# Random walks on intersecting geometries

Reza Sepehrinia,[1,*] Abbas Ali Saberi,[1,2,†] and Hor Dashti-Naserabadi[3]
[1]Department of Physics, University of Tehran, P. O. Box 14395-547, Tehran, Iran
[2]Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln, Germany
[3]School of Physics, Korea Institute for Advanced Study, Seoul 02455, Korea

(Received 20 June 2019; published 29 August 2019)

We present an analytical approach to study simple symmetric random walks on a crossing geometry consisting of a plane square lattice crossed by $n_l$ number of lines that all meet each other at a single point (the origin) on the plane. The probability density to find the walker at a given distance from the origin either in a line or in the plane geometry is exactly calculated as a function of time $t$. We find that the large-time asymptotic behavior of the walker for any arbitrary number $n_l$ of lines is eventually governed by the diffusion of the walker on the plane after a crossover time approximately given by $t_c \propto n_l^2$. We show that this competition can be changed in favor of the line geometry by switching on an arbitrarily small perturbation of a drift term in which even a weak biased walk is able to drain the whole probability density into the line at long-time limit. We also present the results of our extensive simulations of the model which perfectly support our analytical predictions. Our method can, however, be simply extended to other crossing geometries with a single common point.

DOI: 10.1103/PhysRevE.100.022144

## I. INTRODUCTION

Random walks (RWs) are ubiquitous models of stochastic processes playing an essential role in many challenging problems in probability and statistical physics [1–4]. For random walks, the probability density $\rho(r,t)$ to find a walk at time $t$ at a site with distance $r$ from its origin obeys the scaling collapse [5]

$$\rho(r,t) \propto t^{-d_f/d_w} f(r/t^{1/d_w}), \tag{1}$$

with the scaling variable $r/t^{1/d_w}$, where $d_f$ is the dimension of the underlying (possibly fractal) network. On a lattice with translational invariance symmetry in any integer spatial dimension $d_f = d$, it has been shown that the walk is always purely diffusive with $d_w = 2$, with a Gaussian scaling function $f$, which has been the content of many classic textbooks on random walks and diffusion [1,6]. The scaling relation (1) still remains valid when translational invariance is blurred in certain ways or the network is fractal (i.e., for noninteger $d_f$), for which anomalous diffusion with $d_w \neq 2$ may arise in various transport processes [5,7–11].

Using Eq. (1), it is now straightforward to conceive Pólya's recurrence theorem [12] that a simple symmetric RWs on $\mathbb{Z}^d$ lattice is recurrent in $d \leqslant 2$ but transient in $d \geqslant 3$. Also widely known is that the transition between recurrence and transience occurs precisely at $d = 2$ (or more accurately at the spectral dimension $d_s = 2$) rather than at some fractal dimension $2 < d_f < 3$. In this sense, $d = 2$ is the "critical dimension" for intersection of a two-dimensional set (i.e., the path of RWs) and a zero-dimensional set (the origin). Moreover, it is known [13] that the scaling limit (i.e., the limit that the lattice spacing is sent to zero) of the simple RWs in $d$ dimensions converges to the $d$-dimensional standard Brownian motion which has a certain invariance under conformal maps in two dimensions. The conformal invariance in $d = 2$ then provides a powerful tool to exactly determine the values of the involved exponents [13].

Here we consider the RWs on a mixed geometry consisting of an integer lattice $\mathbb{Z}^2$ which is crossed by $n_l$ number of lattices $\mathbb{Z}$ that all share a single common point—the origin (see Fig. 1). The RWs initiate from origin $\mathbf{x} = 0$ at time $t_0 = 0$ and we ask if the statistics of the walks at time $t$ obeys the scaling form (1).

Let us list the main results of this paper:

(1) A competing behavior is observed in which at early times the diffusion along the crossing lines is dominant and becomes less effective in time until a crossover time $t_c$, after which the diffusion on the plane governs the statistics of the RWs.

(2) The probability density to find the random walker at the origin behaves like $\rho(0,t) \propto t^{-\alpha}$ with $1/2 \leqslant \alpha \leqslant 1$ spanning the crossover behavior from early time $t \ll t_c$ with $\alpha = 1/2$ for the line geometry [$d = 1$ and $d_w = 2$ in Eq. (1)] to long-time limit $t \gg t_c$ with $\alpha = 1$ for the plane geometry [$d = 2$ and $d_w = 2$ in Eq. (1)]. Therefore, the symmetric RWs is always recurrent even for arbitrarily large number of crossing lines ($n_l \gg 1$).

