Symmetry of deterministic ratchets

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We consider the overdamped motion of a Brownian particle in an unbiased force field described by a periodic function of coordinate and time. A compact analytical representation has been obtained for the average particle velocity as a series in the inverse friction coefficient, from which follows a simple and clear proof of hidden symmetries of ratchets, reflecting the symmetry of summation indices of the applied force harmonics relative to their numbering from left to right and from right to left. We revealed the conditions under which (i) the ratchet effect is absent; (ii) the ratchet average velocity is an even or odd functional of the applied force, whose dependences on spatial and temporal variables are characterized by periodic functions of the main types of symmetries: shift, symmetric, and antisymmetric, and universal, which combines all three types. These conditions have been specified for forces with those dependences of a multiplicative (or additive-multiplicative) and additive structure describing two main ratchet types, pulsating and forced ratchets. We found the fundamental difference in dependences of the average velocity of pulsating and forced ratchets on parameters of spatial and temporal asymmetry of potential energy of a particle for systems in which the spatial and temporal dependence is described by a sawtooth potential and a deterministic dichotomous process, respectively. In particular, it is shown that a pulsating ratchet with a multiplicative structure of its potential energy cannot move directionally if the energy is of the universal symmetry type in time; this restriction is removed in the inertial regime, but only if the coordinate dependence of the energy does not belong to either symmetric or antisymmetric functions.

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I. INTRODUCTION

Deterministic ratchets include systems in which directed motion of particles arises due to the rectification of cyclically repeating perturbations described by periodic functions of time [1–8]. Unlike a case of stochastic ratchets, which are, as a rule, associated with the functioning of protein engines [9–18], deterministic ratchets are controlled by using human-created processes. The motion of particles suspended in solutions and exposed to a periodic asymmetric potential [19,20] (dielectrophoresis effect [21]), vortices in superconductor systems [22], atoms in dissipative optical lattices [23], and electrons in organic semiconductors [24] are controlled just by such deterministic processes. One of the necessary conditions for the existence of particle current as the ratchet effect is the absence of reflection symmetry of a system in which this motion occurs [13,25]. There are, however, other (less obvious) types of symmetries that regulate the presence or absence of the ratchet effect, and the ability to recognize those symmetries is absolutely necessary for understanding mechanisms of diffusion transport of nanoparticles, as well as for designing nanomachines. The revealing of symmetry properties by the standard symmetry analysis, that is, by finding a group of transformations that leave the equation of motion invariant, was carried out in [26]. Another way to discover symmetry properties of ratchets is to analyze solutions of either equations of motion or stochastic equations for the distribution function describing particle motion in a thermal reservoir. Of course, such a way is also not very easy, since for the symmetry analysis it is necessary to get these solutions or good approximations to them.

In the ratchet theory, the most widely used approximation is the overdamped regime, in which friction is considered to be prevailing over inertia, and the time over which Maxwell velocity distribution is established ($\tau_v = m/\zeta$, m is the particle mass, ζ is the friction coefficient) can be considered as the smallest time parameter of the system ($\tau_v \to 0$) [27]. Next, in the overdamped regime, one can obtain relatively simple analytical expressions for the ratchet average velocity by describing their functioning within the adiabatic [28–31] or high-temperature [32,33] (low-energy) approximations. Moreover, choosing those adiabatic or low-energy approximations as the main approaches, one can return again, within their frameworks, to the inertial dynamics and obtain expressions for the characteristics of such ratchets containing inertial corrections [34,35]. In this way, a number of nontrivial properties of Brownian motors have been discovered, which are a result of the presence or absence of one or another symmetry of particle potential energies [30,31,33,34,36].

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It should be borne in mind that the operation of time reversal, which is often included in the analysis of symmetry properties of a system, entails an additional, fundamental difficulty, since the laws of mechanics are known to be reversible under the time reversal, while the laws of statistical physics that seemingly use these laws of mechanics, are irreversible. Therefore, it is difficult to expect that, in dissipative systems, the probabilities of realization (statistical weights) of forward and backward trajectories of a particle will be identical. All that leads to the necessity of analyzing the symmetry of solutions of statistical equations; it is a difficult problem in view of the difficulty of finding these solutions, which, in addition, may prove to be very cumbersome for any analysis. If, for all that, one succeeded in analyzing the symmetry properties, then the detected symmetry types are called hidden, emphasizing the fact that they cannot be obtained by the standard symmetry analysis. Significant progress in discovering hidden symmetries of ratchet systems was achieved in the recent work of Cubero and Renzoni [37]. The conclusions about the existence of certain hidden symmetries follow there from a rather cumbersome analysis of the symmetry of space-time Fourier components of a resolvent representing the formal solution of the Smoluchowski equation (see the supplemental material in [37]).

In this article, the same hidden symmetries are proved very simply and quite clearly. To do this, we use the approach of Ref. [33], in which the solution of the Smoluchowski equation is obtained as a series in the inverse friction coefficient. This solution allows one to write a compact analytical representation for the ratchet average velocity, which contains products of Fourier components of an applied force, which is periodic both in coordinate and in time. The proof of the hidden symmetries of ratchet systems contained in the expression for the average velocity is carried out in Sec. II (see also Appendixes A and B), and it is based on the fact that there exists an additional symmetry in the numbering of the summation indices of these Fourier components in the products.

Section III gives a summary of the main symmetry types of periodic functions, which include *shift*, *symmetric*, and *antisymmetric* types; this allowed us to introduce an additional, *universal*, type of symmetry that combines all three types. The presence of such symmetry in spatial and/or temporal variables yields special, not typical for other symmetries, properties of ratchets. This section is illustrated with constructive examples of designing all symmetry types on the basis of a single template and gives a comprehensive visual representation of correspondences between symmetries of a function and those of its derivative, which is especially important for comparing properties of ratchets resulting from symmetry features of potential energies and corresponding forces.

Symmetry properties of ratchets with an applied fluctuating force which is a function of coordinate and time of an arbitrary structure are described in Sec. IV. Since the properties of ratchet systems with a force of a multiplicative structure are closest, in certain conclusions, to the case of an arbitrary structure, they are also analyzed in Sec. IV. The most general conclusions obtained here are, to some extent, used in all subsequent sections, which specify the structure of the force, as well as the spatial and temporal dependencies. Namely, in Sec. V, symmetry properties of ratchets with an applied force of a two-component additive-multiplicative structure are considered, and in Sec. VI, an asymmetric dichotomous process of fluctuations of a sawtooth potential. The same section, Sec. VI, presents the results for both pulsating and forced ratchets in terms of temporal and spatial asymmetry parameters, which can be rigorously determined just for a sawtooth-shaped potential relief undergoing dichotomous fluctuations (the most common model in the ratchet theory).

