Detecting nonlinear stochastic systems using two independent hypothesis tests

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Various systems in the real world can be nonlinear and stochastic, but because nonlinear time series analysis has been developed to distinguish nonlinear deterministic systems from linear stochastic systems, there have been no appropriate methods developed so far for testing the nonlinear stochasticity for a given system. Thus, here we propose a set of two hypothesis tests, one for the nonlinearity and one for the stochasticity, independent of each other. The test for the linearity is based on Fourier-transform-based surrogate data with a nonlinear test statistic, while the test for determinism depends on the theory of ordinal patterns or permutations recently developed intensively. We demonstrate the proposed set of tests with time series generated from toy models. In addition, we show that both a foreign exchange market and a temperature series in Tokyo could be nonlinear and stochastic, as well as sometimes with determinism beyond pseudoperiodicity.

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I. INTRODUCTION

Time series analysis is the first step for modeling a given system. At the beginning of the modeling, we would like to know the following properties for properly choosing a class of such a model (Fig. 1): (i) whether a system is stationary or nonstationary, (ii) whether a system is deterministic or stochastic, and (iii) whether a system is linear or nonlinear. If the underlying dynamics is fixed throughout measurements, we call the underlying dynamics stationary; otherwise, we call it nonstationary. If the future state is defined as a function of the current state and/or the past states without probabilistic components, then the underlying dynamics is deterministic; otherwise, we call it stochastic. Namely, when the underlying dynamics is stochastic, there is at least a deviation between such a function and the future state due to some probabilistic component(s). In addition, we call the underlying system with determinism beyond pseudoperiodicity if one rejects the null hypothesis that a time series is a periodic signal perturbed by uncorrelated noise. If such a function is linear in terms of the current and/or past states, we call the underlying dynamics linear. Otherwise, we call it nonlinear.

For specifying a class of such a model rigorously statistically, hypothesis testing is necessary. In such a process, we should be aware of the difference between a statistic and a test [1]: A statistic is obtained from a dataset as a value characterizing a certain aspect of the dataset. A test has additionally a function of whether a null hypothesis, or a hypothesis in question, is likely to be satisfied or not by comparing the statistic with its distribution generated when we assume the null hypothesis. If the statistic deviates much from the distribution assuming the null hypothesis, we reject the null hypothesis. Otherwise, we cannot reject the null hypothesis. See Appendixes A and B for various hypothesis testings and other related analysis, respectively, in nonlinear time series analysis.

Suppose that a given time series is stationary, meaning that the underlying dynamics is fixed throughout the measurements. (If the time series is not stationary, we can detect the nonstationarity using the method of Ref. [1] or identify how the parameters for the underlying dynamics change during the measurements by using the methods described in Refs. [2,3], for example.) There are many methods characterizing nonlinear deterministic systems against linear stochastic systems using surrogate data [4–6] (compare region 1 with region 2 in Fig. 2; see also Appendix A for more surrogate data analysis). However, this main approach in the past cannot test nonlinear stochastic systems (region 3 in Fig. 2) because the test for linearity has been normally evaluated by using a test statistic characterizing the determinism.

Here, we propose to separate the test for linearity from the test for determinism so that the rejections for both tests imply that the underlying dynamics is nonlinear and stochastic. For the test for determinism, we use ordinal patterns [7,8] by extending the ideas in Refs. [9,10]. For the test for linearity, we use Fourier-transform-based surrogate data [4] with a nonlinear test statistic which is not directly linked with the deterministic properties. Therefore, the class of systems that we can identify is a subset of nonlinear and stochastic systems (region 3) as shown in the blue region in Fig. 2. By using pseudoperiodic surrogates together, we will add some depth to the results in our analysis.

II. PROPOSED METHODS

In this section, we propose a test for determinism and a test for linearity.

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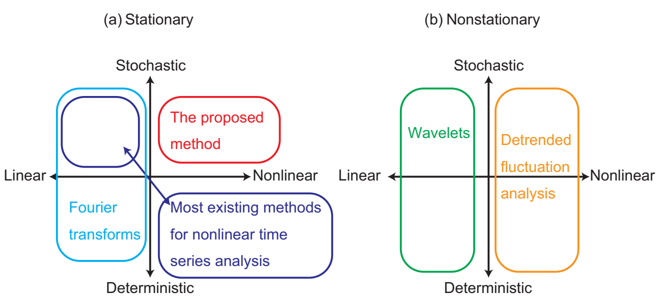

FIG. 1. The classifications of models depending on stationarity-nonstationarity, determinism-stochasticity, and linearity-nonlinearity. Associated methods for time series analysis are also annotated in the corresponding regions of these panels.

A. Test for determinism

Our test for determinism is constructed using ordinal patterns, or permutations [7,8]. The ordinal pattern $\pi(t)$ for a subsequence $x(t), x(t+1), \dots, x(t+L-1)$ of scalar time series $\{x(t)|t = 1, 2, \dots, T\}$ is defined by inequality relations $x(t+i_1) \leqslant x(t+i_2) \leqslant \dots \leqslant x(t+i_L)$ as $\pi(t) = (i_1, i_2, \dots, i_L)$, where $i_1, i_2, \dots, i_L$ are integers from 0 and $L-1$ and unique. Here, we use the ascending order $x(t+i) \leqslant x(t+j)$ of the corresponding time indices if $x(t+i) = x(t+j)$ for $i < j$. In Ref. [9], the authors proposed to count the number of appearing ordinal patterns $\{\pi(t)|t = 1, 2, \dots, T-L+1\}$ for estimating the topological entropy for the underlying dynamics. In particular, it is shown by Amigó and Kennel [9] that if the underlying dynamics is deterministic and expansive, and the given time series is sufficiently long, then the number of appearing ordinal patterns increases exponentially when the length of ordinal patterns increases (Theorem 2 of Ref. [9]). Based on this theorem, Amigó *et al.* [10] identified the existence of forbidden ordinal patterns to infer the determinism for the underlying dynamics.

