Multiplicative measurement noise can facilitate consensus of multiagent networks

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Measurement noise may have an important impact on the collective motion. Here, we investigate the consensus problem of multiagent networks with multiplicative measurement noise. Based on the stability theory of stochastic differential equations and the algebra graph theory, we obtain sufficient conditions for the consensus and nonconsensus. Both of our analytical and numerical results show that the multiplicative measurement noise can facilitate the emergence of the consensus: the convergence rate increases with respect to the noise intensity if the topologies of the underlying networks satisfy some conditions. Our results provide a better understanding of the constructive role of noise. We also report that the convergence rate of multiagent networks is strongly affected by the network topology and the group size.

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I. INTRODUCTION

Recently, the distributed coordinated control of multiagent systems has fascinated more and more researchers from numerous disciplines due to its broad applications in many fields. Examples include cooperative control of an unmanned air vehicle [1], flocking [2–4], collective behavior [5–7], and attitude alignment of clusters of satellites [8]. To reveal the inherent mechanism of multiagent systems, some models have been introduced [2,3,9]. A celebrated self-propelled particles model is the Vicsek model, which assumes that each particle has a tendency to adjust its direction to the average direction of its neighbors [2]. Using the nearest-neighbor rule, the group can move in a common direction eventually.

The consensus problem is one of the fundamental topics in the distributed coordinated control, which focuses on designing a protocol such that the states of all agents reach an agreement. Olfati-Saber and Murray proposed a theoretical framework for analysis of consensus algorithms for multiagent networked systems, and they investigated the consensus problems of multiagent networks with switching topology and communication time delay [10]. One of the most important contributions of their work in Ref. [10] is that two protocols were proposed to solve the cases of continuous-time and discrete-time consensus problems. Further extensions of their work were presented with a look toward noise in Refs. [11–15].

Noise is ubiquitous in natural and man-made systems, which means that the motion of the group is inevitably subjected to noise perturbation in the environment. In a noisy environment, each agent may not measure its neighbors' states accurately, leading to more complicated interactions among agents. Because of the intrinsic nature of noise, it is always regarded as mischievous, i.e., noise will destroy order or add disorder in natural and artificial systems. Additive and multiplicative noise has been extensively used to model the measurement uncertainties in multiagent systems [13,15], and then, the deterministic multiagent system becomes a stochastic system.

The influence of noise has been extensively investigated in the context of synchronization [16], swarming [17], flocking [2,15,18,19], and the consensus [12–14,20]. It was reported that the ordered collective motion of the Vicsek model can occur only when the noise intensity is sufficiently low [2]. The results in Ref. [18] show that the order parameter of the collective motion is a decreasing function of the noise intensity, indicating that noise can destroy the ordered collective motion. Recently, the influence of noise on the consensus dynamics of first-order [12,13] and second-order [14] integrator systems has been investigated. It was shown that strong noise may destroy the emergence of the consensus. Some investigations have shown that noise may enhance the collective behavior of networked systems. It was shown that the multiagent system can achieve the consensus only via noise-based coupling [20]. A quantitative description of how locusts use noise to maintain swarm alignment was given in Ref. [17], showing that noise can facilitate coherence in collective swarm motion.

However, to the best of our knowledge, most of previous studies only show the destructive role of noise, i.e., strong noise may destroy the ordered collective motion. Additionally, some experimental studies have shown that ordered group behavior can be achieved by the coupling of stochastically moving individuals [21–23]. For example, many types of living cells can achieve robust large-scale ordered behavior by combining and coordinating stochastic small-scale components [21,22]. Recent experimental results show that stochasticity offers a promising approach to develop collective robotic systems that exhibit robust deterministic ordered behavior [23]. Inspired by the constructive role of stochasticity in the collective motion of real biological and physical systems, we focus on the constructive role of multiplicative measurement noise in the consensus of multiagent networks.

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We propose a stochastic consensus model in which the intensity of measurement noise is proportional to the error between agents, and the measurement noise will disappear when the states of all agents reach an agreement. We aim to answer the following question: Can multiplicative measurement noise facilitate the emergence of the consensus of multiagent networks? Based on the theory of stochastic differential equations and algebra graph theory, we obtain sufficient conditions which guarantee that all agents can asymptotically reach an average consensus with probability one. Sufficient conditions for the nonconsensus are also established. To proceed, we numerically study the influence of network topologies and group size on the convergence rate. Our results show that the multiplicative measurement noise may facilitate the emergence of the consensus.

II. CONSENSUS MODEL WITH MULTIPLICATIVE MEASUREMENT NOISE

We consider a multiagent system consisting of N agents $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$. Agent v_j is called a neighbor of v_i if v_i can receive the state information of v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i$. The information exchanging among agents can be described by a simple digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{E} = \{(v_i, v_j): v_j \in \mathcal{N}_i, \forall v_i \in \mathcal{V}\}$ is the set of edges and $A \in \mathbb{N}^{N\times N}$ is a weighted adjacency matrix with nonnegative elements. The element $a_{ij} > 0$ if and only if $v_j \in \mathcal{N}_i$. Moreover, we assume $a_{ii} = 0$. Let $D_A = \mathrm{diag}\{\eta_1, \eta_2, \dots, \eta_N\}$, where $\eta_i = \sum_{v_j\in\mathcal{N}_i} a_{ij}$. Then, the Laplacian matrix of the digraph $\mathcal{G}$ can be defined as $L = D_A - A$. Suppose each individual is a dynamical agent which can be described as