In the final section we summarize the results obtained and discuss their significance for analyzing properties of ratchet systems. We emphasize the generality of the results, based on the use of the exact analytical solution of the Smoluchowski equation, and compare them with the symmetry results obtained under some approximations. A comparative analysis is made of situations in which the revealing symmetry properties required using the time reversal operation (properties based on hidden symmetries) or have been found without it. This determines whether the conclusions are valid only in overdamped systems or hold or arise in the inertial regime.

II. SYMMETRY OF THE SOLUTION OF THE SMOLUCHOWSKI EQUATION

We consider, following Refs. [33,38], one-dimensional dynamics of a Brownian particle in a viscous medium, which is characterized by a function $x(t)$ (particle position) that satisfies the Langevin equation:

$$m\ddot{x} = -\zeta\dot{x} + F(x,t) + \xi(t). \tag{1}$$

Here $\dot{x}(t)$ and $\ddot{x}(t)$ are the first and second time derivatives of $x(t)$, m is the particle mass, ζ is the friction coefficient, and $F(x,t) = -\partial U(x,t)/\partial x$ is an applied force corresponding to the potential energy $U(x,t)$. The applied force is considered as a periodic function of the coordinate x and time t, $F(x+L,t) = F(x,t+\tau) = F(x,t)$, where L and τ are the spatial and temporal periods. Thermal fluctuations are modeled by Gaussian white noise $\xi(t)$ with the mean value $\langle\xi(t)\rangle = 0$ and the correlation function $\langle\xi(t)\xi(t')\rangle = 2\zeta k_BT\delta(t-t')$ (k_B and T is the Boltzmann constant and equilibrium absolute temperature, respectively).

For small particles in a sufficiently viscous medium, the inertial term $m\ddot{x}$ can be omitted. Then for statistical description of such (inertialess) motion of a Brownian particle, one can use the distribution function $\rho(x,t)$ that satisfies the Smoluchowski equation [27],

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}J(x,t),$$
$$J(x,t) = -D\frac{\partial}{\partial x}\rho(x,t) + \zeta^{-1}F(x,t)\rho(x,t), \tag{2}$$

and the normalization condition $\int_0^L \rho(x,t)dx = 1$. The expression for the flux $J(x,t)$ contains the diffusion coefficient $D = k_BT/\zeta$. For a steady-state process, after the system forgets its initial condition, the desired average value of particle velocity [motor velocity, which is a functional of the applied force $F(x,t)$] is determined by the double integral:

$$\langle v\rangle = v\{F(x,t)\} = \frac{1}{\tau}\int_0^\tau dt\int_0^L dxJ(x,t). \tag{3}$$

The periodicity of $F(x,t)$, both in coordinate and time, makes it efficient to apply the double Fourier transform to obtain solutions of the Smoluchowski equation: $f(x,t)=\sum_{qj} f_{qj}\exp(ik_q x - i\omega_j t)$, $k_q=(2\pi/L)q$, $\omega_j=(2\pi/\tau)j$, where q and j are integers, k_q and ω_j are wave vectors and frequencies, and $f(x,t)=f(x+L,t+\tau)$ is an arbitrary function, which in the context of the problem may be, for example, a force or a distribution function. To shorten the notation, it is convenient to designate a pair of indices q and j as one index $p=(q,\ j)$ and use it in further transformations. Then the desired functional (3) takes the following form:

$$v\{F(x,t)\}=\zeta^{-1}L\sum_p F_{-p}\rho_p, \tag{4}$$

in which, according to Eq. (2), the Fourier components ρ_p of the distribution function $\rho(x,t)$ satisfy a system of algebraic equations:

$$\rho_p=L^{-1}\delta_{p,0}+\zeta^{-1}a_{-p}\sum_{p'}F_{p-p'}\rho_{p'},$$
$$a_p=a_{qj}=\frac{ik_q}{Dk_q^2+i\omega_j}, \tag{5}$$

where $\delta_{p,0}\equiv\delta_{q,0}\delta_{j,0}$ ($\delta_{q,0}$ and $\delta_{j,0}$ are the Kronecker delta symbols).

The structure of system (5) is such that ρ_p equals the sum of two contributions, independent and dependent on ρ_p, the latter being proportional to ζ^{-1}. Thus, it is advisable to search for its solution by an iterative procedure in the form of a power series in the inverse friction coefficient:

$$\rho_p=\sum_{n=0}^{\infty}\zeta^{-n}\rho_p^{(n)}. \tag{6}$$

Substituting (6) into (5) gives an iterative solution:

$$\rho_p^{(0)}=L^{-1}\delta_{p,0},\quad \rho_p^{(n)}\underset{n\geqslant 1}{=}a_{-p}\sum_{p'}F_{p-p'}\rho_{p'}^{(n-1)}. \tag{7}$$

After $(n-1)$ iterations we get the following result:

$$\rho_p^{(1)}=L^{-1}a_{-p}F_p,$$
$$\rho_p^{(n)}\underset{n\geqslant 2}{=}L^{-1}a_{-p}\sum_{p_2\cdots p_n}F_{p+p_2}a_{p_2}$$
$$\times F_{-p_2+p_3}\cdots a_{p_{n-1}}F_{-p_{n-1}+p_n}a_{p_n}F_{-p_n}. \tag{8}$$

Substituting this result into (6) and (4), together with the replacement of the index $p\to -p_1$ as well as zeroing the Fourier component F_0 to eliminate trivial motion of a particle under the action of a constant applied force (which is not related to the ratchet effect), allows one to write the desired functional as

$$v\{F(x,t)\}=\sum_{n=1}^{\infty}\zeta^{-n-1}R^{(n)}\{F(x,t)\},$$
$$R^{(n)}\{F(x,t)\}=\sum_{p_1\cdots p_n}a_{p_1}\cdots a_{p_n}F_{p_1}F_{p_2-p_1}\cdots F_{p_n-p_{n-1}}F_{-p_n}. \tag{9}$$

Note that introducing new summation variables $p_1'=p_1$, $p_l'=p_l-p_{l-1}$ ($l=2,\dots,n$), which provide equalities $p_l=\sum_{r=1}^{l}p_r'$ ($l=1,\dots,n$), yields an alternative representation for the functional (9) given in [38]:

$$R^{(n)}\{F(x,t)\}$$
$$=\sum_{p_1'\cdots p_n'}a_{p_1'}\cdots a_{p_1'+\cdots+p_n'}F_{p_1'}\cdots F_{p_n'}F_{-p_1'-\cdots-p_n'}. \tag{10}$$

If the force is time independent, $F(x,t)=f(x)$, representation (9) is reduced to a sum equal to zero (see Appendix A),

$$R^{(n)}\{f(x)\}=\left(\frac{i}{D}\right)^n\sum_{q_1\cdots q_n\,(\neq 0)}\frac{f_{q_1}f_{-q_1+q_2}f_{-q_2+q_3}\cdots f_{-q_{n-1}+q_n}f_{-q_n}}{k_{q_1}\cdots k_{q_n}}=0, \tag{11}$$

as it should be.

