Operational approach to quantum stochastic thermodynamics

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We set up a framework for quantum stochastic thermodynamics based solely on experimentally controllable but otherwise arbitrary interventions at discrete times. Using standard assumptions about the system-bath dynamics and insights from the repeated interaction framework, we define internal energy, heat, work, and entropy at the trajectory level. The validity of the first law (at the trajectory level) and the second law (on average) is established. The theory naturally allows one to treat incomplete information and it is able to smoothly interpolate between a trajectory-based and an ensemble level description. We use our theory to compute the thermodynamic efficiency of recent experiments reporting on the stabilization of photon number states using real-time quantum feedback control. Special attention is paid to limiting cases of our general theory, where we recover or contrast it with previous results. We point out various interesting problems, which the theory is able to address rigorously, such as the detection of quantum effects in thermodynamics.

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I. INTRODUCTION

The nonequilibrium thermodynamics of small Markovian systems can be considered to have been well studied for decades if we are interested only in ensemble-average quantities of internal energy, heat, work, or entropy [1–6]. For classical systems it has become clear during the past 25 years that also fluctuations in thermodynamic quantities bear important information and that those fluctuations are constrained by fundamental symmetry relations that are valid arbitrarily far from equilibrium. These symmetry relations are known as fluctuation theorems [7,8]. For a given realization of a stochastic process, an understanding of the fluctuation theorem required the extension of the ensemble-average energetic [9,10] and entropic [11] description to the level of single stochastic trajectories. The resulting theoretical framework is called stochastic thermodynamics [12,13].

Quantum stochastic thermodynamics tries to generalize classical stochastic thermodynamics to systems whose quantum nature cannot be neglected. Obviously, the very definition of a trajectory-dependent quantity is nontrivial as any measurement disturbs the system and the meaning of a trajectory is *a priori* not clear. We note that incomplete and disturbing measurements are also prevalent in classical systems [14], but exploring their consequences for classical stochastic thermodynamics has raised relatively little attention so far [15–21].

Soon after the discovery of classical fluctuation theorems, much effort was devoted to derive fluctuation theorems for quantum systems. A theoretically successful strategy is the two-point measurement approach [22,23], which requires one to measure the energy of the system and the bath at the beginning and at the end of the thermodynamic process. Obviously, for a bath with its prosaic 10^{23} degrees of freedom such a scheme is not even for a classical system practically feasible. In addition, the resulting statistics for internal energy and work cannot fulfill the first law if the initial state is not diagonal in the energy eigenbasis [24]. Nevertheless, within this approach quantum fluctuation theorems can be derived which are formally identical to their classical counterpart. Thus, by measuring the whole universe (system plus bath), the two-point measurement approach circumvents the need to define thermodynamic quantities along a specific system trajectory. Also alternative and complementary approaches based on interferometric measurements [25–29], a single projective measurement [30,31], or no measurement at all [32,33] have been put forward and the semiclassical limit was studied too [34–36]. To conclude, even though those approaches are theoretically powerful, they are experimentally hard to confirm and an important feature of classical stochastic thermodynamics is still missing, namely, the definition of internal energy and entropy along a given quantum trajectory.

Exceptions are quantum systems which, when perfectly observed in the energy eigenbasis, follow a Markovian rate master equation. This is approximately the case in electronic nanostructures (quantum dots) in the sequential tunneling regime [37–40], where the framework of classical stochastic thermodynamics was carried over one by one. Interestingly, trying to adopt this picture to more general quantum dynamics results in unconventional definitions for thermodynamic quantities [41], not to mention the measurement problem. This further demonstrates the need for a radically different approach to quantum stochastic thermodynamics.

One such approach makes use of the framework of repeated interactions [42–44]. Therein, the static bath is replaced by an external stream of ancilla systems which are put in contact with the system one by one and are designed to simulate a thermal bath (arbitrary initial states of the bath were recently treated in Ref. [45]). If the external systems are projectively measured before and after the interaction, a trajectory-based formulation becomes possible similar to classical stochastic thermodynamics. Although such a description yields theoretical insights, in experimental reality a system is usually also in permanent contact with a bath.

An experimentally closer approach uses a technique which was discovered in quantum optics in order to describe the stochastic evolution of a quantum system based on monitoring the environment of the system [46–48]. Given such a measurement scheme, the system dynamics can be unraveled by describing it in terms of a stochastic Schrödinger or master equation. Combined with this dynamical description, researchers recently applied the ideas of stochastic thermodynamics to such quantum systems [49–55]; a completely general picture, however, is still missing. For instance, a trajectory-dependent system entropy was never introduced, making it hard to study entropy production along a single trajectory or on average (specific fluctuation theorems based on a particular choice of the backward dynamics were studied in Refs. [51,53–55]; we will return to this in Sec. VII). Furthermore, the above publications focused only on efficient measurements in which the state of the system along a particular trajectory is always pure (for some specific scenarios first steps were already undertaken to overcome this limitation [50,53]). Finally, only simple protocols excluding feedback control have been studied so far (Refs. [50,51] consider also very simple feedback schemes for specific systems).

To conclude, apart from a few model specific studies, a common feature of all previous approaches is the reliance on a perfectly monitored system and environment such that the system is always in a pure state along every trajectory. In this sense, there is no essential departure from the two-point measurement scheme in which perfect knowledge of every involved degree of freedom is crucial.

A. Results and outline

We here put forward an approach which we call operational quantum stochastic thermodynamics because it places the experimenter in the foreground. A stochastic trajectory, as well as the corresponding thermodynamic quantities of internal energy, heat, work, and entropy along such a trajectory, is defined solely in terms of experimentally meaningful interventions or control operations of the system dynamics. Dynamically, our description rests on recent theoretical progress in describing quantum causal models or quantum stochastic processes [56–65]. Within this picture it is possible to describe the effect of arbitrary control operations happening at arbitrary discrete times applied to an arbitrary quantum system in an experimentally measurable way. It is different from conventional quantum trajectory approaches and we will start the paper by discussing it in Sec. II.

In Sec. III we then connect this approach to the framework of repeated interactions. Partially based on insights from earlier work [66], we will see in Sec. IV that this allows us to find an unambiguous first and second law of thermodynamics for each single control operation.

The only standard assumption we are here using is that the system in the absence of control operations can be modeled by a quantum master equation with a transparent thermodynamic interpretation describing a driven system coupled to a single heat bath.[1] Based on the repeated interaction picture, we will then see in Sec. IV that the definitions of internal energy and system entropy emerge naturally out of the framework if we properly take into account all interacting subsystems. In fact, following the credo "information is physical" [67], we will see that it is necessary to include the full information generated by the measurements in the entropic balance from the beginning on. With this step we also depart from the approaches reviewed above, which need to be modified in the presence of feedback control (see Ref. [68] for an introduction). The first law at the trajectory level and the second law on average are finally verified.

This concludes the first part of the paper, which is about the basic framework of operational quantum stochastic thermodynamics. Its characteristics include the following.

(i) It neither relies on the ability to have control about the environment nor requires continuous measurements.

(ii) By allowing one to treat any kind of incomplete information, it respects experimental reality where every measurement is imprecise and imperfect.

(iii) It shows that any conceivable feedback scenario has a consistent thermodynamic interpretation.[2]

(iv) The notion of stochastic entropy for a quantum system is defined and the second law follows without the need to introduce any backward dynamics.

(v) The framework reveals that quantum stochastic thermodynamics is more than a mere extension of classical stochastic thermodynamics. Any measurement strategy has in general a nontrivial impact on the quantum system and hence there is a plurality of first and second laws in quantum thermodynamics depending on how we measure the system. Note that these many laws of thermodynamics are conceptually different from the many second laws of Ref. [69].

The rest of the paper is about illuminating applications and special cases of the general theory.

(vi) To illustrate points (ii) and (iii), we analyze in Sec. V the quantum stochastic thermodynamics of recent experiments reporting on the preparation and stabilization of photon number states [70,71]. We uncover that the efficiency to prepare such states is remarkably high.

(vii) We consider the case of projective measurements in detail and compare our definitions with the recently introduced notion of quantum heat [51] in Sec. VI A.

(viii) In Sec. VI B we provide a resolution to the no-go theorem derived by Perarnau-Llobet *et al.* [24], which shows that the conventional definition of work used in the two-point measurement scheme [22,23] needs careful reexamination. Indeed, we show that it is inconsistent with our definition of stochastic work.

(ix) Sections VI C–VI E provide important consistency checks. We show that the definitions of standard quantum

[1] An extension beyond this Markovian picture is however possible in some cases (see Sec. VII B).

[2] This includes the case of real-time feedback control, where, in contrast to deterministic feedback control where the times of measurement and feedback are predetermined [68], the control strategy is adapted during the run of the experiment. It also includes the case of time-delayed feedback control.

thermodynamics [3–6] and the repeated interaction framework [66] are contained in our general approach. They arise, however, not by averaging over many trajectories, but by deciding not to do any measurements at all. In the limit of a perfectly observed classical system we recover the definitions of internal energy, heat, and work of standard stochastic thermodynamics. Only our second laws differ because our framework remains valid in the case of feedback control, whereas the conventional framework [10,12,13] needs to be modified then [68].

(x) In Secs. VI F and VI G we discuss particularly interesting cases which allow us to reduce the complexity of our general framework.

The paper ends with some remarks and an outlook. Section VII A discusses the case of multiple heat baths, possible second laws that follow from a time-reversed process, and the necessity to use the repeated interaction framework and to focus on incomplete information from the beginning on. In Sec. VII B we point out interesting future applications such as finding true quantum features in quantum heat engines, relations to Leggett-Garg inequalities, and the detection of non-Markovian effects in thermal machines.

B. Basic notation

The state of a system X at time t is described by a density operator $\rho_X(t)$. The corresponding Hilbert space of the system is denoted by $\mathcal{H}_X$ and the Hamiltonian by H_X or $H_X(\lambda_t)$ if it depends on an externally controlled time-dependent parameter λ_t. The von Neumann entropy of an arbitrary state ρ_X is defined as $S_{\rm vN}(\rho_X) \equiv -\text{tr}_X\{\rho_X \ln \rho_X\}$ and the Shannon entropy of an arbitrary probability distribution $p(x)$ is $S_{\rm Sh}[p(x)] \equiv -\sum_x p(x)\ln p(x)$. To characterize the correlations of a bipartite system XY in state ρ_{XY}, we use the always positive mutual information $I_{X:Y} \equiv S_{\rm vN}(\rho_X) + S_{\rm vN}(\rho_Y) - S_{\rm vN}(\rho_{XY})$. It is closely related to the always positive relative entropy $D[\rho||\sigma] \equiv \text{tr}\{\rho(\ln\rho - \ln\sigma)\}$ by noting that $I_{X:Y} = D[\rho_{XY}\|\rho_X \otimes \rho_Y]$, where $\rho_{X/Y} \equiv \text{tr}_{Y/X}\{\rho_{XY}\}$ denotes the marginal state. Furthermore, we denote superoperators, which map operators onto operators, by calligraphic letters, e.g., $\mathcal{U}$, $\mathcal{V}$, $\mathcal{P}$, etc.

Below we will see that a stochastic trajectory is specified by a sequence of measurement results or outcomes $r_n, \dots, r_1$, which were obtained at times $t_n > \cdots > t_1$. The sequence of outcomes will be denoted by $\mathbf{r}_n \equiv (r_n, \dots, r_1)$. The state of a system X at time $t > t_n$ conditioned on such a sequence will be denoted by $\rho_X(t, \mathbf{r}_n)$. The ensemble-average state is given by $\rho_X(t) = \sum_{\mathbf{r}_n} p(\mathbf{r}_n)\rho_X(t, \mathbf{r}_n)$, where $p(\mathbf{r}_n)$ denotes the probability of obtaining the sequence of outcomes $\mathbf{r}_n$. We will also keep this notation for thermodynamic quantities such as internal energy E, heat Q, work W, and entropy S (which possibly have additional subscripts and superscripts). This means, for instance, that the stochastic internal energy depending on the outcomes $\mathbf{r}_n$ is denoted by $E(t, \mathbf{r}_n)$ whereas the ensemble-average internal energy is written $E(t) = \sum_{\mathbf{r}_n} p(\mathbf{r}_n)E(t, \mathbf{r}_n)$.

II. PROCESS TENSOR

Classical stochastic thermodynamics is based on the theory of classical stochastic processes. A corresponding quantum thermodynamic framework needs to be based on the theory of quantum stochastic processes. Recently, there has been a great deal of progress on this topic and we will here use the process tensor to represent a quantum stochastic process [61–65]. It is the extension of quantum superchannels [72,73] to multiple control operations and it is closely related to the quantum comb framework studied in Refs. [56,57]. Similar frameworks have been developed also within the emergent field of quantum causal modeling [58–60] and even earlier attempts in that direction can be found in Refs. [74,75]. The basic insight behind this formulation is to treat the control operations performed on the system as the elementary objects and not the state of the system itself because the latter in general cannot be fully controlled. Here the terminology "control operation" is used in a wide sense and could describe any action of an external agent such as measurements, unitary kicks, state preparations, noise addition, feedback control operations, etc. Mathematically, we only require that each control operation is described by a completely positive (CP) map. The following review about the basics of the process tensor requires some knowledge about quantum operations and quantum measurement theory; see Refs. [76–80] for introductory texts.

As usual, we consider a system S coupled to a bath B described by an arbitrary initial system-bath state $\rho_{SB}(t_0)$. The composite system-bath state evolves unitarily up to time $t_1 \geqslant t_0$ according to the Liouville–von Neumann equation $\partial_t\rho_{SB}(t) = -i[H_{\rm tot}(\lambda_t), \rho_{SB}(t)]$ ($\hbar \equiv 1$) with global Hamiltonian

$$H_{\rm tot}(\lambda_t) = H_S(\lambda_t) + H_{SB} + H_B. \tag{1}$$

Here the system Hamiltonian H_S might depend on some arbitrary time-dependent control protocol λ_t, but not the interaction Hamiltonian H_{SB} and the bath Hamiltonian H_B. The resulting unitary evolution is described by the superoperator

$$\mathcal{U}_{1,0}\rho_{SB}(t_0) \equiv U(t_1, t_0)\rho_{SB}(t_0)U^\dagger(t_1, t_0), \tag{2}$$

where $U(t_1, t_0) \equiv \mathcal{T}_+ \exp[-i\int_{t_0}^{t_1} dt\, H_{\rm tot}(\lambda_t)]$ with the time-ordering operator $\mathcal{T}_+$.

Then, at time $t_1 > t_0$ we interrupt the evolution by a CP operation $\mathcal{A}(r_1)$, which only acts on the system and yields outcome r_1 (for instance, the result of a projective measurement). Mathematically, we write the operation as

$$\tilde{\rho}_{SB}(t_1^+, r_1) = [\mathcal{A}(r_1) \otimes \mathcal{I}_B]\rho_{SB}(t_1^-). \tag{3}$$

Here $t_1^\pm = \lim_{\epsilon\searrow 0}(t_1 \pm \epsilon)$ denotes a time shortly after or before t_1 and $\mathcal{I}_B$ denotes the identity superoperator acting on B. Note that we assume the control operation to happen instantaneously. It ensures that the experimenter has complete control over the operation: If the control operation takes longer, it would also affect the bath and a clear separation of the dynamics into a dynamics induced by the bath or the external agent becomes problematic. The final state of knowledge after the operation $\tilde{\rho}_{SB}(t_1^+, r_1)$ can explicitly depend on the outcome r_1. Since $\mathcal{A}(r_1)$ is CP, it admits an operator-sum (Kraus) representation of the form

$$\mathcal{A}(r_1)\rho_S = \sum_\alpha A_\alpha(r_1)\rho_S A_\alpha^\dagger(r_1), \tag{4}$$

but we do not require it to be trace preserving. For this reason we have used a tilde in Eq. (3) to emphasize that the state is not normalized. The probability to observe outcome r_1 at time t_1 is $p(r_1) = \mathrm{tr}_{SB}\{\tilde{\rho}_{SB}(t_1^+, r_1)\}$. Then the normalized system state after the control operation at time t_1 becomes $\rho_S(t_1^+, r_1) = \mathcal{A}(r_1)\rho_S(t_1^-)/p(r_1)$. Notice that the map $\mathcal{A}(r_1)/p(r_1)$ is completely positive and trace preserving (CPTP), but nonlinear in the state $\rho_S(t_1^-)$. It is the quantum analog of Bayes' rule. The average system state is accordingly

$$\rho_S(t_1^+) = \sum_{r_1} p(r_1)\rho_S(t_1^+, r_1) = \sum_{r_1} \mathcal{A}(r_1)\rho_S(t_1^-). \quad (5)$$

This would also correspond to our state of knowledge if we ignore the outcome r_1. Notice that the average control operation $\sum_{r_1} \mathcal{A}(r_1)$ is now a CPTP map and can be written as

$$\sum_{r_1} \mathcal{A}(r_1)\rho_S = \sum_{r_1,\alpha} A_\alpha(r_1)\rho_S A_\alpha^\dagger(r_1), \quad (6)$$

with $\sum_{r_1,\alpha} A_\alpha^\dagger(r_1)A_\alpha(r_1) = 1_S$.