$$\dot{x}_i = u_i, \quad i \in \mathcal{I}, \tag{1}$$

where $x_i \in \mathbb{R}$ is the state of agent v_i and $\mathcal{I} = \{1, 2, \dots, N\}$ is the index set. Typically, $u_i \in \mathbb{R}$ is called the protocol which only depends on the states of the neighbors of v_i. Numerous protocols have been designed to investigate the finite-time consensus [24], fixed-time consensus [15], and the consensus with communication delays [14,25]. A typical linear continuous-time consensus protocol,

$$u_i = \sum_{v_j\in\mathcal{N}_i} a_{ij}(x_j - x_i) \tag{2}$$

was proposed by Olfati-Saber and Murray in Ref. [10], where a_{ij} is to account for the reliability of information that v_i received form v_j. Nodes v_i and v_j are called to reach an agreement if and only if $x_i(t) = x_j(t)$ for $i \neq j$. Furthermore, we say that system (1) reaches the consensus if and only if for all different agents $v_i, v_j \in \mathcal{V}$ and any given initial conditions v_i and v_j reach an agreement as $t \to \infty$. Especially, system (1) is said to reach the average consensus if for any $v_i \in \mathcal{V}$ state $x_i(t)$ converges to the consensus value of $x^* = \frac{1}{N}\sum_{i=1}^N x_i(0)$ as $t \to \infty$ [10].

As mentioned above, in a noisy environment, individuals cannot measure their neighbor's state exactly. In this paper, we assume that the state of agent v_i measured by agent i is given by

$$y_{ij} = x_j + \frac{1}{\sqrt{N}}\sigma_{ij}(x_j - x_i)\xi_{ij}(t), \quad v_j \in \mathcal{N}_i, \tag{3}$$

where $\sigma_{ij} \geqslant 0$ represent the noise intensity and $\xi_{ij}(t)$ are independent Gaussian white noise. We note that $\langle\xi_{ij}(t)\rangle = 0$ and $\langle\xi_{ij}(t), \xi_{kl}(t')\rangle = \delta_{ij}\delta_{kl}\delta(t - t')$ where the angular brackets $\langle\cdot\rangle$ denote the average over different realizations of the noise, δ_{ij} is the Kronecker δ function, and δ denotes the Dirac's function. We consider the following protocol, which will be shown to solve the average consensus problem with noise perturbation:

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}(y_{ij} - x_i), \quad i \in \mathcal{I}. \tag{4}$$

With Eqs. (3) and (4), system (1) can be rewritten in the following Itô stochastic differential equation:

$$\begin{aligned}\dot{x}_i = &\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\\ &+ \frac{1}{\sqrt{N}}\sum_{j\in\mathcal{N}_i} a_{ij}\sigma_{ij}(x_j - x_i)\xi_{ij}(t), \quad i \in \mathcal{I}.\end{aligned} \tag{5}$$

It is apparent that, if $\sigma_{ij} = 0$ for all $i, j \in \mathcal{I}$, system (5) is equivalent to the continuous-time consensus model proposed in Ref. [10]. It is worth mentioning that the measurement noise in (5) is multiplicative. Thus, the strength of the noise is proportional to the measurement error between individuals, and the noise will disappear when the states of all agents reach an agreement. Most previous studies of consensus dynamics assume that the interactions of moving individuals are time invariant, however, it has been reported that interactions between individuals of many real biological systems are time varying [26,27]. The noise term in (5) is introduced to model the fluctuating interactions of individuals in real biological systems.

III. STOCHASTIC STABILITY ANALYSES

In order to understand the influence of noise on the consensus dynamics of stochastic multiagent networks, we analytically study the model (5) based on the algebra graph theory and the stability theory of stochastic differential equations. Denoting that $A_\sigma = (a_{ij}\sigma_{ij}) \in \mathbb{R}^{N\times N}$ and $D_\sigma = \mathrm{diag}\{\zeta_1, \zeta_2, \dots, \zeta_N\}$, where $\zeta_i = \sum_{j=1}^N a_{ij}\sigma_{ij}$, we define $G = D_\sigma - A_\sigma$. Obviously, G can be regarded as a Laplacian matrix of digraph $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, A_\sigma)$. For simplicity, we denote that the Gaussian white noises are one dimensional, i.e., $\xi_{ij}(t) = \xi(t)$. Mathematically, the one-dimensional Gaussian white noise can be expressed formally as $\xi(t) = \dot{W}(t)$, where $W(t)$ is a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Thus, system (5) can be rewritten in the following vector form:

$$dx = -Lx\,dt - \frac{1}{\sqrt{N}}Gx\,dW(t). \tag{6}$$

where $x = (x_1, \dots, x_N)^T$. From the definition of the Laplacian matrix, it is apparent that $L\mathbf{1} \equiv \mathbf{0}$ and $G\mathbf{1} \equiv \mathbf{0}$, where $\mathbf{1}^T = (1, 1, \dots, 1)$. Suppose that digraphs $\mathcal{G}$ and $\bar{\mathcal{G}}$ are balanced, then $\mathbf{1}^T L = \mathbf{0}$ and $\mathbf{1}^T G = \mathbf{0}$ [10]. Let $\widehat{x}(t)$

$= \frac{1}{N}\sum_{i=1}^{N} x_i(t) = \frac{1}{N}\mathbf{1}^T x(t), t \geqslant 0$, and we have

$$\begin{aligned} d\widehat{x}(t) &= \frac{1}{N}\mathbf{1}^T\left[-Lx(t)dt - \frac{1}{\sqrt{N}}Gx(t)dW(t)\right] \\ &= -\frac{1}{N}\left[\mathbf{1}^T Lx\,dt + \frac{1}{\sqrt{N}}\mathbf{1}^T Gx\,dW(t)\right] \\ &= 0, \end{aligned}$$

which means $\widehat{x}(t)$ is an invariant quantity. Therefore, the average consensus value $\widehat{x}(t) \equiv x^* = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$. This allows the decomposition of $x = e + x^*\mathbf{1}$, where $e = (e_1, e_2, \dots, e_N)^T$ is a disagreement vector. Thus, $\mathbf{1}^T e = \mathbf{1}^T(x - x^*\mathbf{1}) = \mathbf{1}^T x - Nx^* = 0$, and the disagreement system can be expressed as

$$de = -Le\,dt - \frac{1}{\sqrt{N}}Ge\,dW(t). \tag{7}$$

The multiagent network (5) is said to reach the stochastic average consensus if, for any given initial state $x_i(0)$,

$$P\{\lim_{t\to\infty} |x_i[t, x_i(0)] - x^*| = 0\} = 1, \quad \forall\, i \in \mathcal{I}.$$