Next we demonstrate that the result (9) satisfies the vector and shift symmetry of ratchet systems, which are determined by the following equalities:

$$v\{F(-x,t)\}\underset{\text{(vect)}}{=}-v\{-F(x,t)\}, \tag{12}$$
$$v\{F(x+x_0,t+t_0)\}\underset{\text{(shift)}}{=}v\{F(x,t)\}. \tag{13}$$

The equality (12) means that coordinate inversion (the transformation $x\to -x$) yields inversion of directions of all vector quantities, which are the force and average velocity. The shift symmetry (13) reflects the fact that the average velocity is invariant under arbitrary translations (shifts) of the origin of coordinates and time [shifts by x_0 and t_0 in (13) should be considered independently].

To prove the vector symmetry (12), we note that, in the Fourier components, the transformation $x\to -x$ corresponds to the index transformation $q\to -q$. For this transformation, in terms of the pair of indices $p=(q,\ j)$, we introduce the notation $\bar p=(-q,\ j)$, so that the transformation $p\to\bar p$ corresponds to $x\to -x$, and the Fourier component $F_{\bar p}$ corresponds to the function $F(-x,t)$. With the definition a_p in (5), we have the property $a_{\bar{p}} = -a_p$, so that from (9) follows a chain of equalities:

$$\begin{aligned} R^{(n)}\{F(-x,t)\} &= \sum_{p_1\cdots p_n} a_{p_1}\cdots a_{p_n} F_{\bar{p}_1} F_{\bar{p}_2-\bar{p}_1}\cdots F_{\bar{p}_n-\bar{p}_{n-1}} F_{-\bar{p}_n} \\ &\underset{\bar{p}_l=p'_l}{=} \sum_{p'_1\cdots p'_n} a_{\bar{p}'_1}\cdots a_{\bar{p}'_n} F_{p'_1} F_{p'_2-p'_1}\cdots F_{p'_n-p'_{n-1}} F_{-p'_n} \\ &= \sum_{p'_1\cdots p'_n} (-1)^n a_{p'_1}\cdots a_{p'_n} F_{p'_1} F_{p'_2-p'_1}\cdots F_{p'_n-p'_{n-1}} F_{-p'_n} \\ &= -\sum_{p'_1\cdots p'_n} a_{p'_1}\cdots a_{p'_n} (-F_{p'_1})(-F_{p'_2-p'_1})\cdots(-F_{p'_n-p'_{n-1}})(-F_{-p'_n}) \\ &= -R^{(n)}\{-F(x,t)\}, \end{aligned} \tag{14}$$

from which we have the desired equality (12).

To prove the invariance of (9) under the transformation of the shift symmetry (13), one should take into account that, although under the shifts by x_0 and t_0, each Fourier component F_p acquires a phase factor $\exp(ik_q x_0 - i\omega_j t_0)$, the product of these factors appearing in the expression $F_{p_1}F_{p_2-p_1}\cdots F_{p_n-p_{n-1}}F_{-p_n}$ turns out to be equal to 1. Note that the symmetry properties (12) and (13) are always valid, and here we are only convinced that the resulting solution (9) remains invariant under these transformations, as it should be. There are, however, other symmetry transformations, known as hidden symmetries, which leave the solution invariant. Below one of these hidden symmetries is discussed in detail.

The generality of the expression (9), which is actually an exact solution of the Smoluchowski equation with an arbitrary force $F(x,t)$, periodic in coordinate and time, makes it easy to prove the hidden symmetry found by Cubero and Renzoni (see Ref. [37] and the cumbersome proof of this symmetry in its supplemental material):

$$v\{F(x,t)\} \underset{\text{(C-R)}}{=} v\{F(-x,-t)\}. \tag{15}$$

To prove (15), we use the correspondence between the function $F(-x,-t)$ and its Fourier components F_{-p}, so that the representation (9) yields the following transformation:

$$\begin{aligned} R^{(n)}\{F(-x,-t)\} &= \sum_{p_1\cdots p_n} a_{p_1}\cdots a_{p_n} F_{-p_1} F_{-p_2+p_1}\cdots F_{-p_n+p_{n-1}} F_{p_n} \\ &\underset{\substack{p_l=p'_{n+1-l} \\ (l=1,\dots,n)}}{=} \sum_{p'_1\cdots p'_n} a_{p'_n}\cdots a_{p'_1} F_{-p'_n} F_{p'_n-p'_{n-1}}\cdots F_{p'_2-p'_1} F_{p'_1} = R^{(n)}\{F(x,t)\}. \end{aligned} \tag{16}$$

It proves the presence of the discussed hidden symmetry. Similarly, we can prove the same property starting from representation (10), if we use the following change of summation variables $p'_l = -p''_{n+2-l}$ $(l = 2,\dots,n)$, $\sum_{r=1}^{l} p'_r = \sum_{r=1}^{n+1-l} p''_r$ $(l = 1,\dots,n)$ [38]:

$$\begin{aligned} R^{(n)}\{F(-x,-t)\} &= \sum_{p'_1\cdots p'_n} a_{p'_1}\cdots a_{p'_1+\cdots+p'_n} F_{-p'_1} F_{-p'_2}\cdots F_{-p'_n} F_{p'_1+\cdots+p'_n} \\ &= \sum_{p''_1\cdots p''_n} a_{p''_1+\cdots+p''_n}\cdots a_{p''_1} F_{-p''_1-\cdots-p''_n} F_{p''_n}\cdots F_{p''_2} F_{p''_1} = R^{(n)}\{F(x,t)\}. \end{aligned} \tag{17}$$

Note that the presence of the hidden symmetry (15) [just as the hidden symmetry for forced ratchets, defined by relation (B2) in Appendix B] is a consequence of the symmetry of the solution represented by summing products of space-time harmonics: Renumbering summation indices of the harmonics in the products from right to left gives the same result for the average velocity of a ratchet operating in the overdamped regime as the result in the case of an inverted coordinate and time with numbering of summation indices from left to right.

It should be mentioned that the equality (15) follows formally from the Langevin equation (1): Under the replacement $x \to -x$, $t \to -t$, the quantity $\dot{x}(t)$ does not change its sign, while $\ddot{x}(t)$ changes, so, at $m = 0$, we obtain $\langle \dot{x} \rangle = \zeta^{-1}\langle F(x,t)\rangle = \zeta^{-1}\langle F(-x,-t)\rangle$. However, this observation cannot serve as a rigorous proof, since, as noted above, in dissipative systems, the forward and backward trajectories of a particle contribute to the average velocity with different statistical weights. Therefore, to reveal and prove hidden symmetries, one cannot dispense with analyzing explicit solutions of the Smoluchowski equation. The only benefit from such a comparison of the original Langevin equation and that transformed by the replacement $x \to -x$ and $t \to -t$ is that it leads to the following conclusion: One cannot expect the conservation of the symmetry property (15) at $m \neq 0$.

