Avalanche dynamics in hierarchical fiber bundles

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Heterogeneous materials are often organized in a hierarchical manner, where a basic unit is repeated over multiple scales. The structure then acquires a self-similar pattern. Examples of such structure are found in various biological and synthetic materials. The hierarchical structure can have significant consequences for the failure strength and the mechanical response of such systems. Here we consider a fiber bundle model with hierarchical structure and study the avalanche dynamics exhibited by the model during the approach to failure. We show that the failure strength of the model generally decreases in a hierarchical structure, as opposed to the situation where no such hierarchy exists. However, we also report a special arrangement of the hierarchy for which the failure threshold could be substantially above that of a nonhierarchical reference structure.

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I. INTRODUCTION

A wide variety of materials, both biological [1–3] and synthetic [4], exhibit structural self-similarity. Essentially, a basic structural unit is repeated multiple times, bridging across length scales, for example, from a single carbon nanotube or molecules of microfibrils to fibrils, fibril bundles, and ropes to the design of a space-elevator cable [5]. The smallest units (nanotubes or microfibrils) are coupled together to form one unit of the higher level and so on, eventually reaching a macroscopic scale [1].

The advantage of a hierarchical structure is its ability to carry load due to the inhibitory effect of the hierarchical structure towards damage propagation. It has been observed that a hierarchical structure prevents crack growth [6] and thereby increases the flaw tolerance of the structure compared to a nonhierarchical counterpart of similar system size. Furthermore, the extreme nature of failure statistics, of a chain of fibers, for example, tends towards self-averaging behavior, leading to a more predictable response of the system.

The earliest attempt to take the advantage of a hierarchical structure could be traced back to the construction design of the Eiffel tower [7], which has a three-level hierarchy. There have been multiple other efforts to build efficient hierarchical structures, particularly in view of reducing the material mass and increasing the load-bearing capacity [8–10]. Self-similar "fractal" structures have been studied in view of optimizing strength-to-weight ratio, and their fabrication by additive manufacturing techniques has been discussed [11].

While the failure strengths of hierarchical structures have received considerable attention recently [12], less is known about the intermittent dynamics of such systems in the run up to failure. Particularly, local failure within one unit of the hierarchy (damage nucleation) leads to redistribution of load, and this redistribution and the resulting diffusion of damage are strongly affected by the hierarchical organization of the system. The interplay of the competing effects of local damage nucleation and long-range load redistribution leading to damage diffusion are manifested in the response statistics of the system, i.e., the avalanche dynamics.

Here we consider a fiber bundle model that is arranged in a hierarchical manner, and we study the interplay of damage nucleation and load redistribution in the avalanche response. A fiber bundle model is a generic model for studying response of driven disordered systems leading to fracture [13,14]. Each fiber has a finite failure threshold drawn from a probability distribution and therefore can sustain a particular amount of load. Following the failure of a fiber, the remaining fibers in the bundle share its load, which might lead to further failure and so on. The collective response of the model mimics several features manifested in breakdown experiments in disordered solids [15]. In this particular study, we are interested in the avalanche statistics of the fiber bundle model when the elements or fibers are arranged in a hierarchical structure. We show that the effects of damage nucleation and hierarchically structured load redistribution and damage diffusion are clearly manifested in the avalanche dynamics. Furthermore, we propose a specific hierarchical structure, which increases the total strength of the system substantially from the usual random placing of the fibers.

The paper is organized as follows: First, we describe the hierarchical fiber bundle model. Then we discuss the effects of various levels of hierarchy on the avalanche statistics of the model and note the effect of the hierarchical structure on the overall strength of the system. Then we propose the structure of hierarchy that maximizes the strength of the overall system, for a given distribution of the failure thresholds of the elementary fibers. Finally, we discuss and summarize the results.

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II. MODEL

A fiber bundle model, in its simplest form, is a series of fibers fixed between two rigid parallel plates and the plates are pulled apart by a force (load) leading to the eventual collapse of the system. The individual fibers are considered to be linear elastic and perfectly brittle, i.e., their stress-strain curves are linear up to the irreversible failure point where the fibers cease to support any load. The particular points of failure, however, are not the same for all the fibers but are drawn from a distribution function. Here we consider mainly a uniform distribution between (0,1) unless otherwise mentioned. But similarly well-behaved distributions give similar statistics for the failure dynamics. With the application of a load, the weaker fibers break and the total load is carried by the stronger fibers. This increases the load per fiber acting on the remaining fibers and can lead to further breaking events. The system can eventually reach a stable state, where all fibers are stronger than their respective load per fiber, or it can suffer a catastrophic breakdown if no such stable state exists for the applied load.

The particular load at which stable states cease to exist for a given system of fibers is the critical load for that system. The behavior of the fiber bundle model near its critical load is well studied. A gradual increase of the load from zero to the critical load visits all the stable configurations of the model. The fiber breaking between any two successive stable configurations constitutes one avalanche, and the number of fibers broken in an avalanche defines its size. The size distribution of all the avalanches, from zero to the critical load, is a power-law function. This is similar to what is seen experimentally in disordered systems under time-dependent loading (see, for example, Ref. [16]).

In its simplest form described above, the avalanche statistics of a fiber bundle model is universal with respect to a broad class of failure threshold distributions. The universality, in this case, is guaranteed by the mean-field nature of the model; i.e., following the failure of one fiber, its load is shared equally among the remaining fibers, irrespective of their distance from the failed fiber. A departure from the equal-load-sharing (ELS) mechanism can significantly change the response of the system [14].

Here we consider a hierarchical structure of the model in which the load redistribution is drastically modified from the ELS version. As shown in Fig. 1, we consider a division of the system into hierarchically organized modules such that, at the ith level of hierarchy, each one of the N_i fibers in the ith-level module represents a module of level $i-1$ consisting of a bundle of N_{i-1} fibers, and so on. For a total of k hierarchy levels, the total number of lowest-level fibers is $N = \prod_{i=1}^{k} N_i$.