Note that $\lim_{t\to\infty} |x_i[t, x_i(0)] - x^*| = 0$ is equivalent to $\lim_{t\to\infty} |e_i[t, e_i(0)]| = 0$. One can clearly see that $e(t, 0) \equiv 0$ is a trivial solution of the disagreement system (7). Thus, system (5) could reach the stochastic average consensus if the trivial solution of (7) is almost surely asymptotically stable, i.e., for any initial value $e_0 \in \mathbb{R}^N$, $\|e(t, e_0)\| \to 0$ almost surely as $t \to \infty$.

To proceed, based on the matrix theory, we further simplify the model. Note that $\mathbf{1}$ is the left eigenvector of both L and G associating with the zero eigenvalue. Then, let $U = [U_1, \frac{1}{\sqrt{N}}\mathbf{1}] \in \mathbb{R}^{N\times N}$ be a real orthogonal matrix with $U_1 \in \mathbb{R}^{N\times(N-1)}$. Thus, we have

$$U^T LU = \begin{bmatrix} U_1^T LU_1 & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 0 \end{bmatrix},$$

$$U^T GU = \begin{bmatrix} U_1^T GU_1 & \mathbf{0}_{(N-1)\times 1} \\ \mathbf{0}_{1\times(N-1)} & 0 \end{bmatrix}.$$

Since $\mathbf{1}^T e = 0$, one can see that $U^T e = [U_1, \frac{1}{\sqrt{N}}\mathbf{1}]^T e = [e^T U_1, 0]^T$. Thus, applying the Itô formula, the disagreement system (7) is equivalent to the following system:

$$dz = -\bar{L}z\,dt - \frac{1}{\sqrt{N}}\bar{G}z\,dW(t), \tag{8}$$

where $z = U_1^{\mathrm{T}} e \in \mathbb{R}^{(N-1)}$, $\bar{L} = U_1^T LU_1$, and $\bar{G} = U_1^T GU_1 \in \mathbb{R}^{(N-1)\times(N-1)}$. Furthermore, let digraphs $\mathcal{G}$ and $\bar{\mathcal{G}}$ be strongly connected, the eigenvalues of L and G have non-negative real parts [10]. That means all eigenvalues of $\bar{L}$ and $\bar{G}$ have positive real parts.

In the following, we first derive the sufficient conditions for the consensus of stochastic multiagent networks (5).

Proposition 1. Suppose that digraphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ and $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, A_\sigma)$ are strongly connected and balanced. The multiagent network (5) can reach the stochastic average consensus if

$$\begin{aligned} \gamma &\triangleq 2\lambda_{\min}\left(\frac{\bar{L}^T + \bar{L}}{2}\right) - \frac{1}{N}\lambda_{\max}(\bar{G}^T\bar{G}) \\ &\quad + \frac{2}{N}\lambda_{\min}^2\left(\frac{\bar{G}^T + \bar{G}}{2}\right) > 0. \end{aligned}$$

Proof. Applying the Itô formula [28] to the function $V(z) = \ln z^T z$ along with system (8), we have

$$\begin{aligned} V[z(t)] &= V[z(0)] + \int_0^t \mathcal{L}V[z(s)]ds \\ &\quad - \frac{1}{\sqrt{N}}\int_0^t V_z\bar{G}z(s)dW(s), \quad t \geqslant 0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \mathcal{L}V(z) &= -V_z\bar{L}z + \frac{1}{2}\mathrm{tr}\left[\left(-\frac{1}{\sqrt{N}}\bar{G}z\right)^T V_{zz}\left(-\frac{1}{\sqrt{N}}\bar{G}z\right)\right] \\ &= -V_z\bar{L}z + \frac{1}{2N}\mathrm{tr}[z^T\bar{G}^T V_{zz}\bar{G}z], \end{aligned}$$

and

$$V_z = \frac{2z^T}{z^T z}, \quad V_{zz} = \frac{2I}{z^T z} - \frac{4zz^T}{(z^T z)^2}.$$

Then,

$$\mathcal{L}V(z) = -2\frac{z^T\bar{L}z}{z^T z} + \frac{1}{N}\frac{z^T\bar{G}^T\bar{G}z}{z^T z} - \frac{2}{N}\frac{(z^T\bar{G}z)^2}{(z^T z)^2}. \tag{10}$$

Since digraph $\mathcal{G}$ is strongly connected and balanced, $(L^T + L)/2$ is a valid Laplacian matrix of a undirected graph, and $\mathrm{rank}[(L^T + L)/2) = N - 1$ [10]. By the definition of $\bar{L}$ and $\bar{G}$, we have

$$\begin{aligned} \lambda_{\min}\left(\frac{\bar{L}^T + \bar{L}}{2}\right) &\leqslant \frac{z^T\bar{L}z}{z^T z} \leqslant \lambda_{\max}\left(\frac{\bar{L}^T + \bar{L}}{2}\right), \\ \lambda_{\min}\left(\frac{\bar{G}^T + \bar{G}}{2}\right) &\leqslant \frac{z^T\bar{G}z}{z^T z} \leqslant \lambda_{\max}\left(\frac{\bar{G}^T + \bar{G}}{2}\right). \end{aligned} \tag{11}$$