We then iterate the above procedure by letting the joint system-bath state evolve unitarily up to time $t_2 > t_1$: $\rho_{SB}(t_2^-, r_1) = \mathcal{U}_{2,1}(r_1)\rho_{SB}(t_1^+, r_1)$. Now, however, the unitary operation is allowed to depend on r_1 by changing the control protocol of the system Hamiltonian $H_S[\lambda_t(r_1)]$. This actually corresponds to the simplest form of measurement-based quantum feedback control. Then, at time t_2 we subject the system to another CP control operation $\mathcal{A}(r_2|r_1)$, which is also allowed to depend on r_1 and which gives outcome r_2. Thus, $\rho_{SB}(t_2^+, \mathbf{r}_2) = [\mathcal{A}(r_2|r_1) \otimes \mathcal{I}_B]\rho_{SB}(t_2^-, r_1)$, where $\mathbf{r}_2 = (r_2, r_1)$.

We can reiterate the above procedure by letting the external agent interrupt the unitary system-bath evolution at times $t_n > t_{n-1} > \cdots > t_1$. Let us denote by t an arbitrary time after the nth but before the $(n+1)$th control operation, i.e., $t_{n+1} > t > t_n$. The unnormalized state of the system conditioned on the sequence of outcomes $\mathbf{r}_n$ at such a time t is then given by

$$\begin{aligned}\tilde{\rho}_S(t, \mathbf{r}_n) &= \mathfrak{T}[\mathcal{A}(r_n|\mathbf{r}_{n-1}), \dots, \mathcal{A}(r_1)] \\ &\equiv \mathrm{tr}_B\{\mathcal{U}_{t,n}(\mathbf{r}_n)\mathcal{A}(r_n|\mathbf{r}_{n-1})\cdots\mathcal{U}_{2,1}(r_1) \\ &\quad \times \mathcal{A}(r_1)\mathcal{U}_{1,0}\rho_{SB}(t_0)\}. \end{aligned} \quad (7)$$

Here we have introduced the process tensor $\mathfrak{T}$. Its variable inputs are the set of control operations $\{\mathcal{A}(r_i|\mathbf{r}_{i-1})\}_{i=1}^n$, but not the initial state of the system, the bath, or the composite. The trace of the process tensor gives the probability to observe the sequence of outcomes $\mathbf{r}_n$,

$$p(\mathbf{r}_n) = \mathrm{tr}_S\{\mathfrak{T}[\mathcal{A}(r_n|\mathbf{r}_{n-1}), \dots, \mathcal{A}(r_1)]\}, \quad (8)$$

such that the normalized state of the system can be written as

$$\rho_S(t, \mathbf{r}_n) = \frac{\mathfrak{T}[\mathcal{A}(r_n|\mathbf{r}_{n-1}), \dots, \mathcal{A}(r_1)]}{p(\mathbf{r}_n)}. \quad (9)$$

The process tensor is an operationally well-defined object for any open system dynamics (in particular for any environment) for any possible, physically admissible form of interventions in an experiment. It is different from typical quantum trajectory methods or quantum jump expansions [46–48,78–80], which rely on continuously monitoring the environment of the system. This framework is included as a limiting case in the process tensor, but it does not rely on it: any set of discrete times is allowed and the (often uncontrollable) environment does not need to be monitored. For further research on this topic see Refs. [56–65].

III. PROCESS TENSOR FROM REPEATED INTERACTIONS

In practice, the control operations $\mathcal{A}(r_n|\mathbf{r}_{n-1})$ do not happen spontaneously but require an active intervention from the outside. They are typically implemented by letting the system interact for a short time with an externally prepared apparatus (e.g., a memory or detector). It is the interaction time and the initial state of the apparatus which can be usually well controlled experimentally. This insight will naturally lead us to the framework of repeated interactions, in which we will model at least parts of the external apparatus explicitly.

The main mathematical insight of this section rests on Stinespring's theorem [81], which states that any CPTP map $\mathcal{A}$ can be seen as the reduced dynamics of some unitary evolution in an extended space. More precisely, we can always write

$$\mathcal{A}\rho_S = \mathrm{tr}_U\{V\rho_S \otimes \rho_U V^\dagger\}, \quad (10)$$

where we labeled the additional subsystem by U for unit in view of the thermodynamic framework considered later on and in unison with Ref. [66]. The unit is in an initial state ρ_U and V denotes the unitary operator which acts jointly on SU. Furthermore, any non-trace-preserving CP map $\mathcal{A}(r)$ with outcome r can be modeled as [78]

$$\mathcal{A}(r)\rho_S = \mathrm{tr}_U\{P_U(r)V\rho_S \otimes \rho_U V^\dagger P_U(r)\}, \quad (11)$$

where each positive operator $P_U(r)$ acts only on $\mathcal{H}_U$ and fulfills $\sum_r P_U^2(r) = 1_U$. Notice that Eq. (10) can be recovered from Eq. (11) either by choosing $P_U(r) = 1_U$ or by summing over r. In accordance with our previous superoperator notation, we introduce $\mathcal{P}_U(r)\rho_U \equiv P_U(r)\rho_U P_U(r)$ and $\mathcal{V}\rho_{SU} \equiv V\rho_{SU}V^\dagger$ such that we can write Eq. (11) in the shorter form $\mathcal{A}(r)\rho_S = \mathrm{tr}_U\{\mathcal{P}_U(r)\mathcal{V}\rho_S \otimes \rho_U\}$.

It is worth remarking that the above representation of the control operation is not unique. What we are aiming at here is a minimal consistent thermodynamic description for any given set of control operations. If additional physical insights are available, they have to be taken into account (see Sec. V for a clear experimental example). The only important point, however, is that the general operator-sum representation (4) can be decomposed into more primitive operations (a unitary and a measurement of the unit).

The whole process tensor $\mathfrak{T}[\mathcal{A}(r_n|\mathbf{r}_{n-1}), \dots, \mathcal{A}(r_1)]$ can then be seen as describing the reduced dynamics of a system coupled to a stream of units, which interact sequentially at times $t_n > \cdots > t_1$ with the system (see Fig. 1). This constitutes the framework of repeated interactions. Then the unnormalized joint state of the system and all units, which have interacted with the system up to time t $(t_{n+1} > t > t_n)$ with outcome $\mathbf{r}_n$, can be written as

$$\begin{aligned}\tilde{\rho}_{SU(\mathbf{n})}(t, \mathbf{r}_n) &= \mathrm{tr}_B\{\mathcal{U}_{t,t_n}(\mathbf{r}_n)\mathcal{P}_{U(n)}(r_n|\mathbf{r}_{n-1})\mathcal{V}_{SU(n)}(\mathbf{r}_{n-1})\cdots \\ &\quad \times \mathcal{U}_{2,1}(r_1)\mathcal{P}_{U(1)}(r_1)\mathcal{V}_{SU(1)} \\ &\quad \times \mathcal{U}_{1,0}[\rho_{SB}(t_0) \otimes \rho_{U(n)}(\mathbf{r}_{n-1}) \otimes \cdots \otimes \rho_{U(1)}]\}. \end{aligned} \quad (12)$$

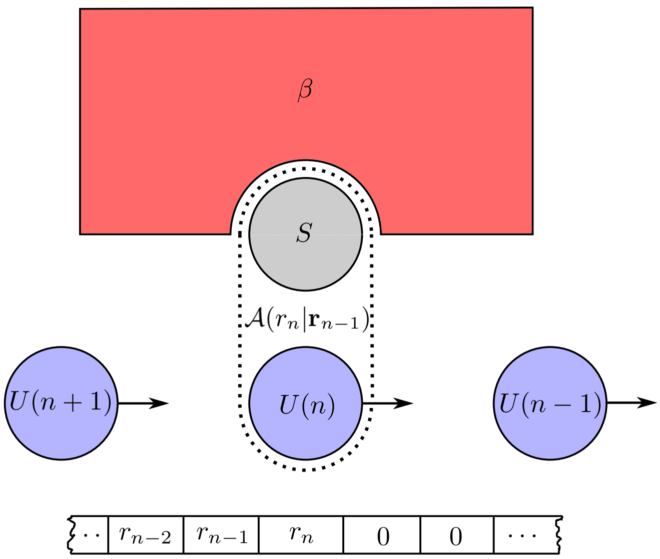

FIG. 1. Sketch of the setup. A system S (gray circle) is in contact with a bath B (red box, later taken to be at inverse temperature β) undergoing in general dissipative dynamics. The evolution of the open quantum system is interrupted at times t_n by control operations $\mathcal{A}(r_n|\mathbf{r}_{n-1})$, which are triggered by the interaction with an external ancilla system called the unit $U(n)$ (blue circles). Each control operation has an outcome r_n, which is recorded in a memory (e.g., a tape of bits), and future control operations are allowed to depend on previous outcomes. The memory for future outcomes is set in a standard state 0.

Except for the unitary system-bath evolution superoperator $\mathcal{U}$ (where the subscripts denote time intervals), subscripts are used to denote the Hilbert space on which the respective (super)operator is acting. In this respect, the joint space of all n units is denoted by $U(\mathbf{n})$. Note that $\mathcal{V}_{SU(n)}(\mathbf{r}_{n-1})$ depends on all previous outcomes $\mathbf{r}_{n-1}$, but due to causality it cannot depend on the nth outcome r_n. The same holds true for the initial state $\rho_{U(n)}(\mathbf{r}_{n-1})$ of the nth unit and also the chosen projection operator $\mathcal{P}_{U(n)}(r_n|\mathbf{r}_{n-1})$ can depend on $\mathbf{r}_{n-1}$. Therefore, the external agent has all the freedom one needs to engineer a desired control operation $\mathcal{A}(r_n|\mathbf{r}_{n-1})$. By construction, after tracing out the units, we obtain the process tensor for the system $\mathfrak{T}[\mathcal{A}(r_n|\mathbf{r}_{n-1}),\dots,\mathcal{A}(r_1)] = \text{tr}_{U(\mathbf{n})}\{\tilde{\rho}_{SU(\mathbf{n})}(t,\mathbf{r}_n)\}$. As it is in most situations obvious from the context which superoperator acts on which object living in which space, we will usually drop the subscripts $S, U(n),\dots$ on superoperators.

IV. OPERATIONAL QUANTUM STOCHASTIC THERMODYNAMICS

A. Preliminary considerations

The process tensor is a formal object which does not make any assumptions about the system-bath dynamics. On the contrary, the standard ensemble-average (or, better, unmeasured) framework of quantum thermodynamics relies on a weakly coupled, memoryless, and macroscopic bath [3–6]. In this section we remain within this weak-coupling paradigm because possible extensions beyond the weak-coupling and Markovian assumption have only recently drawn attention (see also Sec. VII B). Furthermore, we consider in this section only the case of a single heat bath at inverse temperature $\beta = 1/T$ ($k_B \equiv 1$). The extension to multiple heat baths is subtle (see Sec. VII A).

Let us focus on the interval (t_{n-1}, t_n) (excluding the control operations at the boundaries) and let $\rho_S(t)$ be the system state at time $t \in (t_{n-1}, t_n)$ (which later on is allowed to depend on $\mathbf{r}_{n-1}$). The state function internal energy and system entropy for an arbitrary system state $\rho_S(t)$ are defined as

$$E_S(t) \equiv \text{tr}_S\{H_S(\lambda_t)\rho_S(t)\}, \tag{13}$$

$$S_S(t) \equiv S_{\text{vN}}[\rho_S(t)]. \tag{14}$$

According to the first law, the change in system energy $\Delta E_S^{(n)} \equiv E_S(t_n^-) - E_S(t_{n-1}^+)$ can be split into heat and work $\Delta E_S^{(n)} = W_S^{(n)} + Q_S^{(n)}$ by defining

$$W_S^{(n)} \equiv \int_{t_{n-1}^+}^{t_n^-} dt\, \text{tr}_S\left\{\frac{\partial H_S(\lambda_t)}{\partial t}\rho_S(t)\right\}, \tag{15}$$

$$Q_S^{(n)} \equiv \int_{t_{n-1}^+}^{t_n^-} dt\, \text{tr}_S\left\{H_S(\lambda_t)\frac{\partial \rho_S(t)}{\partial t}\right\}. \tag{16}$$

Furthermore, the validity of the second law can also be derived and states that the entropy production is always positive,

$$\Sigma^{(n)} \equiv \Delta S_S^{(n)} - \beta Q_S^{(n)} \geqslant 0, \tag{17}$$

where $\Delta S_S^{(n)} \equiv S_S(t_n^-) - S_S(t_{n-1}^+)$.

Our goal in the rest of this section is to find definitions of internal energy, work, heat, and system entropy along a single trajectory, where a trajectory is defined by the observed sequence of outcomes $\mathbf{r}_n$. The sought-after definitions are required to be intuitively meaningful and to fulfill the first law at the trajectory level and the second law on average. Further appeal to our definitions will be added in Secs. V–VII.

Note that, after tomographic reconstruction of the process tensor (see Sec. II), we know the conditional system states $\rho_S(t_n^\pm, \mathbf{r}_n)$ only right before or right after the nth control operation, but not in between for $t_{n-1} < t < t_n$. To compute the work (15) or heat (16) in between two control operations, additional theoretical input is required, e.g., by solving the master equation for the system or by other forms of inference. This ensures that we recover the standard weak-coupling framework of quantum thermodynamics in the absence of any control operations (see Sec. VI C). Nevertheless, as it increases the computational effort, we present in Sec. VI G possible ways to avoid any additional theory input.

For definiteness, we aim at a stochastic thermodynamic description in the time interval $(t_{n-1}, t_n]$ starting shortly after the $(n-1)$th control operation and ending shortly after the nth control operation. The change in any state function X over the complete interval is denoted by $\Delta X^{(n]}$, whereas $\Delta X^{(n)}$ denotes the change in (t_{n-1}, t_n) (excluding the nth control operation) and ΔX^{ctrl} the change due to the control operation only. Changes in the respective time intervals of any quantity which is not a state function are denoted without a Δ ($X^{(n]}$, $X^{(n)}$, or X^{ctrl}).

B. Stochastic energy and first law

To formulate the first law at the trajectory level correctly, we need to take into account the internal energy of the system and all units. Thus, we define the trajectory-dependent internal energy

$$E_{SU(\mathbf{n})}(t,\mathbf{r}_n) \equiv \text{tr}_{SU(\mathbf{n})}\{H_{SU(\mathbf{n})}(\lambda_t,\mathbf{r}_n)\rho_{SU(\mathbf{n})}(t,\mathbf{r}_n)\}, \quad (18)$$

where $H_{SU(\mathbf{n})}(\lambda_t,\mathbf{r}_n) = H_S(\lambda_t,\mathbf{r}_n) + \sum_{i=1}^n H_{U(i)}$ denotes the sum of the system and all unit Hamiltonians. Since the Hamiltonian is additive, the internal energy splits into its marginal contributions in an obvious way,

$$E_{SU(\mathbf{n})}(t,\mathbf{r}_n) = E_S(t,\mathbf{r}_n) + \sum_{i=1}^n E_{U(i)}(t,\mathbf{r}_n). \quad (19)$$

Notice that it is always simple to get rid of the units in the energetic description by assuming that $H_{U(i)} \sim 1_{U(i)}$. However, the energetic changes of the units can bear some interesting nontrivial features. For instance, it is not sufficient to consider only the actual nth unit in the energetic balance: In our general theory the energy of previous units can change even though they are physically decoupled from the system. This phenomenon does not necessarily require quantum entanglement and simply occurs because our state of knowledge about past units $U(i<n)$ can change depending on the outcome r_n (discussed below).

