Fluctuations in the random-link matching problem

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Using the replica approach and the cavity method, we study the fluctuations of the optimal cost in the random-link matching problem. By means of replica arguments, we derive the exact expression of its variance. Moreover, we study the large deviation function, deriving its expression in two different ways, namely using both the replica method and the cavity method.

I. INTRODUCTION

The application of replica theory [1] to the study of random combinatorial optimization problems (RCOPs) has a long tradition that started more than 30 years ago with the seminal works by Orland [2] and Mézard and Parisi [3]. It became immediately clear that methods borrowed from the theory of disordered systems are very effective to study the average properties of ensembles of RCOPs. Exact results have been obtained, e.g., for mean-field versions of many RCOPs, such as the matching problem [3], the traveling salesman problem [4], K-SAT problems [5], graph partitioning [6], matching enumeration in sparse graphs [7], constraint least-squares problems [8], and many others. In the case of the matching problem, expressions for the finite-size corrections to the average optimal cost have also been obtained [9]. The parallel success of the cavity method [1,10] inspired message-passing algorithms for the solution of specific instances of RCOPs [11].

In this paper, we study the deviations from the average optimal cost (AOC) in a particular RCOP, the random-link matching problem using both the replica and the cavity method. Such an investigation is of methodological interest beyond the analysis of the specific problem. Replica calculations for the study of large deviations of thermodynamic-like functionals in presence of disorder are quite rare in the literature. Indeed, except for the analysis in Ref. [8], where the replica method is applied to the study of large deviations of the minimum, the results cited above mainly concern the typical properties of the solutions only. On the other hand, the application of the cavity method to the study of large fluctuations has not been explored until now, with the exception of Ref. [12]. Here we apply both approaches to the same problem, showing that they lead to the same result.

We will study both the small fluctuations around the AOC and the large deviations from it. In the matching problem we assume that $2N$ vertices, labeled by the index $i \in \{1, \ldots, 2N\}$

are given, along with a positive weight $w_{ij} \in \mathbb{R}^+$ for each pair (i, j). We search for the symmetric matrix $\mathbf{M} = (m_{ij})_{i,j}$ such that the cost

$$C_N[\mathbf{M}] := \sum_{i<j} m_{ij} w_{ij} \qquad (1)$$

is minimized. The minimization has to be performed on the set of matrices $\mathbf{M}$ such that

$$m_{ji} = m_{ij} \in \{0, 1\} \ \forall i, j, \quad \sum_{i=1}^{2N} m_{ij} = 1 \quad \forall j. \qquad (2)$$

In the random-link version of the problem, the quantities w_{ij} are supposed to be independent and identically distributed random variables, with probability density function $\rho(w)$. Using the replica theory, in Ref. [3] it has been proven that, if $\lim_{w \to 0} \rho(w) = 1$, then

$$\mathcal{C} := \lim_{N \to +\infty} \mathbb{E}\left[\min_{\mathbf{M}} C_N[\mathbf{M}]\right] = \frac{\zeta(2)}{2}, \qquad (3)$$

where we have denoted the average over all possible realizations by $\mathbb{E}[\bullet]$, and $\zeta(z)$ is the Riemann ζ function. The calculation was performed introducing a partition function

$$Z_w(\beta) := \sum_{\{m_{ij}\}} \prod_{i=1}^{2N} \mathbb{I}\left(\sum_{j=1}^{2N} m_{ij} = 1\right) e^{-\beta N C_N[\mathbf{M}]}, \qquad (4)$$

where the indicator function $\mathbb{I}(\bullet)$ is equal to 1 if its argument is true and zero otherwise. From the expression above, the replicated free energy can be derived,

$$\Phi(n, \beta) := -\lim_{N \to +\infty} \frac{\ln \mathbb{E}[Z_w^n(\beta)]}{\beta n N}. \qquad (5)$$

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The functional $\Phi(n, \beta)$ has been obtained, in the replica symmetric hypothesis, in Ref. [3] and it is equal to

$$-\beta n \Phi(n, \beta) = -\frac{\beta}{2} \sum_{p=1}^{n} \frac{\Gamma(n+1)}{\Gamma(p)\Gamma(n-p+1)} q_p^2$$

$$+ 2 \ln \left[\iint \frac{dx \, d\eta}{2\pi} (-i\eta)^n \right.$$

$$\left. \times \exp \left(i\eta x + \sum_{p=1}^{\infty} \frac{x^p q_p}{p!} \right) \right], \quad (6a)$$

where the order parameters q_p have to be specified using the saddle-point condition

$$\frac{\partial \Phi(n, \beta)}{\partial q_p} = 0 \quad p \in \mathbb{N}. \quad (6b)$$

The asymptotic AOC is then recovered as the value of Φ in the zero-temperature limit, taking the number of replicas n going to zero,

$$\mathcal{C} = \lim_{\beta \to +\infty} \lim_{n \to 0} \Phi(n, \beta). \quad (7)$$

Observe that we should, in principle, take $n \to 0$ first and then $\beta \to +\infty$, but, as usual in replica calculations, we will assume that the order of two limits can be safely inverted. Here and in the following, we denote by $C_N := \min_{\mathbf{M}} C_N[\mathbf{M}]$ the instance-dependent optimal cost and by $\varrho_N(C)$ its distribution. We expect $\rho_N(C)$ to concentrate, for $N \to +\infty$, on the asymptotic AOC $\mathcal{C}$, $\lim_N \varrho_N(C) = \delta(C - \mathcal{C})$. Moreover, we expect C_N to satisfy a large deviation principle, i.e.,

$$-\frac{\ln \rho_N(C)}{N} = L(C) + O\left(\frac{1}{N}\right), \quad (8)$$

where $L(C)$ is a strictly convex, positive function having $L(\mathcal{C}) = 0$, known as the Cramér rate function. In some sense, $L(C)$ measures how rare it is to find an optimal cost $C \neq \mathcal{C}$ for large values of N.