If a fiber in a ith-level module fails, its load is redistributed equally among all the surviving fibers within the same hierarchical module. If there are no surviving fibers in that module, the module is considered as failed and its load is redistributed among all the surviving level-$(i+1)$ fibers in the next-higher $(i+1\text{st})$-level module. Note that, in this case, the load increments received by the individual level-$(i+1)$ fibers are equal. But it is possible that the numbers of surviving fibers within each of those units are different. In that case, the load shares for the level-i fibers belonging to different units might differ.

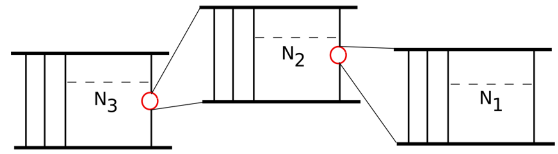

FIG. 1. Schematic illustration of a hierarchical fiber bundle.

It is immediately clear that the dynamics is very different from the ELS dynamics described before. In particular, there is a tendency of accumulation of damage within one unit, until that unit completely fails. This leads to an interesting competition between damage accumulation and load redistribution. When the unit sizes are very small, damage spreads rapidly within a unit because the load of a failed fiber is redistributed very locally. But as the unit size is small, the whole unit breaks quickly, redistributing the load to other units and thereby diffusing the load concentration. On the other hand, if the unit sizes are large, then there is not much damage accumulation to begin with as the individual units behave like ELS fiber bundles. Between these limits one finds a critical size for the units, at which the damage accumulation is enhanced because of load redistribution confinement and the hierarchical structure becomes weakest. This effect is also visible in the avalanche statistics.

Below we investigate the dynamics of such hierarchical fiber bundles. We start with the simplest case of two hierarchy levels and later generalize to three and more levels. At the end we revisit the two-level system but with variable sizes for each unit and show how the overall strength could be increased following a particular prescription for the hierarchical structure which divides the system into units of uneven size in a manner that correlates unit size with strength of the constituent subunits.

III. RESULTS

A. Two-level hierarchy

In its simplest form, we study a two-level hierarchical fiber bundle model. Here we consider a set of N_2 fibers, each of which is made up of N_1 fibers, giving the total number of fibers to be $N = N_1 N_2$. As mentioned above, if N is kept fixed, the various combinations of N_1 and N_2 will give rise to competing effects of damage nucleation and damage diffusion. In Fig. 2, we show the critical load of the model, as a function of N_1, for a fixed total number N of elementary fibers. The limits $N_1 = 1, N_2 = N$ and $N_1 = N, N_2 = 1$ are trivially the ELS model, which has the critical threshold $\sigma_c = 1/4$ [13], when $N \to \infty$. Because load confinement enhances the accumulation of damage in the subunits, for all intermediate sizes the critical load falls below this limit.

For illustration purposes, let us consider the case $N_1 = 2, N_2 = N/2$. This is indeed equivalent to a system of $N/2$ fibers in the ELS setting. But the failure threshold distributions of those $N/2$ fibers are different from the uniform distribution. To calculate the failure threshold of the whole system, the job really is to calculate the failure threshold distribution of the coupled fibers. Assuming that the lowest level fibers are from

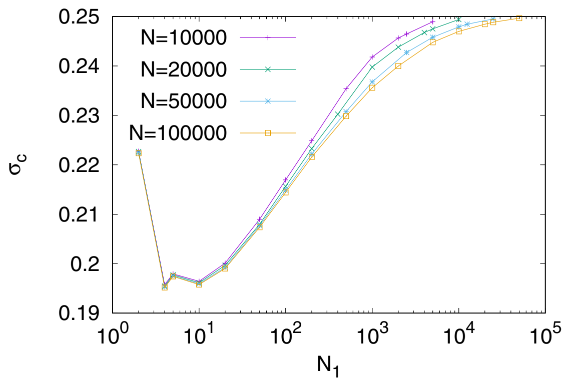

FIG. 2. The critical load of a two-level hierarchical fiber bundle model for fixed system size as a function of the level-one unit size. In the two limiting cases, when the unit size is 1 or equal to the full system size, the critical load is trivially 1/4, which is the result for an ELS fiber bundle with uniform threshold distribution. In all intermediate cases, the critical load is lower due to confinement-enhanced damage accumulation.

a uniform distribution in (0 : 1), if a load σ is applied, the probability that both fibers fail is

(1) For $\sigma \leqslant 0.5$

(a) When both the thresholds are below σ, the probability is σ^2

(b) When one is below σ and the other one is $\sigma \geqslant \sigma_t < 2\sigma$, the probability is $2\sigma(2\sigma - \sigma)$.

(2) For $\sigma > 0.5$

(a) When one of the thresholds is below σ, the probability is $2\sigma - \sigma^2$.

Therefore, considering the total probabilities for the two cases above, the cumulative distribution $P(\sigma)$ of the failure threshold becomes

$$P(\sigma) = 3\sigma^2 \text{ for } \sigma \leqslant 0.5$$
$$= \sigma(2-\sigma) \text{ otherwise,} \quad (1)$$

giving the corresponding probability distribution $p(\sigma)$ as

$$p(\sigma) = 6\sigma \text{ for } \sigma \leqslant 0.5$$
$$= 2 - 2\sigma \text{ for } \sigma > 0.5. \quad (2)$$

For an ELS fiber bundle where the distribution and the cumulative distribution of the failure thresholds are, respectively, $p(\sigma)$ and $P(\sigma)$, the critical elongation per fiber Δ_c follows [13]

$$1 - \Delta_c p(\Delta_c) - P(\Delta_c) = 0, \quad (3)$$

implying in this case

$$1 - 9\Delta_c^2 = 1 \quad (4)$$

with the only physical solution being $\Delta_c = 1/3$. The critical force in turn follows

$$F_c = N(1 - P(\Delta_c))\Delta_c, \quad (5)$$

giving $\sigma_c = F_c/N = 2/9$. Therefore, $\sigma_c^{(2)} = 2/9 < \sigma_c^{(1)} = 1/4$, in the limit $N \to \infty$. This estimate agrees well with the simulation data shown in Fig. 2. As mentioned before, this apparently counterintuitive feature is the result of enhanced damage accumulation due to load confinement within a single unit.