In the absence of any control operations, the first law simply follows from the preceding section and reads

$$\Delta E_S^{(n)}(\mathbf{r}_{n-1}) = W_S^{(n)}(\mathbf{r}_{n-1}) + Q_S^{(n)}(\mathbf{r}_{n-1}), \quad (20)$$

because the marginal state of the units does not change and hence $\Delta E_{U(i)} = 0$ for all i. Note that the work $W_S^{(n)}(\mathbf{r}_{n-1})$ and heat $Q_S^{(n)}(\mathbf{r}_{n-1})$ depend on previous outcomes $\mathbf{r}_{n-1}$ for two reasons: First, the initial system state $\rho_S(t_{n-1}^+,\mathbf{r}_{n-1})$ depends on it, and second, the Hamiltonian $H(\lambda_t,\mathbf{r}_{n-1})$ can be a function of it in case we apply feedback control.

The first law during the control operation at time t_n is more interesting as the internal energy of both the system and units can change. In total, the energetic cost E^{ctrl} of the control operation is defined by

$$E^{\text{ctrl}}(t_n,\mathbf{r}_n) \equiv \Delta E_S^{\text{ctrl}}(t_n,\mathbf{r}_n) + \sum_{i=1}^n \Delta E_{U(i)}^{\text{ctrl}}(t_n,\mathbf{r}_n). \quad (21)$$

It is not a state function and can be split into a worklike and heatlike contribution,

$$E^{\text{ctrl}}(t_n,\mathbf{r}_n) = W^{\text{ctrl}}(t_n,\mathbf{r}_{n-1}) + Q^{\text{ctrl}}(t_n,\mathbf{r}_n). \quad (22)$$

This splitting stems from the convention we used to implement the control operation $\mathcal{A}(r_n|\mathbf{r}_{n-1})$ in the repeated interaction framework: We first applied the unitary operation $\mathcal{V}(\mathbf{r}_{n-1})$ to the joint system-unit state and afterward measured the unit via $\mathcal{P}(r_n)$. In general, we therefore use the definitions

$$\begin{aligned} W^{\text{ctrl}}&(t_n,\mathbf{r}_{n-1}) \\ &= \text{tr}_{SU(\mathbf{n})}\{H_{SU(\mathbf{n})}(\lambda_n,\mathbf{r}_{n-1})[\mathcal{V}(\mathbf{r}_{n-1})\rho_{SU(\mathbf{n})}(t_n^-,\mathbf{r}_{n-1}) \\ &\quad - \rho_{SU(\mathbf{n})}(t_n^-,\mathbf{r}_{n-1})]\}, \end{aligned} \quad (23)$$

$$\begin{aligned} Q^{\text{ctrl}}(t_n,\mathbf{r}_n) = \text{tr}_{SU(\mathbf{n})}\{&H_{SU(\mathbf{n})}(\lambda_n,\mathbf{r}_{n-1})[\rho_{SU(\mathbf{n})}(t_n^+,\mathbf{r}_n) \\ &- \mathcal{V}(\mathbf{r}_{n-1})\rho_{SU(\mathbf{n})}(t_n^-,\mathbf{r}_{n-1})]\}, \end{aligned} \quad (24)$$

with $\lambda_n \equiv \lambda_{t_n}$. Notice that the worklike contribution does not depend on the actual measurement outcome r_n and corresponds to the energetic changes caused by a reversible (unitary) operation. The meaning of the heat injected during the control operation $Q^{\text{ctrl}}(t_n,\mathbf{r}_n)$ will be discussed further below, but we remark that a very similar construction was called quantum heat in Ref. [51]. A difference, which turns out to be crucial, is the fact that Elouard *et al.* applied this definition for the system only without including the unit in the description [51], which causes different interpretations. Furthermore, we are more cautious and do not call it quantum heat. For further discussion on this topic see Sec. VI A.

For now, let us note that both quantities have some additional important properties. First of all, both can be split additively into changes affecting the system or the units,

$$W^{\text{ctrl}}(\mathbf{r}_{n-1}) = W_S^{\text{ctrl}}(\mathbf{r}_{n-1}) + \sum_{i=1}^n W_{U(i)}^{\text{ctrl}}(\mathbf{r}_{n-1}), \quad (25)$$

$$Q^{\text{ctrl}}(\mathbf{r}_n) = Q_S^{\text{ctrl}}(\mathbf{r}_n) + \sum_{i=1}^n Q_{U(i)}^{\text{ctrl}}(\mathbf{r}_n). \quad (26)$$

In particular, the part affecting the system can be expressed solely in terms of the control operation $\mathcal{A}(r_n|\mathbf{r}_{n-1})$ and its average $\mathcal{A}_n \equiv \sum_{r_n}\mathcal{A}(r_n|\mathbf{r}_{n-1})$ and is thus independent of the details of the unit $U(n)$ (see also Ref. [82]). Specifically,

$$W_S^{\text{ctrl}}(\mathbf{r}_{n-1}) = \text{tr}_S\{H_S(\lambda_n,\mathbf{r}_{n-1})(\mathcal{A}_n - \mathcal{I})\rho_S(t_n^-,\mathbf{r}_{n-1})\}, \quad (27)$$

$$Q_S^{\text{ctrl}}(\mathbf{r}_n) = \text{tr}_S\{H_S(\lambda_n,\mathbf{r}_{n-1})[\mathcal{A}(r_n|\mathbf{r}_{n-1}) - \mathcal{A}_n]\rho_S(t_n^-,\mathbf{r}_{n-1})\}. \quad (28)$$

Furthermore, if we use that the marginal state of the previous $n-1$ units does not change during the unitary operation $\mathcal{V}(\mathbf{r}_{n-1})$, we can deduce that the work actually depends only on the energetic changes of the system and the nth unit,

$$W^{\text{ctrl}}(\mathbf{r}_{n-1}) = W_S^{\text{ctrl}}(\mathbf{r}_{n-1}) + W_{U(n)}^{\text{ctrl}}(\mathbf{r}_{n-1}). \quad (29)$$

The previous properties allow us to deduce two separate first laws for the control operation

$$\Delta E_S^{\text{ctrl}}(\mathbf{r}_n) = W_S^{\text{ctrl}}(\mathbf{r}_{n-1}) + Q_S^{\text{ctrl}}(\mathbf{r}_n), \quad (30)$$

$$\Delta E_{U(\mathbf{n})}^{\text{ctrl}}(\mathbf{r}_n) = W_{U(n)}^{\text{ctrl}}(\mathbf{r}_{n-1}) + Q_{U(\mathbf{n})}^{\text{ctrl}}(\mathbf{r}_n). \quad (31)$$

Finally, we can deduce that the average heat injected into the system or the previous units $U(i)$ $(i<n)$ is always zero. Specifically,

$$Q_{S,U(i<n)}^{\text{ctrl}}(t_n,\mathbf{r}_{n-1}) \equiv \sum_{r_n} p(r_n|\mathbf{r}_{n-1}) Q_{S,U(i<n)}^{\text{ctrl}}(t_n,\mathbf{r}_n) = 0, \quad (32)$$

where $p(r_n|\mathbf{r}_{n-1}) \equiv p(\mathbf{r}_n)/p(\mathbf{r}_{n-1})$. Note that Eq. (32) implies $Q_{S,U(i<n)}^{\text{ctrl}}(t_n) = \sum_{\mathbf{r}_n} p(\mathbf{r}_n) Q_S^{\text{ctrl}}(\mathbf{r}_n) = 0$. In contrast, for the actual unit we have $Q_{U(n)}^{\text{ctrl}}(t_n) = 0$ if and only if $[H_{U(n)}, P(r_n|\mathbf{r}_{n-1})] = 0$. We remark that it also appears reasonable to call Q^{ctrl} heat because the emergence of a projector $\mathcal{P}(r_n)$ requires one, in a microscopic picture, to couple the unit to some macroscopic and classical device, which allows the unit to lose information irreversibly due to dissipation and decoherence [83]. This last phenomenological step in quantum measurement theory is sometimes referred to as the

Heisenberg cut [79]. It necessarily entails a certain level of arbitrariness because we do not explicitly model the microscopic interaction between the unit and the final classical environment. It therefore remains unclear how far any notion of temperature is associated with the heat $Q^{\rm ctrl}$ and we will investigate this further in the next section.

To conclude, after adding the first laws with and without a control operation together, we obtain, for the changes over a complete interval,

$$\Delta E_S^{[n]}(\mathbf{r}_n) + \Delta E_{U(\mathbf{n})}^{[n]}(\mathbf{r}_n) = W^{(n]}(\mathbf{r}_{n-1}) + Q^{[n]}(\mathbf{r}_n), \tag{33}$$

where we can split the work and heat into $W^{(n]}(\mathbf{r}_{n-1}) = W^{\rm ctrl}(\mathbf{r}_{n-1}) + W_S^{(n)}(\mathbf{r}_{n-1})$ and $Q^{[n]}(\mathbf{r}_n) = Q^{\rm ctrl}(\mathbf{r}_n) + Q_S^{(n)}(\mathbf{r}_{n-1})$. If we assume trivial Hamiltonians for the units ($H_{U(i)} \sim 1_U$), we get the simplified first law

$$\Delta E_S^{[n]}(\mathbf{r}_n) = W_S^{[n]}(\mathbf{r}_{n-1}) + Q_S^{[n]}(\mathbf{r}_n). \tag{34}$$

For the entropic balance, in general, it will not be that simple.

C. Stochastic entropy and second law

To account for all entropic changes, we need to consider not only the system and all units, but also the entropy of the outcomes $\mathbf{r}_n$ stored in a classical memory (see Fig. 1). This is a crucial point, which distinguishes our theory from standard stochastic thermodynamics where the entropic contribution of the measurement results is neglected (this will play an important role in Sec. VI E). In general, the process tensor depends explicitly on the knowledge of $\mathbf{r}_n$, which cannot be neglected. Furthermore, it is important to also keep the past information of all previous units $U(i<n)$ and outcomes $\mathbf{r}_{n-1}$ because we explicitly allow the current unit and Hamiltonian to depend on all earlier outcomes (this is, for instance, essential if we apply time-delayed feedback control). Thus, we define the stochastic entropy of the process as

$$S_{SU(\mathbf{n})}(t,\mathbf{r}_n) \equiv -\ln p(\mathbf{r}_n) + S_{\rm vN}[\rho_{SU(\mathbf{n})}(t,\mathbf{r}_n)]. \tag{35}$$

Note that the probability $p(\mathbf{r}_n)$ of a particular trajectory can be straightforwardly computed from knowing the unnormalized state of the system [see Eq. (8)]. If this state is not known, evaluation of Eq. (35) requires knowledge of many experimentally sampled trajectories first. Notice that the same is true for the definition of the trajectory-dependent entropy in classical stochastic thermodynamics [11–13].

Next we define the entropy production along a single trajectory over a time interval $(t_{n-1}, t_n]$ by adding to the change in stochastic entropy the heat flow into the system,

$$\Sigma^{[n]}(\mathbf{r}_n) \equiv \Delta S_{SU(\mathbf{n})}^{[n]}(\mathbf{r}_n) - \beta Q_S^{[n]}(\mathbf{r}_n). \tag{36}$$

As in classical stochastic thermodynamics, this expression can have either sign, but on average it is always positive, as we will show below. Crucially, we have taken into account only the heat associated with system changes whereas we did not include $Q_{U(\mathbf{n})}^{\rm ctrl}$ in the entropic balance. This will give us the correct result in all limiting cases and, if we use the commonly made assumption that $H_{U(i)} \sim 1_{U(i)}$, we anyway have $Q_{U(\mathbf{n})}^{\rm ctrl} = 0$ always. Furthermore, as we do not microscopically model the final projective measurement step of the units, it is also unclear which temperature we should associate with heat changes in the units and hence including $Q_{U(\mathbf{n})}^{\rm ctrl}$ in the second law would necessarily imply some ambiguity. While these are all good *a posteriori* arguments, the question whether there exist good *a priori* arguments remains.

To show the positivity of the average entropy production, it is useful to split it into two contributions similar to the first law

$$\Sigma^{[n]}(\mathbf{r}_n) \equiv \Sigma^{\rm ctrl}(\mathbf{r}_n) + \Sigma^{(n)}(\mathbf{r}_{n-1}), \tag{37}$$

with

$$\Sigma^{\rm ctrl}(\mathbf{r}_n) = \Delta S_{SU(\mathbf{n})}^{\rm ctrl}(\mathbf{r}_n) - \beta Q_S^{\rm ctrl}(\mathbf{r}_n), \tag{38}$$

$$\Sigma^{(n)}(\mathbf{r}_{n-1}) = \Delta S_{SU(\mathbf{n})}^{(n)}(\mathbf{r}_{n-1}) - \beta Q_S^{(n)}(\mathbf{r}_{n-1}). \tag{39}$$

We will now show that the second contribution $\Sigma^{(n)}$ is positive even along a single trajectory, whereas the first contribution $\Sigma^{\rm ctrl}$ is positive only on average.

To show $\Sigma^{(n)}(\mathbf{r}_{n-1}) \geqslant 0$ we will use Eq. (17), which holds for an arbitrary initial state $\rho_S(t_{n-1}^+, \mathbf{r}_{n-1})$, together with the fact that the system evolution in between two control operations can be described by a CPTP map independent of the initial state. This is true within the weak-coupling paradigm of quantum thermodynamics [3–6] where the time evolution is governed by a (possible time-dependent) master equation in Lindblad-Gorini-Kossakowski-Sudarshan form. Let us define the CPTP map as $\mathcal{E}_n = \mathcal{E}_n(\mathbf{r}_{n-1})$ such that

$$\rho_S(t_n^-, \mathbf{r}_{n-1}) = \mathcal{E}_n \rho_S(t_{n-1}^+, \mathbf{r}_{n-1}). \tag{40}$$

The inequality $\Sigma^{(n)}(\mathbf{r}_{n-1}) \geqslant 0$ can then be derived along the following lines.

First, by using the mutual information $I_{S:U(\mathbf{n})}$ between the system and the stream of units, we can split the change in joint entropy as

$$\begin{aligned}\Delta S_{SU(\mathbf{n})}^{(n)}(\mathbf{r}_n) = {} & S_{\rm vN}[\rho_S(t_n^-, \mathbf{r}_{n-1})] + S_{\rm vN}[\rho_{U(\mathbf{n})}(t_n^-, \mathbf{r}_{n-1})] \\ & - I_{S:U(\mathbf{n})}(t_n^-) - S_{\rm vN}[\rho_S(t_{n-1}^+, \mathbf{r}_{n-1})] \\ & - S_{\rm vN}[\rho_{U(\mathbf{n})}(t_{n-1}^+, \mathbf{r}_{n-1})] + I_{S:U(\mathbf{n})}(t_{n-1}^+).\end{aligned} \tag{41}$$

Since the marginal state of the units does not change under the action of the CPTP map $\mathcal{E}_n$, their entropic contribution cancels out and we can write in short $\Delta S_{SU(\mathbf{n})}^{(n)}(\mathbf{r}_{n-1}) = \Delta S_S^{(n)}(\mathbf{r}_{n-1}) - \Delta I_{S:U(\mathbf{n})}^{(n)}(\mathbf{r}_{n-1})$. Let us now add the entropy flow $-\beta Q_S^{(n)}(\mathbf{r}_{n-1})$ into the bath to the entropy balance. From the second law (17) we can then infer that

$$\Delta S_{SU(\mathbf{n})}^{(n)}(\mathbf{r}_{n-1}) - \beta Q_S^{(n)}(\mathbf{r}_{n-1}) \geqslant -\Delta I_{S:U(\mathbf{n})}^{(n)}(\mathbf{r}_{n-1}). \tag{42}$$

The positivity of the right-hand side is then guaranteed by contractivity of relative entropy under CPTP maps [84,85]. More specifically, the chain of (in)equalities

$$\begin{aligned} & I_{S:U(\mathbf{n})}(t_{n-1}^+, \mathbf{r}_{n-1}) \\ & = D[\rho_{SU(\mathbf{n})}(t_{n-1}^+, \mathbf{r}_{n-1}) \| (\rho_S \otimes \rho_{U(\mathbf{n})})(t_{n-1}^+, \mathbf{r}_{n-1})] \\ & \geqslant D[\mathcal{E}_n \rho_{SU(\mathbf{n})}(t_{n-1}^+, \mathbf{r}_{n-1}) \| \mathcal{E}_n(\rho_S \otimes \rho_{U(\mathbf{n})})(t_{n-1}^+, \mathbf{r}_{n-1})] \\ & = I_{S:U(\mathbf{n})}(t_n^-, \mathbf{r}_{n-1})\end{aligned} \tag{43}$$

applies, where it is essential that $\mathcal{E}_n$ acts only on S and not on $U(\mathbf{n})$. This concludes the proof of positivity of $\Sigma^{(n)}(\mathbf{r}_{n-1})$.

