c_t}{c_t}, & \text{propagating deletion.} \end{cases} \tag{25}$$

In order for $\pi_t(k)$ to be a solution of Eq. (16), the right-hand sides of Eqs. (22) and (23) must coincide. In the case of random deletion this implies that

$$\frac{1}{c_t}\frac{dc_t}{dt} = -\frac{1}{N_t}. \tag{26}$$

Integrating both sides for $t' = 0$ to t, we obtain the condition

$$c_t = c_0 \frac{N_t}{N_0} = c_0 - \frac{c_0}{N_0} t. \tag{27}$$

Repeating the analysis presented above for the cases of preferential deletion and propagating deletion, it is found that $\pi_t(k)$ solves the master equation (16) for the three network contraction scenarios, while the mean degree c_t decreases linearly in time according to

$$c_t = c_0 - Rt, \tag{28}$$

where the rate R is given by

$$R = \begin{cases} \frac{c_0}{N_0}, & \text{random deletion}, \\ \frac{c_0+2}{N_0}, & \text{preferential deletion}, \\ \frac{c_0}{N_0}, & \text{propagating deletion}. \end{cases} \tag{29}$$

This means that an ER network exposed to any one of the three contraction scenarios remains an ER network at all times, with a mean degree that decreases according to Eq. (28). The network size at time t is $N_t = N_0 - t$, where N_0 is the initial size.

In the case of random deletion the contraction process ends at time $t = N_0$, when the network vanishes completely. In the case of preferential deletion the deleted node at each time step is picked via a randomly selected edge. Therefore, once a node becomes isolated it will never be selected for deletion. As a result, the process of preferential deleted comes to a halt once all the remaining nodes become isolated and $c_t = 0$. Using Eqs. (28) and (29) we find that this happens at time $t_{\rm h} = c_0 N_0/(c_0 + 2)$. Thus, the number of isolated nodes that remain is $N_{\rm h} = 2N_0/(c_0 + 2)$. In the case of propagating deletion one may encounter a situation in which the node deleted at time t becomes isolated, namely it does not have any yet-undeleted neighbors. In this case we continue the deletion process by selecting a random node, randomly selecting one of its neighbors for deletion and continuing the process from there.

V. NUMERICAL INTEGRATION AND COMPUTER SIMULATIONS

To test the convergence of contracting networks towards the ER structure, we study the three network contraction scenarios presented above using numerical integration of the master equation and computer simulations. As an initial network we use the BA network, which is a scale-free network with a power-law degree distribution of the form $P_0(k) \sim k^{-\gamma}$, where $\gamma = 3$ [7,14–16]. To generate the initial networks for the computer simulations, we use the BA growth model, in which at each time step a new node is added to the network and is connected preferentially by undirected edges to m of the existing nodes [14]. The m edges of the new node are added sequentially, under the condition that each existing node can receive at most one of these edges (multiple edges are not allowed, thus the resulting network is a simple graph). The preferential attachment property implies that the probability of an existing node whose degree at time t is k (and is not yet connected to the new node) to receive the next link from the new node is proportional to k. The parameter m may take any nonzero integer value. In the special case of $m = 1$ the resulting network exhibits a tree structure, while for $m \geqslant 2$ it includes cycles. As a seed network for the growth process we use a complete graph of $m + 1$ nodes, such that at time $t = 0$ all the nodes in the seed network are of degree m. Since there are m edges that are added to the network with each new node, and each edge connects two nodes, in the large network limit $N_0 \gg m$ the mean degree is $\langle K \rangle_0 = 2m$. Thus, the network becomes more dense as m is increased. Since the seed network consists of a single connected component, the resulting network consists of a single component at all times. The growth process ends when the network reaches the desired size, denoted by N_0. The degree distribution of a BA network is given by

$$P_0(k) \sim k^{-\gamma}, \tag{30}$$

where $\gamma = 3$. Since the degree of each new node upon formation is m, the lower bound of the degree distribution (30) is $k_{\min} = m$. To generate the initial degree distribution used in the direct integration of the master equation, we use the master equation that describes the BA network growth process [15,16], with the same seed network used in the computer simulations.

In Fig. 1 we present the structure of a BA network with $m = 3$ during growth at an intermediate size of $N = 150$ (left) and at the final size of $N = 200$ (middle). At this point the network starts to contract via preferential node deletion. The structure of the network during the contraction process is presented (right), when its size is down to $N = 150$. To emphasize the variation in the degrees of different nodes, each node is represented by a circle whose area is proportional to the degree of the node. It is apparent that the initial BA network includes several dominant hubs, as expected in a scale-free network, while in the network depicted during contraction there is little variation between the degrees of different nodes. In a supplemental movie [40] we present the evolution of the structure of the same BA network instance during the growth phase and the subsequent contraction phase via random deletion and preferential deletion.