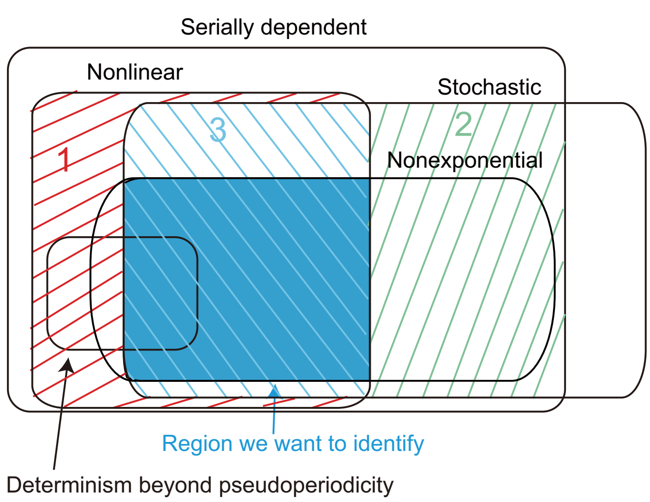

FIG. 2. Classification of dynamic systems. Region 1 corresponds to the class of nonlinear deterministic systems. Region 2 corresponds to the class of linear stochastic systems. Region 3 corresponds to the class of nonlinear stochastic systems. The blue region is the class of systems the proposed approach can identify. In addition, we show the class of systems that pseudoperiodic surrogates [11] can identify.

Therefore, here, we use the contraposition for this theorem: Namely, if the number of appearing ordinal patterns does not increase exponentially, then the underlying dynamics is either stochastic or nonexpansive under the condition that the given time series is sufficiently long. Hence, our test for determinism is constructed in three steps: First, we assume the null hypothesis that the underlying dynamics is deterministic and expansive. Then, a logical consequence is that the number of appearing ordinal patterns $N(l)$ increases exponentially (see the above argument related to Ref. [9]). Thus, second, we test whether the number of appearing ordinal patterns $N(l)$ increases exponentially as the length l of ordinal patterns is varied between $l_{\min}$ and $l_{\max}$. If $\ln N(l)$ can be fitted by a linear model depending on l better than by the corresponding quadratic model of l, then we regard that $N(l)$ increases exponentially depending on l. To judge which fitting is better, we use the F distribution [12] with $(l_{\max} - l_{\min} - 1, l_{\max} - l_{\min} - 2)$ degrees of freedom. (Namely, if the null hypothesis is true, $\ln N(l)$ will fluctuate around $\alpha + \beta l$, where coefficients α and β are specified during this procedure.) We set $l_{\min} = 4$ and $l_{\max} = 8$ in this paper. In addition, we use 5% as a significance level. If the linear model is inferior to the quadratic model even if we take into account the 5% significance level, we call the underlying dynamics nonexponential (see Fig. 2). Otherwise, then we call the underlying dynamics exponential (see Fig. 2).Third, we exclude the case where the underlying dynamics is expansive. A dynamic system is called expansive if two neighboring orbits are separated by more than a predefined distance certainly after some iterations (the number of iterations is not specified here). For showing the expansiveness, we use a recurrence plot [13,14]. The condition for the expansiveness is equivalent to the condition for the sensitive dependence on initial conditions [15]: In a recurrence plot, each diagonal line has one or more interruptions. Therefore, if we can show that the given time series satisfies the condition of the sensitive dependence on initial conditions, we can exclude the possibility for the nonexpansiveness. In the examples below, we use the first 10 000 points to obtain a recurrence plot so that 5% of places excluding the central diagonal line have points plotted, and we evaluate the expansiveness. (This third step is not strictly a hypothesis test yet.)

B. Test for linearity

Our test for linearity is based on an extended version [4] of iterative adjusted Fourier transform surrogates [5]. In the method of Ref. [4], or the symmetrized truncated Fourier transform surrogates, the phases for the top 99% of high-frequency components only are randomized so that trends of the original time series as well as the power spectrum are preserved. Thus, the null hypothesis is that the underlying dynamics is linear noise with trends and is transformed by a static nonlinear monotonic transformation.

Here, we use the average of $x(t)^2 x(t+1)^2$ over the time series as a test statistic. The important point is that this quantity does not have any meaning in terms of determinism.

Therefore, we can evaluate simply the deviation from the linearity by using this test statistic.

We generated 39 such surrogate data unless otherwise mentioned. Thus, the significance level in this paper is $2/(39+1) = 0.05$. Thus, if the average of $x(t)^2x(t+1)^2$ for the original time series is out of the range specified with the corresponding 39 surrogate data, then we reject the null hypothesis that the underlying dynamics is linear noise, implying the nonlinearity for the underlying dynamics.

III. EXAMPLES

We verified the proposed set of methods using nine examples. Then, we applied the proposed set of methods to real data of foreign exchange market and temperature series in Tokyo. In this section, we also use pseudoperiodic surrogates [11] with correlation dimensions [16] to test whether the underlying dynamics has the determinism beyond pseudoperiodicity. For each time series, we used the first 10 000 time points to generate 39 pseudoperiodic surrogates [11] so that each statistic has a significance level of 5%. We used three-dimensional delay coordinates with delay 8 to generate pseudoperiodic surrogates throughout the paper. Since we used 19 dimensions between 2 and 20 and each test statistic has a 5% chance of rejection, we need four rejections out of 19 dimensions to formally reject, as a total, the null hypothesis that a time series is a periodic signal perturbed with uncorrelated noise.