Next we will show that $\Sigma^{\rm ctrl}(\mathbf{r}_n)$ is positive on average. More specifically, we will show that

$$\Sigma^{\rm ctrl}(\mathbf{r}_{n-1}) \equiv \sum_{r_n} p(r_n|\mathbf{r}_{n-1})\Sigma^{\rm ctrl}(\mathbf{r}_n) \geqslant 0. \tag{44}$$

If this holds, then it also follows that $\Sigma^{\rm ctrl}(t_n) = \sum_{\mathbf{r}_n} p(\mathbf{r}_n)\Sigma^{\rm ctrl}(\mathbf{r}_n) \geqslant 0$. After taking the average and using Eq. (32), we are left with three terms

$$\begin{aligned}\Sigma^{\rm ctrl}(t_n, \mathbf{r}_{n-1}) = {} & S_{\rm Sh}[p(r_n|\mathbf{r}_{n-1})] \\ & + \sum_{r_n} p(r_n|\mathbf{r}_{n-1})S_{\rm vN}[\rho_{SU(\mathbf{n})}(t_n^+, \mathbf{r}_n)] \\ & - S_{\rm vN}[\rho_{SU(\mathbf{n})}(t_n^-, \mathbf{r}_{n-1})],\end{aligned} \tag{45}$$

where $S_{\rm Sh}[p(r_n|\mathbf{r}_{n-1})]$ is the Shannon entropy of the conditional probability $p(r_n|\mathbf{r}_{n-1})$.[3] The positivity of $\Sigma^{\rm ctrl}(t_n, \mathbf{r}_{n-1})$ then follows from combining two theorems in quantum measurement theory.

Lemma. Let ρ be an arbitrary state, $\{P_n\}_n$ a set of positive operators fulfilling $\sum_n P_n^2 = 1$, $p_n = \text{tr}\{P_n\rho P_n\}$ the probability to obtain outcome n, and $\rho^{(n)} = P_n\rho P_n/p_n$ the postmeasurement state conditioned on outcome n. Then

$$S_{\rm vN}(\rho) \leqslant S_{\rm Sh}(p_n) + \sum_n p_n S_{\rm vN}(\rho^{(n)}). \tag{46}$$

Proof. We first use that for any such set $\{P_n\}_n$ (see Theorem 11 in Ref. [80], or Ref. [86]),

$$S_{\rm vN}(\rho) \leqslant S_{\rm vN}\left(\sum_n p_n\rho^{(n)}\right), \tag{47}$$

i.e., the average uncertainty after the measurement can only increase. Next we use (see Theorem 11.10 in Ref. [77], or Refs. [87,88])

$$S_{\rm vN}\left(\sum_n p_n\rho^{(n)}\right) \leqslant S_{\rm Sh}(p_n) + \sum_n p_n S_{\rm vN}(\rho^{(n)}). \tag{48}$$

This concludes the proof. ∎

We now apply the Lemma to Eq. (45). If we identify $\{P_n\}$ with $\{P(r_n|\mathbf{r}_{n-1})\}$ acting in the joint system-unit space, the probability p_n with the conditional probability $p(r_n|\mathbf{r}_{n-1})$, and the postmeasurement state $\rho^{(n)}$ with $\rho_{SU(\mathbf{n})}(t_n^+, \mathbf{r}_n)$, we can deduce that

$$\begin{aligned}& S_{\rm Sh}[p(r_n|\mathbf{r}_{n-1})] + \sum_{r_n} p(r_n|\mathbf{r}_{n-1})S_{\rm vN}[\rho_{SU(\mathbf{n})}(t_n^+, \mathbf{r}_n)] \\ & \geqslant S_{\rm vN}[\mathcal{V}\rho_{SU(\mathbf{n})}(t_n^-, \mathbf{r}_{n-1})].\end{aligned} \tag{49}$$

Using that the von Neumann entropy is invariant under unitary transformations, we deduce our desired result. Finally, we remark that the inequality (48) was used before in quantum thermodynamics to show the positivity of the second law for a Maxwell demon employing quantum measurements [89].

[3]This is to be distinguished from the conventional conditional entropy given by $\sum_{\mathbf{r}_{n-1}} p(\mathbf{r}_{n-1})S_{\rm Sh}[p(r_n|\mathbf{r}_{n-1})]$.

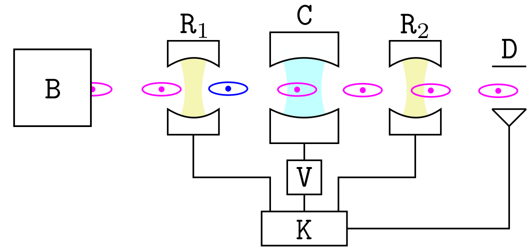

FIG. 2. Sketch of the experimental setup (compare with Fig. 1 from Ref. [71]). We wish to control the central microwave cavity C by a beam of atoms prepared in B. The atoms can be manipulated by the Ramsey cavities R_1 and R_2 and read out by the detector D. The measurement results are sent to a controller K, which decides in real time whether to send a sensor atom to measure the state of the cavity (pink circles) or an emitter or absorber atom to manipulate the state of the cavity (blue circle). In the latter case the atoms are brought into exact resonance with the cavity by applying a voltage V.

V. REAL-TIME PREPARATION AND STABILIZATION OF PHOTON NUMBER STATES VIA QUANTUM FEEDBACK

The ability to control individual quantum systems and to protect them against decoherence have become key challenges in modern quantum science. Recently, experiments in quantum optics reported on the preparation and stabilization of photon number states by using quantum feedback control [70,71]; see also Ref. [90] for previous theoretical work. We will here analyze Ref. [71] (which is very similar to Ref. [70]) within the operational framework of quantum stochastic thermodynamics. We will give insights into the energetic and entropic balances of these experiments by using the timescales and energy scales as reported in Ref. [71]. Moreover, we will see that the efficiency to prepare a pure photon number state is surprisingly high in the experiment (the efficiency to stabilize the pure photon state is zero). However, in order not to overburden the paper, we will leave some experimental imperfections aside. These additional imperfections are listed at the end of this section, but we emphasize here that all of them can be included in the operational framework of quantum stochastic thermodynamics. We will further assume some familiarity of the reader with concepts from quantum optics; for a basic introduction see Ref. [91] and references therein. The notation is chosen close to the original references [70,71].

A. Setup and dynamics

A sketch of the experimental setup is shown in Fig. 2. The system we want to control is a superconducting Fabry-Pérot cavity C with Hamiltonian $\hbar\omega_c a^\dagger a$, where $a^\dagger$ and a denote photon creation and annihilation operators, respectively, and $\omega_c/2\pi = 51.1$ GHz is the experimentally measured frequency of the cavity (in this section we do not set $\hbar \equiv 1$). The cavity is coupled to an outside environment at temperature $T = 0.8$ K, which implies a Bose-Einstein distribution of $N_{\rm th} = (e^{\beta\hbar\omega_c} - 1)^{-1} \approx 0.05$ (we also do not set $k_B \equiv 1$). The dynamics of the cavity are described by the master equation (in a rotating frame)

$$\partial_t \rho_S(t) = \mathcal{L}_0 \rho_S(t) \equiv \frac{1+N_{\text{th}}}{2T_c}\mathcal{D}[a]\rho_S(t) + \frac{N_{\text{th}}}{2T_c}\mathcal{D}[a^\dagger]\rho_S(t). \tag{50}$$

Here the dissipator is defined as $D[a]\rho \equiv a\rho a^\dagger - \{a^\dagger a, \rho\}/2$ and the experimental cavity lifetime is $T_c = 65$ ms.

Due to the interaction with the environment, the cavity tends to thermalize to a Gibbs state, which, for the present parameters, means with probability 0.95 the vacuum state $|0\rangle$ with zero photons. The goal of the feedback loop is to reverse the effect of the dissipation and to stabilize a photon number state $|n\rangle = |n_t\rangle$, where $n_t > 0$ denotes the target number of photons in the following (we will choose $n_t = 2$ in the numerics). To achieve this goal, a beam of atoms created in B via velocity selection and laser excitation is used. The atoms are repeatedly prepared at regular intervals of duration $T_a = 82\ \mu$s and they leave B with a velocity of $v = 250$ m/s. The interaction time of each atom with the cavity can be estimated as $t_{\text{int}} = \sqrt{\pi/2}(\omega_0/v)$, where $\omega_0 = 6$ mm is the waist of the Gaussian cavity mode. This results in an interaction time of roughly $t_{\text{int}} \approx 30\ \mu$s such that $T_c \approx 2000 t_{\text{int}}$. Thus, within a very good approximation we can treat the interactions with the atoms as happening instantaneously as we have assumed in the formal development of our theory. Furthermore, the cavity lifetime is much larger than T_a ($T_c \approx 800 T_a$) such that we will approximate the dissipative time evolution in between two interactions by

$$\mathcal{E} = e^{\mathcal{L}_0 T_a} \approx 1 + \mathcal{L}_0 T_a. \tag{51}$$

To counteract the dissipation by quantum feedback control, we first of all need to measure the state of the cavity. Importantly, this is done in a nondestructive way without absorbing or emitting a photon using a modified Ramsey interferometry scheme. A brief theoretical description works as follows. First of all, the atoms are well described as two-level systems with an energy gap $\hbar\omega_a \approx \hbar\omega_c$ close to the single-photon energy in the cavity. We will denote the two levels by $|g\rangle$ and $|e\rangle$ for ground and excited states, respectively, although both states correspond to highly excited states of the atom, where the orbit of the outer electron is far away from the nucleus creating in turn a large dipole moment [91]. The atoms leave B in the ground state $|g\rangle$ and are afterward subjected to a $\pi/2$ pulse in cavity $\mathtt{R}_1$, which prepares them in the superposition $(|g\rangle + |e\rangle)/\sqrt{2}$. Due to an atom-cavity detuning of $\omega_a - \omega_c \approx 1.5$ MHz, the atom then interacts dispersively with the cavity field, which changes its state to $(|g\rangle + e^{i\phi(n)}|e\rangle)/\sqrt{2}$. Here the n-dependent phase shift $\phi(n) = \Phi_0 n + \varphi_r$ is determined by the phase shift Φ_0 per photon and the phase φ_r, which is adjustable in the Ramsey interferometer. Importantly, no energy is exchanged between the cavity and the atom during the interaction. Then the atom is subjected to another $\pi/2$ pulse in cavity $\mathtt{R}_2$ and finally it is projectively measured in the detector D revealing it to be in either the ground or the excited state. The crux of the setup is that the probability to find the atom in the ground or excited state depends on the number n of photons in the cavity C. If we denote by $r = 0$ the result corresponding to an atom found in the ground state and by $r = 1$ for an atom in an excited state, the conditional probability to obtain outcome r given that there are n photons in the cavity is[4]

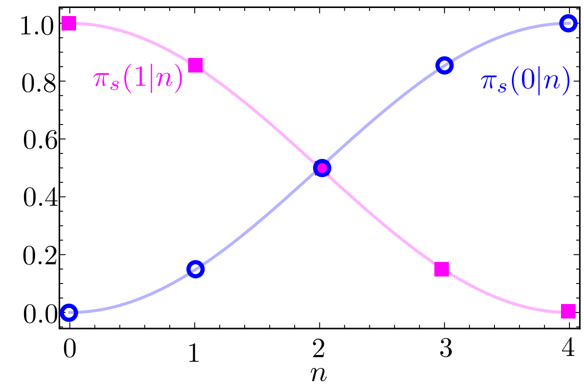

FIG. 3. Plot of the conditional probabilities as a discrete function of n: $\pi_s(0|n)$ (blue circles) and $\pi_s(1|n)$ (closed pink squares). The solid lines serve only as a guide for the eye.

$$\pi_s(r|n) = \frac{1}{2}\left\{1 + \cos\left[\frac{\pi}{4}(n - n_t) + \frac{\pi}{2}(2r-1)\right]\right\}. \tag{52}$$

For $n_t = 2$ this is exemplarily plotted in Fig. 3, showing that it is clearly possible to distinguish between $n > n_t$, $n = n_t$, or $n < n_t$ photons in the cavity, but also demonstrating that we are far away from an ideal projective measurement of the cavity.

If we want to change the number of photons in the cavity, we can send an emitter or absorber atom into the cavity C, which is prepared in either the excited or the ground state, respectively. For this purpose, the energy gap of the atoms is brought into exact resonance with the cavity by applying an external voltage V (Stark shift) such that the atom-cavity dynamics is well described by a Jaynes-Cummings Hamiltonian of the form (interaction picture) $h(a|e\rangle\langle g| + a^\dagger|g\rangle\langle e|)$. We will then ideally choose an effective interaction time $t_e = \pi/2h\sqrt{n_t}$ or $t_a = \pi/2h\sqrt{n_t+1}$ depending on whether we send an emitter or absorber atom, respectively (this is slightly different from the experimental values). In the emitter case, the conditional probability to obtain outcome $r \in \{0, 1\}$ and to observe a transition $n' \to n$ in the state of the cavity reads (compare with, e.g., Sec. 6.2. in Ref. [92])

$$\pi_e(r, n|n') = \sin^2\left(\frac{\pi}{2}\frac{\sqrt{n+r}}{\sqrt{n_t}} + \frac{\pi}{2}r\right)\delta_{n-1+r,n'}, \tag{53}$$

where $\delta_{n,n'}$ denotes the Kronecker delta. For the absorber case we get

$$\pi_a(r, n|n') = \cos^2\left(\frac{\pi}{2}\frac{\sqrt{n+r}}{\sqrt{n_t+1}} + \frac{\pi}{2}r\right)\delta_{n+r,n'}. \tag{54}$$

[4]To deduce Eq. (52), we neglect experimental imperfections in the preparation and readout of the atoms and use in the notation of Ref. [71] $\pi_s(j|n) = [1 + \cos(\Phi_0 n + \varphi_r - j\pi)]/2$, where (opposite to our notation) $j = 0$ ($j = 1$) denotes an atom in the excited (ground) state. After taking this into account, setting the phase shift per atom to $\Phi_0 \approx \pi/4$ [71] and adjusting the variable phase φ_r of the Ramsey interferometer to the optimal value $\varphi_r + \Phi_0 n_t = \pi/2$ [71], we obtain Eq. (52).

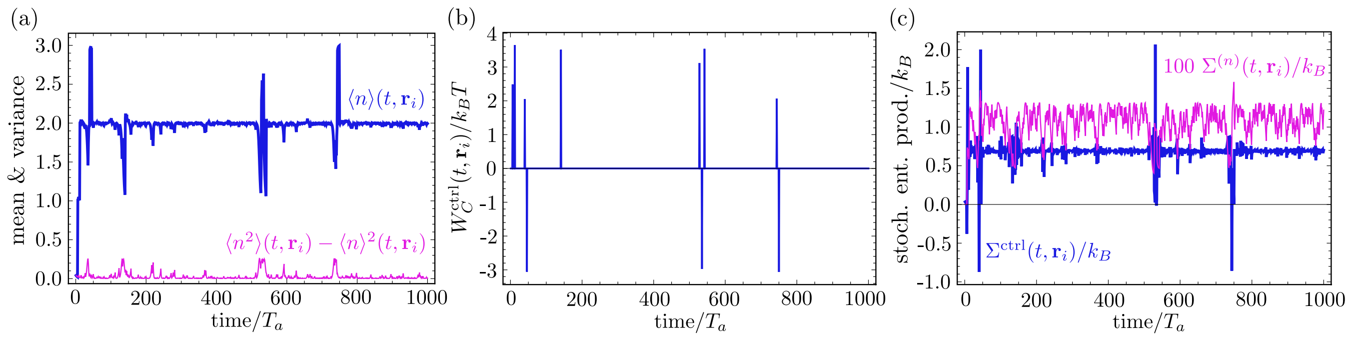

FIG. 4. Stochastic dynamics and thermodynamics of a single realization of the (numerical) experiment over 1000 time steps. (a) Conditional mean value $\langle n\rangle(t, \mathbf{r}_i)$, which fluctuates around the target number of $n_t = 2$ photons (thick blue line on the top), and conditional variance $\langle n^2\rangle(t, \mathbf{r}_i) - \langle n\rangle^2(t, \mathbf{r}_i)$, which is most of the time below 0.1 (thin pink line on the bottom). (b) Dimensionless work invested into the control loop showing spikes exactly at the time when an emitter or absorber atom is sent into the cavity. (c) Dimensionless stochastic entropy production split according to Eq. (37) into the part during the control operation, which can become temporarily negative (thick blue line), and the part in between the control operations, which has been upscaled by a factor of 100 for better visibility (thin pink line mostly on top).