The relations (12), (13), and (15) define symmetry transformations at which the average ratchet velocity changes its sign [as in Eq. (12)] or remains unchanged. The first two transformations are always valid, and the last takes place only in the overdamped regime, when friction prevails over inertia. Additional information on symmetry properties of a ratchet can be obtained when the average velocity turns out to be an even or odd functional of the applied force $F(x,t)$. The presence of these additional properties can be determined by a structure and symmetry of $F(x,t)$; we will discuss that in subsequent sections. We also note here that, in the general case, the functional $v\{F(x,t)\}$ can always be represented as a sum of two functionals, even and odd in $F(x,t)$:

$$v\{F(x,t)\} = v_{\text{even}}\{F(x,t)\} + v_{\text{odd}}\{F(x,t)\}, \quad (18)$$

where

$$v_{\text{even}}\{-F(x,t)\} = v_{\text{even}}\{F(x,t)\},$$
$$v_{\text{odd}}\{-F(x,t)\} = -v_{\text{odd}}\{F(x,t)\}. \quad (19)$$

Turning to the structure of the solution (9), it is easy to conclude that the even functional $v_{\text{even}}\{F(x,t)\}$ contains contributions $R^{(n)}\{F(x,t)\}$ with odd n values, while the odd functional $v_{\text{odd}}\{F(x,t)\}$ contains those with even n values. For example, for systems characterized by the property $v\{F(-x,t)\} = v\{F(x,t)\}$, which holds true for forced ratchets with the additive structure of the force $F(x,t) = g(x) + R(t)$ [37], using the vector symmetry transformation yields $v\{F(x,t)\} = v\{F(-x,t)\} \underset{(\text{vect})}{=} -v\{-F(x,t)\}$; that is, the functional is odd for this class of ratchets. It is clear that the converse is also true: The property $v\{g(x) + R(t)\} = v\{g(-x) + R(t)\}$ follows from the oddness of the functional $v_{\text{odd}}\{g(x) + R(t)\}$. In Appendix B, we show how an additive structure of $F(x,t)$ leads to zero even functionals $v_{\text{even}}\{F(x,t)\}$; that is, we give an alternative proof of the hidden symmetry found in [37] for a forced ratchet.

III. SYMMETRY PROPERTIES OF PERIODIC FUNCTIONS

In this section we discuss symmetry properties that pertain to ratchets with potential energies described by periodic functions. Consider a periodic coordinate function $u(x)$ and its derivative with the opposite sign, $f(x) = -du(x)/dx$, which can be considered as the potential energy of a particle in some periodic structure, with which the particle interacts, and a force corresponding to the energy. Let us choose a coordinate function of an arbitrary shape $\tilde{u}(x)$ on a length interval L [as, for example, $\tilde{u}(x) = \sin(\pi x/L)\exp(-Ax)$ in Fig. 1] and use it as a template. If this function has no symmetry elements, then, being translated with a period L, it gives an example of a periodic function, $u(x)$, also without symmetry elements. We will do the same with the function $\tilde{f}(x) = -d\tilde{u}(x)/dx$ [see Fig. 1(a)] and obtain an example of $f(x)$.

Next, we contract $\tilde{u}(x)$ two times along the horizontal axis and use the result $\hat{u}(x) = \tilde{u}(2x)$ $(0 < x < L/2)$ to construct periodic functions with various symmetry elements. To build symmetric functions of the first type, with so-called *shift* symmetry, we shift the obtained fragment $\hat{u}(x)$ by half a period $L/2$ and reverse its sign. The set of these two constructed fragments is translated further with the period L to obtain a periodic function [Fig. 1(b)]. From the construction, it is clear that the arising periodic function $u(x)$ and its derivative $f(x)$

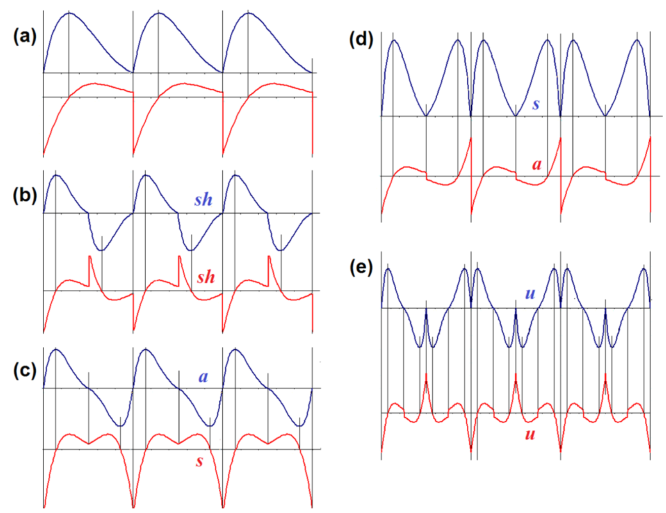


FIG. 1. Examples of periodic functions built with the template $\tilde{u}(x) = \sin(\pi x/L)\exp(-Ax)$ and its derivative (upper and lower curves): asymmetric functions (a), shift-symmetric functions (b), antisymmetric source function and its symmetric derivative (c), symmetric source function and its antisymmetric derivative (d), and universally symmetric functions (e), simultaneously belonging to *sh*, *a*, and *s* types of symmetry.

(with the opposite sign) satisfy identical properties,

$$u_{sh}(x + L/2) = -u_{sh}(x), \quad f_{sh}(x + L/2) = -f_{sh}(x), \quad (20)$$

which are a definition of a *sh*-symmetric function (called *supersymmetric* in [13,25]), when the second half of the function is exactly opposite to its first half. Note that if we write Eq. (20) with an arbitrary shift x_0, for example, $u_{sh}(x + x_0) = -u_{sh}(x)$, then by applying this symmetry transformation twice, we get $u_{sh}(x + 2x_0) = u_{sh}(x)$. On the other hand, since L is the smallest period of $u_{sh}(x)$, then $2x_0 = L$, and the shift exactly by half the period, $x_0 = L/2$, is a fundamental condition for existence of the shift symmetry given by Eq. (20).

