Transparent nonlinear networks

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We consider the reflectionless transport of solitons in networks. The system is modeled in terms of the nonlinear Schrödinger equation on metric graphs, for which transparent boundary conditions at the branching points are imposed. This approach allows to derive simple constraints, which link the equivalent usual Kirchhoff-type vertex conditions to the transparent ones. Our method is applied to a metric star graph. An extension to more complicated graph topologies is straightforward.

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I. INTRODUCTION

Modeling of soliton dynamics in branched structures and networks is relevant to many important tasks in optics, fluid dynamics, condensed matter, biological physics, and polymers. The motivation for such tasks arises from the fact that the highly efficient transfer of information, charge, heat, spin, and optical signals in the form of solitons requires development of effective models providing tools for tunable wave transport in given low-dimensional materials. The topic of soliton transport in branched structures has attracted much attention [1–18].

An effective model that can be used to model soliton dynamics in networks is based on the solution of nonlinear wave equations on metric graphs. These metric graphs are a set of bonds (each assigned a length) connected to each other according to a rule, which is called the topology of a graph. Solving the wave equation in such a domain requires imposing boundary conditions both at the branching points (vertices) and at the ends of each branch. During the past decade, different nonlinear wave equations on networks have become one of the rapidly developing topics in both theoretical and mathematical physics. The early study of the nonlinear Schrödinger equation (NLSE) and soliton dynamics in networks dates back to Refs. [1,2], where the integrability of the NLSE under certain constraints was shown. Later such study was extended to the NLSE on planar graphs [7] and sine-Gordon [10], nonlinear Dirac [14], and nonlinear heat [16] equations. For details on the corresponding stationary problem we refer the reader to [4,9,11,13]. We note that the linear counterparts of these problems, which are called quantum graphs were studied earlier in different contexts (see, e.g., Refs. [19,20]). The extension to the case of nonlinear scattering was discussed in [21–23].

A very important feature of the wave transport in networks is the transmission of solitons through the network branching points, which is usually accompanied by the reflection (backscattering) of a wave at these points. If reflection dominates compared to transmission, then the "resistivity" of a network with respect to the soliton propagation becomes large and this makes such network less effective for the use of signal transfer. Therefore, it is quite important from the viewpoint of practical applications to reduce such resistivity by providing a minimum of reflection, or by its absence. This task leads to the problem of tunable soliton transport in networks, whose ideal result should be reflectionless transmission of the waves through the branching points of the structure. For practical applications in condensed matter, such transmission implies ballistic transport of charge, spin, heat and other carriers in low-dimensional branched materials. The latter is essential for the functionalization of low-dimensional materials having branched structure.

Reflectionless transport of solitons in optical fiber networks is another important problem in fiber optics, as many information-communication devices (e.g., computers, computer networks, telephones, etc.) use solitons for information (signal) transfer. Such networks are also used in different optoelectronic devices. High speed and lossless transfer of information in such devices require minimum backscattering or its absence. Important areas, where the reflectionless or ballistic transport of optical solitons in networks is required, are molecular electronics and conducting polymers [17].

Earlier, the possibility of reflectionless transmission of solitons in networks was considered in several studies. In particular, it was found in Refs. [1,2] that the transmission of solitons through the network branching point can be reflectionless, provided that certain constraints are fulfilled. It was also shown that these constraints provide the integrability of the NLSE on networks. Later, a similar effect was observed for other nonlinear partial differential equations (PDEs), such as the sine-Gordon equation [10] and the nonlinear Dirac equation [14]. In other words, the above studies revealed a conjecture (at least for a few PDEs), which states that if a nonlinear wave equation on a network is integrable, then the transmission of solitons through the branching points becomes reflectionless.

In this paper we give a proof of the above conjecture by showing that the constraints providing such reflectionless transmission and integrability of the NLSE on networks link equivalent usual Kirchhoff-type vertex boundary conditions to the so-called transparent boundary conditions (TBCs). These latter conditions were well studied previously in detail in Refs. [24–39]. The linear counterpart of the problem, i.e., the problem of transparent quantum graphs, was considered in the authors' recent paper [40]. Here we extend the method used in Ref. [40] to the case of the nonlinear Schrödinger equation.