In Fig. 3 the load distribution on the fibers in the last stable configuration before failure (averaged over an ensemble of ∼2000 realizations) is shown for various values of N_1, N_2. As can be seen, the distribution function is widest when the failure threshold is lowest, which implies that some fibers carry much larger loads than the average. Such load concentration leads to enhanced damage accumulation. On either side of this, the load concentration is decreasing due to (1) lack of surviving fibers within one unit (small N_1 limit) and (2) due to wider spreading of the load within one unit (large N_1 limit). As far as the shapes of the distribution functions are concerned, in the two limiting cases mentioned, these are Gaussian. Essentially these two are the widely studied single hierarchy systems. Therefore, their critical loads are randomly fluctuating around the mean-field predicted value 0.5 [13]. One of the simplest cases of two-level hierarchy, therefore, appears when $N_1 = 2, N_2 = N/2$. As calculated above, the critical elongation Δ_c, or for that matter the critical load carried, is 1/3 (note that σ_c is calculated based on the intact bundle). This is for each of the units, some of which have two fibers and others have just one. Consequently, the force distribution splits into two Gaussians, centered at 1/3 and 2/3, with the peak heights determined by the relative abundance of the two type of groups. For more complicated structures, multiple peak positions arise due to a variety of possibilities of the surviving fiber numbers in various levels of hierarchy.

To demonstrate that the multipeaked distributions of load in the last stable configuration are stable with respect to system size variation, in Fig. 4 we have plotted the distributions for $N_1 = 2$ and $N_2 = 100, 300, 500, 1000$. As can be seen, the peak positions remain unaltered at 1/3 and 2/3.

Avalanche size distribution

The effect of damage accumulation is also manifest in the avalanche size distribution of the model. Figure 5 shows the avalanche size distributions for various values of N_1, N_2 with $N = 10\,000$ fixed. The size of an avalanche is defined as the number of fibers failing, in the course of the avalanche progressing at fixed external load, at the lowest level of hierarchy, irrespective of the unit to which they belong. The usual assumption of separation of time scales between slow external loading and fast internal load redistribution has been made. The distributions are averaged over an ensemble of realizations. The generic feature of the avalanche size distribution is a dip that occurs at approximately N_1. This indicates that small avalanches are confined within single units, and their size distribution is essentially that of small ELS fiber bundles with a cutoff at the finite system size N_1. The distributions therefore exhibit three different regimes depending on the N_1/N_2 ratio. First, if $N_1 \ll N_2$ (top row in Fig. 5), single units fail rapidly, and the dynamics is controlled by avalanches involving multiple units. In this limit, the effects of single-unit avalanches are barely visible, the dip at N_1 is weak, and the overall behavior is that of an ELS fiber bundle with its analytically calculable exponent value −5/2 [13]. Second, if N_1 is comparable to N_2, the intermediate dip is most pronounced, and we obtain a bimodal distribution consisting of one peak for small

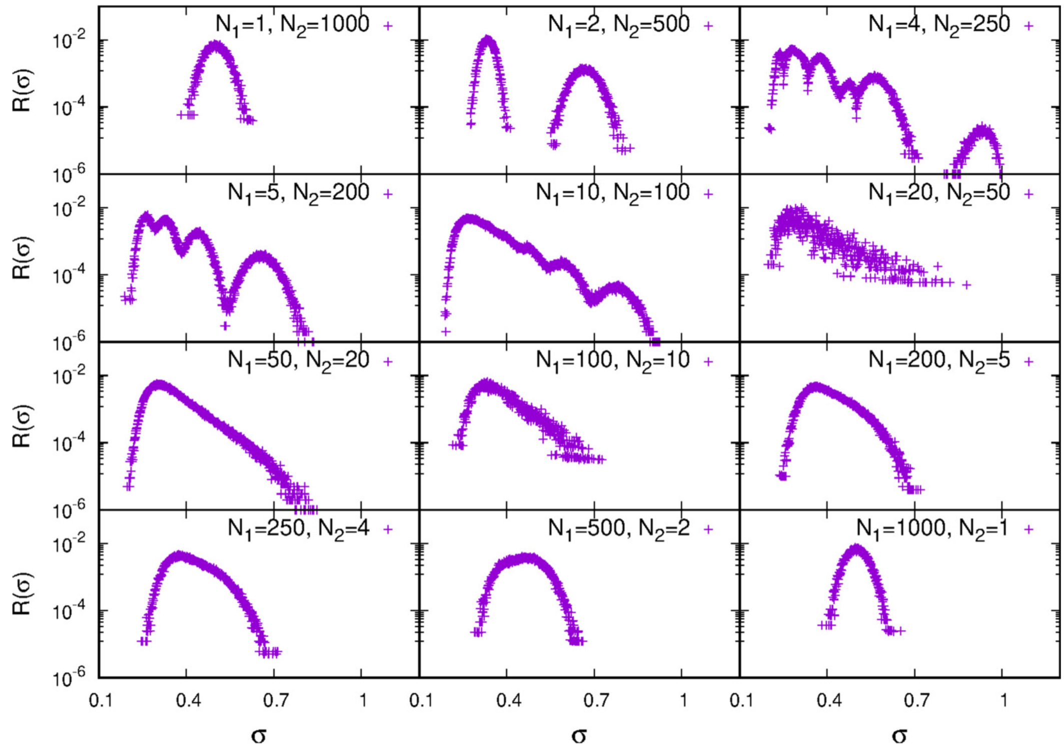

FIG. 3. Distribution of loads on elementary fibers in two-level hierarchical fiber bundles at the point of failure, with system size $N = 10\,000$ fixed; distributions are averaged over an ensemble of fiber bundle realizations. The width of the distributions is largest for intermediate N_1, N_2 values, where the system is weak.

avalanches that are confined within one unit, and a second peak for large avalanches involving multiple units. In the limit of large avalanche sizes, again power-law behavior with an exponent of $-5/2$ is observed (central row in Fig. 5). Third, if $N_1 \gg N_2$ (bottom row in Fig. 5), the peak at large avalanche sizes can hardly be resolved due to limited statistics. In this limit, as N_1 increases we again recover a single power law with the avalanche exponent of the ELS fiber bundle.