In Fig. 2 we present the degree distributions $P(k)$ (solid lines) of a BA network with $m = 50$, obtained from numerical integration of the master equation that describes the growth process [15,16] during growth at an intermediate size of $N = 1300$ (left) and at the final size of $N = 10\,000$ (middle). The resulting degree distributions, presented in a log-log scale, follow a straight line that corresponds to $P(k) \sim k^{-\gamma}$, with $\gamma = 3$. They coincide with the degree distributions obtained from computer simulations of the BA growth process (circles). The corresponding Poisson distributions with the same value of the mean degrees, namely $c = \langle K \rangle$, are also shown (dashed lines). They form narrow and nearly symmetric distributions whose peaks are close to the mean degree c. Clearly, the power-law distribution (solid line) and the Poisson distribution (dashed line) are essentially as different from each other as any two distributions with the same mean degree could be. Starting from $N = 10\,000$ the network contracts via the preferential node deletion scenario. The degree distribution of the contracted network when its size is reduced to $N = 1300$ nodes is shown (right). The results obtained from numerical integration of the master equation (16) and from computer simulations (solid line and circles, respectively) are

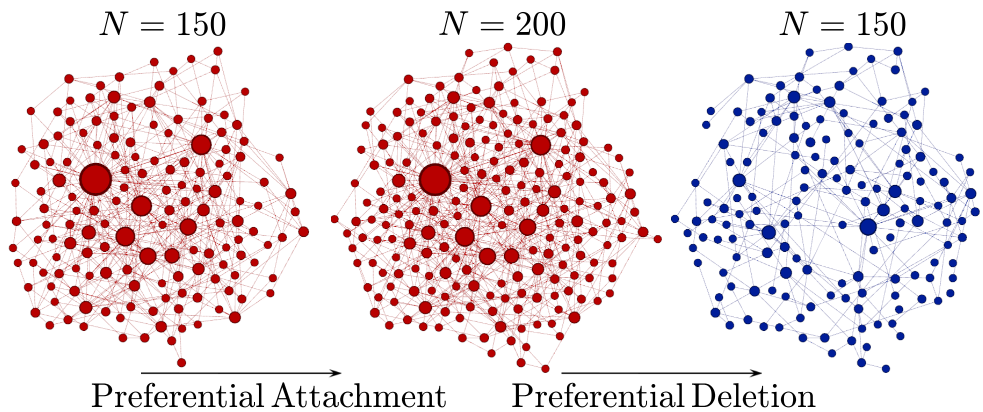

FIG. 1. A BA network is shown during the growth phase, at sizes of $N = 150$ (left) and $N = 200$ (middle), and during the contraction phase when its size is reduced to $N = 150$ (right). There is a striking difference between the structures of the growing network that exhibits large hubs and the contracting network that shows little variation between the degrees of different nodes. In a supplemental movie [40] we present the full evolution of this network during the growth phase and the subsequent contraction phase via random deletion and preferential deletion.

found to be in excellent agreement with the corresponding Poisson distribution with the same mean degree (dashed line).

In Fig. 3 we present the evolution of the mean degree $\langle K \rangle_t$ as a function of time for random deletion (a), preferential deletion (b), and propagating deletion (c). In the random deletion scenario, the mean degree $\langle K \rangle_t$ decreases linearly in time, where $\langle K \rangle_t = \langle K \rangle_0 (1 - t/N_0)$ does not depend on the functional form of $P_0(k)$. In the preferential and propagating deletion scenarios the decay rate of $\langle K \rangle_t$ depends on the initial degree distribution $P_0(k)$. If $P_0(k)$ is a fat tailed distribution such as the power-law distribution, the initial decay of $\langle K \rangle_t$ is fast and then it slows down. This is due to the fact that in these two scenarios an excess of high degree nodes are targeted for deletion in the early stages, enhancing the decrease of $\langle K \rangle_t$.

VI. THE RELATIVE ENTROPY

In order to establish that networks exposed to these contraction scenarios actually converge towards the ER structure, it remains to show that this asymptotic solution is attractive. To this end we quantify the convergence rate of $P_t(k)$ towards the Poisson distribution, using the relative entropy (also referred to as the Kullback-Leibler divergence), defined by [41]

$$S_t = \sum_{k=0}^{\infty} P_t(k) \ln \left[\frac{P_t(k)}{\pi_t(k)} \right], \tag{31}$$

where $\pi_t(k)$ is the Poisson distribution given by Eq. (21). The relative entropy S_t measures the difference between the probability distribution $P_t(k)$ and the reference distribution $\pi_t(k)$. It also quantifies the added information associated with constraining the degree distribution $P_t(k)$ rather than only the mean degree c_t [10,11]. The Poisson distribution is a proper reference distribution because it satisfies $\pi_t(k) > 0$ for all the non-negative integer values of k. The relative entropy is always non-negative and satisfies $S_t = 0$ if and only if $P_t(k) = \pi_t(k)$. Therefore, S_t can be used as a measure of the distance between a given network and the corresponding ER

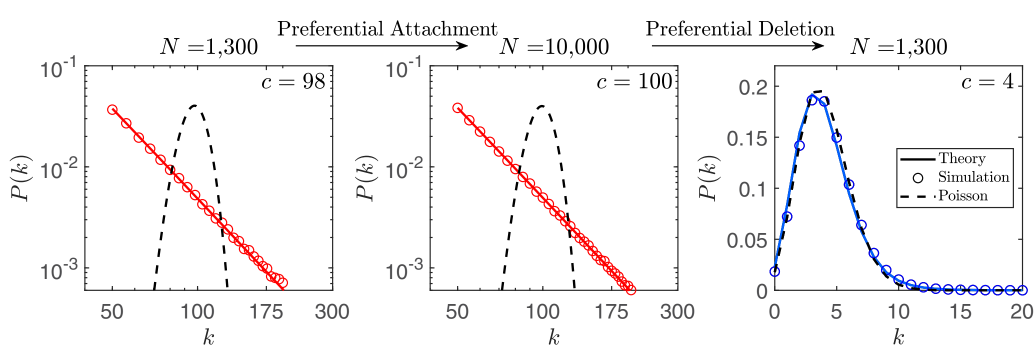

FIG. 2. The degree distributions $P(k)$ of a BA network during growth, obtained from numerical integration of the master equation of Refs. [15] and [16] (solid line) and from computer simulations (circles) at an intermediate size of $N = 1300$ (left) and at the final size of $N = 10\,000$ (middle). The Poisson distribution with the same mean degree is also shown (dashed line). At $N = 10\,000$ the network starts to contract via preferential node deletion. The degree distribution $P(k)$ of the contracted network is shown (right) when it is reduced back to $N = 1300$ nodes. The theoretical results (solid line) obtained from the master equation [Eq. (16)] are in very good agreement with computer simulations (circles) and with the Poisson distribution with the same mean degree (dashed line).