The paper is organized as follows. In the next section we briefly recall the concept of TBCs for the NLSE on the real line. Section III provides an extension of the concept of TBCs to solitons in networks described by the NLSE on metric graphs and presents some numerical results. Finally, Sec. IV presents some concluding remarks.

II. TRANSPARENT BOUNDARY CONDITIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION ON A LINE

The problem of transparent boundary conditions (TBCs) for linear partial differential equations (PDEs) is a well developed topic in mathematical and theoretical physics (see, e.g., [24–39] for reviews). TBCs allow one to formulate PDE problems, originally posed on an unbounded domain, on a bounded domain, making them more accessible to a proper numerical treatment.

However, despite such progress in recent decades, for nonlinear PDEs the topic is not well established yet due to the missing integral transforms (Laplace, Fourier, z transforms) in the nonlinear case. One of the most effective approaches for the nonlinear case is considering the nonlinear term as a potential in a linear PDE; it is called the "potential approach." Below we briefly recall this approach by following Refs. [41,42].

We consider the wave (particle) motion in a one-dimensional (1D) domain $(-\infty, +\infty)$ described by the following time-dependent nonlinear Schrödinger equation (NLSE):

$$i\partial_t\psi + \partial_x^2\psi + \beta|\psi|^2\psi = 0, \quad x \in \mathbb{R}, t > 0, \tag{1}$$

with the initial condition

$$\psi(x,0) = \psi_0(x), \quad x \in \mathbb{R}. \tag{2}$$

In the pioneering papers [43,44], where explicit forms of TBC for the linear wave equations were derived, the main physical condition, to be fulfilled by traveling waves, was the absence of waves going in the left direction at the right boundary ($x = L$) and the absence of right-side going waves at the left boundary ($x = 0$). However, the derivation of TBCs for the nonlinear case is much more complicated than that for the linear one. One can use the so-called potential approach [30] and consider formally Eq. (1) as a linear Schrödinger equation with the potential $V = V(x,t) = \beta|\psi(x,t)|^2$. Then, one can rewrite Eq. (1) in a "linear form" as

$$i\partial_t\psi + \partial_x^2\psi + V\psi = 0, \quad x \in \mathbb{R}, \ t > 0, \tag{3}$$

Let us denote by ψ the solution of Eq. (3) and by v the new unknown defined by the relation ("gauge change")

$$v(x,t) = e^{-i\nu(x,t)}\psi(x,t), \tag{4}$$

where

$$\nu(x,t) = \int_0^t V(x,s)\,ds. \tag{5}$$

We get for the time and space derivatives of ψ in (3)

$$i\partial_t\psi = e^{i\nu}(i\partial_t - V)v, \tag{6}$$

$$\partial_x^2\psi = ie^{i\nu}\left[\partial_x^2 v + 2i\partial_x\nu\partial_x v + i\partial_x^2\nu v - (\partial_x\nu)^2 v\right], \tag{7}$$

and thus v satisfies a Schrödinger-type equation,

$$L_{\mathrm{SE}}(x,t,\partial_x,\partial_t)v := i\partial_t v + \partial_x^2 v + A\partial_x v + Bv = 0, \tag{8}$$

with $A = 2i\partial_x\nu$ and $B = i\partial_x^2\nu - (\partial_x\nu)^2$.

Now expanding the factorization (8), we get

$$\begin{aligned} L_{\mathrm{SE}} &= (\partial_x + i\Lambda^-)(\partial_x + i\Lambda^+) \\ &= \partial_x^2 + i(\Lambda^- + \Lambda^+)\partial_x + i\,Op(\partial_x\lambda^+) - \Lambda^-\Lambda^+, \end{aligned} \tag{9}$$

where $\Lambda^\pm = \Lambda^\pm(x,t,\partial_t)$ are classical pseudodifferential operators [45] and $\lambda^\pm_{1/2}$ are the principal symbols of the operators $\Lambda^\pm$ given by $\lambda^\pm_{1/2} = \mp\sqrt{-\tau}$ with some function τ. The total symbol $\lambda^\pm = \sigma(\Lambda^\pm)$ of $\Lambda^\pm$ admits an asymptotic expansion in inhomogeneous symbols as

$$\lambda^\pm = \sigma(\Lambda^\pm) \sim \sum_{j=0}^{+\infty} \lambda^\pm_{1/2-j/2}. \tag{10}$$