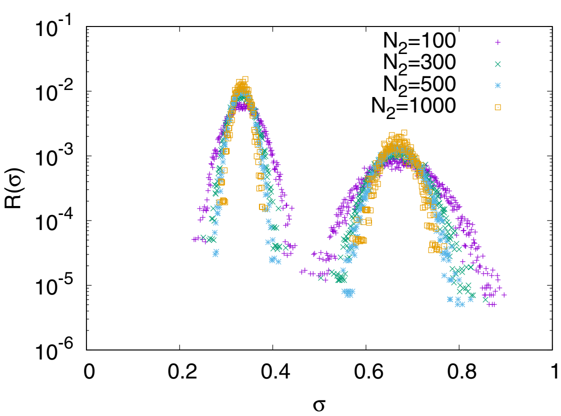

FIG. 4. Distribution of loads on elementary fibers in two-level hierarchical fiber bundles at the point of failure for fixed N_1 value ($N_1 = 2$) and various N_2 values. The peak positions remain unchanged at approximately $1/3$ and $2/3$ (see text).

B. Three-level hierarchy

As the next step of generalization, we study the three-level hierarchical fiber bundle model with N_3 fibers at the highest level, each of which consists of N_2 level-two fibers, of which each is made up of N_1 fibers at the lowest level. As before, we keep $N = N_1 N_2 N_3$ fixed.

The critical load of the system, as before, is affected by the hierarchical structure. As an example, one can calculate the critical load for the case $N_1 = 2, N_2 = 2, N_3 = N/4$. Assuming that at the lowest level the failure threshold distributions of the fibers are drawn from a uniform distribution in (0,1), the probability that one element at the highest level (consisting of two elements in the intermediate level, which in turn are made up of two fibers at the lowest level) fails when a load σ is applied is given by (for $\sigma \leqslant 0.5$)

(1) When both the elements in the intermediate level have failure threshold below σ, $[\int_0^{\sigma} 6\sigma\, d\sigma]^2 = 9\sigma^4$

(2) When one element in the intermediate level has a failure threshold below σ, and the other element has a failure threshold between σ and 2σ, is $2[\int_0^{\sigma} 6\sigma\, d\sigma][\int_{\sigma}^{2\sigma} 6\sigma\, d\sigma] = 27\sigma^4$.

Consequently, the cumulative distribution and the probability distribution for the failure thresholds at the highest level of hierarchy respectively become (for $\sigma \leqslant 0.5$): $P(\sigma) = 63\sigma^4$ and $p(\sigma) = 252\sigma^3$. The critical load follows Eq. (3), giving $\Delta_c = \sqrt[4]{\frac{1}{315}}$ and $\sigma_c = (1 - \frac{63}{315})\sqrt[4]{\frac{1}{315}} \approx 0.1898$. In Fig. 6 we plot σ_c as a function of N_3 when $N_1 = N_2 = 2$. The variation is