(3) We find both analytically and numerically that the crossover time $t_c$ grows with the number of crossing lines $n_l$ with the approximate power-law relation $t_c \propto n_l^2$.

(4) Our analytical prediction for the mean-squared displacement of the RWs on a crossing line at long-time limit $t \gtrapprox t_c \propto n_l^2$, gives $\langle z_l^2 \rangle \approx \sqrt{2/\pi^3} n_l \sqrt{t} \log t$, which is well supported by our results obtained from numerical simulations of the model.

*sepehrinia@ut.ac.ir
†Corresponding author: ab.saberi@ut.ac.ir

 

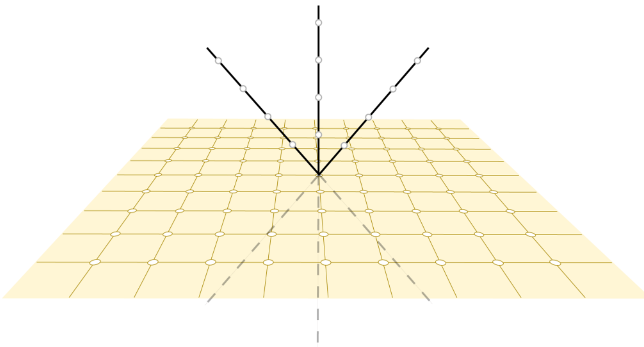

FIG. 1. Illustration of the combined geometry in our model composed of an infinite lattice plane and $n_l = 3$ crossing lattice lines that all share a single common point—the origin.

(5) The probability to find the RWs at a point $\mathbf{r}_p \equiv (x, y)$ on the plane or at a point $z_l$ on a line at time $t$ is provided by the generating function given in Eq. (9).

The rest of this paper is structured as follows. In Sec. II, we will present a general formulation of our model for general combined lattices and briefly discuss its long-time behavior. In Sec. III we will study an interesting nontrivial example of the model composed of a lattice plane and a chain which share a single point. Section IV will discuss the asymmetric RWs on the chain competing with a lattice plane and discuss its asymptotic behavior. In Sec. V we will present the results of our numerical simulations for a plane lattice crossed by $n_l$ number of chains which show perfect agreement with our analytical results. Finally, the last section concludes our discussion.

## II. GENERAL STATEMENT AND FORMULATION OF THE PROBLEM

We consider two general lattices $a$ and $b$, on which the random walk problem is known. We pose the following question: What would the statistics of the random walk motion be like if we connect $a$ and $b$ in a way that they have a single point in common which we call $O$.[1] Consider a classical symmetric random walk that starts from origin $O$. We would like to determine the probability of finding the walker at a given point (on $a$ or $b$) at time step $n$. The most simple quantity to determine for the combined geometry is the first passage probability through the origin. Let us denote the probability of arriving at $O$ for the first time at the $n$th step by $F_{\mathbf{0}}(n)$. For this quantity the walker is required not to visit the point $O$ until the $n$th step. Therefore once it stepped into $a$ or $b$ right after the first step, it should remain there and return to $O$ at the $n$th step. Depending on the geometry of the two lattices which meet each other at $O$, the only shared point between the two domains $a$ and $b$, at the first step the walker would enter into either $a$ or $b$ with probabilities $p_a$ and $p_b$, respectively, where $p_a + p_b = 1$. We now have

$$F_{\mathbf{0}}(n) = p_a F_{\mathbf{0}}^a(n) + p_b F_{\mathbf{0}}^b(n), \tag{2}$$

[1]This problem can well be interpreted as finding the quantum mechanical Green's function of a single particle moving on the underlying lattices.

where $F_{\mathbf{0}}^i(n)$ with $i = a, b$ denotes for the same quantity as $F_{\mathbf{0}}(n)$ for either isolated lattice. Using this simple relation, one can immediately obtain the total return probability $R$ to the origin,

$$R = \sum_{n=1}^{\infty} F_{\mathbf{0}}(n) = p_a R^a + p_b R^b. \tag{3}$$

If the random walk is recurrent on each of the two lattices $a$ and $b$, i.e., $R^a = R^b = 1$, then it will be recurrent on the combined geometry, too, i.e., $R = 1$.