From (9) we deduce the system of operators

$$i(\Lambda^- + \Lambda^+) = A, \tag{11}$$

$$i\,Op(\partial_x\lambda^+) - \Lambda^-\Lambda^+ = i\partial_t + B, \tag{12}$$

which yields the following symbolic system of equations:

$$i(\lambda^+ + \lambda^-) = A, \tag{13}$$

$$i\partial_x\lambda^+ - \sum_{\alpha=0}^{+\infty}\frac{(-i)^\alpha}{\alpha!}\partial_\tau^\alpha\lambda^-\partial_t^\alpha\lambda^+ = -\tau + B. \tag{14}$$

If we identify the terms of order 1/2 in Eq. (13), we obtain $\lambda^-_{1/2} = -\lambda^+_{1/2}$. Then from Eq. (14) we get

$$\lambda^+_{1/2} = \pm\sqrt{-\tau}. \tag{15}$$

The Dirichlet-to-Neumann (DtN) operator corresponds to the choice $\lambda^+_{1/2} = -\sqrt{-\tau}$. From the factorization (8) we have the following TBC applied to the unknown wave function v at the artificial boundaries at $x = 0$ and $x = L$:

$$(-\partial_x + i\Lambda^+)v(0,t) = 0, \tag{16}$$

$$(\partial_x + i\Lambda^+)v(L,t) = 0. \tag{17}$$

Then using Eq. (4) the formal TBCs for ψ at $x = 0$ and $x = L$ can be written as [41]

$$-\partial_x\psi(0,t) + e^{-i\frac{\pi}{4}}e^{i\nu(0,t)}\partial_t^{1/2}(e^{-i\nu(0,t)}\psi(0,t)) = 0, \tag{18}$$

$$\partial_x\psi(L,t) + e^{-i\frac{\pi}{4}}e^{i\nu(L,t)}\partial_t^{1/2}(e^{-i\nu(L,t)}\psi(L,t)) = 0, \tag{19}$$

where the fractional $1/2$ derivative is given by

$$\partial_t^{1/2} f(t) = \frac{1}{\sqrt{\pi}} \partial_t \int_0^t \frac{f(s)}{\sqrt{t-s}}\, ds. \tag{20}$$

Let us note that in (20) the derivative can be directly performed but this would increase the singularity in the convolution kernel. Also, the formulation (20) is better suited for later numerical discretizations. A simple calculation shows that (18) and (19) are equivalent to the so-called impedance boundary condition, [26,30] that has the form of a Neumann-to-Dirichlet operator.

Formally, Eqs. (18) and (19) are similar to those for the linear case. We remark that a detailed treatment of Eqs. (1), (2), (18), and (19) can be found in Refs. [41,42], where the discretization scheme and the numerical method for this problem are also presented. We note that the boundary conditions (18) and (19) hold for both the focusing ($\beta > 0$) and the defocusing ($\beta < 0$) cases. In the next section we will modify these boundary conditions for the NLSE on metric graphs.

III. TRANSPARENT BOUNDARY CONDITIONS FOR NLSE ON METRIC GRAPHS

Soliton dynamics in networks is one of the rapidly evolving topics of the past decade. The early treatment of the problems dates back to Ref. [1], where soliton solutions of the NLSE on metric graphs were obtained and integrability of the problem under certain constraints was shown by proving the existence of an infinite number of conserving quantities.

An interesting feature found in [1] was the fact that, for integrable cases, the transmission of solitons through the graph vertices is reflectionless, i.e., there is no backscattering of solitons at the graph branching point. An explanation of such an effect was given in the recent papers [12,46], where it was stated that if the parameters of the generalized Kirchhoff boundary conditions on a star graph are related to the parameters of the nonlinear evolution equation and satisfy a single constraint, then the nonlinear evolution equation on the star graph can be reduced to the homogeneous equation on the infinite line.