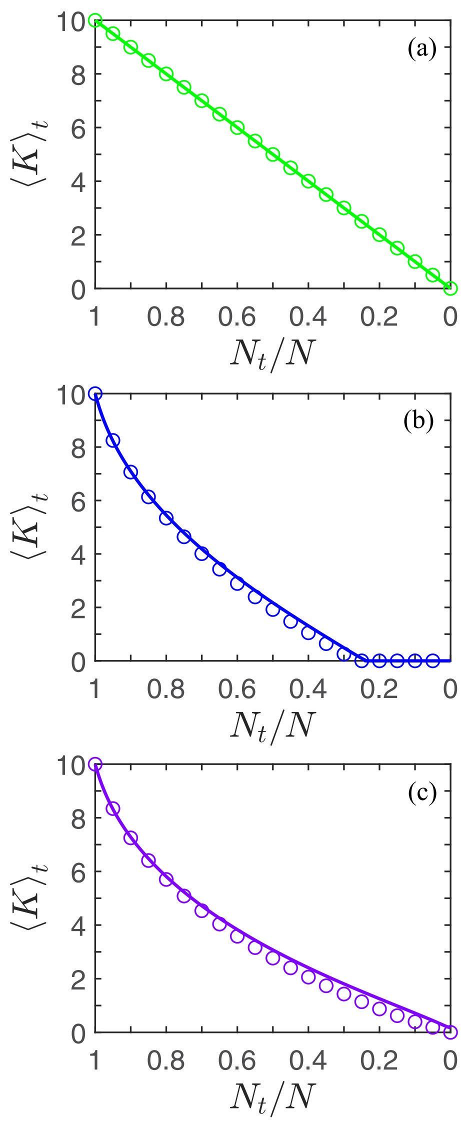

FIG. 3. The mean degrees $\langle K\rangle_t$ vs N_t/N, obtained from numerical integration of the master equation (solid lines), for networks that contract via random deletion (a), preferential deletion (b), and propagating deletion (c), starting from a BA network with $m = 5$ of size $N = 10\,000$. The master equation results are in very good agreement with computer simulation results (circles). In the case of random node deletion $\langle K\rangle_t$ decreases linearly in time according to $\langle K\rangle_t = \langle K\rangle_0 - Rt$, where $R = c_0/N$ is independent of the degree distribution of the initial network. In the preferential deletion and propagating deletion scenarios the time dependence of $\langle K\rangle_t$ during the contraction process depends on the degree distribution of the initial network. If the initial network exhibits a power-law distribution, it is found that in the early stages $\langle K\rangle_t$ quickly decreases due to the preferential deletion of high degree nodes. The decay rate of $\langle K\rangle_t$ gradually slows down and approaches a constant slope as $P_t(k)$ converges towards a Poisson distribution.

network with the same mean degree. In each of the network contraction processes, the degree distribution $P_t(k)$ evolves in time according to Eq. (16). As a result, the relative entropy S_t of the network also evolves as the network contracts. The time derivative of S_t is given by

$$\frac{d}{dt}S_t = \sum_{k=0}^{\infty} \ln\left[\frac{P_t(k)}{\pi_t(k)}\right]\frac{d}{dt}P_t(k) + \sum_{k=0}^{\infty}\frac{d}{dt}P_t(k) - \sum_{k=0}^{\infty}\frac{P_t(k)}{\pi_t(k)}\frac{d}{dt}\pi_t(k). \quad (32)$$

Replacing the order of the summation and the derivative in the second term on the right-hand side of Eq. (32), we obtain

$$\sum_{k=0}^{\infty}\frac{d}{dt}P_t(k) = \frac{d}{dt}\left[\sum_{k=0}^{\infty}P_t(k)\right] = 0. \quad (33)$$

Inserting the derivative $d\pi_t(k)/dt$ from Eq. (22) into the third term on the right-hand side of Eq. (32), and recalling that $c_t = \langle K\rangle_t$, we obtain

$$\sum_{k=0}^{\infty}\frac{P_t(k)}{\pi_t(k)}\frac{d}{dt}\pi_t(k) = -\frac{dc_t}{dt}\sum_{k=0}^{\infty}\left(1-\frac{k}{c_t}\right)P_t(k) = 0. \quad (34)$$

Since the second and third terms in Eq. (32) vanish, the time derivative of the relative entropy is given by

$$\frac{d}{dt}S_t = \sum_{k=0}^{\infty}\ln\left[\frac{P_t(k)}{\pi_t(k)}\right]\frac{d}{dt}P_t(k). \quad (35)$$

This is a general equation that applies to any network contraction scenario in which the Poisson distribution $\pi_t(k)$ is a solution. In order to obtain a more specific equation for a given network contraction scenario, one should insert the expression for the derivative $dP_t(k)/dt$ from the corresponding master equation into Eq. (35).