With the first passage probability in hand, one can find the site $\mathbf{x}$ occupation probability $P_{\mathbf{x}}(n)$ as well. But let us first consider the case for the origin, i.e., $\mathbf{x} = \mathbf{0}$, by letting $P_{\mathbf{0}}(n)$ denote the probability of finding the walker at origin at the $n$th step. This can be expressed in terms of the first passage probability through $O$ with the following relation [3,15]:

$$P_{\mathbf{0}}(n) = \delta_{0n} + \sum_{i=1}^{n} F_{\mathbf{0}}(i) P_{\mathbf{0}}(n-i), \tag{4}$$

in which the summation is assumed to be zero for $n = 0$. Using a $z$ transform, as a discrete-time equivalent of the Laplace transform, on both sides of Eq. (4) by multiplying both sides by $z^n$ and summing over $n$, one can find a simple equation for the generating function

$$P_{\mathbf{0}}(z) = 1 + F_{\mathbf{0}}(z) P_{\mathbf{0}}(z), \tag{5}$$

in which we have used $P_{\mathbf{0}}(z) = \sum_n z^n P_{\mathbf{0}}(n)$ and similar relation for $F_{\mathbf{0}}(z)$. Using Eqs. (5) and (2) gives

$$\begin{aligned} P_{\mathbf{0}}(z) &= [1 - F_{\mathbf{0}}(z)]^{-1}, \\ &= \left[1 - p_a F_{\mathbf{0}}^a(z) - p_b F_{\mathbf{0}}^b(z)\right]^{-1}, \end{aligned} \tag{6}$$

which together with the normalization condition $p_a + p_b = 1$ leads to the following result:

$$\frac{1}{P_{\mathbf{0}}(z)} = \frac{p_a}{P_{\mathbf{0}}^a(z)} + \frac{p_b}{P_{\mathbf{0}}^b(z)}, \tag{7}$$

that is very akin to the reciprocal of the total equivalent resistance of two parallel resistors.

Now let us calculate the site occupation probability $P_{\mathbf{x}}(n)$ at a given site $\mathbf{x}$ other than the origin. This quantity can be determined in terms of the solutions in the individual geometries. The probability to arrive at $\mathbf{x}$ at the $n$th step can be considered as the sum of the probability of being at the origin at any earlier time $i < n$ and arriving to the destination without visiting the origin on the remaining time $n - i$. The latter is known as the "taboo" probability [3], denoted by $T_{\mathbf{x}}(n - i)$, in which the walker avoids the origin. One can therefore find that $P_{\mathbf{x}}(n)$ can be cast into the following form:

$$P_{\mathbf{x}}(n) = \sum_{i=0}^{n-1} P_0(i) T_{\mathbf{x}}(n-i), \tag{8}$$

for which the generating function is $P_{\mathbf{x}}(z) = P_{\mathbf{0}}(z) T_{\mathbf{x}}(z)$. Since the walker has to avoid the origin, it should stay in one of the either $a$ or $b$ geometries during the time interval $(i, n]$. This means one can write

$$P_{\mathbf{x}}(z) = P_{\mathbf{0}}(z) \begin{cases} p_a T_{\mathbf{x}}^a(z) & \mathbf{x} \in a \\ p_b T_{\mathbf{x}}^b(z) & \mathbf{x} \in b \end{cases}. \tag{9}$$

For a translationally invariant lattice one can show that for $\mathbf{x} \neq 0$, $T_{\mathbf{x}}^{a,b}(z) = F_{\mathbf{x}}^{a,b}(z) = P_{\mathbf{x}}^{a,b}(z)/P_{\mathbf{0}}^{a,b}(z)$.

### Long-time asymptotics

In order to determine the probabilities as function of time from their generating functions, one needs to do inverse $z$ transform which means to find the coefficients of Taylor series about $z = 0$. It is often of more interest to look at the behavior in the long-time limit which is encoded in $z \to 1^-$ limit of the corresponding generating function. For instance, if the lattice $a$ has the more rapidly decreasing probability then its generating function is less divergent. As a result, Eq. (7) gives $P_{\mathbf{0}}(z) \overset{z\to 1^-}{\approx} P_{\mathbf{0}}^a(z)/p_a$, meaning that the long-time behavior is governed by the lattice $a$. Then the Tauberian theorem [3,14] can be used to obtain the asymptotic behavior in time domain.