A. Test examples: Maps

We used the following seven models based on maps for testing the validity for the proposed set of methods. Although we understand that the accuracies for the parameters are different from model to model, we use the parameters as described here so that these models can retain typical properties which were demonstrated in their original papers.

1. Autoregressive linear model

The first model is the autoregressive linear model [17]. We used the following model,

$$x(t+1) = -0.8x(t) + \eta_t, \quad (1)$$

where η_t follows the Gaussian distribution with mean 0 and standard deviation 1. This model corresponds to a case where the underlying dynamics is linear and stochastic. We generated 20 time series of length 1 000 000 by using different initial conditions and stochastic components prepared randomly, unless otherwise mentioned.

Some of the results are presented in Figs. 3, 4(a), 5(a), and 6(a). The logarithm for the number of appearing permutations $\ln N(l)$ is concave, meaning that the underlying dynamics should be classified as nonexponential (Fig. 3). In addition, each diagonal line for the recurrence plot has at least an interruption [Fig. 4(a)], meaning that the underlying dynamics is expansive and thus stochastic. Moreover, the statistic $E[x(t)^2x(t+1)^2]$ obtained from the original data is between the minimum and the maximum of those obtained from the surrogate data, meaning that the underlying dynamics is likely to be linear [Fig. 5(a)]. Furthermore, there is no determinism beyond pseudoperiodicity because there was no rejection on Fig. 6(a).

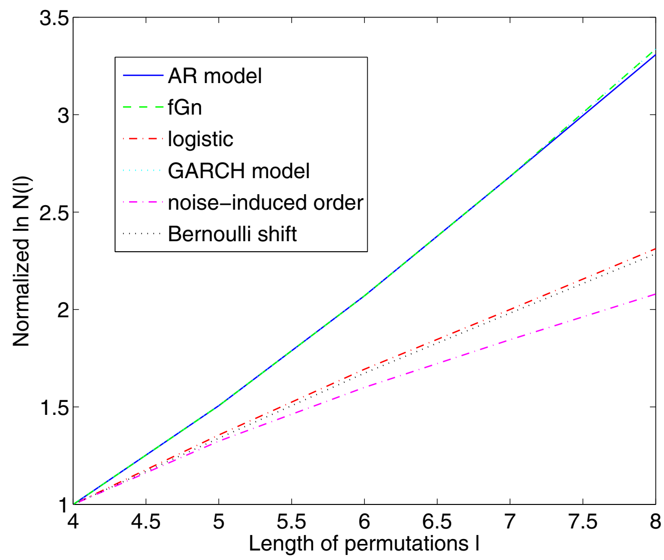

FIG. 3. The increases of the number of appearing permutations $N(l)$ given the length l of permutations. In this figure, the values $\ln N(l)$ for each model are aligned so that the values for $l = 4$ become the same point. To evaluate the straightness, concavity, or convexity of these lines, use a straight edge.

2. Fractional Gaussian noise

The second model is the fractional Gaussian noise (fGn) [18]. The fractional Gaussian noise $\{\xi_t, t \geqslant 0\}$ with the Hurst

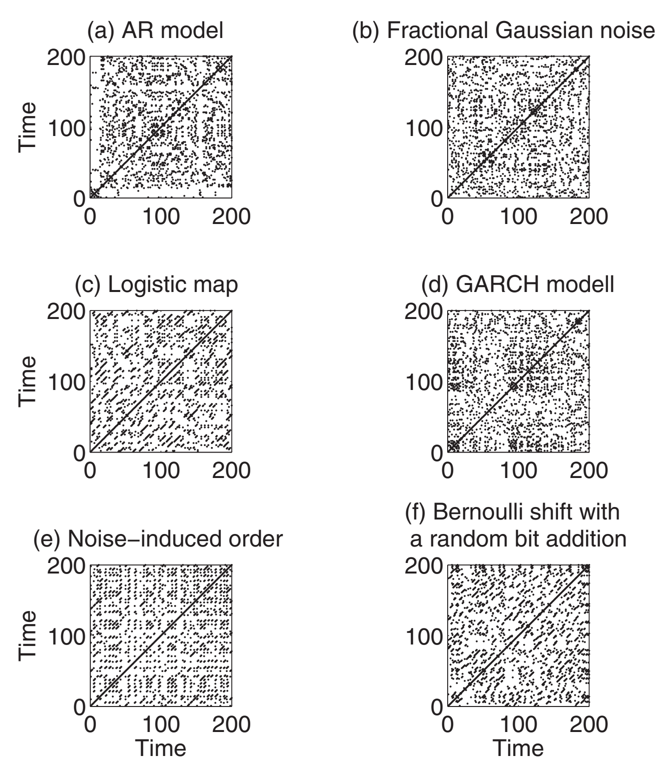

FIG. 4. Parts of recurrence plots for five models.