In Fig. 4 we present the relative entropy S_t obtained from numerical integration of the master equation (solid lines) for random deletion (a), preferential deletion (b), and propagating deletion (c), starting from a BA network with $m = 5$ and size $N = 10\,000$. The master equation results are in very good agreement with the results obtained from computer simulations (circles). The + symbols mark the points at which S_t decays to $1/e$ of its initial values. In the case of random deletion this occurs around $N_t/N \simeq 0.4$, while in the other two scenarios it occurs much earlier, at $N_t/N \simeq 0.9$, following the deletion of only about 10% of the nodes. Note that in the preferential and the propagating deletion scenarios S_t decays very quickly and practically vanishes when more than a half of the nodes still remain.

VII. THE DEGREE-DEGREE CORRELATION FUNCTION

An important distinction in network theory is between uncorrelated random networks and networks that exhibit degree-degree correlations. These correlations are positive (negative) in assortative (disassortative) networks, in which high degree nodes tend to connect to high (low) degree nodes and low degree nodes tend to connect to low (high) degree nodes [42,43]. To quantify the degree-degree correlations, we define the joint degree distribution $P_t(k, k')$ of pairs of nodes that reside on both sides of a randomly selected edge. The marginal degree

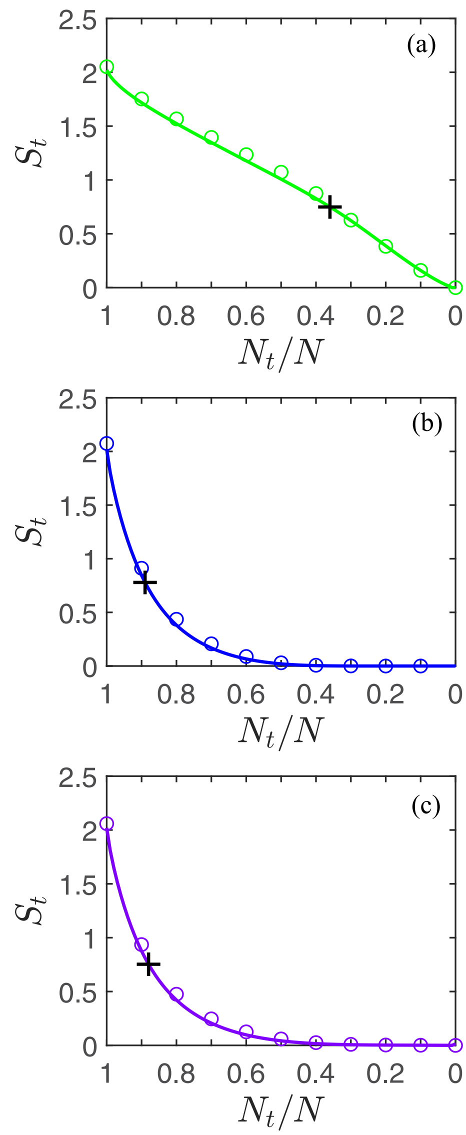

FIG. 4. The relative entropy S_t vs N_t/N, obtained from numerical integration of the master equation (solid lines) for random deletion (a), preferential deletion (b), and propagating deletion (c), starting from a BA network with $m = 5$ and size $N = 10\,000$. The master equation results are in very good agreement with the results obtained from computer simulations (circles). The + symbols mark the points at which S_t decays to $1/e$ of its initial values. In the case of random deletion this occurs around $N_t/N \simeq 0.4$, while in the other two scenarios it occurs much earlier, at $N_t/N \simeq 0.9$, following the deletion of only about 10% of the nodes. Note that in the preferential deletion and the propagating deletion S_t decays very quickly and practically vanishes when more than a half of the nodes still remain.

distribution, obtained by tracing over all possible values of k', is given by

$$\widetilde{P}_t(k) = \frac{k}{\langle K \rangle_t} P_t(k). \tag{36}$$

The degree-degree correlation function is given by

$$\Gamma_t = \langle KK' \rangle_t - \langle \widetilde{K} \rangle_t \langle \widetilde{K} \rangle_t. \tag{37}$$

The first term in Eq. (37) is a mixed second moment of the form

$$\langle KK' \rangle_t = \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} kk' P_t(k, k'), \tag{38}$$

where the sums run over all the possible combinations of the degrees of pairs of adjacent nodes. In the second term of Eq. (37), the mean degree $\langle \widetilde{K} \rangle_t$ of the degree distribution $\widetilde{P}_t(k)$ of nodes adjacent to a randomly selected edge is given by

$$\langle \widetilde{K} \rangle_t = \sum_{k=1}^{\infty} k \widetilde{P}_t(k). \tag{39}$$

If there are no degree-degree correlations, the joint degree distribution of pairs of adjacent nodes is given by

$$P_t(k, k') = \widetilde{P}_t(k) \widetilde{P}_t(k'), \tag{40}$$

and the correlation function satisfies $\Gamma_t = 0$. This is indeed the case in configuration model networks. However, BA networks exhibit degree-degree correlations and are disassortative, namely high degree nodes tend to connect to low degree nodes and vice versa [44].

The master equation (16) follows the time evolution of the degree distribution $P_t(k)$ during the contraction process, but does not account for degree-degree correlations. Therefore, it cannot be used to explore the time dependence of the degree-degree correlation function Γ_t. To examine the effect of network contraction processes on degree-degree correlations, we use computer simulations.