Here, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalues of a symmetric matrix, respectively. Also, we have

$$\frac{z^T\bar{G}^T\bar{G}z}{z^T z} \leqslant \lambda_{\max}(\bar{G}^T\bar{G}). \tag{12}$$

Substituting (11) and (12) into (10), we obtain

$$\begin{aligned} \mathcal{L}V[z(t)] &\leqslant -2\lambda_{\min}\left(\frac{\bar{L}^T + \bar{L}}{2}\right) + \frac{1}{N}\lambda_{\max}(\bar{G}^T\bar{G}) \\ &\quad - \frac{2}{N}\lambda_{\min}^2\left(\frac{\bar{G}^T + \bar{G}}{2}\right), \quad \forall\, t \geqslant 0. \end{aligned}$$

Let

$$\begin{aligned} \gamma &\triangleq 2\lambda_{\min}\left(\frac{\bar{L}^T + \bar{L}}{2}\right) - \frac{1}{N}\lambda_{\max}(\bar{G}^T\bar{G}) \\ &\quad + \frac{2}{N}\lambda_{\min}^2\left(\frac{\bar{G}^T + \bar{G}}{2}\right), \end{aligned} \tag{13}$$

then

$$\int_0^t \mathcal{L}V[z(s)]ds \leqslant -\gamma t, \quad \forall t \geqslant 0.$$

From (9), we obtain the following estimation:

$$\frac{V[z(t)]}{t} \leqslant \frac{V[z(0)]}{t} - \gamma + \frac{\mathcal{M}(t)}{t}, \quad \forall t \geqslant 0,$$

where

$$\mathcal{M}(t) = \int_0^t V_z \bar{G} z(s) dW(s)$$
$$= -\frac{2}{\sqrt{N}} \int_0^t \frac{z^T \bar{G} z}{z^T z} dW(s) \tag{14}$$

is a continuous martingale with $\mathcal{M}(0) = 0$. Calculating the quadratic variation of $\mathcal{M}(t)$, we have

$$\langle \mathcal{M}(t), \mathcal{M}(t) \rangle = \frac{4}{N} \int_0^t \left(\frac{z^T \bar{G} z}{z^T z} \right)^2 ds$$
$$\leqslant \frac{4}{N} \lambda_{\max}^2 \left(\frac{\bar{G}^T + \bar{G}}{2} \right) t, \quad \forall t \geqslant 0.$$

Thus,

$$\lim_{t\to+\infty} \frac{\langle \mathcal{M}(t), \mathcal{M}(t) \rangle}{t} < \infty.$$

Applying the strong law of large numbers [28], we have

$$\lim_{t\to+\infty} \frac{\mathcal{M}(t)}{t} = 0, \quad \text{a.s.}$$

Here "a.s." is an abbreviation for "almost surely". In probability theory, one says that an event happens almost surely if it happens with probability one. Thus, we get

$$\lim_{t\to+\infty} \sup \frac{V[z(t)]}{t} \leqslant -\gamma, \quad \text{a.s.}$$

Consequently,

$$\lim_{t\to+\infty} \sup \frac{\ln \|z(t)\|}{t} \leqslant -\gamma/2, \quad \text{a.s.}$$

If γ is a positive number, the solution $z(t)$ of system (8) is almost surely exponentially convergent to 0. This indicates that the multiagent network (5) reaches the stochastic average consensus. ■

To get an intuition of the conditions in Proposition 1, we consider a special case that graph $\mathcal{G}$ is undirected. Assuming that the noise intensity matrix A_σ is relative with the adjacency matrix A, we obtain the following corollary.

Corollary 1. Suppose that graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is undirected and connected and $\sigma_{ij} \equiv \sigma_0 > 0$ when $a_{ij} > 0$. System (5) can reach the stochastic average consensus if

$$\mu \triangleq 2\lambda_{\min}(\bar{L}) + \frac{\sigma_0^2}{N}\left[2\lambda_{\min}^2(\bar{L}) - \lambda_{\max}^2(\bar{L})\right] > 0.$$

Proof. According to the definition of the balanced digraph [10], the undirected graph $\mathcal{G}$ is a balanced graph. Since $\sigma_{ij} \equiv \sigma_0$, we have $G = \sigma_0 L$, where L is the Laplacian matrix of graph $\mathcal{G}$, and G is the Laplacian matrix of graph $\bar{\mathcal{G}}$. Thus, graph $\bar{\mathcal{G}}$ is balanced and connected when the undirected graph $\mathcal{G}$ is connected. Note that the Laplacian matrices L and G are symmetric. Thus, it is obvious that $\bar{L}$ and $\bar{G}$ are symmetric as well. Therefore,

$$2\lambda_{\min}\left(\frac{\bar{L}^T + \bar{L}}{2}\right) - \frac{1}{N}\lambda_{\max}(\bar{G}^T\bar{G}) + \frac{2}{N}\lambda_{\min}^2\left(\frac{\bar{G}^T + \bar{G}}{2}\right)$$
$$= 2\lambda_{\min}(\bar{L}) + \frac{\sigma_0^2}{N}\left[2\lambda_{\min}^2(\bar{L}) - \lambda_{\max}^2(\bar{L})\right]$$
$$\triangleq \mu.$$

From Proposition 1, if $\mu > 0$ then the multiagent network (5) can reach the stochastic average consensus. ■

From the proof of Proposition 1, one can see that the parameter γ characterizes the decay rate of the group error. Thus, the convergence rate of consensus is proportional to the parameter γ. For undirected networks, the results in Corollary 1 show that, if $2\lambda_{\min}^2(\bar{L}) > \lambda_{\max}^2(\bar{L})$, the convergence rate of consensus will increase with respect to the noise intensity. Thus, from the above analysis we can claim that the multiplicative measurement noise may facilitate the emergence of the consensus of multiagent networks if some conditions on the network topologies are satisfied.