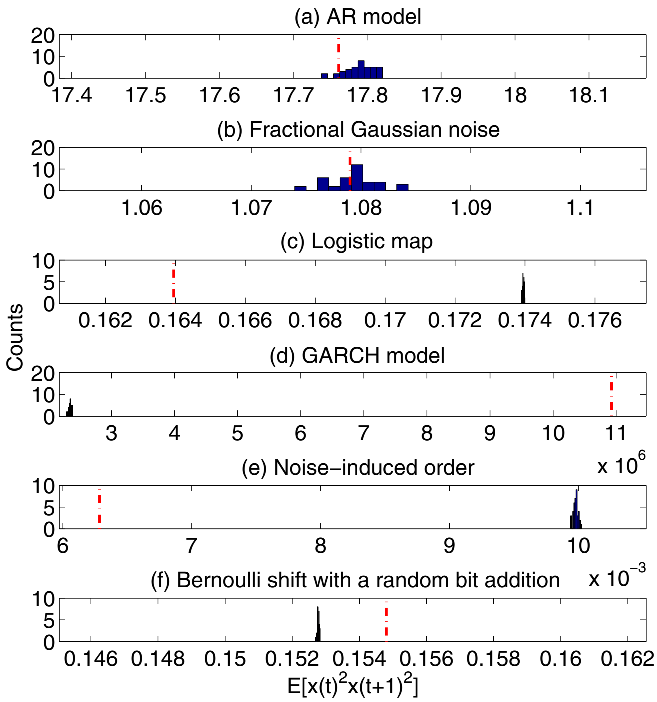

FIG. 5. The results for surrogate data analysis on testing linearity. In each panel, the histogram shows the distribution for the surrogate data of Ref. [4] and the red dash-dotted vertical line corresponds to the value of the test statistic for the original data.

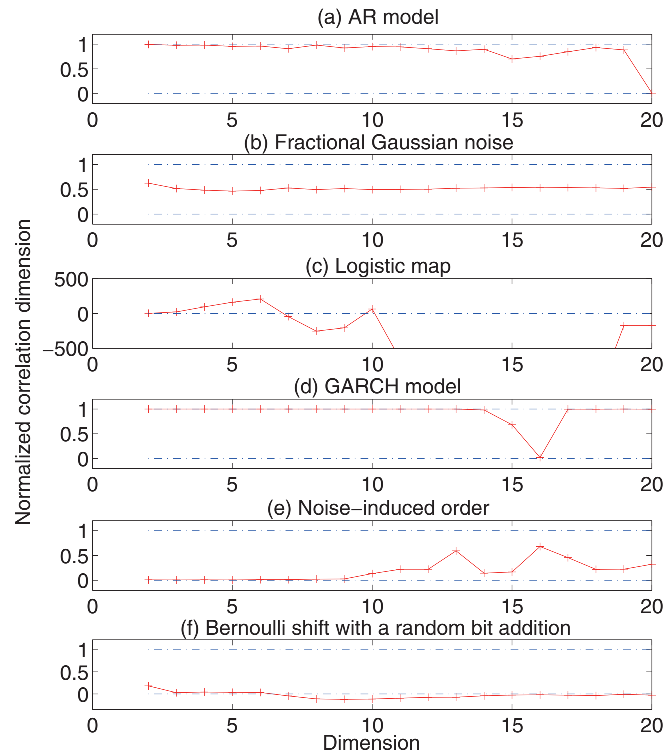

FIG. 6. The results for surrogate data analysis on testing determinism beyond pseudoperiodicity. In each panel, the test statistics are normalized so that the minimum and the maximum for the surrogate data become 0 and 1, respectively.

parameter $H \in (0, 1)$ has the following property in the covariance function [18]:

$$E[\xi_1 \xi_{n+1}] = \tfrac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}]. \quad (2)$$

When we generated each time series, we chose a Hurst parameter $H \in [0.1, 0.9]$ randomly from the uniform distribution. We used the program in Ref. [18] to generate 20 time series of length 2^{20} using different realizations of stochastic components initially if the length is not explicitly specified. This second model is also a case where the underlying dynamics is linear and stochastic.

Some of the results are shown in Figs. 3, 4(b), 5(b), and 6(b). In this case, the underlying dynamics is nonexponential (Fig. 3), expansive [Fig. 4(b)], and thus stochastic, while it is linear [Fig. 5(b)]. In this case, there is no determinism beyond pseudoperiodicity [Fig. 6(b)].

3. The logistic map

The third model is the logistic map [19]. We used the following model:

$$x(t+1) = 3.8x(t)[1 - x(t)]. \quad (3)$$

By using different initial conditions prepared randomly, we generated 20 time series of length 1 000 000 unless otherwise mentioned. This model corresponds to a model of nonlinear determinism.

Some of the results are presented in Figs. 3, 4(c), 5(c), and 6(c). These figures show that the underlying dynamics is exponential (Fig. 3) and nonlinear [Fig. 5(c)], agreeing with the properties for the logistic map. In addition, there is the determinism beyond pseudoperiodicity because there were a sufficient number of rejections happening in Fig. 6(c), which is consistent with the nature of deterministic chaos for the logistic map.

4. GARCH model

The fourth model is the generalized autoregressive conditional heteroscedasticity (GARCH) model [20]. We used the model and parameters shown in Ref. [20], namely,

$$h(t) = 14.4038 + 0.095\epsilon_{t-1}^2 + 0.895h(t-1), \quad (4)$$

$$\epsilon_t = \sqrt{h(t)}\eta_t, \quad (5)$$

$$y(t) = 0.409933 + 0.095y(t-1) + \epsilon_t, \quad (6)$$

where η_t follows the Gaussian distribution of mean 0 and standard deviation 1. We initialized the dynamics by $y(0) = 0$, $h(0) = 1$, and $\epsilon_t = 0$. By observing $y(t)$, we generated 20 time series of length 1 000 000 by using different realizations of stochastic components unless otherwise mentioned. This model corresponds to a case where the underlying dynamics is nonlinear and stochastic.