In Fig. 5 we present the degree-degree correlation function Γ_t obtained from computer simulations (circles) of the contraction process of BA networks of size $N = 10\,000$ with $m = 5$, via random deletion (a), preferential deletion (b), and propagating deletion (c). In the case of random deletion the simulation results are very well fitted by $\Gamma_t \sim (N_t/N)^2$, while the simulation results of the preferential deletion and the propagating deletion processes are very well fitted by an exponential fit (dashed lines). The + symbols mark the points at which Γ_t decays to $1/e$ of its initial values. In the case of random deletion this occurs around $N_t/N \simeq 0.6$, while in the other two scenarios it occurs at $N_t/N \simeq 0.9$, following the deletion of only about 10% of the nodes. Note that in the preferential deletion and the propagating deletion Γ_t decays very quickly and practically vanishes when more than a half of the nodes still remain.

Putting together the results of the last two sections, the convergence of the degree distribution towards a Poisson distribution (as demonstrated by the decay of S_t) and the decay of the degree-degree correlations (measured by Γ_t) imply that networks that contract via one of the three node deletion scenarios discussed in this paper converge towards the ER structure.

VIII. DISCUSSION

The timescales involved in network contraction processes span many orders of magnitude, from fractions of a second in computer networks to months and years in social networks

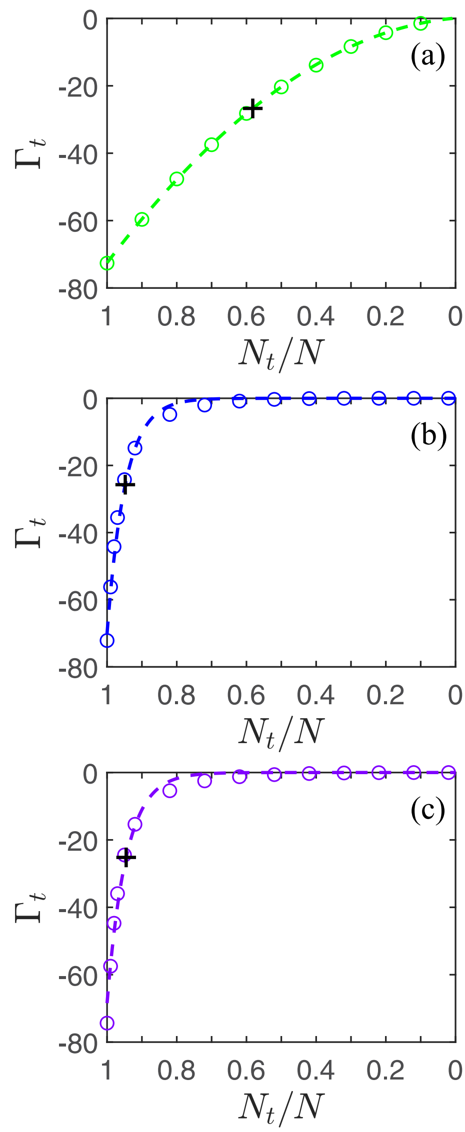

FIG. 5. The correlation function Γ_t vs N_t/N, obtained from computer simulations (circles) for random deletion (a), preferential deletion (b), and propagating deletion (c), starting from a BA network with $m = 5$ and size $N = 10\,000$. In the case of random deletion the simulation results are very well fitted by $\Gamma_t \sim (N_t/N)^2$, while the simulation results of the preferential deletion and the propagating deletion processes are very well fitted by an exponential fit (dashed lines). The + symbols mark the points at which Γ_t decays to $1/e$ of its initial values. In the case of random deletion this occurs around $N_t/N \simeq 0.6$, while in the other two scenarios it occurs at $N_t/N \simeq 0.9$, following the deletion of only about 10% of the nodes. Note that in the preferential deletion and the propagating deletion Γ_t decays very quickly and practically vanishes when more than a half of the nodes still remain.

to millennia in ecological networks. In some cases the contraction may proceed all the way down to the percolation threshold and into the subpercolating regime. In other cases only limited contraction is possible, either because the faulty nodes are quickly fixed or because the failure of a few nodes is sufficient to cause an unrecoverable damage to the entire system.

It is worth mentioning that there are other network dismantling processes that involve optimized attacks, which maximize the damage to the network for a minimal set of deleted nodes [26]. Such optimization is achieved by first decycling the network, namely by selectively deleting nodes that reside on cycles, thus driving the giant component into a tree structure. The branches of the tree are then trimmed such that the giant component is quickly disintegrates. Clearly, networks that are exposed to these optimized dismantling processes do not converge towards an ER structure.

The convergence of a contracting network towards the ER structure takes place over a limited range of network sizes and densities, bounded from above by the initial size N and mean degree $\langle K\rangle_0$ and from below by the size at which the remaining network becomes fragmented and consists of small isolated components and isolated nodes. However, this range can be extended indefinitely by starting the contraction process from a larger and denser network.

Network contraction processes belong to a broad class of dynamical processes that exhibit intermediate asymptotics [45,46]. The ubiquity of such processes is expressed in the following quotation from the opening paragraph of Ref. [45]: "In constructing the idealizations the phenomena under study should be considered at 'intermediate' times and distances [...]. These distances and times should be sufficiently large for details and features which are of secondary importance to the phenomenon to disappear. At the same time they should be sufficiently small to reveal features of the phenomena which are of basic value."