Our computation of the large deviation function $L(C)$ for the random-link matching problem starts exactly from Eq. (6a). It is well known that the replicated average free-energy $n\Phi(n, \beta)$ contains information not only on the average free energy but also on its fluctuations [13]. In particular, using Eq. (5) it is possible to show that, for finite n, $-n\beta\Phi(n, \beta)$ is the cumulant generating function for the free energy $\ln Z_w(\beta)$ [14]. This fact has been used, for example, by Parisi and Rizzo [15] to extract the large deviation function in the Sherrington-Kirkpatrick model. They confirmed the anomalous scaling of fluctuations of the free energy in the RSB phase predicted, near the critical temperature, by Crisanti et al. [14], on the basis of a previous result by Kondor [16]. In the present paper, we are interested in the fluctuation of the "ground-state free energy" C_N in the random-link matching problem, and therefore we have to take $\beta \to +\infty$. The cumulant generating function of the optimal cost is then obtained as

$$\alpha \Phi(\alpha) := \lim_{\substack{\beta \to +\infty \\ n\beta = \alpha}} n\beta \Phi(n, \beta) = -\lim_{N \to +\infty} \frac{\ln \mathbb{E}[e^{-\alpha N C_N}]}{N}$$

$$= \lim_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\kappa_k \alpha^k}{k!}, \quad (9)$$

where κ_k is the kth cumulant of the random variable $N C_N = \lim_{\beta \to +\infty} \beta^{-1} \ln Z_w(\beta)$ [14], the first order in α simply being the asymptotic AOC, $\alpha \Phi(\alpha) = \alpha \mathcal{C} + o(\alpha)$. If $\lim_N N^{-1} \kappa_2$ is finite and different from zero, i.e., $\kappa_2 = 2\sigma^2 N$ with $\sigma^2 = O(1)$, then this implies that $\mathbb{E}[(C_N - \mathcal{C})^2] = \sigma^2 N^{-1}$, i.e., small fluctuations of the optimal cost are Gaussian.

If $\alpha \Phi(\alpha)$ is differentiable and convex, by the Gärtner-Ellis theorem the Cramér function of $\varrho_N(C)$ is obtained as the Legendre-Fenchel transform of $\alpha \Phi(\alpha)$,

$$L(C) := -\lim_{N \to +\infty} \frac{\ln \varrho_N(C)}{N} = \inf_{\alpha}[\alpha \Phi(\alpha) - \alpha C]$$

$$= \alpha_C \Phi(\alpha_C) - \alpha_C C, \quad (10a)$$

with α_C such that

$$C = \frac{\partial[\alpha \Phi(\alpha)]}{\partial \alpha} \bigg|_{\alpha = \alpha_C}. \quad (10b)$$

Conversely, the cumulant generating function can be obtained from the rate function using, once again, a Legendre-Fenchel transform,

$$\alpha \Phi(\alpha) = \inf_{C} [L(C) + \alpha C] = L(C_\alpha) + \alpha C_\alpha, \quad (11a)$$

with C_α such that

$$\alpha = \frac{\partial L(C)}{\partial C} \bigg|_{C = C_\alpha}. \quad (11b)$$

Intuitively, therefore, the quantity α^{-1} plays the role of a "fluctuation scale" with respect to the deviation $C - \mathcal{C}$. Indeed, assuming $L(C)$ strictly convex and non-negative, such that $L(\mathcal{C}) = 0$, then $\alpha = 0$ corresponds to $C_\alpha = \mathcal{C}$. On the other hand, being $\partial_C L(C) < 0$ for $C < \mathcal{C}$, it is clear that positive values of α corresponds to negative fluctuations of the cost, and negative values of α correspond to positive deviations from the asymptotic optimal cost. In the following, we will derive the exact value of σ^2, proving that the small fluctuation of the optimal cost are indeed Gaussian, and we will obtain an expression for the function $\Phi(\alpha)$ that we will solve numerically.

II. SMALL FLUCTUATIONS

Let us start from the computation of the variance of the optimal cost. The small α expansion of $\alpha \Phi(\alpha)$ up to $o(\alpha^2)$ terms will provide us the first and the second cumulant of the optimal cost, i.e., its mean and its variance. Due to the fact that we are performing an expansion around $\alpha = 0$, it is useful to recall the expression of the saddle-point value of the order parameter $q_p \equiv Q_p$ in this particular case, i.e., for $n \to 0$ and $\beta \to +\infty$. It has been shown in Ref. [3] that introducing the

function

$$G_0(t) := \sum_{p=1}^{\infty} \frac{(-1)^{p-1} e^{\beta pt}}{p!} Q_p, \qquad (12)$$

the saddle-point condition for $n \to 0$ and $\beta \to +\infty$ reads

$$G_0(x) = 2 \int_{-x}^{\infty} e^{-G_0(t)} dt \Rightarrow G_0(x) = \ln(1 + e^{2x}). \qquad (13)$$