Here we provide a proof of this conjecture, by showing that vertex boundary conditions in the form of weight continuity and generalized Kirchhoff rules become equivalent to transparent boundary conditions, if the parameters of the problem fulfill the integrability condition given in the form of the sum rule. To do this, we will apply the above method for imposing TBCs to the NLSE on metric graphs. Before doing this, let us briefly recall the treatment of the NLSE on metric graphs, cf. Ref. [1].

Before, this was done for quantum graphs described by the linear Schrödinger equation on metric graphs. We consider the star graph with three bonds B_j (see Fig. 1), for which a coordinate x_j is assigned. Choosing the origin of coordinates at the vertex, 0, for bond B_1 we put $x_1 \in (-\infty, 0]$ and for $B_{1,2}$ we fix $x_{2,3} \in [0, +\infty)$. In what follows, we use the shorthand notation $\Psi_j(x)$ for $\Psi_j(x_j)$ where x is the coordinate on the bond j to which the component Ψ_j refers. The nonlinear Schrödinger equation on each bond B_j of such a graph can be written as

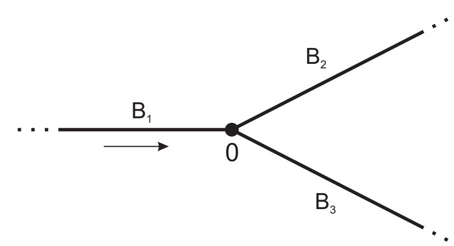

FIG. 1. Sketch of a star graph with three semi-infinite bonds.

$$i\,\partial_t \psi_j + \partial_x^2 \psi_j + \beta_j |\psi_j|^2 \psi_j = 0, \quad x \in B_j, t > 0. \tag{21}$$

Solving Eq. (21) requires imposing initial conditions and boundary conditions at the branching point. The latter can be derived from the fundamental physical laws, such as norm and energy conservation, which are given as

$$\frac{dN}{dt} = 0, \quad \frac{dE}{dt} = 0, \tag{22}$$

where

$$N(t) = \int_{-\infty}^0 |\psi_1|^2\, dx + \int_0^\infty |\psi_2|^2\, dx + \int_0^\infty |\psi_3|^2\, dx$$

and

$$E = E_1 + E_2 + E_3,$$

with

$$E_j = \int_{B_j} \left[\left| \frac{\partial \psi_j}{\partial x} \right|^2 - \frac{\beta_j}{2} |\psi_j|^4 \right] dx.$$

It was shown in Ref. [1] that the conservation laws (22) lead to the vertex conditions

$$\alpha_1 \psi_1(0) = \alpha_2 \psi_2(0) = \alpha_3 \psi_3(0) \tag{23}$$

and the generalized Kirchhoff rules

$$\frac{1}{\alpha_1} \frac{\partial \psi_1}{\partial x}\bigg|_{x=0} = \frac{1}{\alpha_2} \frac{\partial \psi_2}{\partial x}\bigg|_{x=0} + \frac{1}{\alpha_3} \frac{\partial \psi_3}{\partial x}\bigg|_{x=0}, \tag{24}$$

where α_j are nonzero real constants. The asymptotic conditions for Eq. (21) are imposed as

$$\lim_{|x| \to +\infty} \psi_j = 0. \tag{25}$$

The single soliton solutions of Eq. (21) fulfilling the vertex boundary conditions (23) and (24) and the asymptotic condition (25) can be written as [1]

$$\psi_j(x,t) = a \sqrt{\frac{2}{\beta_j}} \frac{\exp\left[i\frac{vx}{2} - i\left(\frac{v^2}{4} - a^2\right)t\right]}{\cosh[a(x - l - vt)]}, \tag{26}$$

where the parameters β_j fulfill the sum rule

$$\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3}. \tag{27}$$

Here v, l, and a are bond-independent parameters characterizing velocity, initial center of mass, and amplitude of a soliton, respectively.