IX. SUMMARY

In summary, we analyzed the evolution of network structure during generic contraction processes, using the master equation, the relative entropy, and degree-degree correlations. We showed that in generic contraction scenarios, namely random, preferential, and propagating deletion processes, the network structure converges towards the ER structure, which exhibits a Poisson degree distribution and no degree-degree correlations. These results have important implications in real world situations. For example, in cascading failures they imply that the part of the network that continues to function is likely to converge towards an ER structure. In the context of ecological networks, they imply that mass extinctions not only reduce the number of species but may also alter the structure of the networks describing the interactions between them from scale-free-like networks to ER-like networks. To conclude, while scale-free network structures with power-law degree distributions are predominant in a world of growing or expanding networks, the uncorrelated Poisson-distributed ER structures are expected to be widespread in a world of contracting networks.

ACKNOWLEDGMENT

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APPENDIX A: DETAILED DERIVATION OF THE MASTER EQUATION

Below we derive the master equation describing the temporal evolution of the degree distribution $P_t(k)$ during network contraction via random node deletion, preferential node deletion, and propagating node deletion.

1. Random node deletion

In the random node deletion scenario at each time step a random node is deleted from the network together with its links. To derive an equation for the time dependence of $N_t(k)$ one needs to account for the primary effect of the deletion of a node of degree k and for the secondary effect in which nodes of degrees k and $k+1$ lose a link due to the deletion of an adjacent node. The probability that the deleted node is of degree k is given by $N_t(k)/N_t$. Therefore, the contribution of the primary effect of node deletion to the time derivative of $N_t(k)$ is given by $R_t(k \to \varnothing)$ [Eqs. (5) and (9)]. Regarding the secondary effect, if the node deleted at time t is of degree k', it affects k' other nodes, which lose one link each. Among these k' nodes, the probability of each one of them to be of degree k is given by $kN_t(k)/[N_t\langle K\rangle_t]$. Summing up over all the possible values of the degree k' of the deleted node and evaluating the expectation value of the number of nodes of degree k that are connected to the deleted node, we obtain the secondary effect of random node deletion on nodes of degree k. The rate at which nodes of degree k lose one link and are reduced to degree $k-1$ is given by

$$W_t(k \to k-1) = \sum_{k'=1}^{\infty} \frac{N_t(k')}{N_t} \frac{k'kN_t(k)}{N_t\langle K\rangle_t} = \frac{kN_t(k)}{N_t}. \quad \text{(A1)}$$

Similarly, the rate at which nodes of degree $k+1$ lose one link and are reduced to degree k is given by

$$W_t(k+1 \to k) = \frac{(k+1)N_t(k+1)}{N_t}. \quad \text{(A2)}$$

Combining the results for the primary and the secondary effects, it is found that the time dependence of $N_t(k)$ is given by

$$\frac{d}{dt}N_t(k) = \frac{(k+1)}{N_t}[N_t(k+1) - N_t(k)]. \quad \text{(A3)}$$

Inserting this result into Eq. (14), we obtain the master equation

$$\frac{d}{dt}P_t(k) = \frac{1}{N_t}[(k+1)P_t(k+1) - kP_t(k)]. \quad \text{(A4)}$$

In Appendix B we present an exact solution of Eq. (A4), which provides the time dependent degree distribution $P_t(k)$ for any initial degree distribution $P_0(k)$.

2. Preferential node deletion

In the scenario of preferential node deletion, at each time step a node is selected for deletion with probability proportional to its degree. The probability that the node selected for deletion at time t is of degree k is given by $kN_t(k)/[N_t\langle K\rangle_t]$. If the node selected for deletion at time t is of degree k', there are k' other nodes that will be affected, losing one link each. The probability of each one of these k' nodes to be of degree k is given by $kN_t(k)/[N_t\langle K\rangle_t]$. Summing up over all the possible values of the degree k' and evaluating the expectation value of the number of nodes of degree k that are connected to the deleted node, we obtain that the secondary effect on nodes of degree k is given by

$$W_t(k \to k-1) = \sum_{k'=1}^{\infty} \left[\frac{k'N_t(k')}{N_t\langle K\rangle_t}\right]\left[\frac{k'kN_t(k)}{N_t\langle K\rangle_t}\right] = \frac{\langle K^2\rangle_t}{\langle K\rangle_t^2 N_t} kN_t(k). \quad \text{(A5)}$$

Similarly, the secondary effect on nodes of degree $k+1$ is given by

$$W_t(k+1 \to k) = \frac{\langle K^2\rangle_t}{\langle K\rangle_t^2 N_t}(k+1)N_t(k+1). \quad \text{(A6)}$$

Summing up the contributions of the primary and the secondary effects, we obtain the time derivative of $N_t(k)$, which is thus given by

$$\frac{d}{dt}N_t(k) = \frac{\langle K^2\rangle_t}{\langle K\rangle_t^2 N_t}[(k+1)N_t(k+1) - kN_t(k)] - \frac{k}{\langle K\rangle_t N_t}N_t(k). \quad \text{(A7)}$$

Inserting this result into Eq. (14), we obtain the master equation

$$\frac{d}{dt}P_t(k) = \frac{\langle K^2\rangle_t}{\langle K\rangle_t^2 N_t}[(k+1)P_t(k+1) - kP_t(k)] - \frac{k - \langle K\rangle_t}{\langle K\rangle_t N_t}P_t(k). \quad \text{(A8)}$$