It is worth noting that the conditions in Proposition 1 are not necessary. Thus, the multiagent networks may diverge if the conditions in Proposition 1 are not satisfied. The following proposition shows that the multiagent network (5) cannot reach the stochastic average consensus if some conditions on the network topologies and the noise intensity are satisfied.

Proposition 2. Suppose that digraphs $\mathcal{G}$ and $\bar{\mathcal{G}}$ are strongly connected and balanced. The multiagent network (5) is exponentially divergent in a mean square if

$$\upsilon \triangleq -2\lambda_{\max}\left(\frac{\bar{L}^T + \bar{L}}{2}\right) + \frac{1}{N}\lambda_{\min}(\bar{G}^T\bar{G}) > 0.$$

Proof. Applying the Itô formula [28] for the function $U(z) = z^T z$ along with the system (8), we have

$$U[z(t)] = U[z(0)] + \int_0^t \mathcal{L}U[z(s)]ds$$
$$- \frac{1}{\sqrt{N}} \int_0^t U_z \bar{G} z(s) dW_s, \quad t \geqslant 0, \tag{15}$$

where

$$\mathcal{L}U(z) = -U_z\bar{L}z + \frac{1}{2}\mathrm{tr}\left[\left(-\frac{1}{\sqrt{N}}\bar{G}z\right)^T U_{zz}\left(-\frac{1}{\sqrt{N}}\bar{G}z\right)\right]$$
$$= -2z^T\bar{L}z + \frac{1}{N}(z^T\bar{G}^T\bar{G}z).$$

It is easy to estimate that

$$z^T\bar{G}^T\bar{G}z \geqslant \lambda_{\min}(\bar{G}^T\bar{G})U(z). \tag{16}$$

Thus, we obtain from (11) and (16) that

$$\mathcal{L}U(z) \geqslant \left[-2\lambda_{\max}\left(\frac{\bar{L}^T + \bar{L}}{2}\right) + \frac{1}{N}\lambda_{\min}(\bar{G}^T\bar{G})\right]U(z),$$
$$\triangleq \upsilon U(z). \tag{17}$$

Obviously, the last term on the right hand of Eq. (15) is a continuous martingale. Taking the mathematical expectation of both sides of (15), we obtain from (17) that

$$\mathbb{E}[U(t)] = \mathbb{E}[U(0)] + \int_0^t \mathbb{E}\{\mathcal{L}U[z(s)]\}ds$$
$$\geqslant \mathbb{E}[U(0)] + \upsilon \int_0^t \mathbb{E}[U(s)]ds.$$

Therefore,

$$\frac{d\mathbb{E}[U(t)]}{dt} \geqslant \upsilon \mathbb{E}[U(t)].$$

Clearly, we have

$$\frac{d\{e^{-\upsilon t}\mathbb{E}[U(t)]\}}{dt} = e^{-\upsilon t}\left[-\upsilon\mathbb{E}[U(t)] + \frac{d\mathbb{E}[U(t)]}{dt}\right] \geqslant 0.$$

Then, we get

$$e^{-\upsilon t}\mathbb{E}[U(t)] \geqslant \mathbb{E}[U(0)], \quad \forall t > 0,$$

i.e.,

$$\mathbb{E}[U(t)] \geqslant \mathbb{E}[U(0)]e^{\upsilon t}, \quad \forall t > 0.$$

Therefore $\mathbb{E}[\|z(t)\|^2] \to \infty$ as $t \to \infty$ if υ is positive, which yields that the solution of system (8) is exponentially divergent in a mean square. Thus, individuals in system (5) cannot reach the average consensus in a mean square. ■

IV. SIMULATIONS

To validate the validity of the theoretical analysis, we perform numerical simulations for multiagent networks (5) with different topologies. We use the Euler-Maruyama numerical scheme [29] in our numerical simulations to deal with the stochastic differential equations. In all the simulations, the initial conditions are randomly taken from the interval $[-1, 1]$. The adjacency matrix A of all network topologies is limited to the 0-1 matrix. For simplicity, we set $\sigma_{ij} = \sigma_0$ for all $i, j \in \mathcal{I}$. The group error $E(t) = \|e(t)\|$ is chosen to clarify the evolution process. We say multiagent networks reach the average consensus at time $\mathcal{T}$ if $E(t) \leqslant 10^{-4}, \forall t \geqslant \mathcal{T}$. Thus, the convergence time is defined as $T = \inf\{\mathcal{T}: E(t) \leqslant 10^{-4}, \forall t \geqslant \mathcal{T}\}$. In the following simulations, the convergence rate of the consensus is assessed by the convergence time T. And the convergence time is computed by averaging over 100 simulations.

We begin with numerical simulations of multiagent network (5) with topology as shown in Fig. 1(a). Obviously, the undirected graph in Fig. 1(a) is connected. It is easy to calculate that $\lambda_{\min}(\bar{L}) = 6$, $\lambda_{\max}(\bar{L}) = 8$, and $\gamma = 12.25$ when $\sigma_0 = 0.5$. Thus, the conditions in Corollary 1 are satisfied. Figure 1(b) displays the evolutions of the group error $E(t)$ (main plot) and individuals' states (insert plot). One can clearly see that system (5) reaches the average consensus asymptotically as predicted by Corollary 1. To further verify the influence of noise on the convergence rate of the consensus, we performed numerical simulations of system (5) with different values of σ_0. Figure 1(c) shows a plot of the convergence time as a function of σ_0, computed in networks with topology as shown in Fig. 1(a). From this figure, it is apparent that the convergence rate of the consensus strictly increases with σ_0, which confirms that noise can accelerate the emergence of the consensus.