Note that, depending on the number n of photons in the cavity, an absorber (emitter) atom will not always absorb (emit) a photon.

Finally, it is important to realize that the atoms are detected at a time delay, as indicated also in Fig. 2. This means that, before the ith atom is registered with outcome r_i at the detector D, there have been already $d = 5$ atoms which have interacted or are about to interact with the cavity such that we cannot influence their initial state anymore. This point is important for the design of the feedback control law. In order to decide at time $t_i \equiv iT_a$ what kind of atom to send into the cavity, we can only use the state estimate $\rho_S(t_{i-d}^+, \mathbf{r}_{i-d})$ at time $t_{i-d} = (i-d)T_a$. Then, finally, the feedback control law is simply to sent an absorber atom as soon as we estimate $\sum_{n>n_t} p_n(t_{i-d}, \mathbf{r}_{i-d}) > p_{n_t}(t_{i-d}, \mathbf{r}_{i-d})$ and an emitter atom if we estimate $\sum_{n<n_t} p_n(t_{i-d}, \mathbf{r}_{i-d}) > p_{n_t}(t_{i-d}, \mathbf{r}_{i-d})$, where $p_n(t, \mathbf{r}_n)$ denotes the probability to have n photons in the cavity at time t given a measurement record $\mathbf{r}_n$. Otherwise we keep measuring the system. After each feedback operation we also wait d time steps before we apply the feedback control law again. This simple feedback control law is slightly different from the experiment, but as we will see now, it works well.

B. Quantum stochastic thermodynamics

In the preceding section we stated all necessary ingredients to apply our framework. The state of the cavity is conveniently described by the probability $p_n(t, \mathbf{r}_i)$ because coherences between different photon number states never play a role. The control operations $\mathcal{A}(r_i|\mathbf{r}_{i-1})$ are either measurements or feedback operations (which can be emissive or absorbative). Its effect on the cavity field can be described by the conditional probabilities (52)–(54). Due to the time delay, we can set $\mathcal{A}(r_i|\mathbf{r}_{i-1}) = \mathcal{A}(r_i|\mathbf{r}_{i-d-1})$. Furthermore, the atoms always leave B in the ground state $|g\rangle$ and they are always projected at the end of the interaction such that the sequence of outcomes $\mathbf{r}_i$ is simply a sequence of zeros and ones. Finally, the evolution in between two interactions is modeled by Eq. (51). Also, numerically, all parameters have been fixed in the preceding section.

Figure 4 shows various quantities for a single realization of the process over 1000 time intervals. Figure 4(a) shows the evolution of the conditional mean photon number $\langle n\rangle(t, \mathbf{r}_i) \equiv \sum_n np_n(t, \mathbf{r}_i)$ (blue thick line) and the conditional variance $\langle n^2\rangle(t, \mathbf{r}_i) - \langle n\rangle^2(t, \mathbf{r}_i)$ (pink thin line). For perfect stabilization around $n_t = 2$ one would expect $\langle n\rangle(t, \mathbf{r}_i) = n_t$ and zero variance. As the plot shows, we are not far from that limit. The variance stays most of the time below 0.1 and only significantly deviates from it when our knowledge about the mean changes. This can be caused by the emission or absorbtion of a photon into or from the environment or by erroneous detection events as it is not perfectly possible to distinguish between one, two, or three photons (recall Fig. 3). Whenever our estimate about the mean changes significantly, the external agent performs a feedback control operation, which changes the energy of the cavity in a deterministic way and entails a work cost $W_C^{\rm ctrl}(\mathbf{r}_{i-1})$ as depicted in Fig. 4(b). Note that the work cost associated with the measurement of the cavity is zero, $W_C^{\rm ctrl}(\mathbf{r}_{i-1}) = 0$, although there is always a work cost $W_A^{\rm ctrl}(\mathbf{r}_{i-1})$ associated with the preparation of the atoms except for the case of absorbative feedback (discussed below; keep in mind that we use the subscripts C and A here instead of S and U as in Sec. IV). As the plot demonstrates, the experiment is not very costly in terms of the work invested in the cavity, which is roughly a few k_BT. Note that we also sometimes gain work as indicated by negative values and that also the work costs fluctuate due to the fact that the state of the system can be different at different times. What is more costly is the generation of the information needed to estimate the state of the cavity. This is shown in Fig. 4(c), where the entropy production $\Sigma^{\rm ctrl}(t, \mathbf{r}_i)$ (thick blue line) is roughly $0.7k_B$ at each time step with some rare exceptions and strong fluctuations at the times where we perform feedback control operations. As a closer inspection reveals (not shown here), the main cause of this is the generation of information in the memory quantified by $-k_B \ln p(r_i|\mathbf{r}_{i-1})$. In comparison, the entropy produced in between two control operation $\Sigma^{(n)}(t, \mathbf{r}_i)$ (thin pink line), upscaled by a factor of 100, is much smaller than $\Sigma^{\rm ctrl}(t, \mathbf{r}_i)$ due to the fact that in the short time interval T_a not much is happening. Also note that $\Sigma^{(n)}(t, \mathbf{r}_i)$ is always positive, as predicted by our theory.

The previous observations are also confirmed by the average description. Figure 5 shows the (thermo)dynamics for

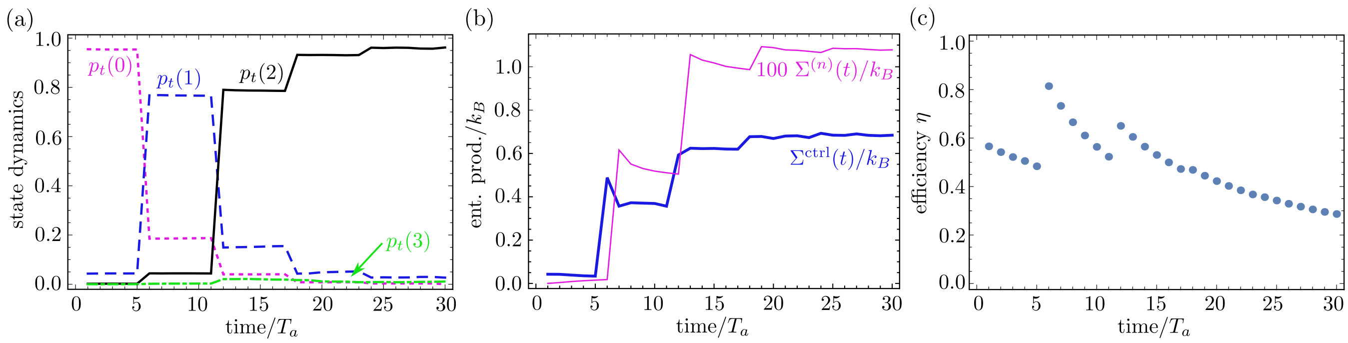

FIG. 5. Averaged dynamics and thermodynamics of 2000 repetitions of the (numerical) experiment over 30 time steps. (a) Dynamics of the cavity population displaying the probability to find zero photons (pink dotted line), one photon (blue dashed line), two photons (black solid line), and three photons (green dash-dotted line). (b) Dimensionless entropy production again split according to Eq. (37). On average, the part associated with the control operations is now positive (thick blue line). The part in between the control operations is again upscaled by a factor 100 for better visibility (thin pink line). (c) Time-dependent efficiency (57) of the experiment.

30 time steps averaged over 2000 numerical realizations. Figure 5(a) depicts the time evolution of the probabilities $p_0(t) = \sum_{\mathbf{r}_i} p_0(t, \mathbf{r}_i)p(\mathbf{r}_i)$ (dotted pink line), $p_1(t)$ (dashed blue line), $p_2(t)$ (solid black line), and $p_3(t)$ (dash-dotted green line) to have zero, one, two, or three photons in the cavity. The effect of the time delay $d = 5$ can be clearly recognized as well as the success of the feedback loop to reach a pure photon state with probability $p_{n_t=2}(t) \approx 0.96$. Note that the shown time interval of $30T_a \approx 0.04T_c$ is too small to have a significant probability for a quantum jump induced by the environment, but to better see the impact of the time delay we have decided to show here only a short time window. Furthermore, Fig. 5(b) shows the entropy production $\Sigma^{\rm ctrl}$ (thick blue line) and $\Sigma^{(n)}$ (thin pink line, scaled by a factor 100). In accordance with the previous plot, we can conclude that the maintenance of the measurement and feedback loop is the thermodynamically most costly part with the most dominant contribution stemming from the recording of the outcomes $\mathbf{r}_i$ in a classical memory (not shown). In addition, the plot also demonstrates that $\Sigma^{\rm ctrl}$ is positive on average, as predicted by our theory.

Finally, Fig. 5(c) shows the efficiency of the experiment in terms of generating a nonequilibrium state of the cavity with respect to the resources invested in the feedback loop. If we sum up the entropy production (37) in each time step and use the first laws (20) and (30), we can confirm that the average integrated entropy production after N time steps becomes

$$\sum_{i=1}^{N} \Sigma^{(i)} = \frac{W_C^{\rm tot} - \Delta F_C}{T} + k_B S_{\rm Sh}[p(\mathbf{r}_N)] \geqslant 0. \tag{55}$$

Here $W_C^{\rm tot}$ is the average integrated work invested in the cavity during the feedback loop [also see Fig. 4(b)] and $\Delta F_C = F_C(NT_a) + k_B T \ln Z_C$ is the change in nonequilibrium free energy starting from a cavity state in equilibrium with partition function $Z_C = {\rm tr}_C\{e^{-\beta\hbar\omega_c a^\dagger a}\}$. The average free energy after N time steps is computed by averaging over the energy and entropy of the cavity state, i.e.,

$$F_C(NT_a) = \sum_{\mathbf{r}_N} p(\mathbf{r}_N)\{E_C(\mathbf{r}_N) - k_B T S_{\rm vN}[\rho_C(\mathbf{r}_N)]\}. \tag{56}$$

Finally, the last term in Eq. (55) denotes the entire average information content $S_{\rm Sh}[p(\mathbf{r}_N)]$ associated with the outcomes of the experiment. It follows from the second law (55) that the following efficiency is bounded by one:

$$\eta \equiv \frac{\Delta F_C}{W_C^{\rm tot} + k_B T S_{\rm Sh}[p(\mathbf{r}_N)]} \leqslant 1. \tag{57}$$

As Fig. 5(c) demonstrates, we can achieve remarkably high efficiencies peaked around values of 0.8 and 0.65 before they decay in the long run to zero. This decay is due to the fact that stabilizing the photon number state does not change its free energy anymore while the measurement and feedback loop still consume resources. Thus, the preparation of the photon number state is very efficient, but the stabilization of it has by definition an efficiency of zero. While we focused on the average efficiency here, we remark that our framework is also ideally suited to study efficiency fluctuations [93].

One might wonder why only the work invested in the cavity enters the definition of the efficiency (57), but not the work $W_A^{\rm ctrl}$ invested to prepare the state of the atoms. The latter is non-negligible: Since the atoms leave B in the ground state and the outgoing stream of atoms is roughly an equal mixture of atoms in the excited and ground states (which follows from Fig. 3 once we have stabilized the state around n_t photons), the work invested per atom is $W_A^{\rm ctrl} \approx \hbar\omega_a/2$. This has its origin in the initial creation of the superposition in cavity $\mathtt{R}_1$.[5] However, what we are interested in here is how efficiently can we use a given amount of nonequilibrium resources (i.e., atoms in a pure state) to perform some task (creation of Fock states). That efficiency should be the same independent of, for instance, the question of whether the atoms leaving B are in the ground or excited state (in the latter case we would additionally extract

[5]This simple argument neglects the atoms used for the feedback control, which, however, constitute only a small fraction of the atoms used in the experiment. Furthermore, the proportion of outgoing atoms in the excited state is not exactly 0.5, but depends on the question whether the target photon number n_t is above or below the thermal equilibrium value, which determines whether a state with n_t photons tends to absorb or emit a photon into or from the environment.

work $W_A^{\rm ctrl} \approx -\hbar\omega_a/2$ from the atoms). The second law of thermodynamics is concerned only with changes in entropy, which are zero for the incoming and outgoing stream of atoms. In fact, the outgoing stream of atoms can be used again, e.g., for the next experiment. To make sense of this argument, it is important to note that the state of the atoms is not in a mixture of ground and excited states because in each experimental run we know exactly the state of the atoms by looking at the measurement record $\mathbf{r}_N$. There is thus zero uncertainty associated with their state.

The thermodynamic description would not be complete if we did not mention that the experimental implementation also involves other costs, e.g., the cooling of the environment down to less than 1 K, the laser preparation of the atoms in B, or the electronics associated with the controller K. What we have provided here is a minimal thermodynamic description of the system, which involves all essential contributions. Similar to other idealizations in thermodynamics, it is possible to imagine that the hidden thermodynamic costs of running the laboratory equipment can be made arbitrarily small in an ideal world.

C. Further experimental imperfections

Finally, we mention that the experiment is a little more complicated than described here. For instance, the number of atoms interacting with the cavity at a given time is not fixed to one, but rather Poisson distributed (with an average number of 0.6 atoms) such that there could be zero, one, or two atoms per interaction. Furthermore, it can also happen that the detector D fails to detect an atom. Those and other small imperfections are the reason why the experimentally observed probability $p_t(n_t)$ is around 0.8 [71].

VI. SPECIAL CASES

A. Projective measurement

We start this section by considering the case of a single projective measurement. Not only is this an illustrative example, but we will also need it in Secs. VI B and VI E.

We denote the outcome of the projective measurement by r and the associated projector by $|r\rangle\langle r|_S$. We assume no degeneracies in the measured observable here. Within our repeated interaction framework, we use a unit with Hilbert space of dimension $\dim\mathcal{H}_U = \dim\mathcal{H}_S$, the initial state is taken to be the pure state $|1\rangle\langle 1|_U$, and we assume a trivial unit Hamiltonian $H_U \sim 1_U$. The dynamical aspects of the Stinespring dilation are then fixed by the unitary V,

$$V = \sum_{r,u} |r, u + r - 1\rangle\langle r, u|_{SU} \tag{58}$$

(where the sum in the ket has to be interpreted modulo $\dim\mathcal{H}_U$), which is followed by a projective measurement of the unit in the basis $\{|r\rangle_U\}$. It is interesting to look at the states at the different steps of the process given an arbitrary initial system state $\rho_S(t^-) = \sum_s \lambda_s|s\rangle\langle s|_S$. After the unitary V the marginal states of $\rho_{SU}^* \equiv V\rho_S(t^-)\rho_U V^\dagger$ are

$$\rho_S^* = \sum_r p(r)|r\rangle\langle r|_S, \quad \rho_U^* = \sum_r p(r)|r\rangle\langle r|_U. \tag{59}$$

Here we have introduced the probability $p(r) = \sum_s |\langle r|s\rangle|^2\lambda_s$ to obtain result r. After the projective measurement, the unnormalized state reads

$$\tilde{\rho}_{SU}(r, t^+) = p(r)|r, r\rangle\langle r, r|_{SU}, \tag{60}$$

from which it is easy to read the marginal states and the average state after the control operation.

Let us now look at the thermodynamic interpretation of a projective measurement within our framework. The work (23) and heat (24) of the control operation become

$$W_S^{\rm ctrl} = \sum_r p(r)\langle r|H_S|r\rangle - \sum_s \lambda_s\langle s|H_S|s\rangle, \tag{61}$$

$$Q_S^{\rm ctrl}(r) = \langle r|H_S|r\rangle - \sum_{r'} p(r')\langle r'|H_S|r'\rangle. \tag{62}$$

We easily confirm $\sum_r p(r)Q_S^{\rm ctrl}(r) = 0$. Moreover, the work vanishes whenever the measured basis coincides with the eigenbasis of the initial system state $\rho_S(t^-)$, albeit the fluctuating heat does not. Finally, we can also confirm Eq. (46), which in this case is essentially $S_{\rm Sh}[p(r)] \geqslant S_{\rm Sh}(\lambda_s)$.

It is instructive to compare these results with the framework of Ref. [51]. There the quantum stochastic thermodynamics of projective measurements was also considered and the authors called the sum $W_S^{\rm ctrl} + Q_S^{\rm ctrl}(r)$ quantum heat and justified it by the fact that a quantum measurement is intrinsically stochastic unless the measured basis coincides with the basis of $\rho_S(t^-)$ (we add that only pure states were considered in Ref. [51], thus leaving any classical uncertainty aside). Remarkably, we reach exactly the opposite conclusion on average: Since $\sum_r p(r)Q_S^{\rm ctrl}(r) = 0$, we infer that the average energetic change is purely work $W_S^{\rm ctrl}$ instead of heat.