Some of the results are shown in Figs. 3, 4(d), 5(d), and 6(d). These figures mean that the underlying dynamics is nonexponential (concave; Fig. 3) and expansive [Fig. 4(d)], leading to the stochasticity, while the underlying dynamics is nonlinear [Fig. 5(d)]. Furthermore, there is no

TABLE I. Results for time series generated from maps. These results are for time series of length 1 000 000 (2^{20} for the fractional Gaussian noise (fGn) case) without observational noise. The AR model and fGn are linear and stochastic, and thus we were expecting them to be classified to linear stochastic (nonlinear = 0, nonexp. = 1, and expan. = 1) or linear exponential (nonlinear = 0 and nonexp. = 0). The logistic map is nonlinear deterministic, and thus we were expecting it to be classified into the category of nonlinear exponential (nonlinear = 1 and nonexp. = 0). The GARCH model and noise-induced order are mostly nonlinear and stochastic, and thus we were expecting initially them to be classified into nonlinear stochastic (nonlinear = 1, nonexp. = 1, expan. = 1) or nonlinear exponential (nonlinear = 1 and nonexp. = 0). Therefore, the results shown in this table mean that time series generated from these model examples were classified almost perfectly. We note here that the lines with count 0 are omitted here so that we can save space.

Tests				Models					
Nonlinear	Deter. beyond pseudo-period	Nonexp.	Expan.	AR model	fGn	logistic	GARCH	Noise-induced order	Bernoulli shift
1	0	0	1	0	0	9	0	6	2
1	1	0	1	0	0	11	0	0	2
0	0	1	1	19	19	0	0	0	0
1	0	1	1	1	1	0	15	12	10
1	1	1	1	0	0	0	5	2	6
			Total	20	20	20	20	20	20

determinism beyond pseudoperiodicity [Fig. 6(d)]. Therefore, these parts are consistent with the properties of the GARCH model.

5. Noise-induced order

The fifth model is the model of noise-induced order [21]. We used the following model:

$$x(t+1) = \begin{cases} \{-[0.125 - x(t)]^{1/3} + 0.506\,735\,7\}\exp[-x(t)] + b + \phi_t & \text{if}\, x(t) < 0.125, \\ \{[x(t) - 0.125]^{1/3} + 0.506\,073\,57\}\exp[-x(t)] + b + \phi_t & \text{if}\, 0.125 < x(t) < 0.3, \\ 0.121\,205\,692 \times \left\{10x(t)\exp\left[\frac{10}{3}x(t)\right]\right\}^{19} + b + \phi_t, & 0.3 < x(t), \end{cases} \quad (7)$$

where $b = 0.023\,288\,527\,9$ and each ϕ_t follows the uniform distribution between $-10^{-2.5}$ and $10^{2.5}$. We generated 20 time series of length 1 000 000 by using different realizations of stochastic components. This model also corresponds to a case where the underlying dynamics is nonlinear and stochastic.

One of the results is presented in Figs. 3, 4(e), 5(e), and 6(e). The function $\ln N(l)$ is convex (Fig. 3), meaning that the underlying dynamics is nonexponential. Together with the expansiveness [Fig. 4(e)], the underlying dynamics should be classified as stochastic. On the other hand, the underlying dynamics is nonlinear because the statistic $E[x(t)^2x(t+1)^2]$ obtained from the original data is out of the interval specified with the minimum and the maximum of 39 surrogates [Fig. 5(e)]. Moreover, there is no determinism beyond pseudoperiodicity [Fig. 6(e)]. As a total, the underlying dynamics is nonlinear and stochastic, which is consistent with the assumptions for the above model.

6. The Bernoulli shift with a random bit addition

Our sixth model is that of the Bernoulli shift. To run the Bernoulli shift appropriately with a digital computer, we need to add a random bit as follows so that it does not stick to zero:

$$x(t+1) = 2x(t) \bmod 1 + 2^{-20}b_t, \quad (8)$$

where b_t takes 0 or 1 randomly with an equal probability of 0.5.

Some of such results are shown in Figs. 3, 4(f), 5(f), and 6(f). The logarithm for the number of appearing ordinal patterns increases in a slightly convex manner (Fig. 3). The underlying dynamics is expansive [Fig. 4(f)]. Therefore, the underlying dynamics is stochastic. Moreover, the underlying dynamics is likely to be nonlinear [Fig. 5(f)]. In addition, there is the determinism beyond pseudoperiodicity [Fig. 6(f)]. Therefore, based on the time series generated from the Bernoulli shift, the underlying dynamics is classified as nonlinear and stochastic with the determinism beyond pseudoperiodicity. This classification is consistent with our assumptions for generating the time series.

7. Results of each single example up to the previous subsubsections

By combining the other 19 results for each model, the overall results so far are summarized in Table I. The proposed set of methods worked well with two exceptions: The first exception is that a time series for each of AR model and fGn was wrongly classified as nonlinear stochastic. But this case is a chance level. The second exception is that 6 out of 20 time series for noise-induced order were classified as nonlinear and exponential. This case implies that when a time series is classified as stochastic, it really came from a stochastic source, while when a time series is classified as with an exponential increase of appearing permutations,

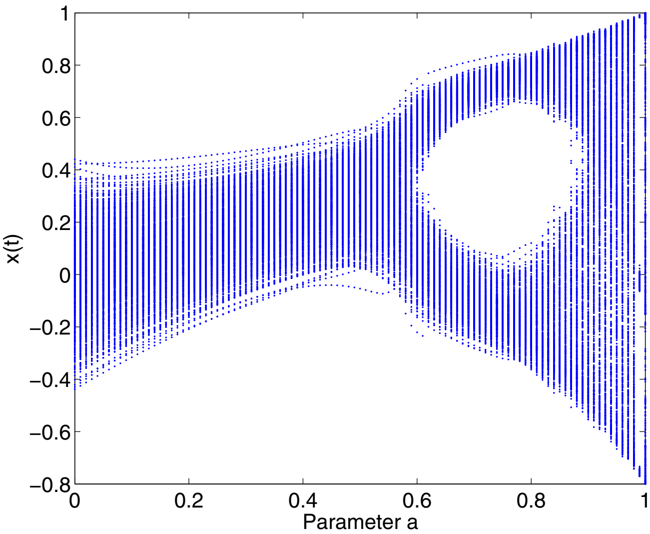

FIG. 7. Bifurcation diagram for the AR model converted into the logistic map.

there are some cases where the underlying dynamics may still contain stochastic components. The GARCH model and the Bernoulli shift were statistically classified as stochastic with the determinism beyond pseudoperiodicity. This classification could be interesting for further studies. The other time series for these numerical experiments were finely correctly.