To obtain the expansion of $\Phi(\alpha)$, let us start from the double integral appearing in the argument of the logarithm in Eq. (6a), to be evaluated on the saddle-point Q_p for small α. We have

$$\iint \frac{dx\, d\eta}{2\pi} (-i\eta)^n e^{i\eta x + \sum_{p=1}^{\infty} \frac{x^p Q_p}{p!}}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{\beta^k k!} \iint \frac{dx\, d\eta}{2\pi} \ln^k (-i\eta) e^{i\eta x + \sum_{p=1}^{\infty} \frac{x^p Q_p}{p!}}. \qquad (14)$$

Using now the representation of the logarithm

$$\ln(x) = \int_0^{+\infty} \frac{e^{-t} - e^{-xt}}{t} dt$$

we can write

$$\lim_{\substack{\beta \to +\infty \\ \beta n = \alpha}} \iint \frac{dx\, d\eta}{2\pi} (-i\eta)^n e^{i\eta x + \sum_{p=1}^{\infty} \frac{x^p Q_p}{p!}}$$

$$= 1 + \alpha \int [\theta(-h) - e^{-G_0(h)}] dh$$

$$+ \frac{\alpha^2}{2} \iint dh_1 dh_2 [\theta(-h_1)\theta(-h_2) - \theta(-h_1)e^{-G_0(h_2)}$$

$$- \theta(-h_2)e^{-G_0(h_1)} + e^{-G_0(\max\{h_1,h_2\})}] + o(\alpha^2)$$

$$= 1 + \frac{\zeta(2)}{4} \alpha^2 + o(\alpha^2). \qquad (15)$$

Similarly, the first term in Eq (6a) can be written as

$$\frac{\beta}{2} \sum_{p=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(p)\Gamma(n-p+1)} Q_p^2$$

$$= \frac{\alpha}{2} \sum_{p=1}^{\infty} (-1)^{p-1} \left[1 - \frac{\alpha}{\beta} H_{p-1} + o\left(\frac{\alpha}{\beta}\right) \right] Q_p^2$$

$$= \frac{\alpha}{2} \zeta(2) - \frac{\alpha^2}{2\beta} \sum_{p=1}^{\infty} (-1)^{p-1} H_{p-1} Q_p^2 + o(\alpha^2). \qquad (16)$$

where H_p is the pth harmonic number. If we now use the fact that $H_{p-1} = \int_0^{\infty} \frac{e^{-t} - e^{-pt}}{1 - e^{-t}} dt$, then

$$\lim_{\substack{\beta \to +\infty \\ n\beta = \alpha}} \frac{\beta}{2} \sum_{p=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(p)\Gamma(n-p+1)} Q_p^2$$

$$= \alpha \frac{\zeta(2)}{2} + \alpha^2 \int_{-\infty}^{+\infty} dh e^{-G_0(h)} \int_0^{+\infty} G_0(h-t) dt + o(\alpha^2)$$

$$= \alpha \frac{\zeta(2)}{2} + \alpha^2 \frac{\zeta(3)}{2} + o(\alpha^2). \qquad (17)$$

Collecting all results we get

$$\Phi(\alpha) = \frac{\zeta(2)}{2} - \frac{\zeta(2) - \zeta(3)}{2} \alpha + o(\alpha), \qquad (18)$$

implying that small fluctuations of the optimal cost in the random-link matching problem are Gaussian, with a variance given by

$$\mathbb{E}[(C_N - C)^2] = \frac{\zeta(2) - \zeta(3)}{N} + o\left(\frac{1}{N}\right). \qquad (19)$$

Recently Wästlund [17] has derived the variance of the random-link assignment problem with exponentially distributed random weights on a complete bipartite graph. His result, obtained using a purely probabilistic approach, coincides with Eq. (19), apart from a global factor 4, due to the fact that the optimal cost of the assignment problem is twice the optimal cost of the matching problem for $N \to +\infty$. In our calculation, only the assumption $\lim_{w \to 0} \rho(w) = 1$ for the weight probability density function has been used.

III. LARGE DEVIATIONS VIA REPLICAS

In the previous section we have performed a small α expansion to extract the variance of the optimal cost C_N for $N \gg 1$. To get instead the large deviation function $L(C)$, we have to keep α finite. We follow two different approaches and we compare then our results with numerical simulations. Let us start from the replica method. Following our general recipe, we derive the large deviation function starting from Eq. (6a), writing down the saddle-point equation in the $\beta \to +\infty$ limit, taking $n\beta = \alpha$ fixed. We start considering $\alpha < 0$ (we will relax this assumption in our cavity calculation). Using the fact that, for $n \in (-1, 0)$,

$$\int (-i\eta)^n e^{i\eta x} d\eta = -\frac{2\Gamma(n+1)\sin(n\pi)}{(-x)^{n+1}} \theta(-x), \qquad (20)$$

and defining as in Eq. (12)

$$G_{n,\beta}(t) := \sum_{p=1}^{\infty} \frac{(-1)^{p-1} Q_p e^{\beta pt}}{p!} \qquad (21)$$

the argument of the logarithm can be rewritten as

$$\iint \frac{dx\, d\eta}{2\pi} (-i\eta)^n \exp \left(i\eta x + \sum_{p=1}^{\infty} \frac{x^p Q_p}{p!} \right)$$

$$= -\frac{\beta \Gamma(n+1)\sin(n\pi)}{\pi} \int_{-\infty}^0 \exp \left(\sum_{p=1}^{\infty} \frac{x^p Q_p}{p!} \right) \frac{dx}{(-x)^{n+1}}$$

$$= -\frac{\beta \Gamma(n+1)\sin(n\pi)}{\pi} \int_{-\infty}^{+\infty} e^{-n\beta t - G_{n,\beta}(t)} dt. \qquad (22)$$