## III. 1*d* LATTICE AND SQUARE LATTICE

In this section, we are going to study an interesting nontrivial example of the general formulation presented in the previous section by taking $a$ to be a square lattice in the $x$-$y$ plane and $b$ to be $n_l$ number of one-dimensional (1$d$) lattices which cross $a$ at a single point—the origin $O$ (Fig. 1). The RW problem is exactly solvable on the line ($l$) and plane ($p$) with known results [3,15]

$$P_{\mathbf{0}}^l(z) = \frac{1}{\sqrt{1-z^2}} \overset{z\to 1^-}{\approx} [2(1-z)]^{-\frac{1}{2}}, \tag{10}$$

$$P_{\mathbf{0}}^p(z) = \frac{2}{\pi} K(z^2) \overset{z\to 1^-}{\approx} \frac{1}{\pi} \log[1/(1-z)], \tag{11}$$

where $K(x)$ is the elliptic integral of first kind. Long-time $n \to \infty$ behavior of the occupation probability of the origin is decreasing algebraically with time given by $P_{\mathbf{0}}^l(2n) \approx \frac{1}{\sqrt{\pi n}}$ and $P_{\mathbf{0}}^p(2n) \approx \frac{1}{\pi n}$ on the chain and the square lattice, respectively.

Due to symmetry, the motion of random walk on the lines will be the same as if there was a single line. Therefore we can replace them with a single line but with different probability of hopping to line at the origin. That is, $p_l = n_l/(n_l + 2)$ and $p_p = 2/(n_l + 2)$. Using Eq. (7) for the combined geometry and noting that the second term in the right-hand side is dominant in the limit $z \to 1^-$ we have $P_{\mathbf{0}}(2n) \approx \frac{n_l+2}{2\pi n}$. Since each lattice is translationally invariant as we mentioned for $\mathbf{x} \neq 0$ we have $T_{\mathbf{x}}^{l,p}(z) = P_{\mathbf{x}}^{l,p}(z)/P_{\mathbf{0}}^{l,p}(z)$. Therefore it is enough to know $P_{\mathbf{x}}^{l,p}(z)$ in order to calculate the probabilities on other lattice points of combined lattice. For linear and square lattices it is also analytically available. For example, $P_{\mathbf{x}}^l(z) = (1-z^2)^{-1/2}(1-\sqrt{1-z^2})^{|x|}z^{-|x|}$ and $P_{\mathbf{x}}^p(z)$ can be expressed in terms of hypergeometric functions [16]. Using Eq. (9) we obtain the site probability for a given lattice point. A plot of probability as a function of position at a given time is shown in Fig. 2 for line and plane. To see how the diffusion takes place on the line, for example, we can calculate the moments of the probability distribution. Total probability of finding the walker, i.e., the zeroth moment on the line, $P^l(n) = \sum_{\mathbf{x}\in l} P_{\mathbf{x}}(n)$, is given by

$$P^l(z) = p_l \frac{2z}{1-z+\sqrt{1-z^2}} P_{\mathbf{0}}(z). \tag{12}$$

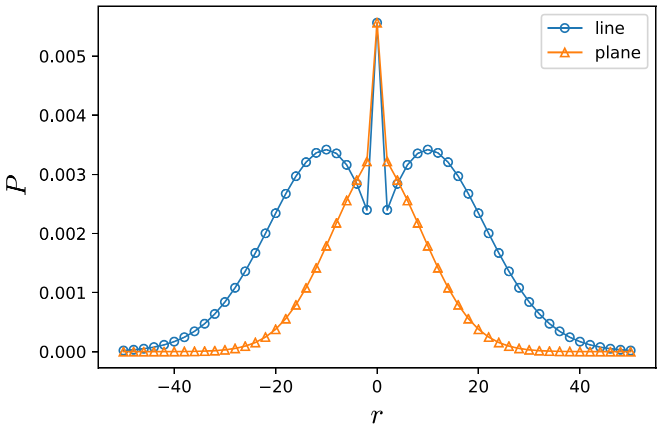


FIG. 2. Probability of finding the RWs at distance $r$ from the origin either on a line (circles) or on the plane (triangles) at time $t = 200$ for $n_l = 1$ obtained from numerical inverse $z$ transform of Eq. (9). Both probability functions get wider in time.