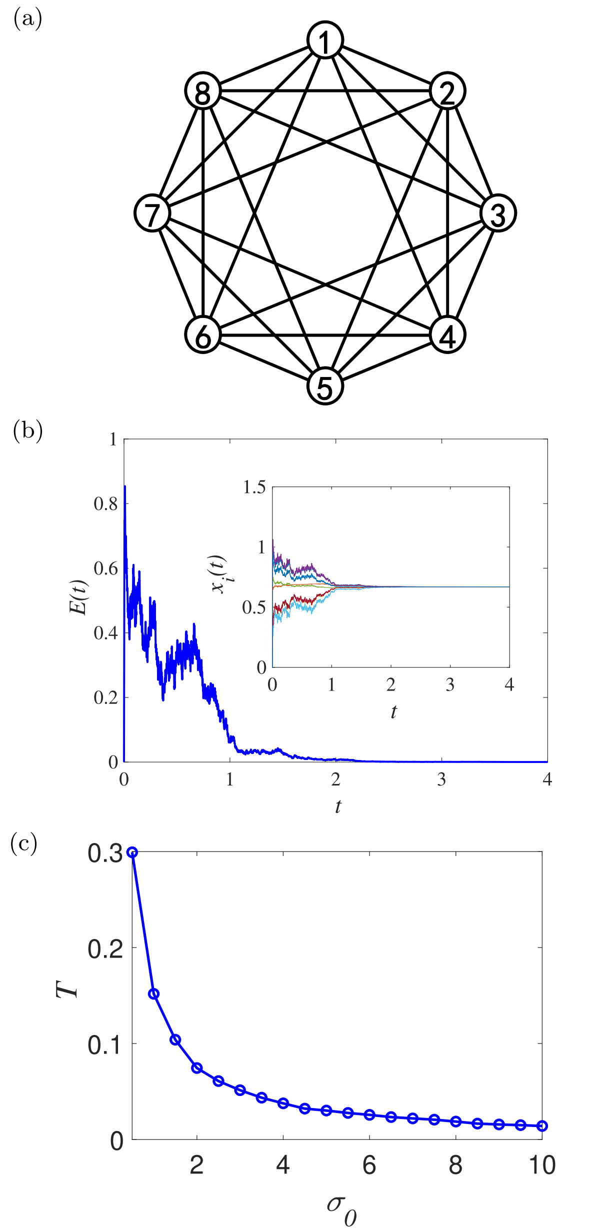

FIG. 1. (a) A connected undirected network with eight nodes. (b) The evolution of each individual's state $x_i(t)$ (the inset) and the variation of group error $E(t)$ (main) where the noise intensity $\sigma_0 = 0.5$. (c) The convergence time as a function of noise intensity σ_0 for a multiagent network with topology as shown in (a).

To proceed, we investigate the influence of network topologies on the convergence rate. We show the simulation results for multiagent networks with two well-known topologies: the small-world graph [30] and the scale-free graph [31]. The small-world network model was first introduced by Watts and Strogatz, which has two important structural characteristics: a high clustering coefficient and a short average path length [30]. These topological features underly the collective dynamics processes on networks, such as synchronization and coordination processes [32]. Using a topological randomness parameter p, the small-world network $(0 < p \ll 1)$ is capable of interpolating between a regular network $(p = 0)$ and a

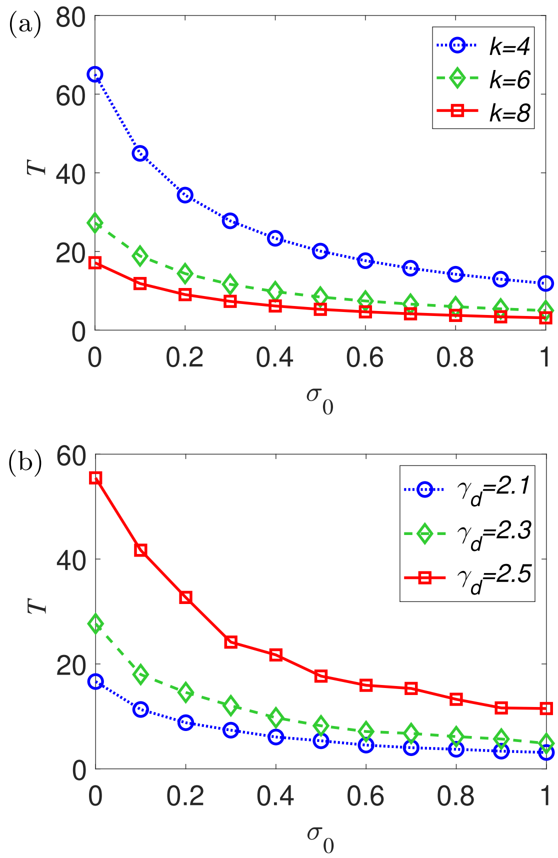

FIG. 2. The impact of the noise intensity on the convergence time of multiagent networks with small-world and scale-free topologies. (a) The convergence time as a function of noise intensity σ_0 for small-world networks with $N = 500$, $p = 0.3$ and different values of the mean degree $k = 4$ (dotted line), $k = 6$ (dashed line), and $k = 8$ (solid line). (b) The convergence time as a function of noise intensity σ_0 for scale-free networks with $N = 500$ and different values of the degree exponent $\gamma_d = 2.1$ (dotted line), $\gamma_d = 2.3$ (dashed line), and $\gamma_d = 2.5$ (solid line).