The saddle-point equation (6b) becomes

$$\beta p \binom{n}{p} Q_p = 2 \frac{(-1)^p}{p!} \frac{\int_{-\infty}^{+\infty} e^{(p-n)\beta t - G_{n,\beta}(t)} dt}{\int_{-\infty}^{+\infty} e^{-n\beta t' - G_{n,\beta}(t')} dt'} \Longrightarrow$$

$$G_{n,\beta}(x) = -2 \frac{\int_{-\infty}^{+\infty} K_{n,\beta}(x+t) e^{-n\beta t - G_{n,\beta}(t)} dt}{\int_{-\infty}^{+\infty} e^{-n\beta t' - G_{n,\beta}(t')} dt'}, \qquad (23)$$

where the function

$$K_{n,\beta}(u) := \sum_{p=1}^{\infty} \frac{e^{\beta u p}}{\binom{n}{p} \beta p (p!)^2} \tag{24}$$

appears. Observing now that (see Appendix)

$$\lim_{\substack{\beta \to +\infty \\ n\beta = \alpha}} K_{n,\beta}(u) = \theta(u) \left(\frac{1}{\alpha} - u \right), \tag{25}$$

the saddle-point equation can be written as

$$\begin{aligned} G_\alpha(x) &:= \lim_{\substack{\beta \to +\infty \\ n\beta = \alpha}} G_{n,\beta}(x) \\ &= -\frac{2\alpha}{\mathcal{Z}_\alpha} \int_{-x}^{+\infty} \left(x + t - \frac{1}{\alpha} \right) e^{-\alpha t - G_\alpha(t)} dt, \end{aligned} \tag{26a}$$

where we have introduced

$$\mathcal{Z}_\alpha := -\alpha \int_{-\infty}^{+\infty} e^{-\alpha t' - G_\alpha(t')} dt'. \tag{26b}$$

The equation above implies that $\lim_{x \to +\infty} x^{-1} G_\alpha(x) = 2$ for any value of α, i.e., $G_\alpha(x) \sim 2x$ for large x. Assuming $\alpha > -2$, this also implies that $\lim_{x \to -\infty} G_\alpha(x) = 0$. By consequence, the integral appearing in the expression of $\mathcal{Z}_\alpha$ converges for $\alpha > -2$ only and diverges otherwise. Using the saddle-point equation (23) in Eq. (6a) we finally get

$$\alpha \Phi(\alpha) = \frac{\alpha}{\mathcal{Z}_\alpha} \int_{-\infty}^{+\infty} G_\alpha(t) e^{-\alpha t - G_\alpha(t)} dt - 2 \ln \mathcal{Z}_\alpha. \tag{26c}$$

Equation (26a) can be solved numerically for a given value of α, allowing then to evaluate $\alpha \Phi(\alpha)$ in Eq. (26c), whose Legendre transform is the desired large deviation function $L(C)$. Using the properties of G_α derived above, it can be seen that $\lim_{\alpha \to -2^+} \alpha \Phi(\alpha) = -\infty$: The presence of such a singularity gives us information on the large C behavior of the Cramér function, i.e., it implies that $\lim_{C \to +\infty} C^{-1} L(C) = 2$. As anticipated, Eqs. (26) have been derived assuming $-2 < \alpha < 0$. To get an expression that can be prolonged to positive values of α we will use the cavity method.

IV. LARGE DEVIATIONS VIA CAVITY

The equation for $\Phi(\alpha)$ given by the replica method for $\alpha \in (-2, 0]$ can be also obtained using the cavity method and actually extended to positive values of α. The starting point is the cavity condition for the occupancy of an edge in the random-link matching problem. In particular, in the cavity approach, each edge (i, j) is associated to its weight w_{ij} and to two cavity fields, ϕ_i and ϕ_j on its vertices, containing information on the rest of the graph, in such a way that the occupancy m_{ij} of the the edge is distributed as

$$P(m_{ij}) = \frac{\exp[-\beta m_{ij}(N w_{ij} - \phi_i - \phi_j)]}{1 + \exp[-\beta (N w_{ij} - \phi_i - \phi_j)]}. \tag{27}$$

In the $\beta \to +\infty$ limit, an edge is occupied if, and only if, $N w_{ij} < \phi_i + \phi_j$. At zero temperature and in the large-N limit,

the cavity fields satisfy the following equation [4,18,19]:

$$\phi_0 = \min_{k \in \partial 0} (N w_{k0} - \phi_k). \tag{28}$$

Here $\partial 0$ is the set of neighbors of the node 0. The mate node i^* of 0 is such that

$$i^* = \arg \min_{k \in \partial 0} (N w_{k0} - \phi_k). \tag{29}$$

In Refs. [4,18,19] the previous equations have been studied and solved. The AOC predicted by the cavity method coincides with the one obtained using the replica approach.