By using the pseudoperiodic surrogates together with the proposed methods, we can have more depths for obtained results.

8. AR model converted into the logistic map

In addition, we constructed a model with a parameter which connects a linear stochastic model with a nonlinear deterministic model. Namely, we consider the following model:

$$x(t+1) = a(1 - 1.8x(t)^2) + (1-a)[-0.8x(t) + \eta_t], \quad (9)$$

where η_t follows the Gaussian distribution with mean 0 and standard deviation 1. When $a = 0$, the above model coincides with an autoregressive linear model, while the model matches with a logistic map when $a = 1$. In between, we may observe nonlinear and stochastic behavior. We generated a time series for each parameter a. A numerically obtained bifurcation diagram for the model of Eq. (9) is shown in Fig. 7.

Figure 8 shows the results when we applied the proposed set of methods to the model of Eq. (9). These panels for the figure show that the underlying dynamics for Eq. (9) is nonlinear and stochastic when the parameter a is between 0.13 and 0.4, at 0.63 and 0.66, and between 0.89 and 0.98. In particular, the intervals of 0.13 and 0.4 and of 0.89 and 0.98 correspond to the cases where the attractor of Eq. (9) is represented by an interval for the values of $x(t)$. On the other hand, the interval for the parameter a between 0.41 and 0.88 seems to be the region where the period 2 behavior is dominant. This behavior will explain why there is a certain difference between the interval and the left and right sides of the interval. This behavior is also consistent with our

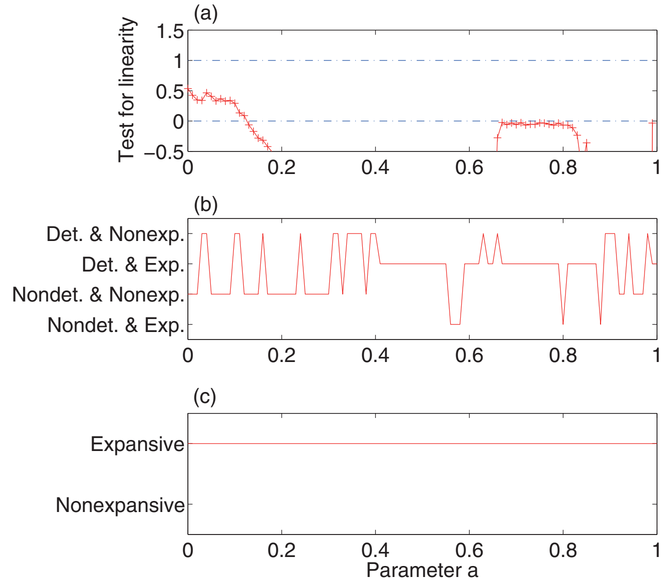

FIG. 8. Results for the AR model converted into the logistic map. Panel (a) shows the results for the test of linearity. Panel (b) shows the combined results for surrogate data analysis on determinism beyond pseudoperiodicity and the test of exponential growth for the number of appearing permutations. Panel (c) shows the results on the test of expansiveness. In panel (a), the test statistic for each a is scaled so that the minimum and the maximum for the surrogate data become 0 and 1, respectively.

previous observation that sometimes a stochastic source may be classified as with an exponential increase of appearing ordinal patterns.

With the pseudoperiodic surrogates, the underlying dynamics was most likely to be classified with the determinism beyond pseudoperiodicity when $0.4 \leqslant a \leqslant 0.87$ and some other discrete values of a. Thus, a mixture of nonlinear dynamics, stochasticity, and determinism beyond pseudoperiodicity was observed only at some discrete values of a such as 0.16, 0.24, 0.31, 0.32, 0.34, 0.35, 0.36, 0.37, 0.39, 0.40, 0.63, 0.66, 0.89, 0.90, 0.91, 0.94, and 0.98 in our simulations here.

B. Example: Flows

Moreover, we tested the proposed set of methods with flows, or systems where time is continuous.

1. The Rössler models

We followed Ref. [11] to use the Rössler model [22] under conditions that dynamical noise is added or not and that the original underlying dynamics without dynamical noise is chaotic or periodic. The Rössler model follows the next equations:

$$dx = -(y+z)dt + dW_x(t), \quad (10)$$

$$dy = (x+ay)dt + dW_y(t), \quad (11)$$

$$dz = [2 + z(x-4)]dt + dW_z(t), \quad (12)$$

TABLE II. Results for time series generated from flows of the Rössler models.

Tests				Stochastic		Deterministic	
Nonlinear	Deter. beyond pseudo-period	Nonexp.	Expan.	Chaotic	Periodic	Chaotic	Periodic
1	1	0	0	0	0	0	7
1	0	0	1	0	0	10	0
0	1	0	1	0	0	2	0
1	1	0	1	0	0	8	13
1	0	1	1	14	19	0	0
1	1	1	1	6	1	0	0
			Total	20	20	20	20

where we use the parameter $a = 0.398$ for the chaotic case and the parameter 0.3909 for the periodic case. We set $dt = 0.2$ unit times. When we apply the dynamical noise, we generated $dW_x(t)$, $dW_y(t)$, and $dW_z(t)$ so that each of them follows the Gaussian distribution of mean 0 and standard deviation 0.05. Otherwise, we set $dW_x(t) = dW_y(t) = dW_z(t) = 0$. For analyzing generated time series, we used the first minimum for the mutual information to decide a time interval with which we evaluate the ordinal patterns we discussed above.