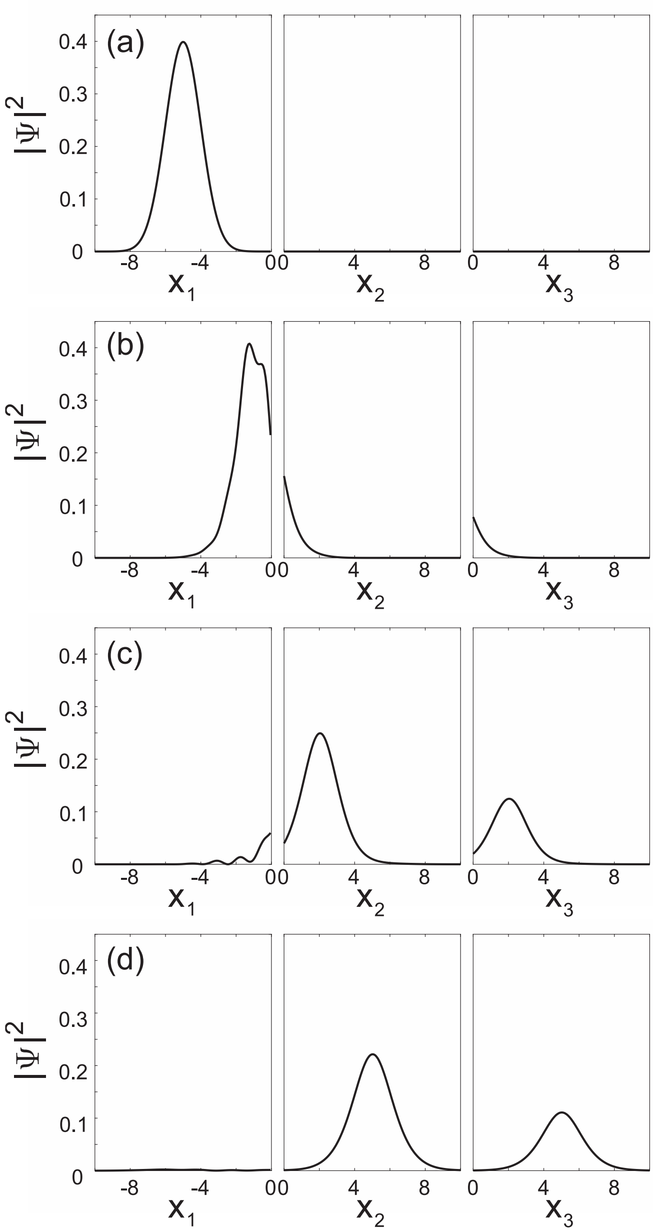

FIG. 2. The profile of the wave function plotted at different time moments $t = 0$ (a), $t = 1.6$ (b), $t = 2.8$ (c), and $t = 4$ (d) for the regime in which the sum rule is fulfilled (no reflection occurs): $\alpha_1 = \sqrt{\beta_1} = 1/\sqrt{1/2 + 1/4}$, $\alpha_2 = \sqrt{\beta_2} = \sqrt{2}$, and $\alpha_3 = \sqrt{\beta_3} = \sqrt{4}$. Each column number (from left to right) corresponds to a bond number.

Equation (27) represents the conditions for integrability of the problem given by Eqs. (21), (23), (24), and (25), i.e., the integrability of the NLSE on a metric star graph plotted in Fig. 1. It was shown in [1] that under the constraint (27) the problem has an infinite number of constants of motion. Below we show an additional consequence of Eq. (27), which can be formulated as follows: If the parameters β_j in Eq. (21) fulfill the condition (constraint) (27), then the vertex boundary conditions (23) and (24) become equivalent to the TBCs at the point 0.