random network ($p = 1$). Using the method in Ref. [30], we generated small-world networks with $p = 0.3$ and $N = 500$. It was shown that the second smallest eigenvalue of the Laplacian matrix, also known as the algebraic connectivity, of the underlying networks characterizes the performance of consensus dynamic of deterministic multiagent networks [10]. The results in Corollary 1 show that the convergence rate of stochastic multiagent networks (5) also depends on the algebraic connectivity. Grabow *et al.* showed that, for fixed network size, the algebraic connectivity of small-world networks increases with respect to the mean degree k (see Fig. 2(a) in Ref. [33]). Therefore, it is intuitively imaginable that a higher k leads to faster convergence rate of consensus. In what follows, we will assess the effect of the mean degree and noise intensity on the convergence rate of the consensus of a multiagent network (5) with small-world topologies. The convergence time T is plotted in Fig. 2(a) as a function of σ_0, computed in small-world networks with different values of mean degree k. First, we observe that the network with larger mean degree yields a higher convergence rate of the consensus. This observation can be easily explained by the

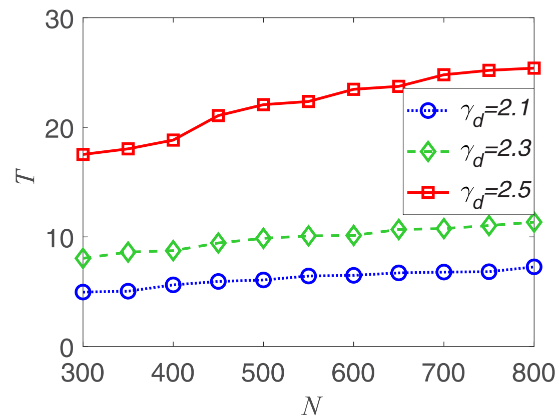

FIG. 3. The impact of network size on convergence time computed in a scale-free network with different degree exponents $\gamma_d = 2.1$ (dotted line), $\gamma_d = 2.3$ (dashed line), and $\gamma_d = 2.5$ (solid line). The noise intensity σ_0 is taken as 0.4.

effect of mean degree k on the algebraic connectivity of small-world networks. Second, we observe that the convergence rate of the multiagent network (5) monotonously increases with respect to the noise intensity. This observation supports the conclusion that noise can facilitate the emergence of the consensus.

The scale-free network is characterized by a highly heterogeneous degree distribution $P(k)$, defined as the probability that a randomly chosen node is connected to k other nodes, showing a power-law form $P(k) \sim k^{-\gamma_d}$ [31]. The presence of a scale-free degree distribution can have an important impact on the performance of some dynamic processes on networks, such as flocking of multiagent networks [27] and propagation of infectious agents [34]. We generated the scale-free networks by using the uncorrelated configuration model [35] in which the average nearest-neighbor degree of each node is almost the same. It was shown that networks with small degree exponents γ_d have relatively large average nearest-neighbor degrees. Average nearest-neighbor degrees can give a more refined measure of the connectivity of the network. It was shown that, in the absence of noise, the convergence rate of a multiagent system strictly increases with the number of topological neighbors [36]. Thus, it is expected that stochastic multiagent networks with a larger average nearest-neighbor degrees still display higher rate of convergence to the consensus. In Fig. 2(b), we plot the convergence time as a function of σ_0, computed in scale-free networks with different values of degree exponent γ_d. We observe that the smaller the degree exponent γ_d (the average nearest-neighbor degree is large) of the scale-free network is, the faster the convergence rate of the consensus is. The results in Fig. 2(b) also illustrate that the convergence rate of the consensus significantly increases with respect to the noise intensity, which confirms the constructive role of noise.

In the following, we focus on the influence of the network size on the convergence rate of the consensus. In Fig. 3, for a given noise intensity, we plot the convergence time as a function of network size N for scale-free networks with different values of degree exponent γ_d. From this figure, it

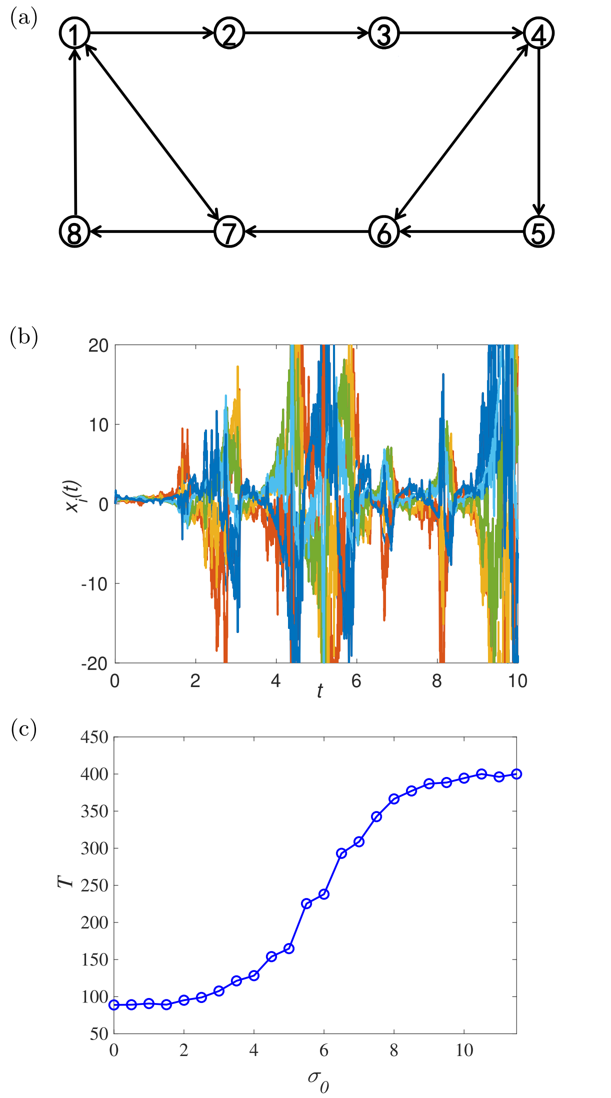

FIG. 4. (a) A connected directed network with eight nodes. (b) The evolution of each individual's state $x_i(t)$ where the noise intensity $\sigma_0 = 12$. (c) The convergence time as a function of noise intensity σ_0.