The results are summarized in Table II. The results demonstrate that the proposed methods work well even for flows. The misclassifications for two cases for chaotic and deterministic models are a chance level.

2. *Lorenz '96II model*

We also tested the proposed set of methods using Lorenz '96II model [23,24], which is a 240-dimensional model for the atmosphere. The model equations are as follows:

$$\dot{x}_i = -x_{i-2}x_{i-1} + x_{i-1}x_{i+1} - x_i + F - \frac{h_x c}{b}\sum_{j=1}^{n} y_{j,i}, \quad (13)$$

$$\dot{y}_{j,i} = -cby_{j+1,i}y_{j+2,i} + cby_{j-1,i}y_{j+1,i} - cy_{j,i} + \frac{h_y c}{b}x_i, \quad (14)$$

with periodic boundary conditions of

$$x_i = x_{i+m} \quad (15)$$

and

$$y_{j+n,i} = y_{j,i+1}, \quad (16)$$

where we set $m = 40$, $n = 5$, $F = 8$, $b = 10$, $c = 10$, $h_x = 1$, and $h_y = 1$. The variables x_i correspond to the higher layer of the atmosphere, while the variables $y_{j,i}$ correspond to the lower layer of the atmosphere. We observed $y_{1,1}(t)$ every 0.01 unit times for yielding 20 time series of length 1 000 000 consecutively from a random initial condition.

In this case, our numerical results show that this high-dimensional dynamics is classified as nonlinear and exponential with the determinism beyond pseudoperiodicity (see Table III). Therefore, the proposed set of methods seem to work well even for a high-dimensional dynamics.

C. Dataset of foreign exchange market

We applied the proposed approach to a dataset of a foreign exchange market compiled by the Thomson Reuters Corporation. This dataset covers the quotes for exchanges between the United States dollar and the Japanese yen between January 2006 and December 2015. We prepared a series of interquote intervals and applied the above two tests.

The results are presented in Table IV. The results mean that the underlying dynamics for the interquote intervals is mostly nonlinear and stochastic, and sometimes statistically significantly with the determinism beyond pseudoperiodicity.

D. Temperature at Tokyo

We also analyzed the real dataset for the temperature at Tokyo for every minute between January 2006 and December 2015. We followed the procedures similar to the flows above.

The results presented in Table IV mean that the underlying dynamics for the temperature was mostly nonlinear and stochastic with the determinism beyond pseudoperiodicity.

TABLE III. Results for time series generated from Lorenz '96II model.

Tests				Data	
Nonlinear	Deter. beyond pseudo-period	Nonexp.	Expan.	Without obs. noise	With obs. noise
1	1	0	1	20	20
			Total	20	20

TABLE IV. Results for time series in the real world. For the dataset of foreign exchange market, we extracted a segment of length 1 000 000 and applied the proposed set of methods for each segment. For the dataset of temperature at Tokyo, we extracted a part of the dataset every 2 years and applied the proposed set of methods.

Tests				Data	
Nonlinear	Deter. beyond pseudo-period	Nonexp.	Expan.	USD/JPY (2006–2015)	Temperature at Tokyo (2006–2015)
0	0	1	1	7	0
1	0	1	1	137	1
1	1	1	1	22	4
			Total	166	5

IV. DISCUSSIONS

A. Robustness for the proposed methods

We evaluated robustness for the proposed methods from two viewpoints: observational noise and length of time series.

When we added 5% observational noise, we almost correctly identified nonlinearity and stochasticity for the AR model, the fractional Gaussian noise, the logistic map, the GARCH model, and the model of noise-induced order (see Table V). Even if the underlying dynamics is high dimensional as the Lorenz '96II model with 5% observational noise, then these time series were classified as exponential and nonlinear, which is consistent with nonlinear deterministic. If the level of observational noise becomes as large as the original signal, the proposed methods may classify such a time series as a stochastic one.

When we shortened the time series to 100 000, we observed that the stochasticity cannot be identified well in the AR model and was misidentified in the logistic map (Table VI). Thus, the time series needed are relatively long for the proposed methods to work robustly.

B. Comparison with the methods in the existing literature

There are many methods that characterize stochasticity for the underlying dynamics compared with determinism [10,25–30], but these methods are either qualitative or quantitative compared with independent noise or linear surrogates. Therefore, the proposed test for stochasticity would be uniquely applied for identifying more general stochastic systems even including nonlinearity, although the systems should have a number of permutations increasing in a nonexponential manner.

C. Limitations

The limitation of the proposed methods is that we need relatively long time series due to the combinatorial nature of ordinal patterns. If we set $l_{\min}$ and $l_{\max}$ smaller, we may be able to shorten the length of time series. This point should be investigated further in the future.

Another limitation is that the third step for the test of determinism is not a hypothesis test yet. We should implement this step, in the future, as a hypothesis test so that we can more rigorously judge whether the underlying dynamics is expansive statistically.

Currently, we cannot analyze a nonautonomous system driven by noise since we regard such a system with a time-varying component and thus nonstationary. In the future, we will try to extend the current machinery toward such a direction.

TABLE V. Results for time series generated from maps. These results are for time series of length 1 000 000 (2^{20} for the fractional Gaussian noise case) with 5% observational noise.

Tests				Models					
Nonlinear	Deter. beyond pseudo-period	Nonexp.	Expan.	AR model	fGn	logistic	GARCH	Noise-induced order	Bernoulli shift
1	0	0	1	0	0	6	0	1	19
1	1	0	1	0	0	14	0	0	1
0	0	1	1	18	19	0	0	0	0
1	0	1	1	1	1	0	15	18	0
1	1	1	1	1	0	0	5	1	0
			Total	20	20	20	20	20	20

TABLE VI. Results for time series generated from maps. These results are for time series of length 100 000 (2^{17} for the fractional Gaussian noise case) with 5% observational noise.