Without loss of generality of the approach, we can assume that $\alpha_j = \sqrt{\beta_j}$. To impose TBCs for the NLSE on the metric graph shown in Fig. 1, we split the whole domain (graph) into two domains called "interior" ($-\infty < x < 0$) and "exterior" ($0 < x < \infty$) ones (see, e.g., Refs. [25–28,40] for details). Correspondingly, we have interior and exterior problems. The interior problem is given on B_1 by the equations

$$i\,\partial\psi_1 + \partial_x^2\psi_1 + \beta_1|\psi_1|^2\psi_1 = 0, \quad x < 0,\ t > 0,$$
$$\psi_1|_{t=0} = \Psi^I(x), \quad \partial_x\psi_1|_{x=0} = (T_+\psi_1)|_{x=0}.$$

The exterior problems for $B_{2,3}$ can be written as

$$i\,\partial\psi_{2,3} + \partial_x^2\psi_{2,3} + \beta_{2,3}|\psi_{2,3}|^2\psi_{2,3} = 0, \quad \psi_{2,3}|_{t=0} = 0,$$
$$\psi_{2,3}|_{x=0} = \Phi_{2,3}(t), \quad \Phi_{2,3}(0) = 0,$$
$$(T_+\Phi_{2,3})|_{x=0} = \partial_x\psi_{2,3}|_{x=0}.$$

We rewrite the NLSE of exterior problems for $B_{2,3}$ as

$$i\,\partial\psi_{2,3} + \partial_x^2\psi_{2,3} + V_{2,3}\psi_{2,3} = 0, \tag{28}$$

with the potentials $V_{2,3} = \beta_{2,3}|\psi_{2,3}|^2$. Furthermore, we introduce the new functions $v_{2,3}$ given as

$$v_{2,3}(x,t) = e^{-i\nu_{2,3}(x,t)}\psi_{2,3}(x,t), \tag{29}$$

where

$$\nu_{2,3}(x,t) = \int_0^t V_{2,3}(x,s)\,ds. \tag{30}$$

Then from the factorization in Eq. (9) we have the following TBCs for the wave functions $v_{2,3}$:

$$(-\partial_x + i\Lambda^+)v_{2,3}(0,t) = 0. \tag{31}$$

Using Eq. (29) we can write the formal TBCs for $\psi_{2,3}$ at $x = 0$ as

$$-\partial_x\psi_{2,3}(0,t) + e^{-i\frac{\pi}{4}}\,e^{i\nu_{2,3}(0,t)}\partial_t^{1/2}(e^{-i\nu_{2,3}(0,t)}\psi_{2,3}(0,t)) = 0. \tag{32}$$

Using the vertex boundary condition (23) we have

$$\begin{aligned}\partial_x\psi_{2,3}|_{x=0} =& \frac{1}{\sqrt{\pi}}e^{-i\frac{\pi}{4}+i\beta_{2,3}\int_0^t|\psi_{2,3}(0,s)|^2ds}\\ &\times\partial_t\int_0^t\frac{\psi_{2,3}(0,\tau)e^{-i\beta_{2,3}\int_0^\tau|\psi_{2,3}(0,s)|^2ds}}{\sqrt{t-\tau}}d\tau\\ =&\frac{1}{\sqrt{\pi}}\sqrt{\frac{\beta_1}{\beta_{2,3}}}e^{-i\frac{\pi}{4}+i\beta_1\int_0^t|\psi_1(0,s)|^2ds}\\ &\times\partial_t\int_0^t\frac{\psi_1(0,\tau)e^{-i\beta_1\int_0^\tau|\psi_1(0,s)|^2ds}}{\sqrt{t-\tau}}d\tau.\end{aligned} \tag{33}$$

From the vertex boundary condition (24) and (33) we get

$$\begin{aligned}\partial_x\psi_1|_{x=0} =& \frac{\sqrt{\beta_1}}{\sqrt{\beta_2}}\partial_x\psi_2|_{x=0} + \frac{\sqrt{\beta_1}}{\sqrt{\beta_3}}\partial_x\psi_3|_{x=0}\\ =&\frac{1}{\sqrt{\pi}}\beta_1\left(\frac{1}{\beta_2}+\frac{1}{\beta_3}\right)e^{-i\frac{\pi}{4}+i\beta_1\int_0^t|\psi_1(0,s)|^2ds}\\ &\times\partial_t\int_0^t\frac{\psi_1(0,\tau)e^{-i\beta_1\int_0^t|\psi_1(0,s)|^2ds}}{\sqrt{t-\tau}}d\tau.\end{aligned} \tag{34}$$