The recurrence relation for the cavity fields can be also used to extract information on the fluctuations and evaluate $\alpha \Phi(\alpha)$. For the sake of simplicity, let us start from a different version of the problem, i.e., the random-link matching problem on a sparse graph, and in particular on a Bethe lattice, and let us follow the approach of Rivoire [12] for the study of large deviation on sparse topologies. In this case, we are interested in solving our problem on a graph having $2N$ vertices, each one of them having coordination z: We will later take the limit $z \to 2N - 1$. Taking this limit might sound dangerous, because we apply a result obtained for a sparse topology to a dense one. However, the random-link matching problem is an "effectively sparse" problem: Given a fully connected topology, the probability that a given node is connected, in the optimal matching, to its nth-nearest neighbor is exponentially small in n [19]. We will denote by $L_z(C)$ the large deviation function for random-link matching problem on the Bethe lattice, so that $L(C) = \lim_{z \to +\infty} L_z(C)$.

To obtain an expression for it, we proceed as usual in the cavity approach, i.e., starting from an intermediate graph having z randomly chosen (cavity) nodes with coordination $z - 1$, and all the other nodes with coordination z.
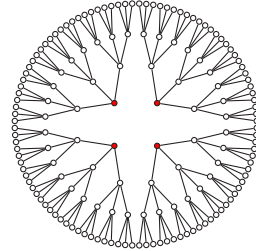
Let us denote by $\hat{L}_z$ the large deviation function corresponding to the random-link matching problem on such a topology.

We can recover the "correct" Bethe lattice topology in two ways. We can, for example, connect a new node to the z cavity nodes.
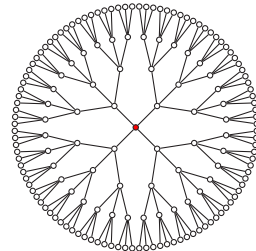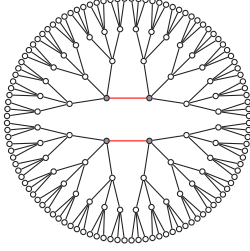

The optimal cost will be shifted by a certain amount $N^{-1} \varepsilon$ in such a way that the probability density function of the optimal

cost satisfies the equation

$$e^{-(N+1/2)L_z(C)} = \int e^{-N\hat{L}_z\left(\frac{N+1/2}{N}C-\varepsilon\right)} p_v(\varepsilon)d\varepsilon$$

$$\asymp e^{-N\hat{L}_z(C)+\frac{\alpha}{2}C} \int e^{-\alpha N\varepsilon} p_v(\varepsilon)d\varepsilon, \quad (30)$$

where $\alpha := -\partial_C \hat{L}_z(C)$ and $p_v(\varepsilon)$ is the distribution of the energy shift ε due to a vertex addition.

Another possibility is to add $z/2$ edges.

We obtain in this case the following relation:

$$e^{-NL_z(C)} = \left[\prod_{k=1}^{z/2} \int p_e(\epsilon_k)d\epsilon_k\right] e^{-N\hat{L}_z(C-\sum_k \epsilon_k)}$$

$$\asymp e^{-N\hat{L}_z(C)} \left[\int p_e(\epsilon)e^{-\alpha N\epsilon}d\epsilon\right]^{\frac{z}{2}}, \quad (31)$$

where $p_e(\epsilon)$ is the distribution of the energy shift ϵ due to an edge addition. Taking the ratio of the two expressions above, we obtain an equation for $L_z(C)$ at the leading order in N, namely

$$\alpha\Phi(\alpha) \equiv L_z(C) + \alpha C = \lim_N \left\{ z \ln\left[\int p_e(\epsilon)e^{-\alpha N\epsilon}d\epsilon\right]\right.$$

$$\left. - 2 \ln\left[\int e^{-\alpha N\varepsilon} p_v(\varepsilon)d\varepsilon\right]\right\}. \quad (32)$$

Taking $z = 2N - 1 \approx 2N$ we obtain the expression for our case. To evaluate the previous quantity, let us introduce the joint distribution $p_v(\phi, \varepsilon)$ of the cavity field entering in the added node and of the energy shift, such that $p_v(\varepsilon) = \int p_v(\phi, \varepsilon)d\phi$. We define the reweighted distribution of the cavity field as

$$p_\alpha(\phi) := \frac{1}{\mathcal{Z}_\alpha} \int e^{-\alpha N\varepsilon} p_v(\phi, \varepsilon)d\varepsilon, \quad (33)$$

with $\mathcal{Z}_\alpha$ proper normalization constant. In our case

$$p_v(\phi, \varepsilon) \equiv p_v(\varepsilon)\delta\left(\frac{\phi}{N} - \varepsilon\right),$$

because the cost shift due to the addition of a node coincides with the incoming cavity field, see Eq. (28). If we denote by

$$\pi_\alpha(u) := \int_0^{+\infty} p_\alpha(\hat{w} - u)d\hat{w},$$

then the distribution of the cavity field can be rewritten as

$$p_\alpha(\phi) = \frac{1}{\mathcal{Z}_\alpha} \int p_v(\phi, \varepsilon)e^{-\alpha N\varepsilon}d\varepsilon = \frac{p_v(\phi/N)e^{-\alpha\phi}}{\mathcal{Z}_\alpha}$$

$$= \frac{2e^{-\alpha\phi}\pi_\alpha(\phi)}{\mathcal{Z}_\alpha}\left[1 - \frac{1}{N}\int_{-\infty}^\phi \pi_\alpha(\chi)d\chi\right]^{2N-1}, \quad (34)$$

where we have used Eq. (28) to express p_v in terms of π_α. In the $N \to +\infty$ limit we obtain an equation for p_α,

$$p_\alpha(\phi) = \frac{2e^{-\alpha\phi}\pi_\alpha(\phi)}{\mathcal{Z}_\alpha} \exp\left[-2\int_{-\infty}^\phi \pi_\alpha(\chi)d\chi\right]. \quad (35)$$