Tests				Models					
Nonlinear	Deter. beyond pseudo-period	Nonexp.	Expan.	AR model	fGn	logistic	GARCH	Noise-induced order	Bernoulli shift
0	0	0	1	19	4	0	0	0	0
1	0	0	1	1	1	0	0	4	0
0	0	1	1	0	13	0	0	0	0
1	0	1	1	0	1	4	14	16	18
0	1	1	1	0	1	0	0	0	0
1	1	1	1	0	0	16	6	0	2
			Total	20	20	20	20	20	20

D. Implications for real datasets

In the field of econophysics, there are many papers in the existing literature where researchers employ the Fokker-Planck equation for quantifying the deterministic and stochastic parts for the underlying dynamics [31,32]. In addition, there is a work showing the nonlinearity of economic indices [33]. Therefore, our finding that a foreign exchange market sometimes contains some nonlinear deterministic and stochastic components itself is not new and is consistent with the statements in the existing literature, although we believe that our approach is a demonstration that the underlying dynamics of a foreign exchange market is nonlinear and has a time-dependent mixture of deterministic and stochastic nature through rigorous hypothesis tests. Since our previous work on a similar dataset shows that a foreign exchange market has a chaotic nature [34] from the viewpoint of a marked point process, we should investigate more closely whether the underlying dynamics for a foreign exchange market is stochastic chaos [35].

As for the weather data, Paluš and Novotná [36] showed in 1994 using daily measurements that the temperature and the pressure recordings are linear and nonlinear, respectively, using Fourier-transform based-surrogate data [6] with an extended version of mutual information, which they call the redundancy. Judging from our current results as well as our previous results [37], daily measurements might be too coarse to show that their underlying dynamics is nonlinear and stochastic. There was an observation that the monthly Southern Oscillation Index is more stochastic rather than chaotic using correlation dimensions and Lyapunov exponents [38]. In addition, there is a beautiful theory explaining the annual cycles of climate by the stability for the underlying stochastic dynamics [39]. Thus, the stochastic aspects on the weather should be investigated further to, for example, integrate more renewable energy resources into grids [40] and improve the quality of our lives.

Therefore, we should investigate more deeply in the future when there can coexist the nonlinearity, stochasticity, and determinism beyond pseudoperiodicity.

E. Conclusions

In sum, we have proposed two methods, one which tests linearity and one which tests determinism. The test of linearity uses Fourier-transform-based surrogate data with a nonlinear statistic, while the test of determinism judges whether the number of appearing ordinal patterns increases exponentially. The set of methods needs a long time series of length 1 000 000, but the set of methods may be used under 5% observational noise. We showed that a foreign exchange market is mostly nonlinear and a time-fluctuating mixture of stochasticity and determinism beyond pseudoperiodicity, while the temperature in Tokyo is also nonlinear and quite often contains both stochastic components as well as deterministic components which affect beyond pseudoperiodicity. The coexistence of nonlinearity, stochasticity, and determinism beyond pseudoperiodicity were formally shown for a foreign exchange market and a weather variable using hypothesis tests. Therefore, the proposed set of methods and the method of Ref. [11] are complementary because the proposed set of methods is looking for an intersection between nonlinearity and stochasticity, while the method of Ref. [11] is trying to narrow down the class of nonlinear systems by looking for the determinism beyond pseudoperiodicity as a subset of the nonlinear class. Hence, if we use these methods together, the methods will help each other to provide the deeper results such as formally identifying a mixture of stochasticity and determinism beyond pseudoperiodicity, only one of which cannot characterize the GARCH model correctly [41]. The contents of this paper have been substantially extended from the contents [42] we presented at a conference in 2018.

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APPENDIX A

In this Appendix, we summarize surrogate data analysis in the context of nonlinear time series analysis. Surrogate data analysis was used by Osborne *et al.* [43] for testing the nonlinearity for the underlying dynamics of a given time series. A test of nonlinearity using surrogate data has been extended further in Refs. [4–6] toward a nonstationary time series [44]. Although all of the above approaches are based on Fourier transform, there are some approaches that are based on wavelet transform (for example, see Ref. [45]). There is even a method for testing the linearity that does not use the Fourier transform or the wavelet transform [46].

There are some surrogate data that are testing the properties other than the linearity for the underlying dynamics, for example, that of Ref. [47] for serial dependence. This approach has been extended to test short-term dynamics by small shuffle surrogates [48]. The other null hypotheses we can find in the existing literature include (i) there is no determinism beyond pseudoperiodicity [11,49], (ii) the underlying dynamics is stationary [50], (iii) there is no singularity within an oscillation [51], (iv) only short-term firing rates have meaning [52], and (v) two time series are independent [53].

There is another way to use surrogate data, which is to obtain a likely distribution for a statistic. This way of surrogate data was used in Ref. [54] for multiscale processes.

APPENDIX B

Advancements for nonlinear time series analysis in the past three decades can be also observed in characterizing nonstationarity and fractalness. The key achievement is the detrended fluctuation analysis [55,56] for characterizing long-term dependence in a nonstationary time series. Please see Ref. [57] if you are interested in further recent developments in this direction. Since the current work is focusing on stationary time series, we cannot relate the current work with the targets for the detrended fluctuation analysis, which is originally for nonstationary time series (see Fig. 1). Thus, we will attempt to include nonstationary time series within our targets and use the method of the detrended fluctuation analysis within our framework in the future communication.

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