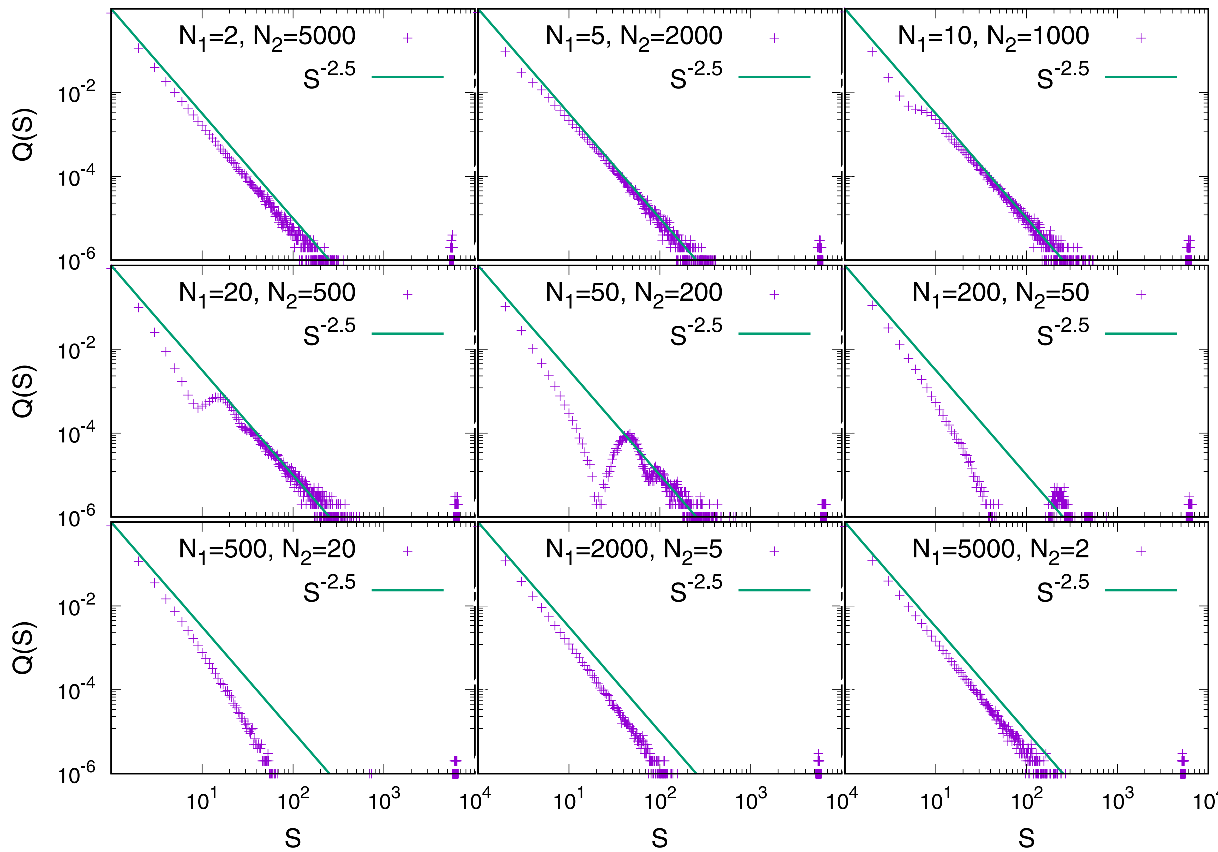

FIG. 5. Size distribution of avalanches for the same systems as in Fig. 3. Large avalanches that span multiple units show a power-law decay with exponent value $-5/2$, and smaller avalanches which are confined in one unit show an exponentially truncated power-law decay. The crossover occurs for avalanche sizes approximately equal to the unit size N_1 and is marked by a dip in the distribution.

of the form $\sigma_c(N_3) = \sigma_c(\infty) + AN_3^{-1/\nu}$, where $\nu = 3/2$ gives a reasonable fit. This is exactly the scaling form and exponent value seen for the usual ELS model [17].

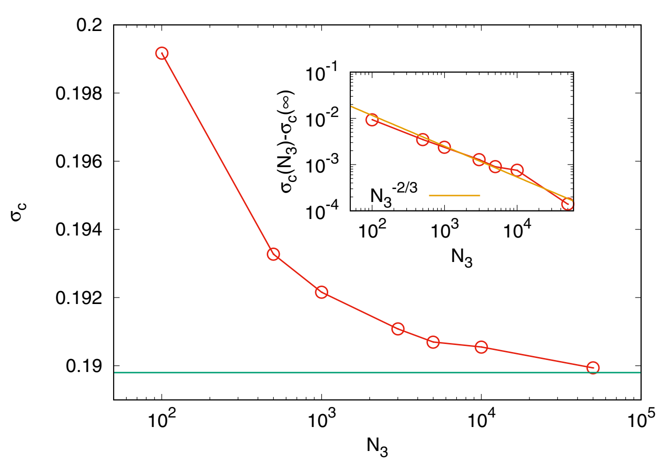

FIG. 6. The horizontal line shows the analytical estimate $\sigma_c \approx 0.1898$ for the critical load of a three-level hierarchical fiber bundle model where $N_1 = 2$, $N_2 = 2$, and N_3 is large. The points, joined by a curve, are the simulation results for various values of N_3. For large values, the critical load approaches the analytical estimate according to the finite-size scaling relation $\sigma_c(N_3) = \sigma_c(\infty) + AN_3^{-1/\nu}$. This is verified in the inset plot, and $\nu = 3/2$, which is the finite-size scaling exponent of the ELS fiber bundle model [17], gives a reasonable fit.

For the total range of $N_1, N_2, N_3 = N/(N_1N_2)$, the critical loads tend to the ELS limit $1/4$ if either $N_1 = N$, $N_2 = N$, or $N_3 = N$. The load-bearing capacity is smaller for all the intermediate configurations where the effect of confinement-enhanced damage accumulation is prominent. Notably, in all hierarchical configurations, the critical load is lower than for the ELS model.

Figure 7 shows the distribution of the load per lowest order fiber in the last stable configuration. The width of the distribution is maximum for the lowest failure thresholds, confirming the effect of confinement-enhanced damage nucleation. As before, the extreme limits show Gaussian distributions (not shown), and in all intermediate steps, multiple peaks arise due to various numbers of surviving fibers in different levels.

Avalanche size distribution

The avalanche size distribution, shown in Fig. 8, reflects the damage nucleation process described above. As before, large avalanches which span many units of the lowest and/or the intermediate hierarchy level define a $-5/2$ power-law tail. But the effect of damage nucleation is visible in smaller avalanches that do not span multiple units and are exponentially decaying in size, leading to dips in the avalanche size distribution. There are two length scales, corresponding to the two levels of lower hierarchies, where such decays are seen.