3. Propagating node deletion

The propagating node deletion scenario describes network contraction processes such as cascading failures, in which the damage propagates from a deleted node to its neighbors. In this scenario, at each time step we delete a random neighbor of the node deleted in the previous step. If the last deleted node does not have any yet-undeleted neighbor, we pick a random node, select a random neighbor of this node for deletion and continue the process from there. The probability that the node deleted at time t will be of degree k' is given by $k'N_t(k')/[N_t\langle K\rangle_t]$. One of these k' edges connects it to the node deleted in the previous time step and another edge connects it to the node to be deleted in the next time step. Apart from these two neighbors, there are $k'-2$ neighbors that lose one link each upon deletion of a node of degree k'. The probability of each one of these k' nodes to be of degree k is given by $kN_t(k)/[N_t\langle K\rangle_t]$. Summing up over all the possible degrees k' of the node deleted at time t, we obtain the secondary effect on nodes of degree k, which is given by

$$W_t(k \to k-1) = \frac{\langle K^2\rangle_t - 2\langle K\rangle_t}{\langle K\rangle_t^2 N_t} kN_t(k). \quad \text{(A9)}$$

Similarly, the secondary effect on nodes of degree $k+1$ is

$$W_t(k+1 \to k) = \frac{\langle K^2\rangle_t - 2\langle K\rangle_t}{\langle K\rangle_t^2 N_t}(k+1)N_t(k+1). \quad \text{(A10)}$$

The complete equation describing the time dependence of $N_t(k)$ is thus given by

$$\frac{d}{dt}N_t(k) = \frac{\langle K^2\rangle_t - 2\langle K\rangle_t}{\langle K\rangle_t^2 N_t}[(k+1)N_t(k+1) - kN_t(k)] - \frac{k}{N_t\langle K\rangle_t}N_t(k). \quad \text{(A11)}$$

Inserting this result into Eq. (14), we obtain the master equation

$$\frac{d}{dt}P_t(k) = \frac{\langle K^2\rangle_t - 2\langle K\rangle_t}{\langle K\rangle_t^2 N_t}[(k+1)P_t(k+1) - kP_t(k)] - \frac{k - \langle K\rangle_t}{\langle K\rangle_t N_t}P_t(k). \quad \text{(A12)}$$

APPENDIX B: EXACT SOLUTION OF THE MASTER EQUATION FOR RANDOM NODE DELETION

Below we solve Eq. (A4) for a general initial degree distribution, given by $P_0(k)$. To this end, we define the generating function

$$G(x,t) = \sum_{k=0}^{\infty} x^k P_t(k). \quad \text{(B1)}$$

The initial condition of the generating function is denoted by

$$G(x,0) = G_0(x) = \sum_{k=0}^{\infty} x^k P_0(k), \quad \text{(B2)}$$

while $G(1,t) = 1$ at all times due to the normalization of $P_t(k)$. Multiplying Eq. (A4) by x^k and taking a sum over all values of k, we obtain the following differential equation for $G(x,t)$:

$$\frac{\partial}{\partial t}G(x,t) = \left(\frac{1-x}{N-t}\right)\frac{\partial}{\partial x}G(x,t). \quad \text{(B3)}$$

In general, the solution of Eq. (B3) must take the form

$$G(x,t) = f[t + (N-t)x]. \quad \text{(B4)}$$

Inserting $t = 0$ in Eq. (B4), we find that $f(y) = G_0(y/N)$. Therefore,

$$G(x,t) = G_0\left[\frac{t}{N} + \left(1 - \frac{t}{N}\right)x\right]. \quad \text{(B5)}$$

Using the expression of $G_0(x)$ in terms of $P_0(k)$, we obtain

$$G(x,t) = \sum_{k=0}^{\infty}\left[\frac{t}{N} + \left(1 - \frac{t}{N}\right)x\right]^k P_0(k). \quad \text{(B6)}$$

Using the binomial expansion of $[t/N + (1 - t/N)x]^k$, we obtain

$$G(x,t) = \sum_{\ell=0}^{\infty} x^\ell \left(\frac{N-t}{t}\right)^\ell \sum_{k=\ell}^{\infty}\binom{k}{\ell}\left(\frac{t}{N}\right)^k P_0(k). \quad \text{(B7)}$$

Therefore,

$$P_t(k) = \left(1 - \frac{t}{N}\right)^k \sum_{k'=k}^{\infty}\binom{k'}{k}\left(\frac{t}{N}\right)^{k'-k} P_0(k'). \quad \text{(B8)}$$

No such solution exists for the master equations describing the preferential deletion and for the propagation deletion scenarios, which are presented above, in Appendix A. Therefore, one needs to rely on numerical integration of the master equations.