Moreover, because of Eq. (27), the energy cost due to the addition of an edge

$$p_e(\epsilon) = \iint d\phi_1 d\phi_2\, p_\alpha(\phi_1)p_\alpha(\phi_2)$$

$$\times \int_0^{+\infty} dw\, \rho(w)\delta\left[\epsilon - \min\left(0, w - \frac{\phi_1 - \phi_2}{N}\right)\right], \quad (36)$$

and therefore, by means of an integration by parts, we get for large N

$$\alpha\Phi(\alpha) := \alpha C + L(C) = -2\ln\mathcal{Z}_\alpha$$

$$+ \frac{\alpha}{\mathcal{Z}_\alpha}\iint d\phi_1 d\phi_2\, p_\alpha(\phi_1)p_\alpha(\phi_2)$$

$$\times \int_0^\infty dw\, w e^{-\alpha(w-\phi_1-\phi_2)}\theta(\phi_1 + \phi_2 - w). \quad (37)$$

Note that, until now, no assumptions have been made on the range of values of α, except the implicit ones about the fact that the quantities above are well defined and convergent: The cavity expression can be used therefore for both positive and negative values of α, provided that the involved quantities are finite.

It can be seen that the expression in Eq. (37) is equivalent to the one given in Eq. (26c) in its range of validity. Indeed, introducing the function

$$G_\alpha(\phi) := 2\int_0^{+\infty} \hat{w} p_\alpha(\hat{w} - \phi)d\hat{w}, \quad (38)$$

Eq. (35) simplifies as

$$p_\alpha(\phi) = \frac{1}{\mathcal{Z}_\alpha}\frac{dG_\alpha(\phi)}{d\phi}e^{-\alpha\phi - G_\alpha(\phi)}, \quad (39)$$

and therefore we can write, using Eq. (38), a self-consistent equation for the function $G_\alpha(\phi)$ that is found to be identical to Eq. (26a), proving that the function G_α introduced here is the same appearing in the replica approach. Repeating the arguments presented in the replica derivation, we obtain that the integrals are finite for $\alpha > -2$. Substituting Eq. (39) in Eq. (37), simple manipulations give us the same expression presented in Eq. (26c), the main difference being that, in the obtained formula,

$$\mathcal{Z}_\alpha = \int e^{-\alpha\phi} p_v(\phi)d\phi = \int e^{-\alpha\phi}\frac{dG_\alpha(\phi)}{d\phi}e^{-G_\alpha(\phi)}d\phi. \quad (40)$$

If we further restrict ourselves to negative values of α, then an integration by parts allows us to write $\mathcal{Z}_\alpha$ in the same form given in Eq. (26a), proving the equivalence of the two
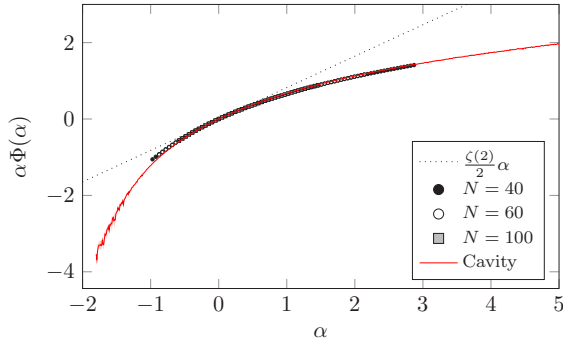
FIG. 1. Cavity and numerical results for the $\alpha\Phi(\alpha)$ in the random-link matching problem. The cavity results have been obtained by means of a population dynamics algorithm, using a population of 10^5 fields. The numerical data have been obtained instead from 10^{10} instances for each value of α and N: For large N the finiteness of the number of instances makes the curves going to $\alpha\zeta(2)/2$ because of the concentration of the measure.

approaches and implicitly providing us a prolongation of the replica result.

V. NUMERICAL RESULTS

We have integrated Eq. (37) by means of a population dynamics algorithm, and we have compared our results with the value of $\alpha\Phi(\alpha)$ obtained by numerical simulations, performed solving a large number of instances of the problem. In our numerical simulations, we assumed $\rho(w) = e^{-w}$. The quantity $\alpha\Phi(\alpha)$ can be evaluated directly, using the fact that $\alpha\Phi(\alpha) = -\lim_{N\to\infty} N^{-1} \ln \mathbb{E}[e^{-\alpha N C_N}]$. The agreement is very good in the neighborhood of the origin, as can be seen in Fig. 1, where the cavity and numerical results are compared. For reference, we have plotted the tangent to the curve in the origin, whose slope coincides with the AOC, $\zeta(2)/2$. We have discarded the data points where finite-sample effects appear for larger values of $|\alpha|$; obviously, better estimates can be obtained running an exponentially large number of instances in the size of the system. Finite-size effects still appear in the evaluation of the derivative of $\alpha\Phi(\alpha)$ (that has been also
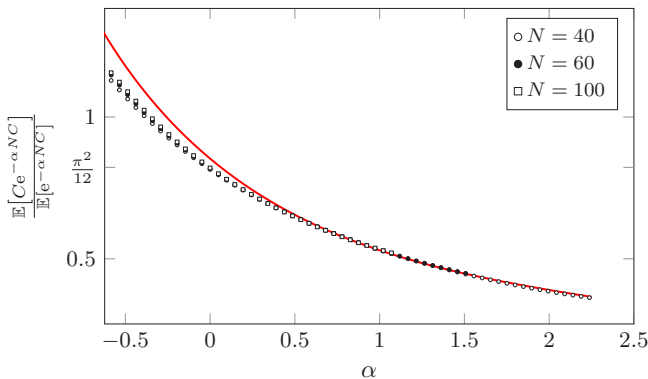