This discrepancy can be traced back to the fact that we model the projective measurement in a larger space using Stinespring's theorem, which was not done in Ref. [51]. Remarkably, in this larger space we also called the energetic changes caused by the final measurement $P_U(r)$ heat (although not quantum heat because as soon as classical uncertainty is considered too, it also plays a role, e.g., in classical stochastic thermodynamics; see Sec. VI E). Thus, we applied a philosophy somewhat similar to that of Elouard *et al.* [51], but reached the opposite conclusion. This shows that the thermodynamic interpretation of a quantum measurement depends on where we put the Heisenberg cut. To defend the present approach, we want to highlight a number of key differences.

First, by using Stinespring's theorem we respect the fact that a quantum measurement does not happen spontaneously but requires an active intervention by the experimentalist, who brings two systems (the system to be measured and the detector) into contact. However, bringing two different physical systems into contact in general requires work (compare also with the switching work in Ref. [66]).

Second, our second law differs from the one derived in Ref. [51] as soon as multiple projective measurements are considered. In our case, the entropy production is on average given by the Shannon entropy of the entire sequence of measurement results $S_{\rm Sh}[p(\mathbf{r}_n)] = \sum_\ell S_{\rm Sh}[p(r_\ell|\mathbf{r}_{\ell-1})]$ [with $p(r_1|r_0) \equiv p(r_1)$]. In Ref. [51] the entropy production is instead quantified by the Shannon entropy of the last measurement result only, $S_{\rm Sh}[p(r_n)]$, and also the quantum heat does not enter their second law.

Finally, we mention that the thermodynamic cost of quantum measurements was also explicitly studied elsewhere [94–98]. In particular, Refs. [94–96,98] reached similar conclusions by noting that performing a quantum measurement allows the external agent to extract work. Hence, the average energetic cost of the measurement should be counted as work. Also, in a recent proposal of a Maxwell demon based only on projective measurements it was noted that the fields, which are controlled to implement the measurement, provide the energy for the demon [99].

B. Two-point measurement approach

The two-point measurement approach, which is closely related to the theory of full counting statistics, has become the primarily used approach to derive quantum fluctuation relations in various open quantum systems [22,23,40]. While theoretically powerful, we already discussed the practical weakness of this approach in the Introduction: Experimental confirmations so far have only been achieved for work fluctuation relations in isolated systems [27,31,100] or in electronic nanocircuits when the electrons behave according to a classical rate master equation [37–39].

We here critically reexamine the two-point measurement approach from a foundational perspective. We also view it in the context of Ref. [24], which proves that there exists no measurement strategy of work whose statistics fulfill (i) a quantum work fluctuation theorem and (ii) reproduce, when averaged, the unmeasured first law for arbitrary initial states. This important no-go theorem proves that quantum stochastic thermodynamics is distinct from its classical counterpart: It is in general impossible to make the averaged picture coincide with the unmeasured picture in quantum thermodynamics. Nevertheless, within our framework we will find that the no-go theorem does not apply in the sense that the work defined in the two-point measurement approach is not even work according to our framework.

We consider the following standard scenario, where the unitary evolution of an isolated system is interrupted by two projective measurements. We assume that the projective measurements are described as in Sec. VI A with energetically neutral units. Since the system is isolated, we will also drop the subscript S on all quantities.

First, the system is prepared in a Gibbs state such that

$$\rho(t_0^-) = \frac{e^{-\beta H(\lambda_0)}}{Z(\lambda_0)} = \frac{1}{Z(\lambda_0)}\sum_{\epsilon_0} e^{-\beta\epsilon_0}|\epsilon_0\rangle\langle\epsilon_0|, \tag{63}$$

where $Z(\lambda_0) = \text{tr}\{e^{-\beta H(\lambda_0)}\}$ denotes the partition function. Its internal energy is denoted by $E(t_0^-) = \text{tr}\{H(\lambda_0)\rho(t_0^-)\}$. Then, at time t_0 we projectively measure the energy and obtain outcome r_0, which is uniquely associated with one energy eigenvalue $\epsilon_0(r_0)$. Since the measurement basis coincides with the eigenbasis, the work during this measurement is zero. However, the internal energy clearly changes along a single trajectory and this is due to heat:

$$\epsilon_0(r_0) - E(t_0^-) = Q^{\rm crtl}(r_0). \tag{64}$$

In the next step we let the isolated system evolve according to an arbitrary time-dependent Hamiltonian $H(\lambda_t)$. The state at time $t_1 > t_0$ is given by $|\psi(t, r_0)\rangle = U(t)|\epsilon_0(r_0)\rangle$, where $U(t)$ denotes the unitary time evolution operator generated by $H(\lambda_t)$. As the system is completely isolated, the change in internal energy is purely given by work

$$\langle\psi(t_1, r_0)|H(\lambda_1)|\psi(t_1, r_0)\rangle - \epsilon_0(r_0) = W^{(1)}(r_0). \tag{65}$$

Finally, there is another projective measurement in the eigenbasis of $H(\lambda_1)$ with outcome r_1, uniquely associated with some eigenenergy $\epsilon_1(r_1)$. The change in internal energy now has in general a work and a heat contribution:

$$\begin{aligned}\epsilon_1(r_1) &- \langle\psi(t_1, r_0)|H(\lambda_1)|\psi(t_1, r_0)\rangle \\ &= W^{\rm ctrl}(r_0) + Q^{\rm ctrl}(r_1, r_0).\end{aligned} \tag{66}$$

To derive an explicit form for it, we expand the prior state with respect to the final measurement basis: $|\psi(t_1, r_0)\rangle = \sum_{\epsilon_1} c_{\epsilon_1}|\epsilon_1\rangle$. Then we obtain

$$W^{\rm ctrl}(r_0) = \sum_{\epsilon_1}|c_{\epsilon_1}|^2\epsilon_1 - \langle\psi(t_1, r_0)|H(\lambda_1)|\psi(t_1, r_0)\rangle, \tag{67}$$

$$Q^{\rm ctrl}(r_1, r_0) = \epsilon_1(r_1) - \sum_{\epsilon_1}|c_{\epsilon_1}|^2\epsilon_1. \tag{68}$$

Both contributions differ from zero unless in the classical case where $|\psi(t_1, r_0)\rangle = |\epsilon_1(r_1)\rangle$. Thus, in that scenario it would be justified to call $Q^{\rm ctrl}(r_1, r_0)$ "quantum" heat.

Now consider the probability for the sequence of outcomes

$$p(r_1, r_0) = |\langle\epsilon_1(r_1)|U(t)|\epsilon_0(r_0)\rangle|^2\frac{e^{-\beta\epsilon_0(r_0)}}{Z(\lambda_0)}. \tag{69}$$

It is a straightforward exercise to show that this probability distribution implies the so-called quantum work theorem or quantum Jarzynski equality, first derived in Refs. [101–103]:

$$\begin{aligned}\langle e^{-\beta[\epsilon_1(r_1)-\epsilon_0(r_0)]}\rangle &\equiv \sum_{\epsilon_1,\epsilon_0} p(r_1, r_0)e^{-\beta[\epsilon_1(r_1)-\epsilon_0(r_0)]} \\ &= \frac{Z(\lambda_1)}{Z(\lambda_0)}.\end{aligned} \tag{70}$$

Now, in analogy to the classical Jarzynski equality, the fluctuating quantity $\epsilon_1(r_1) - \epsilon_0(r_0)$ in the exponent is called work in the two-point measurement approach [22,23]. However, our framework reveals that

$$\epsilon_1(r_1) - \epsilon_0(r_0) = W^{(1)}(r_0) + W^{\rm ctrl}(r_0) + Q^{\rm ctrl}(r_1, r_0). \tag{71}$$

That is to say, the fluctuating quantity in the exponent is not work alone. Hence, one better calls the quantum work theorem a quantum internal energy theorem.

We end this section by pointing out that we are not the first to criticize the notion of work within the two-point measurement approach. For instance, Deffner *et al.* also criticize this approach for not being "thermodynamically consistent as it does not account for the thermodynamic cost of these measurements" [97]. Remarkably, they were able to derive a modified quantum Jarzynski equality for the work (65) done in between the two projective measurements [97].

C. Standard framework of quantum thermodynamics

If we perform no control operations at all, our framework obviously reproduces the standard framework of quantum thermodynamics mentioned at the beginning of Sec. IV A. This fact might seem so obvious that it is not worth stressing. However, it is important to realize that the standard framework of quantum thermodynamics cannot be recovered by performing an ensemble average over $p(\mathbf{r}_n)$, but only by deciding not to apply any control operation at all (apart from maybe preparing a certain initial state and reading out the final state). That is to say, in order to recover standard quantum thermodynamics, it is important to have a framework which can cope with incomplete information and allows us to do "nothing" to the system. All previous frameworks of quantum stochastic thermodynamics, which rely on a perfectly measured system in a pure state, fail to reproduce the picture without control operations because any measurement disturbs the process in general. In fact, in almost all previous works the notion of a stochastic entropy along a single trajectory was not even defined. The only exceptions are Refs. [42,43] where, however, the definition of stochastic entropy depends on the initial state chosen and therefore needs to be adapted in each experiment. The reason why classical stochastic thermodynamics reproduces the average picture (see Sec. VI E) is the fact that there is always one fixed basis and no coherences are possible. The current framework therefore fills an important conceptual gap between quantum and classical stochastic thermodynamics.

D. Conventional repeated interaction framework

The framework of repeated interactions gives rise to a generalized thermodynamic theory by realizing that the stream of external units can act in the most general scenario as a resource of nonequilibrium free energy, which encompasses many previously considered theories [66] (see also Ref. [104] for important earlier work). However, the repeated interaction framework considered previously differs from our framework by avoiding any measurement on the units. In order to recover this thermodynamic framework, it is important to realize (as in Sec. VI C) that a simple ensemble average of the process tensor over the outcomes $\mathbf{r}_n$ will not do the job. The only correct way to recover previous results from our framework is to not perform any measurement, i.e., in the language of Sec. III to choose the projector $P(r_n|\mathbf{r}_{n-1}) = 1_U$ throughout. In this case, the process tensor can be written as $\mathfrak{T}[\mathcal{A}_n, \dots, \mathcal{A}_1]$, where $\mathcal{A}_i$ is a CPTP map acting at time t_i. The control operations, and hence the process tensor, do not depend on any outcome $\mathbf{r}_n$ anymore (alternatively, one could say that each control operation at time t_i has only one possible outcome). Furthermore, every incoming unit is decorrelated from the previous units as in Ref. [66].

Our thermodynamic framework of the process tensor is therefore much more general and flexible than the previous framework apart from one important difference. In Ref. [66] the units were allowed to interact with the system for a finite duration whereas here we consider only instantaneous interactions (or more precisely interaction times where the effect of the bath can be neglected to leading order). From a thermodynamic point of view, this is not necessary. However, to be able to clearly distinguish between control operations on the system and system-bath dynamics, this assumption is necessary (compare with the discussion in Sec. II).

We now show that our thermodynamic framework is not in contradiction to the one of Ref. [66], if we avoid any measurements of the units. Since no quantity depends on $\mathbf{r}_n$ anymore, the internal energy is simply

$$E_{SU(\mathbf{n})}(t) = E_S(t) + \sum_{i=1}^{n} E_{U(i)}(t). \tag{72}$$

However, the internal energy of all previous units $U(i < n)$ never enters the first law and thus can be neglected. In fact, in the absence of any control operation this is evident from Eq. (20). During the control operations, because there is no final measurement, $Q^{\text{ctrl}}(t_n) = 0$ and only $W^{\text{ctrl}}(t_n)$ can differ from zero. However, the work only depends on the state of the nth unit and not on previous units [cf. Eq. (29)]. Hence, the first law during the control operation becomes $W^{\text{ctrl}}(t_n) = \Delta E_S(t_n) + \Delta E_{U(n)}(t_n)$ because the marginal state of all other units does not change. We therefore obtain the same first law over one interaction period $(t_{n-1}, t_n]$:

$$\Delta E_S^{[n]} + \Delta E_{U(n)}^{[n]} = W^{\text{ctrl}}(t_n) + W^{(n)} + Q^{(n)}. \tag{73}$$

Finally, note that $W^{\text{ctrl}}(t_n)$ would be identified in the context of Ref. [66] with the switching work W_{switch} required to turn on and off the system-unit interaction.

We now turn to the second law. Without any outcomes $\mathbf{r}_n$ we obtain from Eq. (35) the entropy $S_{SU(\mathbf{n})}(t) = S_{\text{vN}}[\rho_{SU(\mathbf{n})}(t)]$. Again, this differs from Ref. [66] by explicitly taking into account the joint entropy of all units and the system. To recover Ref. [66], we start again with the situation without control operation. From Eq. (17) we know that $\Delta S_S^{(n)} - \beta Q_S^{(n)} \geqslant 0$ and, since the marginal unit states do not change, we can extend this to

$$\Delta S_S^{(n)} + \Delta S_{U(n)}^{(n)} - \beta Q_S^{(n)} \geqslant 0. \tag{74}$$

Next, our second law during the control operation becomes

$$\Sigma^{\text{ctrl}}(t_n) = S_{\text{vN}}[\rho_{SU(\mathbf{n})}(t_n^+)] - S_{\text{vN}}[\rho_{SU(\mathbf{n})}(t_n^-)] = 0, \tag{75}$$

because the von Neumann entropy is invariant under unitary transformation. If we use the two facts that the unitary $\mathcal{V}$ acts only locally on the system and the nth unit and that the initial state of the unit is decorrelated from the system, we immediately confirm that Eq. (75) can be rewritten as

$$\Sigma^{\text{ctrl}}(t_n) = \Delta S_{\text{vN}}^{\text{ctrl}}(\rho_S) + \Delta S_{\text{vN}}^{\text{ctrl}}(\rho_U) - I_{S:U(n)}(t_n^+) = 0. \tag{76}$$

Taking the mutual information to the other side of the equation and combining it with Eq. (74), we can confirm for an entire interaction interval that

$$\Delta S_S^{[n]} + \Delta S_{U(n)}^{[n]} - \beta Q^{(n)} \geqslant I_{S:U(n)}(t_n^+) \geqslant 0. \tag{77}$$

This reproduces the generalized second law from Ref. [66]. The reason why the final mutual information between the system and the previous units was discarded in Ref. [66] becomes clear by recalling that every unit which has already interacted with the system does not have the chance to interact with the system again. All final mutual information will therefore be lost. This is in contrast to the general framework developed here where we allowed for all kinds of feedback control. Under these more general circumstances, the remaining mutual information after the interaction represents a valuable thermodynamic resource, which cannot be neglected.

E. Standard classical stochastic thermodynamics

A tacitly made assumption in classical stochastic thermodynamics is the ability to measure perfectly (i.e., without error and without disturbance) the state of the system [12,13]. These assumptions can be completely overcome by using the operational approach to stochastic thermodynamics, but attention has to be paid to the fact that the classical version of Stinespring's theorem does not follow from the quantum version stated in Sec. III [82].