Figure 3 shows the plot of this moment as a function of time. Using Eq. (12) we can see that the long-time behavior of the total probability on the line is $P^l(n) \approx \frac{n_l}{\pi^{3/2}\sqrt{2n}} \log(8n)$. The first moment vanishes because of symmetry under $\mathbf{x} \to -\mathbf{x}$. To see how fast the particle diffuses on the line, it is also worth calculating second moment $\langle z_l^2 \rangle = \sum_{\mathbf{x}\in l} |\mathbf{x}|^2 P_{\mathbf{x}}(n)$. We find

$$\langle z_l^2 \rangle = p_l \frac{z(1+z)^{1/2}}{(1-z)^{3/2}} P_{\mathbf{0}}(z). \tag{13}$$

The time dependence at large time is $\langle z_l^2 \rangle \approx \sqrt{2/\pi^3} n_l \sqrt{n} \log n$.

We now ask the following question: Could the asymptotic behavior, which we obtained above, be changed in favor of the line by increasing the number of lines $n_l$? The answer is no, because changing $p_l$ does not change the $z$ dependence of $P_{\mathbf{0}}(z)$ at the limit $z \to 1$ and therefore the asymptotic behavior will be dominated by the plane. However, the probability $p_l$ will set a timescale before which the behavior is effectively one dimensional and then crosses over to two dimensional. The timescale tends to infinity as the probability $p_l$ tends to

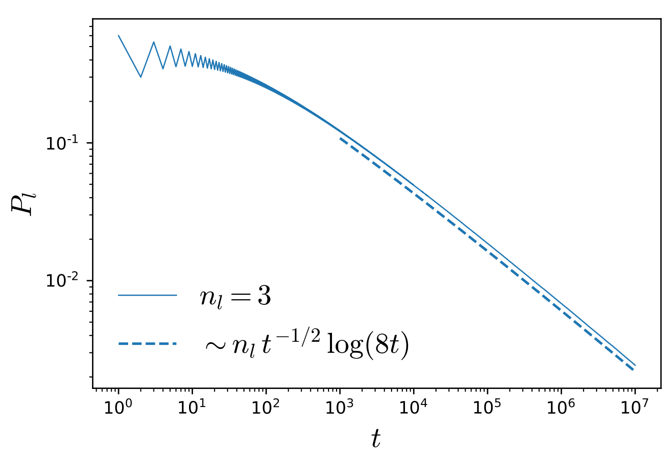


FIG. 3. Total probability for the walker for being on the line geometries as a function of time. The dashed line represents the asymptotic approximation predicted by our analytical result.

1. We define the cross-over time $t^* = -1/\ln z^* \approx 1/(1-z^*)$, where the $z^*$ is the value at which two terms in the brackets in Eq. (7) become of the same order, $\frac{p_p}{P_{\mathbf{0}}^p(z^*)} = \frac{p_l}{P_{\mathbf{0}}^l(z^*)}$. For small $p_p$ this condition is fulfilled at a value of $z^*$ very close to 1. At this limit we use the approximate forms of these probabilities (10) and (11) which gives $\frac{1}{\pi}\sqrt{2(1-z^*)}\ln[1/(1-z^*)] = p_p/p_l$. This is a transcendental equation for $z^*$ and thus the solution is not algebraic; however, it can be easily shown that $1-z^*$ approaches zero faster than $(p_p/p_l)^2$ and slower than $(p_p/p_l)^{(2+\delta)}$ for any positive $\delta$.

## IV. 1*d* BIASED WALK AND SQUARE LATTICE

A simple generalization of previous case which turns out to be important is to consider asymmetric walk on the line. We denote different jump probabilities to the right by $p$ and to the left by $q = 1 - p$. This also can be a representation of the walk on the Bethe lattice [17–19]. We should only replace the generating function of site occupation probabilities of the line with

$$P_{\mathbf{x}}^l(z) = (1-4pqz^2)^{-1/2}\left(1-\sqrt{1-4pqz^2}\right)^{|x|} \times \begin{cases}(2qz)^{-|x|} & x>0\\ (2pz)^{-|x|} & x<0\end{cases}. \tag{14}$$