is apparent that multiagent networks with larger size need more time to achieve the consensus. This observation can be easily explained by the fact that the stochastic interactions in system (5) decrease with respect to the network size. Thus, the convergence rate of the consensus mainly depends on the algebraic connectivity of the underlying networks when the network size is too large. Previous investigation has shown that the algebraic connectivity of the scale-free graph is a decreasing function of the graph size [37]. In addition, the theoretical analysis in Sec. III shows that the decay rate of the group error is proportional to the value of γ defined in (13) which decreases with the network size N. Therefore, the observation in Fig. 3 is in accord with the above theoretical analysis.

All the above observations together with the theoretical results in Proportion 1 confirm that the multiplicative measurement noise may facilitate the emergence of the consensus dynamics of multiagent networks if some conditions on the network topology and the noise intensity are satisfied. It should be noted that the conditions in Proposition 1 are not necessary. Thus, it is expected that the multiagent networks may diverge if the conditions in Proposition 1 are not satisfied. We then turned to show the numerical results for the nonconsensus. Figure 4(a) shows a directed graph with eight nodes. It is easy to calculate that $\lambda_{\max}[(\bar{L}+\bar{L}^T)/2] = 3.34$ and $\lambda_{\min}[(\bar{L}+\bar{L}^T)/2] = 0.34$. Taking $\sigma_{ij} = \sigma_0$, then $\lambda_{\min}(\bar{G}^T\bar{G}) = 0.39\sigma_0^2$. Thus, the conditions in Proposition 2 are satisfied if $\sigma_0 > 11.7$. Noting that the conditions in Proposition 2 are only sufficient, it may give an overestimation of the critical noise intensity for the nonconsensus. The numerical results show that the consensus cannot occur for the multiagent network (5) if $\sigma_0 > 11.5$ [$\sigma_0 = 12$ in Fig. 4(b)]. For $0 \leqslant \sigma_0 \leqslant 11.5$, Fig. 4(c) displays the convergence time as a function of noise intensity. The results in this figure illustrate that noise may inhibit the emergence of the consensus if the network topology satisfies the conditions in Proposition 2.

V. CONCLUSION AND DISCUSSION

To summarize, we have investigated the influence of noise on the consensus dynamics of multiagent networks. By means of extensive numerical simulations and the theory of stochastic analysis, we find that multiplicative measurement noise can facilitate the emergence of the consensus. Our results are in sharp contrast with most of the previous studies [12–14] that strong perturbation of noise can destroy the consensus behavior of multiagent systems. In the consensus models with additive measurement noise, the strength of the noise is independent of individuals' states. Multiagent systems only rely on the attractive interactions of the deterministic coupling to achieve ordered group behavior [12–14]. The additive measurement noise will not disappear even if the states of all individuals reach an agreement. Thus, the ordered group behavior will not occur if the perturbation of noise is too strong. In our model, we assume that the measurement noise is multiplicative. The constructive role of multiplicative measurement noise can be explained by regarding the noise term as additional stochastic interactions imported into the deterministic system. In this way, individuals in our model are subjected to two interactions, a deterministic attractive coupling and a white-noise-based stochastic coupling. The stochastic interactions in our model are attractive (positive) at some times and repulsive (negative) at other times. The transition between attractive and repulsive interactions is stochastic. With the stochastic coupling, the interactions between individuals are fluctuating. This is in accord with many real biological systems, e.g., fish school [26] and bird flocks [27]. Although the repulsive interactions caused by the stochastic coupling can weaken the mutual attraction between individuals, the attractive interactions from both deterministic and stochastic couplings will drive the group to the coordinated collective motion. And the stochastic interactions in model (5) will become weak and eventually disappear when the group gradually converges to the consensus.

To get some insight into the mechanism of the constructive role of noise, we carried out the theoretical analysis, and sufficient conditions for the consensus are obtained. It is worthwhile to note that the consensus dynamics of model (5) depends on both the deterministic and the stochastic couplings. Thus, in order to realize the consensus, some conditions on the Laplacian matrix of the network topologies and the noise intensity matrix should be satisfied. And we also show that the multiagent system cannot reach the consensus if the conditions in Proposition 2 are satisfied. The influence of network topologies on the convergence rate of the consensus has also been investigated. It shows that, for the multiagent networks with small-world topologies, the larger the mean degree, the faster the convergence rate. And for the multiagent networks with scale-free topologies, a smaller degree exponent is better for the system to reach the consensus. Furthermore, our results show that the convergence time is a decreasing function of the network size.

Our paper highlights the constructive role of multiplicative measurement noise in the emergence of the consensus of multiagent networks. In addition to the consensus dynamics, flocking is another important collective behavior exhibited by many living beings, such as birds and fish. In 1995, Vicsek *et al.* numerically investigated the influence of additive measurement noise on the flocking dynamics of a three-dimensional self-propelled particle model. It was shown that the flocking occurs only when the strength of noise is sufficiently small [2]. However, the role of multiplicative measurement noises in the flocking dynamics of Vicsek type models is still not clear. We hope that the proposed analytical framework in this paper can help to investigate the constructive role of noise in the flocking dynamics.

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