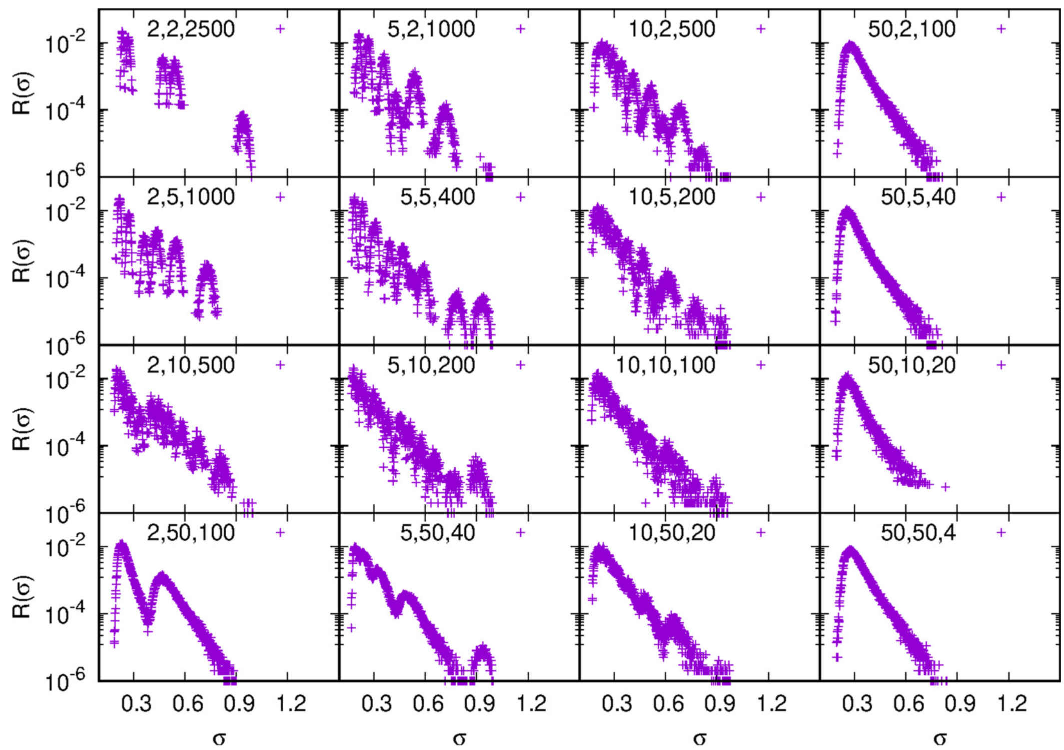

FIG. 7. Distribution of loads on elementary fibers in three-level hierarchical fiber bundle models at the point of failure, system size $N = 10000$ fixed, for different values of N_1, N_2 and $N_3 = N/(N_1N_2)$. The effects of stress confinement are evident for intermediate values of N_1, N_2, where the stress distribution is widest.

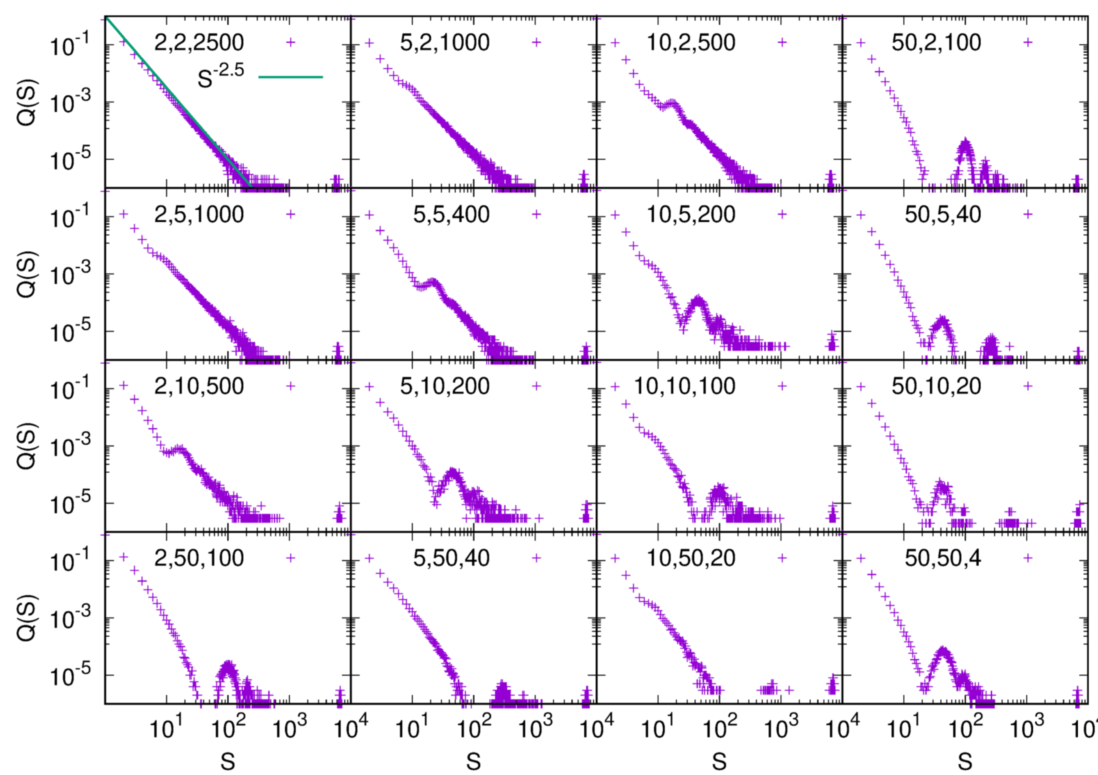

FIG. 8. Size distribution of avalanches for the same systems as in Fig. 7. Large avalanches that span multiple units show a power-law decay with exponent value $-3/2$, and smaller avalanches which are confined in units of lower hierarchy show an exponential decay.

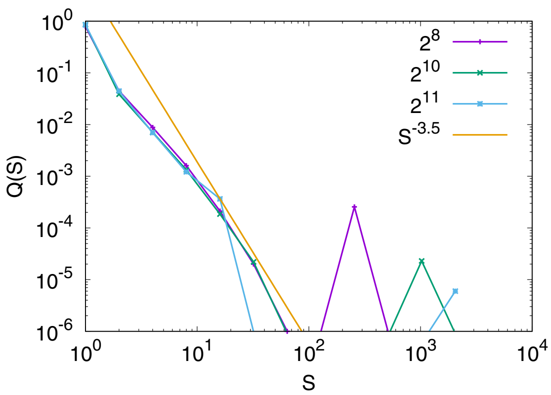

FIG. 9. Size distributions of avalanches for a hierarchical structure with two fibers in each unit and multiple levels of hierarchy; the peaks at large avalanche sizes correspond to system-spanning avalanches at the point of failure.