It can be shown that the biased walk on the chain is not recurrent. More quantitatively $R^l = 1 - |2p-1|$ which is less than 1 if $p \neq \frac{1}{2}$. As a result, the RW on the combined geometry will no longer be recurrent as we have $R = 1 - \frac{1}{3}|2p-1|$. In contrast to the previous case, the occupation of the origin is dominated by the behavior of line. We can see that $P_{\mathbf{0}}^l(z)$ is convergent in the limit $z \to 1^-$. As a result, Eq. (7) gives $P_{\mathbf{0}}(z) \approx \frac{1}{3}P_{\mathbf{0}}^l(z)$. Now it can easily be shown that $P_{\mathbf{0}}(2n) \approx \frac{1}{3}(\pi n)^{-1/2}(4pq)^n$. It is also interesting to note that the first moment, i.e., the probability of finding the walker on the line is approaching one which means that even an infinitesimal amount of drift on the line will pull the walker into the line.

## V. NUMERICAL SIMULATIONS

In this section we present the results of our extensive numerical simulations of the model discussed in the previous sections and compare them with our analytical predictions. We consider systems of combined geometries composed of a lattice plane and various number $n_l$ of lattice lines $n_l = 1, 2, 3, 4, 5, 10, 20, 30, 40, 50, 100, 200, 300$. The total simulation time for the cases $n_l = 1, 2, 3, 4, 5, 10$ and $20, 30, 40, 50$ and $100, 200, 300$ are taken to be $10^5$, $10^6$, $10^7$, respectively, to be able to capture their corresponding asymptotic behavior. All measured quantities are averaged over more than $5 \times 10^8$ independent samples for each case. We assume that the random walker starts moving from origin at $t_0 = 0$ in all computations.

The first natural and standard quantity of interest is the mean-squared displacement (MSD) of the RWs on the combined geometries. In order to see the individual contribution of the lattice plane and the lines, we have computed MSD for the plane (i.e., $\langle \mathbf{r}_p^2 \rangle$) and the lines (i.e., $\langle z_l^2 \rangle$) at time $t$ separately. Figures 4 and 5 show the corresponding dynamical evolution

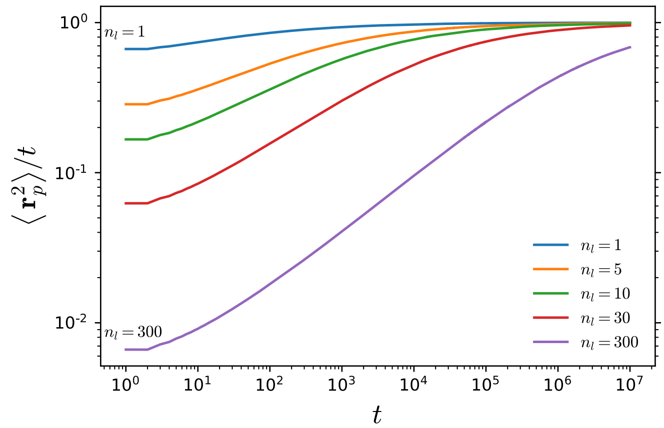


FIG. 4. Mean-squared displacement of the RWs over time $t$ on the plane, i.e., $\langle \mathbf{r}_p^2 \rangle/t$, as function of $t$ in logarithmic scale for different number of crossing lines $n_l = 1$ to 300 from top to bottom. All data converge to the diffusion constant $D_p = 1$ on the plane at long-time limit.

of MSD for various number of crossing lines. As shown in Fig. 4, the walker spends more time in the line geometries for $n_l \gg 1$ at the beginning times. It happens because at small $t$, the walker visits the origin so often and there is a higher probability $p_l$ for the walker to go to one of the line geometries $[p_l = n_l/(2+n_l)]$. This leads to the decrease in MSD on the plane for $n_l \gg 1$. At very large times, instead, the plane geometry will become dominant and the asymptotic behavior of the walker converges to a normal diffusion on a plane with the diffusion constant $D_p = 1$. This explains why all plots for different $n_l$ converge to the same asymptotic value in Fig. 4.