Here we restrict ourselves to study the standard case of stochastic thermodynamics assuming perfect continuous measurements and no feedback control. We focus only on a classical discrete system, which makes random jumps between a finite set of states $s \in \{1, \dots, d\}$. Its dynamics are described by a rate master equation

$$\frac{d}{dt}p_s(t) = \sum_{s'} R_{s,s'}(\lambda_t)p_{s'}(t). \tag{78}$$

Here $p_s(t)$ is the probability to find the system in state s at time t, whose energy we denote by $H(s, \lambda_t)$ (dropping the subscript S on H). The rate matrix $R_{s,s'}(\lambda_t)$ can depend on an external control parameter λ_t. It is required to fulfill the local detailed balance condition

$$\frac{R_{s,s'}(\lambda_t)}{R_{s',s}(\lambda_t)} = e^{-\beta[H(s,\lambda_t)-H(s',\lambda_t)]}, \tag{79}$$

which allows us to link energetic changes in the system to entropic changes in the bath. Due to the assumptions of standard stochastic thermodynamics, one knows at each time t the state s of the system without any uncertainty (denoted s_t in the following). The stochastic energy and entropy at time t are then defined by

$$E_{\rm ST}(s_t) \equiv H(s_t, \lambda_t), \quad S_{\rm ST}(s_t) \equiv -\ln p_{s_t}(t), \tag{80}$$

where we used a subscript ST to denote definitions used in standard stochastic thermodynamics. Note that the stochastic entropy $S_{\rm ST}(s_t)$ is determined by evaluating the solution of the rate master equation along a particular stochastic trajectory [11]. Work and heat for a sufficiently small time step dt are defined as[6]

$$W_{\rm ST}(s_t) \equiv H(s_{t-dt}, \lambda_t) - H(s_{t-dt}, \lambda_{t-dt}), \tag{81}$$

$$Q_{\rm ST}(s_t) \equiv H(s_t, \lambda_t) - H(s_{t-dt}, \lambda_t) \tag{82}$$

such that $E_{\rm ST}(s_t) - E_{\rm ST}(s_{t-dt}) = W_{\rm ST}(s_t) + Q_{\rm ST}(s_t)$. Furthermore, using rather complicated algebraic manipulations, one can compute the change of stochastic entropy along a particular trajectory [11–13] (we will see below that evaluating the quantities in discrete time steps simplifies the algebra significantly). In the resulting expression it is then possible to single out a term related to the entropy production, which on average yields the always positive expression

$$\Sigma_{\rm ST}(t) \equiv \Delta S_{\rm ST}(t) - \beta Q_{\rm ST}(t) \geqslant 0, \tag{83}$$

where $\Delta S_{\rm ST}(t) \equiv S_{\rm Sh}[p_s(t)] - S_{\rm Sh}[p_s(t-dt)]$ turns out to be the (infinitesimal) change in Shannon entropy of the solution $p_s(t)$ of the rate master equation and $Q_{\rm ST}(t) = \sum_s H(s, \lambda_t)[p_s(t) - p_s(t-dt)]$ is the average heat entering the system per time step dt.

Our goal is now to show the following: (i) how a perfect nondisturbing measurement arises in our context; (ii) that we obtain identical expressions for the stochastic heat, work, and internal energy in this limit; (iii) that we obtain a different expression for stochastic entropy, which yields a different but meaningful second law; and (iv) how the entropy production of standard stochastic thermodynamic arises in our context when we change the definition of stochastic entropy.

(i) To obtain a perfect measurement, we can basically use the same steps as in Sec. VI A. We start with a classical probability $p_U = \delta_{u,1}$ and view the unitary (58) as a permutation matrix. Then the result is that the state of the system gets copied onto the state of the unit. Next we consider the limit where we measure the system continuously, i.e., in small time steps $dt = t_n - t_{n-1}$ such that the probability for a jump in each interval is very small: $R_{s,s'}(\lambda_t)dt \ll 1$. Furthermore, we assume that all units are identical and uncorrelated initially. In this limit, the sequence of measurement outcomes $\mathbf{r}_n$ is identical to the state of the units, which is identical to the trajectory taken by the system. This is the essence of a perfect classical and continuous measurement. As a consequence, the state of the system at time $t \geqslant t_n^+$ only depends on the last measurement outcome r_n, but not on any of the previous outcomes $\mathbf{r}_{n-1}$. Furthermore, the state of the system during the interval $(t_{n-1}, t_n]$ changes from $\mathbf{p}(t_{n-1}^+, r_{n-1}) = |r_{n-1}\rangle$ at the beginning to $\mathbf{p}(t_n^-, r_{n-1}) = |r_{n-1}\rangle + dt\sum_s R_{s,r_{n-1}}(\lambda_t)|s\rangle$ shortly before the control operation and to $\mathbf{p}(t_n^+, r_n) = |r_n\rangle$ at the end after the nth control operation. Below we will identify $t_n = t$ and $t_{n-1} = t - dt$.

(ii) We now turn to the energetic description. As in standard stochastic thermodynamics, we neglect the energetics associated with the memory, that is, we set $H_U \sim 1_U$ for all units. This implies that we can replace our stochastic energy $E_{SU(\mathbf{n})}(t, r_n)$ by $E_S(t, r_n)$. Then the stochastic energy at the beginning of the interval is simply $H(r_{n-1}, \lambda_{t-dt})$ and at the end it reads $H(r_n, \lambda_t)$, which is identical to the definition used in classical stochastic thermodynamics. Furthermore, in the absence of control, we obtain from Eq. (15)

$$\begin{aligned} W^{(n)}(r_{n-1}) &= \sum_s [H(s, \lambda_t) - H(s, \lambda_{t-dt})]p_s(t_{n-1}^+, r_{n-1}) \\ &= H(r_{n-1}, \lambda_t) - H(r_{n-1}, \lambda_{t-dt}), \end{aligned} \tag{84}$$

which is identical to Eq. (81).[7] Furthermore, the work during the control step [Eq. (23)] is zero because the marginal state

[6]In stochastic thermodynamics, one usually writes δW or $đW$ to denote the infinitesimal character of the quantity. Often, one also represents quantities defined for single trajectories with a small letter, e.g., w. We here decided to stick closer to our notation from Sec. IV, keeping in mind that we are interested only in small time steps dt.

[7]We remark that there is a certain degree of freedom involved in the evaluation of the integral in Eq. (15). However, this degree of freedom is also there in the identification (81) and (82) and it is only important to stick consistently to one choice.

of the system does not change (see also Sec. VI A). Thus, we conclude that the definition of the total work $W^{(n]}(r_{n-1})$ during one full interval is identical to the definition used in classical stochastic thermodynamics. It remains to look at the change of heat during one full interval $Q_S^{(n]}(r_n, r_{n-1})$. First of all, from Eq. (16) the heat exchanged during the interval without control becomes

$$Q_S^{(n)}(r_{n-1}) = \sum_s H(s, \lambda_t) p_s(t_n^-, r_{n-1}) - H(r_{n-1}, \lambda_t), \quad (85)$$

which is different from the definition (82). However, it is now also important to take into account the heat exchanged during the control step [Eq. (24)], in which we update our knowledge about possible system changes. It is simple to see that this quantity reduces to

$$Q_S^{\text{ctrl}}(r_n, r_{n-1}) = H_S(r_n, \lambda_t) - \sum_s H(s, \lambda_t) p_s(t_n^-, r_{n-1}) \quad (86)$$

such that $Q_S^{(n]}(r_n, r_{n-1}) = Q_S^{\text{ctrl}}(r_n, r_{n-1}) + Q_S^{(n)}(r_{n-1})$ is identical to the standard definition in classical stochastic thermodynamics. To conclude, our definitions for stochastic internal energy, work, and heat are identical to the ones used in classical stochastic thermodynamics.

(iii) We now take a look at the entropic balance. The change in stochastic entropy (35) over a full interval becomes

$$\Delta S_{SU(\mathbf{n})}^{(n]}(r_n, r_{n-1}) = -\ln p(r_n|r_{n-1}), \quad (87)$$

where we used that the system and units are, after each measurement, in a pure state and their entropy vanishes. Furthermore, we used that the system dynamics are Markovian and hence $p(r_n|\mathbf{r}_{n-1}) = p(r_n|r_{n-1})$. The stochastic entropy production (36) over one interval then becomes

$$\Sigma^{(n]}(r_n, r_{n-1}) = -\ln p(r_n|r_{n-1}) - \beta Q_S^{(n]}(r_n, r_{n-1}), \quad (88)$$

which can have either sign. As deduced in Sec. IV, it is positive after averaging over $p(r_n|r_{n-1})$:

$$\Sigma^{(n]}(r_{n-1}) = \sum_{r_n} p(r_n|r_{n-1}) \Sigma^{(n]}(r_n, r_{n-1})$$
$$= S_{\text{Sh}}[p(r_n|r_{n-1})] - \beta Q_S^{(n]}(r_{n-1}) \geqslant 0. \quad (89)$$

Note that this second law is identical to the conventional one of stochastic thermodynamics if we apply Eq. (83) to an initially pure state $p_s(t-dt) = \delta_{s,r_{n-1}}$, which implies $\Delta S_{\text{ST}}(t) = S_{\text{Sh}}[p(r_n|r_{n-1})]$ and $Q_{\text{ST}}(t) = Q_S^{(n]}(r_{n-1})$. Unfortunately, although $S_{\text{Sh}}[p(r_n|r_{n-1})]$ is infinitesimal small, it is of $O(dt^\nu)$ with $\nu < 1$. Therefore, the rate of entropy production diverges:

$$\lim_{dt\to 0} \frac{\Sigma^{(n]}(r_{n-1})}{dt} = \infty. \quad (90)$$

Although seldomly stated [105], this is related to the fact that the Shannon entropy $S_{\text{Sh}}[p_s(t)]$ is not differentiable when the kernel of $p_s(t)$ changes. Furthermore, by averaging Eq. (89) also over $p(r_{n-1})$, we obtain

$$\Sigma^{(n]} = S_{\text{Sh}}(r_n|r_{n-1}) - \beta Q_S^{(n]} \geqslant 0. \quad (91)$$

Here $S_{\text{Sh}}(r_n|r_{n-1}) = \sum_{r_{n-1}} p(r_{n-1}) S_{\text{Sh}}[p(r_n|r_{n-1})]$ defines the conditional Shannon entropy. This second law is different from the conventional one (83). Instead of containing the change in Shannon entropy of the system state, it contains the conditional Shannon entropy, which is nothing but the entropy rate of the stochastic process [106]. Of course, if we divide Eq. (91) by dt, it still diverges. Furthermore, the difference in the two entropy productions is precisely given by $\Sigma^{(n]} - \Sigma_{\text{ST}}(t) = S_{\text{Sh}}(r_{n-1}|r_n)$. Here the backward conditional entropy $S_{\text{Sh}}(r_{n-1}|r_n) = \sum_{r_n} p(r_n) S_{\text{Sh}}[p(r_{n-1}|r_n)]$ can be computed via Bayes' rule: $p(r_{n-1}|r_n) = p(r_n|r_{n-1})p(r_{n-1})/p(r_n)$.

We emphasize that our second law (91) has a transparent physical interpretation. It consists of the entropic change in the bath quantified by the Clausius-like term $-\beta Q_S^{(n]}$ plus the change in entropy in our memory for the measurement outcomes. As we measure perfectly and continuously, the rate of information generation in the memory is infinite (in reality, every sampling rate is finite and no divergence arises). Therefore, even in equilibrium where $Q_S^{(n]} = 0$, we will have a positive entropy production $\Sigma^{(n]} > 0$ due to the fact that we measure the system and continuously generate information. In stochastic thermodynamics, one instead finds $\Sigma_{\text{ST}} = 0$ at equilibrium. The discrepancy of the two second laws is rooted in the fact that standard stochastic thermodynamics keeps the observer out of the construction. This works well if one only perfectly monitors a classical system, but if one starts to apply feedback control one needs to modify the theory [68]. By following the credo that information is physical [67] and by treating the measurement and the system on an equal footing, no modification is necessary in our framework. We remark that our second law (91) was very recently experimentally confirmed [107] (see also the discussion in Ref. [82]).

(iv) In addition, we can recover the conventional second law of stochastic thermodynamics if we redefine entropy. Namely, if we replace our definition of entropy by the conventional one (80), the stochastic entropy production becomes, in our notation,

$$-\ln p(r_n) + \ln p(r_{n-1}) - \beta[H(r_n, \lambda_t) - H(r_{n-1}, \lambda_t)]. \quad (92)$$

If we average over $p(\mathbf{r}_n)$ and use that the measured probabilities are identical to the probabilities of the system $p(r_n = s) = p_s(t)$ and $p(r_{n-1} = s) = p_s(t-dt)$, we obtain

$$S_{\text{Sh}}[p_s(t)] - S_{\text{Sh}}[p_s(t-dt)]$$
$$- \beta \sum_s H(s, \lambda_t)[p_s(t) - p_s(t-dt)]. \quad (93)$$

This is identical to Eq. (83).

F. Getting rid of units in the thermodynamic description

We used the external stream of units to guide our thermodynamic analysis along the framework of repeated interactions. In many important realistic situations it is also clear how to model the units physically. This is, for instance, the case for the micromaser, the experimental setup studied in Sec. V, or for certain mesoscopic devices where tunneling electrons and Cooper pairs could be identified as units [108,109]. Therefore, the framework of repeated interactions allows us to treat a larger class of physically relevant scenarios.

Nevertheless, there are also scenarios where the exact microscopic nature of the units is not known or hard to model. Furthermore, as also the process tensor relies only on specifying CP maps $\mathcal{A}(r_n|\mathbf{r}_{n-1})$ acting on the system, it is worth asking whether we can get rid of the sometimes rather artificial units in the thermodynamic description. Energetically, we have already seen that simply setting $H_{U(n)} \sim 1_U$ for all n cancels out all unit contributions from the first law. To get rid of the units from the entropic considerations, we will need to restrict ourselves to efficient control operations [79,80]. Efficient control operations are defined by the requirement that they can be written as

$$\tilde{\rho}_S(r) = \mathcal{A}(r)\rho_S = A(r)\rho_S A(r)^\dagger \tag{94}$$

as opposed to the more general form (4). They have the specific property that any initially pure state ρ_S gets mapped to a pure state again. Mathematically, every efficient control operation can be modeled by an initially pure unit state $\rho_U = |\psi\rangle\langle\psi|_U$, which interacts unitarily via V with the system and is finally projectively measured using $P(r) = |r\rangle\langle r|$. This implies that

$$\tilde{\rho}_S(r) = \mathcal{A}(r)\rho_S = \langle r|\mathcal{V}[\rho_S \otimes |\psi\rangle\langle\psi|_U]|r\rangle_U. \tag{95}$$

This construction extends to multiple operations conditioned on previous results $\mathbf{r}_{n-1}$ in the obvious way.

To see that the units also do not enter the entropic balance in this case, notice that the unit state is pure and decorrelated from the system after every operation. This follows from the fact that we perform a rank-1 projective measurement on the units after each control operation. The joint state of the system and all units after obtaining the sequence of outcomes $\mathbf{r}_n$ is simply $\rho_{SU(\mathbf{n})}(t, \mathbf{r}_n) = \rho_S(t, \mathbf{r}_n) \otimes |\mathbf{r}_n\rangle\langle\mathbf{r}_n|_{U(\mathbf{n})}$, with $|\mathbf{r}_n\rangle\langle\mathbf{r}_n|_{U(\mathbf{n})} \equiv |r_n\rangle\langle r_n|_{U(n)} \otimes \cdots \otimes |r_1\rangle\langle r_1|_{U(1)}$. The joint entropy for this state becomes $S_{\text{vN}}[\rho_{SU(\mathbf{n})}(t, \mathbf{r}_n)] = S_{\text{vN}}[\rho_S(t, \mathbf{r}_n)]$. Also, before the interaction at time t_n, we have

$$S_{\text{vN}}[\rho_{SU(\mathbf{n})}(t_n^-, \mathbf{r}_n)] = S_{\text{vN}}[\rho_S(t_n^-, \mathbf{r}_n)], \tag{96}$$

where we used that the initial unit state is pure and hence always decorrelated from the system. We note that the ensemble-average system unit state $\sum_{\mathbf{r}_n} p(\mathbf{r}_n)\rho_{SU(\mathbf{n})}(t, \mathbf{r}_n)$ is in general classically correlated.

To summarize, in the case of energetically neutral units and efficient control operations, the stochastic internal energy and entropy can be reduced to

$$E_S(t, \mathbf{r}_n) = \text{tr}_S\{H_S(\lambda_t, \mathbf{r}_n)\rho_S(t, \mathbf{r}_n)\}, \tag{97}$$

$$S_S(t, \mathbf{r}_n) = -\ln p(\mathbf{r}_n) + S_{\text{vN}}[\rho_S(t, \mathbf{r}_n)]. \tag{98}$$

Note, however, that we are still using the external units to model the control operations dynamically. In fact, we will discuss in Sec. VII A how far it is possible to get completely rid of the units.

G. Quantum stochastic thermodynamics without theory input

To set up our framework of quantum stochastic thermodynamics, we needed to be able to know the work (15) and heat (16) exchanged with the bath in between two control operations. Those are path-dependent quantities [i.e., they are not determined alone by the state at the boundary $\rho_S(t_n^\pm, \mathbf{r}_n)$] and estimating them requires additional theoretical input. Although this is necessary to recover the average picture in general (see Sec. VI C), it is instructive to discuss cases which do not require any additional theoretical modeling.