Functions of the second and third types of symmetry can be realized, respectively, by reflection of the fragment $\hat{u}(x)$ about an axis drawn through a point $x = L/2$, or by using this point as a center of symmetry. Next, one should place the transformed fragment near the source one and then translate the resulting pair with the period L. As a result, we obtain periodic functions of *symmetric* and *antisymmetric* types (s and a types), which satisfy the equations,

$$u_s(x + x_s) = u_s(-x + x_s), \quad u_a(x + x_a) = -u_a(-x + x_a), \quad (21)$$

where x_s and x_a set, respectively, the position of a symmetry axis and symmetry center. Hereinafter, the indices s, a, and sh denote symmetric, antisymmetric, and shift-symmetric functions. Term-by-term differentiation of Eqs. (21) by coordinate and substitution $x_s \leftrightarrow x_a$ leads to equalities,

$$f_a(x + x_a) = -f_a(-x + x_a), \quad f_s(x + x_s) = f_s(-x + x_s), \quad (22)$$

from which it follows that s-symmetric functions become a symmetric, and vice versa [Figs. 1(c) and 1(d)]. By choosing the origin at a point corresponding to a symmetry element which exists in a system (x_s or x_a), s and a functions are reduced, respectively, to even or odd. The three symmetry types listed here correspond to classification of periodic functions given in Ref. [26].

From the diagrams in Figs. 1(c) and 1(d), one can see that, for s or a functions, each period L contains, respectively, two axes or two symmetry points spaced apart from each other by half a period. Let us prove the validity of this observation by a s-symmetric function. If the argument $x + x_s$ of $u_s(x + x_s)$ is denoted by x', then the first equality of (21) takes the form $u_s(x') = u_s(-x' + 2x_s)$, which can be considered as an equivalent definition of s symmetry [similarly, a symmetry can be defined as $u_a(x') = -u_a(-x' + 2x_a)$]. Next, we consider a point $x'_s = x_s - L/2$ shifted by half a period relative to the symmetry axis x_s. For this point we have $u_s(x') = u_s(-x' + 2x'_s + L)$; that is, $u_s(x') = u_s(-x' + 2x'_s)$, and x'_s is the position of the second symmetry axis shifted by half a period, as was to be proved.

It is important to note that there exists a class of functions that possess all three types of symmetries listed; that is, they are simultaneously s, a, and sh functions. The simplest example here is a sinusoid, which can be "transformed" into a certain even or odd function by a chosen shift of the origin, and also demonstrates sh-type symmetry: Shifting it by half a period and a sign change accompanying the shift leads to the same curve. We will consider such functions as a separate class of functions with the *universal* type of symmetry and introduce the label u for it. The importance of this (universal) class lies in the fact that all properties inherent separately to ratchets with symmetries s, a, and sh are simultaneously inherent to ratchets with u-type symmetry in spatial and temporal dependencies of their potential energies. This leads to a number of interesting and not always obvious conclusions presented in this article.

To construct a periodic u-type function, one can take, for example, an antisymmetric function $u_a(x)$ [which uses our template $\tilde{u}(x)$] and choose its L-width segment, $\tilde{u}_a(x)$. It will serve as a new pattern, with a-type symmetry. Next, one should contract the pattern $\tilde{u}_a(x)$ twice along the horizontal axis, $\hat{u}_a(x) = \tilde{u}_a(2x)$, place it into the left half period, and in the right half period place its reflection about the axis $x = L/2$. After translating the resulting pair of fragments by the period L, one obtains the periodic function shown in Fig. 1(e). Graphically, it is easy to see that this function, in addition to the properties (21), also satisfies the sh-symmetry property (20). It is easy to prove rigorously that if a function simultaneously has two symmetries, then it also has the third one. For instance, in the example considered, s and a symmetries arose by construction, while sh symmetry is proved as follows. In the first expression of Eq. (21), we make the replacement $x' = x + x_s$, so that we have $u_s(x') = u_s(-x' + 2x_s)$. Next, we similarly rewrite the second expression of Eq. (21), $u_a(x') = -u_a(-x' + 2x_a)$ (with $x' = x + x_a$). Since we consider $u(x)$ that simultaneously possesses s and a symmetries, then $u(x) = u_a(x') = u_s(x')$, and the equality $u(-x' + 2x_s) = -u(-x' + 2x_a)$ holds. With the replacement $x'' = -x' + 2x_s$, we get $u(x'') = -u(x'' - 2x_s + 2x_a)$. In accordance with our construction, coordinates of a symmetry axis and a symmetry center are as follows: $x_s = L/2$ and $x_a = L/4$. Therefore, $x_s - x_a = L/4$, $u(x'') = -u(x'' - L/2) = -u(x'' + L/2)$, and we arrive at property (20), which was to be proved.

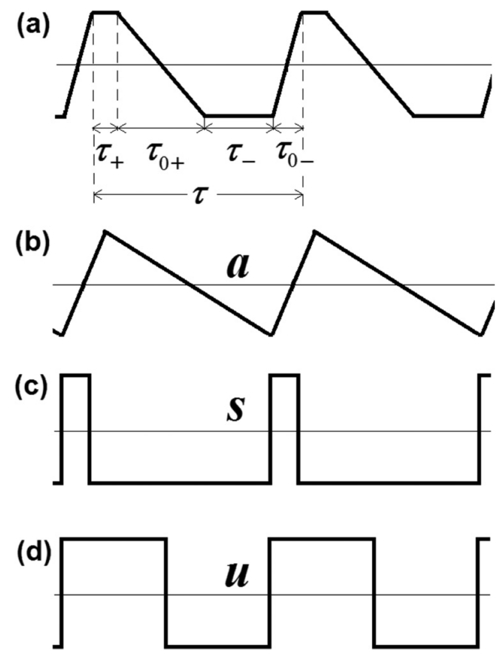


FIG. 2. Examples of periodic piecewise linear functions: nonsymmetric of a general form (a), antisymmetric (b), symmetric (c), and universally symmetric (d).

The fact that all symmetry types can be obtained by using the same fragment of an arbitrary curve (pattern), which, by means of shifts, contractions, and reflections, completes in the whole period a periodic function belonging to a certain symmetry type, says that the cardinality of a set of functions of s, a, sh, and u types are the same. The arbitrariness of a choice of the pattern suggests an infinitely large number of realizations of each symmetry type. Therefore, the conclusions of this article, which formulate symmetry properties of ratchets with potential energies of one or another symmetry type in coordinate and time variables, are very common and seem essentially important.

In conclusion of this section, we give examples, by a time function $\sigma(t)$, of concrete implementations of the discussed symmetry types in case of piecewise linear functions (Fig. 2). Such functions are widely used in the theory of Brownian motors. The shape of curve (a) illustrates the most general case, which does not belong to any of the symmetries under consideration [like the curves (a) in Fig. 1], but it is reduced to a curve with a symmetry of the types listed by a certain choice of the parameters: s type takes place at $\tau_{0+} = \tau_{0-}$, a type at $\tau_+ = \tau_-$, and u type at simultaneous fulfillment of equalities $\tau_+ = \tau_-$ and $\tau_{0+} = \tau_{0-}$ (sh type itself is not realized). The shape of curve (b) [a special case of (a) at $\tau_+ = \tau_- = 0$] is often found in ratchet models with sawtooth coordinate dependencies of potential energies (sawtooth potentials). Finally, the step functions (c) and (d) describe, respectively, asymmetric (with s-type symmetry) and symmetric (with u-type symmetry) deterministic dichotomous processes of potential energy change in time (also known as a two-state model, which implies that each state does not depend on time).