[1] S. Havlin and R. Cohen, *Complex Networks: Structure, Robustness and Function* (Cambridge University Press, New York, 2010).
[2] M. E. J. Newman, *Networks: An Introduction, Second Edition* (Oxford University Press, Oxford, 2018).
[3] E. Estrada, *The Structure of Complex Networks: Theory and Applications* (Oxford University Press, Oxford, 2011).
[4] P. Erdős and A. Rényi, On random graphs I, Publ. Math. Debrecen **6**, 290 (1959).
[5] P. Erdős and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[6] P. Erdős and A. Rényi, On the evolution of random graphs II, Bull. Int. Stat. Inst. **38**, 343 (1961).
[7] B. Bollobás, *Random Graphs*, 2nd ed. (Academic, London, 2001).
[8] M. Bauer and D. Bernard, Maximal entropy random networks with given degree distribution, arXiv:cond-mat/0206150.
[9] L. Bogacz, Z. Burda, and B. Waclaw, Homogeneous complex networks, Physica A **366**, 587 (2006).
[10] A. Annibale, A. C. C. Coolen, L. P. Fernandes, F. Fraternali, and J. Kleinjung, Tailored graph ensembles as proxies or null models for real networks I: Tools for quantifying structure, J. Phys. A **42**, 485001 (2009).
[11] E. S. Roberts, T. Schlitt, and A. C. C. Coolen, Tailored graph ensembles as proxies or null models for real networks II: Results on directed graphs, J. Phys. A **44**, 275002 (2011).
[12] R. Albert, H. Jeong, and A.-L. Barabási, Diameter of the World-Wide Web, Nature (London) **401**, 130 (1999).
[13] S. Redner, How popular is your paper? An empirical study of the citation distribution, Eur. Phys. J. B **4**, 131 (1998).
[14] A.-L. Barabási and R. Albert, Emergence of scaling in random networks, Science **286**, 509 (1999).
[15] P. L. Krapivsky, S. Redner, and F. Leyvraz, Connectivity of Growing Random Networks, Phys. Rev. Lett. **85**, 4629 (2000).
[16] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Structure of Growing Networks with Preferential Linking, Phys. Rev. Lett. **85**, 4633 (2000).
[17] B. Bollobás, O. Riordan, J. Spencer, and G. Tusnády, The degree sequence of a scale-free random graph process, Random Struct. Algorithms **18**, 279 (2001).
[18] R. Albert and A.-L. Barabási, Statistical mechanics of complex networks, Rev. Mod. Phys. **74**, 47 (2002).

[19] A.-L. Barabási, Scale-free networks: A decade and beyond, Science **325**, 412 (2009).
[20] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Resilience of the Internet to Random Breakdowns, Phys. Rev. Lett. **85**, 4626 (2000).
[21] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Breakdown of the Internet under Intentional Attack, Phys. Rev. Lett. **86**, 3682 (2001).
[22] P. Crucitti, V. Latora, and M. Marchiori, Model for cascading failures in complex networks, Phys. Rev. E **69**, 045104(R) (2004).
[23] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Catastrophic cascade of failures in interdependent networks, Nature (London) **464**, 1025 (2010).
[24] R. Pastor-Satorras, C. Castellano, P. Van Mieghem, and A. Vespignani, Epidemic processes in complex networks, Rev. Mod. Phys. **87**, 925 (2015).
[25] R. Albert, H. Jeong, and A.-L. Barabási, Error and attack tolerance of complex networks, Nature (London) **406**, 378 (2000).
[26] A. Braunstein, L. Dall'Asta, G. Semerjian, and L. Zdeborová, Network dismantling, Proc. Natl. Acad. Sci. USA **113**, 12368 (2016).
[27] L. Daqing, J. Yinan, K. Rui, and S. Havlin, Spatial correlation analysis of cascading failures: Congestions and blackouts, Sci. Rep. **4**, 5381 (2014).
[28] B. Schäfer, D. Witthaut, M. Timme, and V. Latora, Dynamically induced cascading failures in power grids, Nat. Commun. **9**, 1975 (2018).
[29] J. Török and J. Kertész, Cascading collapse of online social networks, Sci. Rep. **7**, 16743 (2017).
[30] L. Lőrincz, J. Koltai, A. F. Győr, and K. Takács, Collapse of an online social network: Burning social capital to create it? Social Networks **57**, 43 (2019).
[31] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (North Holland, Amsterdam, 2007).
[32] C. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Sciences*, 3rd ed. (Springer-Verlag, Berlin, Heidelberg, 2004).
[33] J. C. Butcher, *Numerical Methods for Ordinary Differential Equations* (John Wiley & Sons, Chichester, UK, 2003).
[34] N. C. Wormald, Differential equations for random processes and random graphs, Ann. Appl. Probab. **5**, 1217 (1995).
[35] N. C. Wormald, The differential equation method for random graph processes and greedy algorithms, in *Lectures on Approximation and Randomized Algorithms*, edited by M. Karonski and H. J. Proemel (Polish Scientific, Warsaw, 1999), p. 73.
[36] T. G. Kurtz, Solutions of ordinary differential equations as limits of pure jump Markov processes, J. Appl. Prob. **7**, 49 (1970).
[37] T. G Kurtz, The central limit theorem for Markov Chains, Ann. Probab. **9**, 557 (1981).
[38] L. Warnke, On Wormald's differential equation method, arXiv:1905.08928.
[39] Trickle down: Diffuse downward through some hierarchical structure, from *The American Heritage Dictionary of Phrasal Verbs* (Houghton Mifflin Harcourt, Boston, 2005).
[40] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.100.032314 for a supplemental movie.
[41] S. Kullback and R. A. Leibler, On information and sufficiency, Ann. Math. Stat. **22**, 79 (1951).
[42] M. E. J. Newman, Assortative Mixing in Networks, Phys. Rev. Lett. **89**, 208701 (2002).
[43] M. E. J. Newman, Mixing patterns in networks, Phys. Rev. E **67**, 026126 (2003).
[44] B. Fotouhia and M. G. Rabbat, Degree correlation in scale-free graphs, Eur. Phys. J. B **86**, 510 (2013).
[45] G. I. Barenblatt, *Scaling* (Cambridge University Press, Cambridge, UK, 2003).
[46] G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge University Press, Cambridge, UK, 1996).