Without changing any of our general conclusions, one way would be to consider only a specific subset of control protocols λ_t. These control protocols consist of a sudden switch of the Hamiltonian after each control operation, i.e., the protocol changes instantaneously from λ_{n-1} to λ_n at time t_n^+ and after the switch we keep the protocol constant until the next control operation. Note that the protocol is still allowed to depend on $\mathbf{r}_n$, which we have suppressed for notational convenience. Thus, in short, we can write that $\lambda_t(\mathbf{r}_{n-1}) = \lambda_{n-1}(\mathbf{r}_{n-1})$ if $t \in (t_{n-1}, t_n]$. Those sets of control protocols are characterized by the fact that the work (15) and heat (16) can be computed without any knowledge about the system state in between two control operations:

$$\begin{aligned} W_S^{(n)}(\mathbf{r}_{n-1}) = \text{tr}_S\{[&H_S(\lambda_{n-1}, \mathbf{r}_{n-1}) \\ &- H_S(\lambda_{n-2}, \mathbf{r}_{n-2})]\rho_S(t_{n-1}^+, \mathbf{r}_{n-1})\}, \end{aligned} \tag{99}$$

$$\begin{aligned} Q_S^{(n)}(\mathbf{r}_{n-1}) = \text{tr}_S\{&H_S(\lambda_{n-1}, \mathbf{r}_{n-1}) \\ &\times[\rho_S(t_n^-, \mathbf{r}_{n-1}) - \rho_S(t_{n-1}^+, \mathbf{r}_{n-1})]\}. \end{aligned} \tag{100}$$

Another way to approach this problem is to try to set up an effective thermodynamic description based solely on knowledge of the dynamical map $\mathcal{E}_n$ defined in Eq. (40). Note that the dynamical map can be inferred from knowledge of the process tensor. The very problem of this approach comes from the fact that different physical situations (with different thermodynamic values for $W_S^{(n)}$ and $Q_S^{(n)}$) can give rise to the same dynamical map $\mathcal{E}_n$. Thus, if we try to pursue the second way, we will not be able to recover the results from Secs. VI C and VI D in general. Nevertheless, it could be worthwhile to pursue this direction because the thermodynamic description of dynamical maps was already investigated before [110–113]. In particular, for dynamical maps which have additional properties, such as being Gibbs state preserving, the present framework could be fruitfully combined with the resource theory approach to quantum thermodynamics [114,115].

VII. CONCLUSION

A. Final remarks

We have presented a theoretical framework which is able to cope with arbitrary quantum operations and arbitrary unravelings of them. It uses very natural definitions of internal energy (18) and entropy (35), but in its most general form it can appear quite heavy. In particular, the framework of repeated interactions added another layer of complexity and it is worthwhile to ask whether we can get rid of it completely. For efficient control operations we have seen already in Sec. VI F that the units do not enter the laws of thermodynamics anymore, although they still played a role dynamically. This was important in order to arrive at an unambiguous interpretation of heat and work during the control step. To exemplify this, let us look at an arbitrary efficient operation $\tilde{\rho}_S(r) = A(r)\rho_S A(r)^\dagger$ again. It is tempting to use the polar decomposition theorem $A(r) = U(r)P(r)$, where $U(r)$ is a unitary matrix and $P(r)$ a positive matrix, to define work and heat exchanges. One idea could be to associate changes in the energy caused by $P(r)$ $[U(r)]$ as heat (work). Unfortunately, one then arrives at the conclusion that a projective measurement is on average a heat and not a work source and we have debated this problem already in Sec. VI A. Moreover, there is also a reverse polar decomposition theorem $A(r) = P'(r)U(r)$, where $P'(r) \neq P(r)$ in general. This would then give rise to a different splitting into heat and work for the same control operation. This is even true in the case $P'(r) = P(r)$ because in the reverse decomposition the positive matrix acts after the unitary. By using Stinespring's dilation theorem we have circumvented this difficulty in the repeated interaction framework. In this picture the unitary V must always act first to correlate the system and the unit before it is followed by a measurement of the unit. This fixes the ambiguity of assigning heat and work, which can be conveniently computed by using the control operations only, see Eqs. (27) and (28). Thus, for efficient control operations with energetically neutral units the explicit modeling of the units is no longer necessary.

Another subtle point concerns the definition of an entropy production via a time-reversed process. We have here decided to find meaningful definitions of heat and entropy at the first place and then checked that the entropy production $\Sigma = \Delta S_S - \beta Q$ as known from phenomenological nonequilibrium thermodynamics is positive on average. Remarkably, within the framework of classical stochastic thermodynamics there is an equivalent alternative approach by defining the stochastic entropy production as

$$\tilde{\Sigma}(\mathbf{r}_n) \equiv \ln \frac{p(\mathbf{r}_n)}{p^\dagger(\mathbf{r}_n^\dagger)}. \tag{101}$$

Here $p^\dagger(\mathbf{r}_n^\dagger)$ is the probability to observe the time-revered trajectory in a suitably chosen time-reversed experiment [12,13]. This stochastic entropy production fulfills a fluctuation theorem and a second law and it is linked to the (breaking of) time-reversal symmetry of the underlying microscopic Hamiltonian dynamics [7,8,22,23]. It is tempting to apply a similar strategy also within our framework by defining a suitable time-reversed process to construct the entropy production (101). Unfortunately, for a general quantum operation it is not clear what the corresponding time-reversed process should be. Various proposals have been put forward and used in the literature [42–45,51,52,54,55,112,116,117], resulting in multiple possible second laws for the same physical situation. It is an advantage of the present framework that we are able to derive a second law without taking the detour of defining a time-reversed process, which, at least at the moment, seems to entail an unwanted amount of ambiguity.

As a final remark we comment on the possibility to extend the present framework beyond the case of a single heat bath. In fact, this is even an open problem in classical stochastic thermodynamics from an experimental point of view: As soon as multiple heat baths induce transitions between the same system states, a local measurement of the system only will not reveal which bath has triggered the transition. Classically, a way out of this dilemma is to experimentally ensure that the transition between each pair of states is caused by only a single bath, for instance, by geometrically separating the system into subsystems, where each subsystem interacts with only one bath. This is indeed what happens in transport experiments through quantum dots [37,38]. Quantum mechanically, this separation is more difficult to achieve. At least within the standard approach based on a Born-Markov secular approximation [3–6], the system jumps between energy eigenstates of the composite system which are in general entangled. On the other hand, it was also recently argued that a local approach to the dynamics (where each dissipator in a quantum system acts only on a specific subsystem) is feasible from a thermodynamic point of view [104,118,119]. If that is the case, it should be possible in principle to apply our framework to a situation with multiple baths in some limit. As the proper extension of quantum thermodynamics to the presence of multiple heat baths can already bear surprising difficulties at the average level [120], these investigations are left for the future.

B. Outlook

In this section we outline three promising future applications that allow us to answer in a general and rigorous way open problems in quantum thermodynamics.

1. Quantum coherence and Leggett-Garg inequalities

One primary task of quantum thermodynamics is to unravel how quantum features (such as coherence or entanglement) influence the performance of quantum heat engines and other devices. An introduction to this topic was recently provided in Ref. [121]. While several interesting results have been found (showing that quantum effects can be both beneficial and detrimental), one always has to be cautious when comparing them with classical systems. In fact, it is far from obvious to what extent quantum and classical models can be compared and what are genuine quantum features. For instance, the mere presence of coherences (i.e., off-diagonal elements of the density matrix in the energy eigenbasis) is not sufficient to conclude that the heat engine operates in the quantum regime [122]. As we will show now, our framework allows us to rigorously answer whether a given heat engine uses quantum coherence. Moreover, this is closely related to the violations of Leggett-Garg inequalities [123].

Our analysis is based on recent progress in understanding genuine quantum effects in Markovian systems interrupted by projective measurements at a set of discrete times [124–126]. Concisely, the authors of Ref. [124] have proven that the results $\mathbf{r}_n$ obtained from the projective measurements in an arbitrary nondegenerate basis $\{|r_n\rangle\}$ cannot be generated by a classical stochastic process if and only if the Markovian dynamics are "coherence generating and detecting" for an initially diagonal state in the measurement basis. The notion of coherence generating and detecting is defined by using the dephasing operator $\mathcal{D} = \sum_{r_n} \mathcal{P}(r_n)$, where $\mathcal{P}(r_n)$ denotes the projection superoperator with respect to $|r_n\rangle\langle r_n|$, and by demanding that there exist times $t, \tau \geqslant 0$ such that

$$\mathcal{D} \circ \mathcal{E}(t) \circ \mathcal{D} \circ \mathcal{E}(\tau) \circ \mathcal{D} \neq \mathcal{D} \circ \mathcal{E}(t+\tau) \circ \mathcal{D}, \tag{102}$$

where $\mathcal{E}(t)$ denotes the dynamical map of the system in between the control operations (here assumed to be time homogeneous for simplicity) and $\circ$ the composition of two maps. An extension to inhomogeneous maps and more general (i.e., non-Markovian) dynamics can be found in Refs. [125,126].

This framework fits perfectly into our language as we can deal with projective measurements at discrete times as well as dephasing operations. To give a simple and intuitive example how this framework could be used to detect quantum signatures in thermodynamics, we consider the quantum Otto cycle, which was recently also realized experimentally [127]. The Otto cycle is a four-step process $A \to B \to C \to D$ (see, e.g., Fig. 2 in Ref. [121]). In $A \to B$ the system undergoes isolated (unitary) Hamiltonian evolution, where the system Hamiltonian changes from $H_S(1)$ to $H_S(2)$. In $B \to C$ the system is coupled to a cold bath at temperature T_C and undergoes pure relaxation dynamics, which we here assume to be modeled by a Lindblad master equation as is often done. In $C \to D$ the system is again isolated and its Hamiltonian is changed from $H_S(2)$ back to $H_S(1)$ again. Finally, in $D \to A$ the system is coupled to a hot bath at temperature T_H and undergoes again pure relaxation assumed to be described by a Lindblad master equation. After the unitary strokes at points B and D the density matrix of the system contains coherences in general. If the cycle is performed in finite time such that the heat baths do not fully erase the coherences, then it is possible that coherences are still present at points A and C and it becomes an interesting question whether they change the thermodynamic performance.

To unambiguously answer this question, one could perform a dephasing operation $\mathcal{D}$ in the energy eigenbasis at any of the four points. If this changes the work output or the thermodynamic efficiency,[8] then the machine shows quantum effects. The dephasing operation $\mathcal{D}$ is easily implemented, for instance, by performing a projective measurement of the energy without recording its outcome (see Sec. VI A). Importantly, the energetic cost of this control operation is zero (provided the unit is energetically neutral) and therefore it does not inject or extract any work into the engine. Hence, while the dephasing operation has an entropic cost, this does not play any role in computing the work and heat flows in the Otto cycle, which are essential to compute its performance.

Conversely, by the theorem derived in Ref. [124–126], we know that coherences can only influence the dynamics if the statistics associated with projective energy measurements at any subset of the four points in the Otto cycle show nonclassical signatures by not obeying the Kolmogorov consistency condition. For instance, if the dephasing operation at point B has an influence on the thermodynamic performance, then also

$$\sum_{r_B} p(r_A, r_B, r_C) \neq p(r_A, r_C). \tag{103}$$

Here we have denoted the outcome of the projective measurement at point A by r_A (and analogously for the other points) in the spirit of our previous notation. Thus, instead of looking at the effect of the dephasing operation, we could also alternatively use the process tensor formalism to infer the statistics of the projective measurements directly.

Remarkably, Eq. (103) is a necessary prerequisite to violate the Leggett-Garg inequality [123]. Thus, by probing the multitime correlations of a quantum stochastic process, we can also learn something about its thermodynamic behavior and unravel the regime where it has no analogous classical stochastic thermodynamic process. First results in this direction have been already obtained in Refs. [128,129]. In addition, there are also entropic Leggett-Garg inequalities [123,130,131], which relate Eq. (103) to the entropy of the measurement result $H(r_C, r_B, r_A)$. As this quantity plays a crucial role in our second law, it would be interesting to investigate whether a Maxwell demon can extract more or less work from a system and measurement process able to violate the entropic Leggett-Garg inequalities.

2. Entanglement

Closely related to the previous analysis is the question of how far entanglement can boost the performance of a heat engine. There has been much theoretical progress on understanding the role of entanglement for work extraction (see, e.g., Refs. [132–137]), mostly, however, for extracting work in idealized protocols. A realizable and continuously working heat engine using quantum entanglement has not yet been presented. In contrast, classical correlations are known to be indispensable for autonomous multipartite heat engines such as thermoelectric devices [138–142].

To test whether a thermodynamic process is influenced by entanglement, consider a bipartite system AB existing in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ as the working fluid. One could then follow a similar strategy as above, but this time, instead of applying a dephasing operation, one would apply an entanglement-breaking operation $\mathcal{B}$, which keeps classical correlations. If the reduced state of system A is given by $\rho_A = \sum_i \lambda_i |i\rangle\langle i|_A$, then the control operation

$$\mathcal{B}\rho_{AB} = \sum_i |i\rangle\langle i|_A \rho_{AB} |i\rangle\langle i|_A \tag{104}$$

would destroy any entanglement but keep all classical correlations. Monitoring the response of a multipartite system to such a control operations then allows the experimenter to infer how far quantum correlations play a role thermodynamically. As above, this procedure exemplifies how useful generalized control operation are, not only to control a thermodynamic process but also to unravel specific properties of it.

3. Non-Markovian signatures in heat engines

The last part of this outlook probably requires the largest research effort, but it seems to be necessary in order to obtain a complete framework of stochastic thermodynamics for small quantum systems. Indeed, for sufficiently low temperatures and sufficiently small timescales (i.e., where the standard Born-Markov secular master equation fails) it is expected that generic open quantum systems exhibit non-Markovian behavior. Furthermore, even at room temperature there is evidence that non-Markovianity can drastically effect biochemical

[8] Note that we have not specified here how to actually infer the work output or efficiency. This could be done purely theoretically or purely experimentally, for instance, by doing quantum state tomography at the four points A, B, C, and D after waiting long enough such that the system operates at steady state (actually, state tomography at two points suffices if we are able to accurately compute the effect of the unitary strokes).

processes such as photosynthesis [143,144] and there is evidence that non-Markovian effects can also boost the performance of heat engines [145–147]. Despite the fact that there are several ways to rigorously quantify non-Markovianity in open quantum systems [148,149], establishing a rigorous connection between thermodynamics and non-Markovianity has proven to be challenging so far [150].

Notice that the present framework crucially hinges on the assumptions of a Markovian system evolution. However, it is likely that it is possible to overcome the assumptions from Sec. IV A. One route could be to enlarge the system space by incorporating explicitly the most dominant degrees of freedom of the environment in the dynamics, a strategy which was directly or indirectly proposed in Refs. [146,147, 151–157]. Preliminary results also show that this is not even necessary if we do not consider real-time feedback control [158].

To outline how it would be possible to rigorously detect non-Markovian effects in quantum thermodynamics, we make use of the notion of a causal break. This notion was recently introduced in Ref. [61] to give a general and rigorous definition of non-Markovianity based on the process tensor, which generalizes previous attempts [148,149]. The basic idea is to apply a control operation to the system, which reprepares it in a state independent of all past events. Any dependence of future events on past events then reveals non-Markovian effects.

To have a particular application in quantum thermodynamics in mind, imagine a steadily working heat engine. The details of the machine, i.e., whether it uses multiple heat baths or feedback control as a resource and whether it acts as a refrigerator or thermoelectric device, do not matter for the present consideration. Furthermore, let us denote the steady state of the machine by $\bar{\rho}_S$. Now, as a causal break we apply a control operation which replaces the current state of the system by the steady state $\bar{\rho}_S$. This is always possible: We could, for instance, projectively measure the state of the system and then prepare the state $\bar{\rho}_S$. Since $\bar{\rho}_S$ will be in general mixed, this preparation procedure will be probabilistic (i.e., described by multiple Kraus operators and not a single one). The crux is now to apply this control operation when the machine has already reached steady state, i.e., we effectively replace $\bar{\rho}_S$ by $\bar{\rho}_S$ on average. When the system behaves Markovian, the future statistics of all measurements will not depend on this repreparation procedure, but if the system exhibits non-Markovian behavior, there will be observable consequences as our control operation has destroyed all time correlations of the system with the past. To see whether such a causal break has an influence on the thermodynamics (which does not need to be the case even when the overall dynamics is non-Markovian), one could measure, e.g., the work output of the device or its efficiency. Since the system was assumed to operate at steady state, any change in its thermodynamic behavior after the causal break described above unambiguously reveals non-Markovian effects.

Thus, to summarize, we are only beginning to explore quantum effects in thermodynamics. To access those quantum effects in a laboratory, it is important to be able to apply various control operations to the system. The present paper provides the toolbox to describe these control operations thermodynamically even along a single stochastic trajectory.

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