IV. SYMMETRY PROPERTIES OF RATCHETS WITH AN ARBITRARY STRUCTURE OF $F(x,t)$,

In the most general case, a two-variable function $F(x,t)$ can be expressed in terms of one-variable functions, of coordinate and time, by the following N-component additive-multiplicative form:

$$F(x,t)=\sum_{r=1}^{N} g^{(r)}(x)\sigma^{(r)}(t), \tag{23}$$

in which it is assumed that the N value can tend to infinity [as, e.g., in the case of the function $F(x-at)$ for which the representation (23) is obtained by expanding it into a Taylor series for *at* at $N\to\infty$]. Setting the N value in (23) specifies the dependencies under study, and therefore allows one to give them a clear comparative analysis. For example, an additive-multiplicative form with $N=2$ is frequently used in the theory of ratchets; we consider such systems in the next section. In this section, we will start from formulating the ratchets' symmetry properties without detailing the structure of $F(x,t)$. First we will discuss symmetries of $F(x,t)$ in a spatial variable, and then in time. In this consideration, the transformation of one of the variables is accompanied by the constancy of the other, which can be considered as a parameter. Possible symmetry types, in a spatial or temporal variable, are limited to *s*, *a*, *sh*, and *u* types, which will be enumerated. Symmetry properties obtained in this way include, as a special case, those obtained by considering $F(x,t)$ of a multiplicative form $g(x)\sigma(t)$, in which symmetry types of $g(x)$ or $\sigma(t)$ are separately examined [then, for example, the sign of $F(x,t)$ can be changed by transformation of only one function, $g(x)$ or $\sigma(t)$]. Next we will consider several important cases of mixed symmetry, in which certain symmetry types of $F(x,t)$ are realized only at performing simultaneous transformation of two variables, x and t.

Symmetry in a spatial variable. The first important, intuitively obvious, and known symmetry property is that in systems without a preferential direction the ratchet effect cannot exist [13]. The potential energy of a particle in such systems evidently possesses *s* symmetry, $U(x+x_s,t)=U(-x+x_s,t)$; therefore a corresponding force, $F(x,t)=-\partial U(x,t)/\partial x$, is *a* symmetric, $F(x+x_a,t)=-F(-x+x_a,t)$, and the position of the symmetry axis, $x=x_s$, for the potential becomes the center of symmetry, $x=x_a$, for the force. Let us write the following chain of equalities, in which the second equals sign corresponds to the selected symmetry of $F(x,t)$, and the words in parentheses under other equals signs hereinafter indicate the symmetry operations used [see relations (12), (13), and (15)]:

$$\begin{aligned}v\{F(x,t)\} &\underset{\text{(shift)}}{=} v\{F(x+x_a,t)\}=v\{-F(-x+x_a,t)\}\\ &\underset{\text{(shift)}}{=} v\{-F(-x,t)\}\underset{\text{(vect)}}{=}-v\{F(x,t)\}=0\,.\end{aligned} \tag{24}$$

The equality to zero appears as the $v\{F(x,t)\}$ value is equal to itself with the opposite sign. This zero proves the absence of the ratchet effect in the systems under discussion.

The useful symmetry property, namely, the oddness of the functional $v\{F(x,t)\}$ in $F(x,t)$,

$$v\{F(x,t)\}=-v\{-F(x,t)\}\equiv v_{\text{odd}}\{F(x,t)\}, \tag{25}$$

resides in spatially antisymmetric systems, $U(x+x_a,t)=-U(-x+x_a,t)$, with a *s*-symmetric force, $F(x+x_s,t)=F(-x+x_s,t)$ (a center of symmetry of the potential $x=x_a$ corresponds to a symmetry axis of the force $x=x_s$). Indeed,

$$\begin{aligned}v\{F(x,t)\} &\underset{\text{(shift)}}{=} v\{F(x+x_s,t)\}=v\{F(-x+x_s,t)\}\\ &\underset{\text{(shift)}}{=} v\{F(-x,t)\}\underset{\text{(vect)}}{=}-v\{-F(x,t)\}.\end{aligned} \tag{26}$$

Evenness of the functional $v\{F(x,t)\}$ in $F(x,t)$,

$$v\{F(x,t)\}=v\{-F(x,t)\}\equiv v_{\text{even}}\{F(x,t)\}, \tag{27}$$

takes place for shift-symmetric spatial dependence of the potential energy, $U(x+L/2,t)=-U(x,t)$ [or the corresponding force, $F(x+L/2,t)=-\partial U(x+L/2,t)/\partial x=-F(x,t)$]:

$$v\{F(x,t)\}\underset{\text{(shift)}}{=} v\{F(x+L/2,t)\}=v\{-F(x,t)\}. \tag{28}$$

The listed symmetry properties are of general character in the sense that only the spatial variable is subjected to the symmetry operations, but no transformation of the time dependence of the force is assumed. Since time reversal does not occur in the analysis at all, symmetry properties associated only with coordinate transformations are valid not only in the overdamped mode, when friction prevails over inertia, but also for inertial dynamics. The time t can be any fixed time. In Table I, the "nonsymm" column corresponds to the considered symmetry properties under transformations of the spatial variable. The universal type of symmetry combines *s*, *a*, and *sh* types, so it is not surprising that *u* type is characterized by the absence of the ratchet effect in the same way as *s* type [Eq. (24)]. If we proceed from the properties of the two types, *a* and *sh*, we also get that the velocity vanishes but due to the fact that a functional, which is both even and odd [Eqs. (26) and (28)], can only be zero. Obtaining an identical result (the absence of the effect for a potential of *u* type over the spatial variable) through different considerations is not surprising, since, as noted above, the fulfillment of two standard symmetries automatically entails the fulfillment of the third.

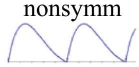
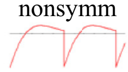
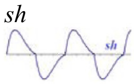
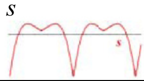
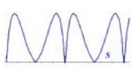
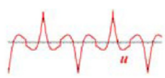
Symmetry in time variable. Among symmetry transformations with respect to time variable, only one operation is not connected with time reversal: It is the shift transformation defining a *sh*-symmetric function as $F(x,t+\tau/2)=-F(x,t)$. Since this type of symmetry is similar to the shift transformation of the spatial variable, $F(x+L/2,t)=-F(x,t)$, then, similarly to (28), we get

$$v\{F(x,t)\}\underset{\text{(shift)}}{=} v\{F(x,t+\tau/2)\}=v\{-F(x,t)\}, \tag{29}$$

and the evenness of the functional (29) also holds when taking into account inertial effects.