Figure 5 also presents MSD on a line for various $n_l$. The dashed lines show our analytical predictions for each $n_l$ for the MSD on a line at the very long-time limit which is $n_l$ dependent and shows perfect agreement between our numerical

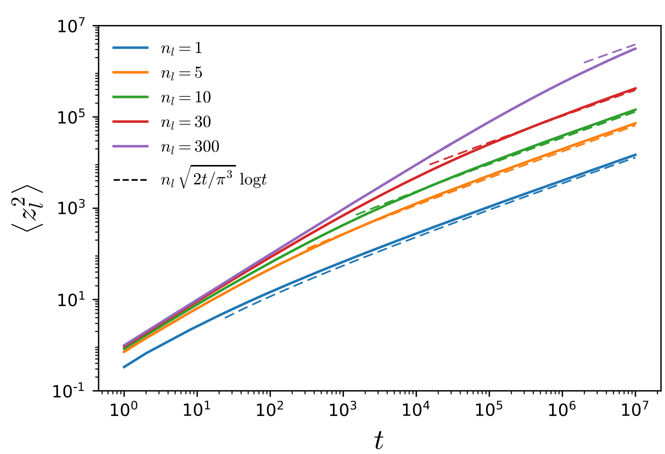


FIG. 5. Mean-squared displacement of the RWs on a crossing line, i.e., $\langle z_l^2 \rangle$, as function of $t$ in logarithmic scale for different number of crossing lines $n_l = 1$ to 300 from bottom to top. The dashed lines show the comparison with our analytical prediction of the behavior at long-time limit, i.e., $\langle z_l^2 \rangle \approx \sqrt{2/\pi^3} n_l \sqrt{t} \log t$ for $t \gtrsim t_c \propto n_l^2$—see Fig. 6.

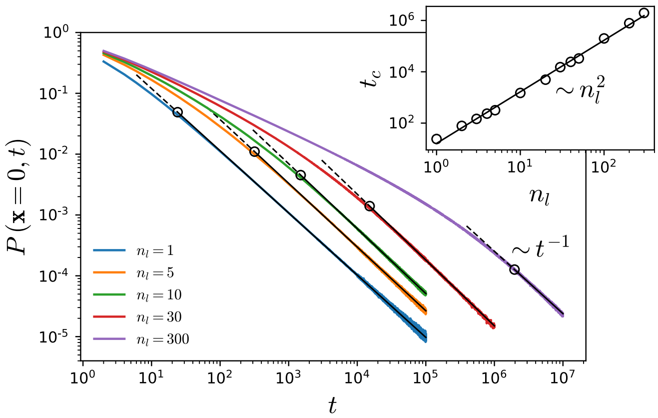


FIG. 6. The probability to find the random walker at the origin $\mathbf{x} = 0$ at time $t$, i.e., $P(\mathbf{x} = 0, t)$, as function of $t$ in logarithmic scale for different number of crossing lines $n_l = 1$ to 300 from left to right. The open circles mark the crossover time $t_c$ after which the behavior is governed by the plane geometry with known scaling relation $\propto t^{-1}$ shown by the solid lines fitted to our data and followed by the dashed lines before $t_c$ when the behavior is still affected by the crossing line geometries. Inset: The crossover time $t_c$ as a function of the number of crossing lines $n_l$. The solid line is the best power-law fit to our data $t_c \propto n_l^2$ in perfect agreement with our analytical prediction—see the text.

simulations and analytical approximations. Notice that, unlike the symmetric diffusion on a single line geometry (which is known to behave as $\langle z_l^2 \rangle \approx t$), the asymptotic behavior of the walks on the lines in our model does not follow the free diffusion and is governed by the square root of time containing a logarithmic correction, i.e., $\langle z_l^2 \rangle \approx \sqrt{2/\pi^3} n_l \sqrt{t} \log t$ for $t \gg 1$.

To better quantify the competition between the plane geometry and the crossing lines, we have computed the probability of finding the random walker at the origin $\mathbf{x} = 0$ at time $t$, i.e., $P(\mathbf{x} = 0, t)$, as function of $t$ for various $n_l$. As shown in Fig. 6, the behavior of $P(\mathbf{x} = 0, t)$ shows two primary and asymptotic regimes roughly for $t < t_c$ and $t > t_c$, respectively, for a $n_l$-dependent crossover time $t_c$. We find that at early times $t \ll t_c$ the behavior is governed by $\propto t^{-1/2}$ for the line geometry and crosses over to the long-time limit $t \geqslant t_c$ with $\propto t^{-1}$ for the plane geometry. For every $n_l$, we define $t_c$ as the (approximately) first time after which the scaling behavior of $P(\mathbf{x} = 0, t)$ is given by $\propto t^{-1}$ (marked by the open circle symbols in the Fig. 6). The inset of Fig. 6 shows the scaling relation between the crossing time and the number of crossing lines as $t_c \propto n_l^2$, which is in close agreement with our analytical approximations discussed at the end of Sec. III.