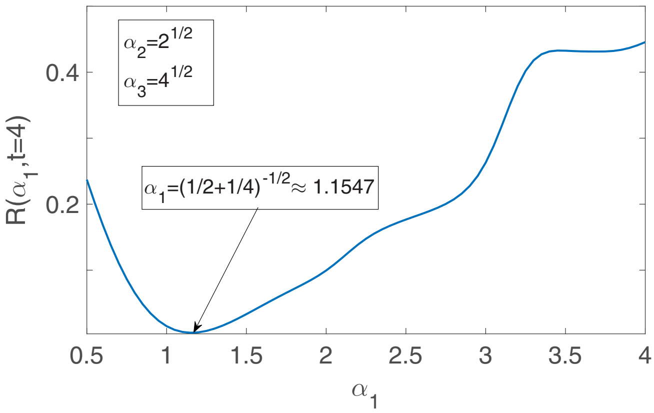

FIG. 3. Dependence of the vertex reflection coefficient R on the parameter α_1 when time elapses ($t = 4$).

It is clear that if the sum rule given by Eq. (27) is fulfilled, i.e.,

$$\beta_1\left(\frac{1}{\beta_2}+\frac{1}{\beta_3}\right)=1,$$

then the boundary condition given by Eq. (34) coincides with that in Eq. (19). Thus fulfilling the sum rule (27) implies that the vertex boundary conditions (23) and (24) become equivalent to the TBCs at the graph vertex. This can be shown by direct numerical solution of Eq. (21) for the boundary conditions (23) and (24). In Fig. 2 the profile of the soliton $|\psi_j|^2$ obtained numerically is plotted at different time moments for the regime in which the sum rule (27) is fulfilled. Numerical simulations are performed for the right traveling Gaussian wave packet given by

$$\Psi^I(x) = (2\pi)^{-1/4}\exp[2.5ix - (x+5)^2/4]$$

at four consecutive time steps.

To show that, for the case in which the sum rule is violated the transmission of soliton is accompanied by reflections, in Fig. 3 we plotted the reflection coefficient, which is determined as the ratio of the partial norm for the first bond to the total norm,

$$R = \frac{N_1}{N_1+N_2+N_3},$$

as a function of α_1 for the fixed values of α_2 and α_3. It is clear from this plot that the reflection coefficient becomes zero at the value of α_1, which fulfills the sum rule (27). This also can be considered as additional confirmation of the vertex boundary conditions in Eqs. (23) and (24) becoming equivalent to the transparent ones. It is clear that such a conjecture can be derived for a star graph with an arbitrary number of bonds. Finally, we note that the above constraint for TBCs given by Eq. (27) is applicable not only for solitons, but for arbitrary solutions of the NLSE on graphs.

IV. CONCLUSIONS

In this paper we have studied the problem of reflectionless soliton transport in network branching points by modeling the soliton dynamics in networks in terms of the nonlinear Schrödinger equation on metric graphs. By combining the concept of transparent boundary conditions with the Kirchhoff-type boundary conditions at the vertex, we derived constraints which make the latter conditions equivalent to the transparent ones. This gives a clear explanation of the previously observed [1] conjecture on the absence of soliton backscattering when the NLSE on metric graphs is integrable and the integrability is provided in terms of the above constraint.

Also, solving the problem numerically, we have shown for the star graph a reflectionless transmission of solitons through the vertex in the case in which the parameters fulfill the sum rule. We note that this approach can be directly extended to arbitrary graph topologies which contain any subgraph connected to two or more outgoing, semi-infinite bonds. Moreover, we believe that our approach can be extended to other PDEs, where a similar regime of reflectionless vertex transmission of sine-Gordon [10] and Dirac [14] solitons has been observed. We remark that our approach can be directly extended to other graphs topologies, such as tree, loop, triangle, etc., provided the graph consists of finite subgraphs and two or more semi-infinite outgoing bonds.

The above model for reflectionless soliton transport through the network branching points may have direct and important applications for different, practically important problems of optics, condensed matter, and polymers. Among such applications one can consider optical fiber networks widely used in computing and communication technologies, where the signal transfer is done in the form of soliton transport.

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