When using time reversal in symmetry analysis, one should take into account the presence of Cubero-Renzoni hidden symmetry in ratchet systems (15), because of which the

TABLE I. Symmetry properties of ratchets with a multiplicative structure of potential energy, $U(x,t) = w(x)\sigma(t)$, and the corresponding force, $F(x,t) = g(x)\sigma(t)$, where $g(x) = -dw(x)/dx$. The information indicated in the table's cells is as follows: "0", "even," and "odd" denote the absence of the ratchet effect, and evenness and oddness of the functional $v\{F(x,t)\}$, respectively. The "(o-d)" stresses mean that the property is valid only in the overdamped regime. The results in the rows and columns of "nonsymm" are valid for the function $F(x,t)$ of the general form, without detailing its structure, since one of the variables does not undergo symmetry transformations.

x-symmetries		*t*-symmetries ($\sigma(t)$)				
$w(x)$	$g(x)$	nonsymm 	*sh* 	*s* 	*a* 	*u* 
nonsymm 	nonsymm 		even	 odd (o-d)	 0 (o-d)	even 0 (o-d)
sh 	*sh* 	even	even	even 0 (o-d)	even 0 (o-d)	even 0 (o-d)
a 	*s* 	odd	0	odd	odd 0 (o-d)	0
s 	*a* 	0	0	0	0	0
u 	*u* 	0	0	0	0	0

obtained symmetry properties of ratchets will hold valid only in the inertialess overdamped mode. We now turn to the consideration of such properties, in which the above symmetries under coordinate transformation will be replaced by their time analogs. For an *a*-symmetric function, $F(x,t+t_a) = -F(x,-t+t_a)$, the transformation similar to (24) is as follows:

$$v\{F(x,t)\} \underset{\text{(shift)}}{=} v\{F(x,t+t_a)\}$$
$$= v\{-F(x,-t+t_a)\} \underset{\text{(shift)}}{=} v\{-F(x,-t)\}$$
$$\underset{\text{(C-R)}}{=} v\{-F(-x,t)\} \underset{\text{(vect)}}{=} -v\{F(x,t)\} = 0. \quad (30)$$

The oddness of the functional, similar to (25), holds for a *s*-symmetric in time function, $F(x,t+t_s) = F(x,-t+t_s)$. The chain of transformations, constituting the proof of this property, contains all the links included in (26) and one additional—the use of Cubero-Renzoni symmetry:

$$v\{F(x,t)\} \underset{\text{(shift)}}{=} v\{F(x,t+t_s)\}$$
$$= v\{F(x,-t+t_s)\} \underset{\text{(shift)}}{=} v\{F(x,-t)\}$$
$$\underset{\text{(C-R)}}{=} v\{F(-x,t)\} \underset{\text{(vect)}}{=} -v\{-F(x,t)\}. \quad (31)$$

In Table I, the line "nonsymm" corresponds to the considered symmetry properties related to transformations of the time variable. Just as when considering symmetries related to transformations of a spatial variable, the *u* type of symmetry combines *s*, *a*, and *sh* types, and therefore the functional $v\{F(x,t)\}$ is even in $F(x,t)$ in the general case and equal to zero in the absence of inertia. This means that the ratchet effect, which is prohibited in the overdamped regime, can be permitted by taking inertia into account. The evenness of the functional [described by Eq. (29)] indicates that the existence of an additional spatial symmetry of *s* or *a* types together with the multiplicative structure of $F(x,t)$ leads to the absence of the ratchet effect even for inertial particles (for the symmetry of *s* type, it follows from the fact that the functional is simultaneously even and odd). If the multiplicative structure is not assumed, then conclusions of this kind will be justified only in the case when they are based on proofs including parallel symmetry transformations both in coordinates and time. In the cells of Table I, at the intersections of rows and columns corresponding to *s*, *a*, *sh*, and *u* types of symmetry, we present information on the properties of the functional $v\{F(x,t)\}$ for these cases.

Mixed symmetry types. Consider some important symmetry types of $F(x,t)$ concerning simultaneous transformations of two variables x and t. Let us prove that there is no ratchet effect in a system with a potential that has the property $U(x,t+t_s) = -U(x+L/2,-t+t_s)$ (in Refs. [13,25] such a potential was called a supersymmetric potential). It can be interpreted as a potential with *sh*-symmetric coordinate dependence [corresponding to the same *sh*-symmetric coordinate dependence of a force, $F(x,t+t_s) = -F(x+L/2,-t+t_s)$] and with *s*-symmetric dependence on time. The proof of the symmetry prohibition of the ratchet effect in such systems implies the time reversal transformation and (due to this) the use of the hidden symmetry (15), namely,

$$v\{F(x,t)\} \underset{\text{(shift)}}{=} v\{F(x,t+t_s)\} = v\{-F(x+L/2,-t+t_s)\} \underset{\text{(shift)}}{=} v\{-F(x,-t)\} \underset{\text{(C-R)}}{=} v\{-F(-x,t)\} \underset{\text{(vect)}}{=} -v\{F(x,t)\} = 0\,. \quad (32)$$

As Cubero-Renzoni symmetry is used, the ratchet effect disappears (in general, and not as a result of a special tuning of system parameters) only in the overdamped regime. If the force has a similar property, $F(x+x_s,t) = -F(-x+x_s,t+L/2)$, but with inverted symmetries, which can be interpreted as s-symmetric coordinate dependence of the force (corresponding to a-symmetric coordinate dependence of the potential) and sh-symmetric dependence on time, then the ratchet effect is absent in the general case, in both inertial and inertialess dynamics, since the time reversal operation is not involved in the proof:

$$v\{F(x,t)\} \underset{\text{(shift)}}{=} v\{F(x+x_s,t+\tau/2)\} = v\{-F(-x+x_s,t)\} \underset{\text{(shift)}}{=} v\{-F(-x,t)\} \underset{\text{(vect)}}{=} -v\{F(x,t)\} = 0\,. \quad (33)$$

Note that simultaneous transformation of two variables is not equivalent to two separate transformations of each variable, if we assume, for example, a multiplicative form of $F(x,t)$. Due to this, we cannot generally relate the sign inversion of $F(x,t)$ to any particular operation in coordinate and time. Therefore, the used interpretation (and others of a similar kind) of the mixed transformation as a combination of two separate transformations of coordinate and time is rather conditional and can only serve as an intuitive explanation of the meaning of the transformations being carried out. We illustrate this with a few examples.