FIG. 2. Plot of the derivative of $\alpha\Phi(\alpha)$ obtained from numerical simulation compared with the theoretical prediction obtained using cavity.
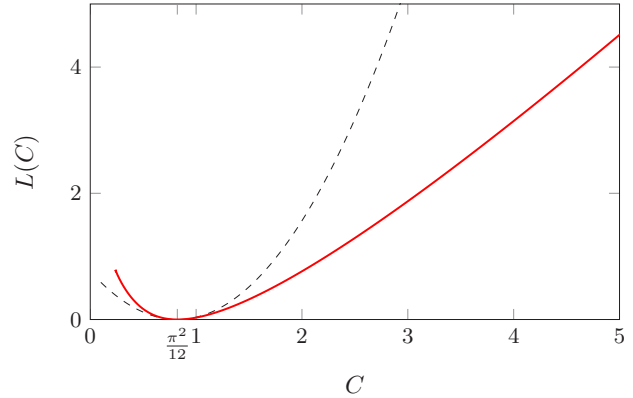

FIG. 3. Function $L(C)$ obtained performing the Legendre transform of our cavity results, compared with the quadratic approximation (dashed) provided by the results in Sec. II.

directly evaluated as $\partial_\alpha[\alpha\Phi(\alpha)] = \lim_{N\to\infty} \frac{\mathbb{E}[C_N e^{-\alpha N C_N}]}{\mathbb{E}[e^{-\alpha N C_N}]}$) for $\alpha < 0$, larger sizes being closer to the theoretical predictions, see Fig. 2. Finally, the theoretical prediction near $\alpha = -2$ becomes noisy and less reliable, because of the approaching of the divergence. In this regime, the convergence of the cavity fields distribution fails and an accurate estimation of $\alpha\Phi(\alpha)$ is more challenging. In Fig. 3 we have plotted the Legendre transform of our cavity results, that is, the desired large deviation function. The curve is compared with the quadratic approximation in the origin, corresponding to the Gaussian small-fluctuation behavior.

VI. CONCLUSIONS

Using the replica approach we have evaluated the variance of the average optimal cost for the random-link matching problem on the complete graph, assuming a distribution of the weights such that $\lim_{w\to 0} \rho(w) = 1$. Our result is in agreement with a previously obtained expression by Wästlund, proving that the small fluctuations of the optimal cost around its asymptotic AOC are Gaussian. We have then derived an expression for the Legendre transform $\alpha\Phi(\alpha)$ of the Cramér function $L(C)$ using both the replica theory (for positive cost fluctuation) and the cavity method. The cavity formula, in particular, has been obtained for the random-link matching problem on a generic sparse graph having fixed coordination, and provides a recipe for the numerical evaluation of $\alpha\Phi(\alpha)$. In the fully connected case, our results also show that $\alpha\Phi(\alpha)$ diverges for $\alpha \to -2$, implying that $\lim_{C\to+\infty} C^{-1} L(C) = 2$.

APPENDIX: DERIVATION OF Eq. (25)

To prove the limit in Eq. (25), we can start using the fact that $\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}$. The series can be written as

$$\sum_{p=1}^{\infty} \frac{1}{\binom{\alpha/\beta}{p}} \frac{e^{\beta u p}}{(p!)^2} = \frac{\Gamma(-\alpha/\beta)}{\beta} \sum_{p=1}^{\infty} \frac{1}{\Gamma(p-\alpha/\beta)} \frac{(-e^{\beta u})^p}{p!\, p}. \quad \text{(A1)}$$

We have to compute the $\beta \to +\infty$ limit at $n\beta = \alpha$ fixed. The coefficient in front of the series has limit $\lim_{\beta \to +\infty} \beta^{-1} \Gamma(-\alpha/\beta) = -1/\alpha$. On the other hand,

$$\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{1}{\Gamma(p-\alpha/\beta)} \frac{(-e^{\beta u})^p}{p!\, p} + \theta(u)$$

$$= \lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \left[\frac{\Gamma(p)}{\Gamma(p-\alpha/\beta)} - 1 \right] \frac{(-e^{\beta u})^p}{(p!)^2}, \quad \text{(A2)}$$

where we have used the fact that

$$\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{(-e^{\beta u})^p}{(p!)^2} = -\theta(u). \quad \text{(A3)}$$

We can write the following expansion:

$$\frac{\Gamma(p)}{\Gamma(p-\alpha/\beta)} - 1$$

$$= \sum_{k=1}^{\infty} \frac{(-\alpha/\beta)^k}{k!} B_k(\hat{\psi}_0(p), \hat{\psi}_1(p), \dots, \hat{\psi}_{k-1}(p)), \quad \text{(A4)}$$

where we have introduced the (opposite) polygamma function $\hat{\psi}_n(z) := -\psi_n(z)$

$$\psi_n(z) := \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) = (-1)^{n+1} \int_0^{\infty} t^n \frac{e^{-tz}}{1-e^{-t}} dt \quad \text{(A5)}$$

and the (complete) exponential Bell polynomials

$$B_k(x_1, \dots, x_k) := \left(\frac{\partial}{\partial t} \right)^k \exp\left(\sum_{j=1}^{\infty} \frac{x_j t^j}{j!} \right) \Bigg|_{t=0}. \quad \text{(A6)}$$