C. Higher level hierarchy

In cases where the number of levels is high, but the size of each level is small, the effect of damage accumulation is rather intriguing. As shown before, there will be a length-scale cutoff for each level, but the number of such cutoffs is high. The overall distribution, therefore, is affected at multiple scales. Due to the exponential growth in the system size with the number of levels, only a limited range is accessible for numerical analysis. In analyzing this case, we use the same value $N_i = N_1 = 2$ value for all hierarchy levels, producing a self-similar structure. Figure 9 shows avalanche size distributions for various values of the number of levels. The avalanche size distributions have power-law characteristics but are much steeper than what is expected for a ELS reference structure. The simulations are shown for a Weibull distribution of thresholds, and results are similar for the uniform distribution.

D. Maximizing the global failure threshold

A common observation for the hierarchical fiber bundle models studied here is that the failure threshold of any hierarchical structure is lower than the failure threshold of a system with equal size and one level of hierarchy (i.e., the usual FBM). In this section, we deal with the question whether the failure threshold of the hierarchical structure could be increased beyond the limit of the single-level arrangement.

The idea is to use a set of elements with failure thresholds drawn from a given distribution, to classify them according to their failure threshold, and then to arrange them in a suitable hierarchical structure to maximize the effective failure threshold of the system. Such situations could arise in various design problems with shared load, such as the power grid network, computer redundancies, traffic networks, and so on. The crucial feature is to group the elements into units by correlating the unit size with the failure thresholds of the individual elements. (Of course, in practice this depends on availability of a method to determine the failure threshold of a unit without destroying it). For simplicity, we consider a two-level hierarchy.

The key to increase the failure strength in fiber bundles is to distribute the total load in such a way (a) that the fibers receive loads according to their failure threshold (see, e.g., Ref. [18]) or (b) to make sure that all fibers are of equal strength. In the second case, even an equal load redistribution scheme would give the maximum failure threshold (assuming that the total strength of all fibers together is constant). In a hierarchical fiber bundle model, we use both of these ways together. Sticking to the two-level hierarchy, the basic idea is that on the lower level the fibers of approximately equal strength are partitioned into groups. These groups are made of unequal size, such that also total strength of each group is approximately equal; i.e., both on the lower and on the higher hierarchical level we create bundles consisting of elements of approximately equal strength. As a consequence, weaker fibers are partitioned into larger groups and stronger fibers into smaller groups. The precise values of the group sizes will depend on (1) the total number of groups to be created and (2) the failure threshold distribution. After this partitioning, in the higher level, one ends up with fibers of equal strength, for which a uniform load redistribution gives the maximum failure threshold, as mentioned before. Note that due to the unequal sizes of the groups, the stronger fibers need to carry higher load (since they belong to smaller size groups) and weaker fibers carry smaller load (since they belong to larger groups). In this way, the two mechanisms mentioned before work together in hierarchical systems to maximize the global failure threshold. A computational procedure to achieve this objective is the following.

Let the elementary fibers be arranged in the ascending order of their failure thresholds. Then divide this ordered list into m groups with population fractions n_i, with $\sum_i^m n_i = 1$. If the average value of the failure thresholds in each group is f_i, then

$$\sum_{i=1}^{m} n_i f_i = \frac{1}{2} \tag{6}$$

for a uniform distribution in (0,1). We now require that the numbers and failure thresholds in each group fulfill the relation

$$\frac{n_i}{n_{i+1}} = \frac{f_{i+1}}{f_i}. \tag{7}$$

Combining Eq. (6) and Eq. (7), we can see that this requirement is tantamount to

$$n_i f_i = \frac{1}{2m}; \tag{8}$$

i.e., each group has the same average load-bearing capacity.

Now, if the partition between $i-1$st and ith group in terms of threshold values is at x_i, then

$$n_i = x_i - x_{i-1} \tag{9}$$

and

$$f_i = x_{i-1} + \frac{x_i - x_{i-1}}{2}. \tag{10}$$

Putting Eq. (9) and Eq. (10) into Eq. (8), we get

$$(x_i - x_{i-1})\frac{(x_i - x_{i-1})}{2} = \frac{1}{2m} \tag{11}$$

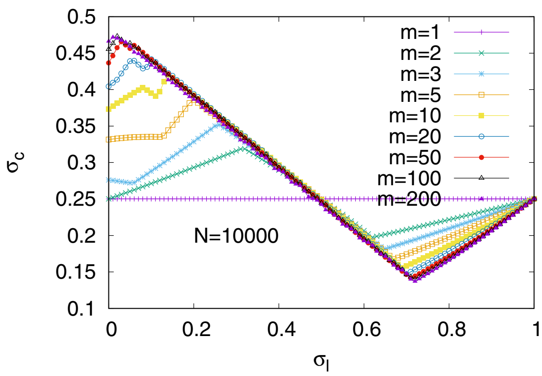

FIG. 10. The critical load σ_c, as a function of the partitioning parameter σ_l in Eq. (14). The maximum strength is obtained for large m values when the groups are well partitioned into almost equal strengths and is substantially higher than the single hierarchy limit 1/4, which is in turn higher than all other hierarchical arrangements.

or

$$x_i^2 - x_{i-1}^2 = \frac{1}{m} \quad \text{with} \quad x_m = 1, \quad x_0 = 0. \tag{12}$$

This implies $x_i = \sqrt{\frac{i}{m}}$.

Next, we make use of the fact that, in a ELS fiber bundle, the weakest fibers may well be rather useless in terms of the strength of the system. This is especially true when the number of partitions is small. This can be easily seen by looking at a ELS fiber bundle with N fibers and uniformly distributed thresholds on the interval (0,1) that has a critical stress of $\sigma_c = 0.25$ and thus carries a load of $0.25N$: Throw away the weaker half of the fibers to create a bundle with $N/2$ fibers and thresholds on the interval (0.5,1), which has a critical stress of 0.5 and carries a load of $0.25N$ still. This simply follows from the fact that at the critical point, the weaker half of the system is broken, and it is the stronger half that carries the total load. We apply a similar strategy by lumping, prior to partitioning, all level-one fibers with a threshold below σ_l into a single level-two fiber $i = 0$, which means that in the thermodynamic limit $N, m \to \infty$ they become irrelevant to the dynamics of the system.