## VI. CONCLUSIONS

We have studied analytically the random walks problem on a combined lattice geometry composed of two generalized lattices with a single common point. After a general formulation of the problem, we illustrated the consequences in some nontrivial interesting examples by considering a lattice plane crossed by $n_l$ number of lattice lines at the origin. We have found that the probability of returning to the starting point at a long-time limit is governed by the plane. The total probability of being in the line geometry increases first at the beginning time and then starts to decrease at larger times. Mean-squared displacement asymptotically converges to the normal diffusion on the plane but it behaves like $\sqrt{t} \ln t$ on the line geometries. We have shown that the crossover time from the primary to the asymptotic regimes scales approximately as $t_c \propto n_l^2$. Rather simple corollary is that the walk will be recurrent if it is recurrent on both lattices and will be transient if it is transient on at least one of them. We also examined the stability of the asymptotic behavior of the walk by introducing a perturbation to the model with a drift term along the line geometry (for $n_l = 1$). We have found that even an infinitesimal amount of drift can totally change the asymptotic behavior of the walk in a way that the line geometry will dominate the long-time behavior of the perturbed model.

Our problem can also be viewed as a normal diffusion on a lattice plane with a single defect (or trapping) site of variable waiting time. In this context, there has been studied [20] a random reset problem on a $d$-dimensional lattice containing one trapping site with an exponential waiting time at the defect which exhibits a localization-delocalization phase transition. In our case, however, the waiting time is a power law given by the diffusion along the crossing lines tuned by their number. Theory of diffusion in disordered media has a wide range of applications. Among the most important practical applications related to our present study are the kinetics of trapping processes which include various dynamical processes such as polymers in solutions [21], electron-hole recombination in random surfaces, and exciton trapping and annihilation [22,23]. The results presented here could also motivate applications in random search processes and diffusions on geometries with junctions such as heat conduction process in metals.

## ACKNOWLEDGMENTS

R.S. and A.A.S. acknowledge partial financial support from the research council of the University of Tehran. A.A.S. also acknowledges support from the Alexander von Humboldt Foundation (DE). We also thank the High Performance Computing (HPC) center in the University of Cologne, Germany, where a part of computations have been carried out.

[1] W. Feller, *An Introduction to Probability Theory and its Applications*, 3rd ed. (John Wiley & Sons, Hoboken, NJ, 1968), Vol. I.

[2] W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed. (John Wiley & Sons, Hoboken, NJ, 1971), Vol. II.

[3] B. D. Hughes, *Random Walks and Randon Environments, Volume 1: Random Walks* (Oxford University Press, Oxford, UK, 1995).

[4] N. Masuda, M. A. Porter, and R. Lambiotte, Phys. Rep. **716**, 1 (2017).

[5] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).

[6] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).
[7] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[8] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, 2001).
[9] C. Van den Broeck, Phys. Rev. A **40**, 7334 (1989).
[10] J. Klafter, G. Zumofen, and A. Blumen, J. Phys. A **24**, 4835 (1991).
[11] L. Acedo and S. B. Yuste, Phys. Rev. E **57**, 5160 (1998).
[12] G. Pólya, Math. Ann. **84**, 149 (1921).
[13] C. F. Lawler and V. Limic, *Random Walk: A Modern Introduction* (Cambridge University Press, Cambridge, UK, 2010), Vol. 123.
[14] G. H. Hardy, *Divergent Series* (Oxford University Press, Oxford, 1949).
[15] E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).
[16] K. Ray, arXiv:1409.7806.
[17] B. D. Hughes and M. Sahimi, J. Stat. Phys. **29**, 781 (1982).
[18] D. Cassi, Europhys. Lett. **9**, 627 (1989).
[19] C. Monthus and C. Texier, J. Phys. A: Math. Gen. **29**, 2399 (1996).
[20] A. Falcón-Córtes, D. Boyer, L. Giuggioli, and S. N. Majumdar, Phys. Rev. Lett. **119**, 140603 (2017).
[21] M. D. Ediger and M. D. Fayer, Macromolecules **16**, 1839 (1983).
[22] J. Klafter and A. Blumen, J. Lumin. **34(1-2)**, 77 (1985).
[23] S. Shatz and V. Halpern, Chem. Phys. **91**, 237 (1984).