The expansion above is obtained applying the Faà di Bruno formula for the derivative of a composed function $\partial_x^n (f \circ g)(x)$ with $f(x) \equiv e^x$ and $g(x) \equiv -\ln \Gamma(x)$. If we now assume $B_0 \equiv 1$, then the Bell polynomials satisfy the recurrence relations [20]

$$B_{k+1}(x_1, \dots, x_{k+1}) = \sum_{i=0}^{k} \binom{k}{i} B_{k-i}(x_1, \dots, x_{k-i}) x_{i+1}. \quad \text{(A7)}$$

We will show now by induction that, for $k \geqslant 2$,

$$\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{B_k(\hat{\psi}_0(p), \dots, \hat{\psi}_{k-1}(p))}{\beta^k} \frac{(-e^{\beta u})^p}{(p!)^2} = 0. \quad \text{(A8)}$$

For $k=1$ we have

$$\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{\hat{\psi}_0(p)}{\beta} \frac{(-e^{\beta u})^p}{(p!)^2}$$

$$= \lim_{\beta \to +\infty} \int_0^{+\infty} \frac{J_0(2e^{\beta \frac{u-t}{2}}) - 1}{1 - e^{-\beta t}} dt = -u\theta(u). \quad \text{(A9)}$$

The $k=2$ case follows straightforwardly. Indeed,

$$\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{B_2(\hat{\psi}_0(p), \hat{\psi}_1(p))}{\beta^2} \frac{(-e^{\beta u})^p}{(p!)^2}$$

$$= \lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{\hat{\psi}_0^2(p) + \hat{\psi}_1(p)}{\beta^2} \frac{(-e^{\beta u})^p}{(p!)^2}$$

$$= -\int_0^{+\infty} dt \int_0^{+\infty} d\tau\, \theta(u - t - \tau)$$

$$+ \int_0^{+\infty} dt\, t\theta(u - t) = 0. \quad \text{(A10)}$$

Let us suppose now that the statement is true for generic $k > 2$ and let us consider it for $k+1$. Then we have

$$\sum_{p=1}^{\infty} \frac{B_{k+1}(\hat{\psi}_0(p), \dots, \hat{\psi}_k(p))}{\beta^k} \frac{(-e^{\beta u})^p}{(p!)^2} = \sum_{i=0}^{k} \binom{k}{i} \sum_{p=1}^{\infty} \frac{B_{k-i}(\hat{\psi}_0(p), \dots, \hat{\psi}_{k-i-1}(p)) \hat{\psi}_i(p)}{\beta^{k+1}} \frac{(-e^{\beta u})^p}{(p!)^2}. \quad \text{(A11)}$$

The inner sum can be written as

$$\int_0^{+\infty} dt\, \frac{(-1)^i t^i}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \frac{B_{k-i}(\hat{\psi}_0(p), \dots, \hat{\psi}_{k-i-1}(p))}{\beta^{k-i}} \frac{(-e^{\beta(u-t)})^p}{(p!)^2}, \quad \text{(A12)}$$

implying that

$$\sum_{p=1}^{\infty} \frac{B_{k+1}(\hat{\psi}_0(p), \dots, \hat{\psi}_k(p))}{\beta^k} \frac{(-e^{\beta u})^p}{(p!)^2} = \sum_{i=0}^{k-2} \binom{k}{i} \int_0^{+\infty} dt\, \frac{(-1)^i t^i}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \frac{B_{k-i}(\hat{\psi}_0(p), \dots, \hat{\psi}_{k-i-1}(p))}{\beta^{k-i}} \frac{(-e^{\beta(u-t)})^p}{(p!)^2}$$

$$+ k(-1)^{k-1} \int_0^{+\infty} dt\, \frac{t^{k-1}}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \frac{\hat{\psi}_0(p)}{\beta} \frac{(-e^{\beta(u-t)})^p}{(p!)^2} + (-1)^k \int_0^{+\infty} dt\, \frac{t^k}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \frac{(-e^{\beta(u-t)})^p}{(p!)^2}. \quad \text{(A13)}$$

The last two terms in the previous expression in the $\beta \to \infty$ tend to zero

$$-k(-1)^{k-1} \iint_0^{+\infty} t^{k-1} \theta(u - t - \tau) d\tau dt + (-1)^{k-1} \int_0^{+\infty} dt\, t^k \theta(u - t) = 0. \quad \text{(A14)}$$

By the induction hypothesis, the remaining $k-2$ contributions are infinitesimal as well for $\beta \to +\infty$, and the thesis is proved. We can restrict therefore the expansion in Eq. (A4) to the $k=1$ term in the $\beta \to +\infty$ hypothesis. We have

$$\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \left[\frac{\Gamma(p)}{\Gamma(p - \alpha/\beta)} - 1 \right] \frac{(-e^{\beta u})^p}{(p!)^2} = \alpha u \theta(u). \tag{A15}$$

The relations above finally give us the asymptotic in Eq. (25).

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