The above calculation can, then, be repeated for the remaining fraction $(1 - \sigma_l)$ of fibers, with new variables x_i', which are related to the old variables x_i as

$$x_i' = \frac{x_i - \sigma_l}{1 - \sigma_l}. \tag{13}$$

Therefore, the optimized partitioning of the fibers in hierarchical modules under a threshold σ_l is

$$x_i = \sqrt{\frac{i}{m}(1 - \sigma_l) + \sigma_l}, \tag{14}$$

when there are m units in the second level of hierarchy, in a two-level hierarchical system. Finally, to answer the question for what value of σ_l an optimal configuration may be obtained, Fig. 10 shows the value of the critical load σ_c for two-level hierarchical fiber bundles partitioned according to Eq. (14) with various values of the truncation parameter σ_l and the number of groups m. For $m = 1$, the critical load is trivially at 0.25. For all other values of m, there are at least some values of σ_l for which the critical load is higher than this reference value. Indeed, the gain of minimizing the effects of weaker fibers, as described above, could be seen for small σ_l values. On the other hand, for large m, the system gets well partitioned into m equally strong groups. Under that circumstance, given that the average threshold of the whole system cannot change, the only option is that the strength of each group tends to that average value, i.e., 0.5. Therefore we see that the critical strength tends to 0.5 for high m at low σ_l values. This implies a 100% increase in the strength and is much higher than other attempts to maximize strength [18].

Therefore, we find that for a suitable arrangement of the hierarchical structure, it is possible to reach close to the maximum possible failure threshold under uniform loading and uniform load redistribution.

IV. DISCUSSION AND CONCLUSIONS

The hierarchical nature of a heterogeneous material can have major consequences on its dynamics and load-bearing capacity. We consistently find that, for fiber bundle models, a hierarchical arrangement generally reduces the global failure threshold as opposed to the one without such an arrangement. The reason behind the lower failure threshold for hierarchical fiber bundle structures is simply the mean-field nature of the fiber bundle model. Due to the mean-field nature, the broadest possible range of load redistribution is the one where all fibers can equally share the load of a failed fiber. For any hierarchical arrangement, there is necessarily a load localization, since the load of a failed fiber must be shared by fibers belonging to the same hierarchical unit. This will always make a part of the system inaccessible for load redistribution following the failure of a given fiber, and the local confinement of load enhances local damage accumulation. This effect of load or damage localization due to hierarchical structures is also manifest in the avalanche dynamics of the system. In the avalanche dynamics, large avalanches that span several hierarchical units behave like in the mean-field model. However, smaller avalanches, that are confined within single units, differ significantly from the mean-field limit. Indeed, looking at the avalanche size distributions, is a possible way to understand the hierarchical nature of a material for which the detailed internal structures are not known.

It is important to note, however, that the hierarchical structure works (shows interesting dynamics) only when there are fluctuations within the individual units. In this case that fluctuation comes from the failure threshold distributions. If the set of failure threshold values for each fiber at the lowest level were identical, then we would end up with multiple copies of the same unit at the lowest level. Similarly, if the unit sizes at each level become large, then we end up with multiple copies of an usual equal load-sharing fiber bundle model. It is the fluctuations in the failure threshold values that come from relatively small sizes of the units that give the interesting dynamics of the hierarchical fiber bundles. The so-called thermodynamic limit here needs to be only $\prod_i N_i \to \infty$, rather than taking each $N_i \to \infty$ individually. Even in the simplest case of the two-level hierarchy, the limits $N_1 \to \infty$, finite N_2 and finite $N_1, N_2 \to \infty$ do not give the same resulting dynamics. It has been analytically shown here that $N_1 = 2$, $N_2 \to \infty$ gives $\sigma_c = 2/9$, while $N_1 \to \infty$, $N_2 = 2$ will retain the usual fiber bundle mean-field result of $\sigma_c = 1/4$.

It is interesting to discuss the relation between hierarchical FBM as studied here and fiber bundle models with local load sharing (LLS). By imposing load redistribution to occur within a unit of the same hierarchical level only, the hierarchical fiber bundles imply a certain degree of locality in their load-sharing rule. At the same time, we note that the adjacency structure that is imposed by the hierarchy (adjacent are fibers that belong to the same hierarchical unit) is distinct from adjacency in an Euclidean or network sense. Both concepts of locality might be combined, e.g., by embedding the hierarchical structure into an Euclidean space, to produce LLS hierarchical fiber bundle models for which there exists (depending on hierarchical structure and load-sharing rule) a huge space of possible variations.

For the ELS hierarchical FBM as considered here, we have been able to show that the adjacency structure imposed by the hierarchical organization, with its confinement of load redistribution after a local failure within the same unit, can be used to enhance the load-bearing capacity of the entire system. The key idea is to make all units have the same total capacity, i.e., combining a large number of weaker fibers in one group and a small number of stronger fibers in another and intermediate sizes in between. A specific recipe of such a division is proposed which can significantly increase the total capacity. In practice this is contingent upon knowledge of the fiber failure thresholds, which depends on availability of a suitable nondestructive testing method.

In conclusion, we have studied the avalanche dynamics of hierarchical fiber bundles and have shown that its nature bears the signature of the underlying hierarchy. While the failure thresholds of the fiber bundles in usual hierarchical structures are less than the one without such structures, we propose a mechanism in which the critical load could be made much higher. The observation that hierarchy does not necessarily make a structure stronger matches findings on the failure strength of hierarchical fuse models, where hierarchically organized structures were found to be slightly weaker than nonhierarchical reference structures [6]. On the other hand, the main advantage of hierarchical structures, namely, that they effectively suppress crack propagation driven by crack-tip stress concentrations and therefore possess a high degree of flaw tolerance [6], has no counterpart in fiber bundle models, which by their nature are devoid of spatial structure.

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