One can try to interpret the property $F(x+x_0,t) = F(-x+x_0,t+\tau/2)$ as a symmetry in coordinate with $x_a = x_0$, due to which the sign of $F(x,t)$ undergoes a first change, and sh symmetry in time causing the second sign change. But, as a result, there will be no sign change for $F(x,t)$. For a multiplicative force, $F(x,t) = g_a(x)\sigma_{sh}(t)$, the functional $v\{F(x,t)\}$ will be equal to zero due to the asymmetry of $g_a(x)$ [see (24) and Table I]. However, an arbitrary structure of $F(x,t)$ together with simultaneous transformation of the variables yields

$$v\{F(x,t)\} \underset{\text{(shift)}}{=} v\{F(x+x_0,t+\tau/2)\} = v\{F(-x+x_0,t)\} \underset{\text{(shift)}}{=} v\{F(-x,t)\} \underset{\text{(vect)}}{=} -v\{-F(x,t)\}\,, \quad (34)$$

that is, only the oddness of the functional $v\{F(x,t)\}$ in $F(x,t)$ [see (25)], but not the absence of a current at all. The property under discussion can also be regarded as both s symmetry with respect to coordinate with $x_s = x_0$ and a time shift by half a period without a change in the sign of $F(x,t)$. But this interpretation will not change the above result, $v\{F(x,t)\} = v_{\text{odd}}\{F(x,t)\}$.

One can try to interpret the property $F(x+x_0,t+t_0) = F(-x+x_0,-t+t_0)$ as a set of two s symmetries or two a symmetries. For $F(x,t)$ of an arbitrary structure, this property will not yield any symmetry consequences, whereas for a multiplicative structure of $F(x,t)$, the result will be $v\{F(x,t)\} = 0$. Finally, in the case of $F(x+x_0,t+t_0) = -F(-x+x_0,-t+t_0)$ we have

$$v\{F(x,t)\} \underset{\text{(shift)}}{=} v\{F(x+x_0,t+t_0)\} = v\{-F(-x+x_0,-t+t_0)\} \underset{\text{(shift)}}{=} v\{-F(-x,-t)\} \underset{\text{(C-R)}}{=} v\{-F(x,t)\}\,, \quad (35)$$

so $v\{F(x,t)\} = v_{\text{even}}\{F(x,t)\}$ in the overdamped regime [the corresponding results for multiplicative forces $F(x,t) = g_a(x)\sigma_s(t)$ or $F(x,t) = g_s(x)\sigma_a(t)$ are given in Table I].

V. TWO-COMPONENT ADDITIVE-MULTIPLICATIVE FORM OF THE FUNCTION $F(x,t)$,

Consider symmetry properties of ratchets for which the function $F(x,t)$ contains only two summands, $N=2$, in the representation (23) with $g^{(1)}(x) = f(x)$, $g^{(2)}(x) = g(x)$, $\sigma^{(1)}(t) = 1$, and $\sigma^{(2)}(t) = \sigma(t)$. This structure of a two-variable function is most often used in the theory of ratchets, since the potential energy and the corresponding force are sums of stationary and fluctuation contributions,

$$U(x,t) = u(x) + w(x)\sigma(t), \quad F(x,t) = f(x) + g(x)\sigma(t),$$
$$f(x) = -du(x)/dx, \quad g(x) = -dw(x)/dx, \quad (36)$$

so that the functioning of ratchets can be quite effectively analyzed analytically within standard methods of theoretical physics, for example, by using Green's functions describing the diffusion dynamics in the potential profile $u(x)$ (see Ref. [39] and the literature cited therein).

The analysis of symmetry properties of ratchets with the force $F(x,t)$ of (36) is complicated by the fact that the functions $f(x)$ and $g(x)$ can relate not only to different symmetry types, but, even within the same symmetry type, have different (noncoinciding) axes or centers of symmetry. Since the time dependence of $F(x,t)$ is controlled, according to (36), by a single function of time, $\sigma(t)$, it would be expedient to consider first various symmetry types of $\sigma(t)$, based on the above properties (29)–(31) for $\sigma(t) = \sigma_{sh}(t)$, $\sigma_a(t)$, and $\sigma_s(t)$. The consideration yields the following relations:

$$v\{f(x) + g(x)\sigma_{sh}(t)\} = v\{f(x) - g(x)\sigma_{sh}(t)\},$$
$$v\{f(x) + g(x)\sigma_a(t)\} \underset{\text{(o-d)}}{=} -v\{-f(x) + g(x)\sigma_a(t)\},$$
$$v\{f(x) + g(x)\sigma_s(t)\} \underset{\text{(o-d)}}{=} v_{\text{odd}}\{f(x) + g(x)\sigma_s(t)\} \quad (37)$$

(the "(o-d)" under the equals signs mean that the equalities are valid only in the overdamped regime). The first of these equalities reflects the fact that the ratchet velocity turns out to be an even functional of $g(x)$ for the sh-symmetric time dependence. The second and third equations, occurring only in the overdamped regime, mean that the velocity will be an odd functional of $f(x)$ at $\sigma(t) = \sigma_a(t)$ and an odd functional of the total force $F(x,t) = f(x) + g(x)\sigma(t)$ at $\sigma(t) = \sigma_s(t)$ [in the derivation of the third equality, the transformation chain is similar to (31)].

For the function $\sigma(t)$ of the universal symmetry, $\sigma(t) = \sigma_u(t)$, all three properties (37) are realized simultaneously. We comment on this situation as follows. In the overdamped regime, the oddness of $v\{F(x,t)\}$ with respect to $F(x,t)$ of the additive-multiplicative structure (36), realized with $\sigma(t) = \sigma_s(t)$, means that the functional $v\{F(x,t)\}$ can be represented, in the general case, as a sum of two contributions. The first one is an odd functional of $f(x)$ and even of $g(x)$, and the second, on the contrary, is an even functional of $f(x)$ and odd of $g(x)$ (the product of two even or two odd components is "rejected" by the required oddness with respect to the total force). If, in addition to the s symmetry of the time dependence, its sh symmetry takes place (a symmetry is then realized automatically), then the second contribution should be absent [according to the first equality in (37)], so that only the first one remains. In the general case, taking inertia into account, only the first equality in (37) will be true.

It is interesting to analyze symmetry properties of ratchets with two s-symmetric functions of coordinate, $u(x)$ and $w(x)$, characterized by different positions of their symmetry axes. The corresponding forces then refer to a-symmetric functions satisfying equalities $f(x + x_a^{(f)}) = -f(-x + x_a^{(f)})$ and $g(x + x_a^{(g)}) = -g(-x + x_a^{(g)})$ with different symmetry centers $x_a^{(f)}$ and $x_a^{(g)}$. Using the a symmetry of the function $g(x)$ leads to the following chain of equalities:

$$\begin{aligned} &v\{f(x) + g(x)\sigma(t)\} \\ &\underset{\text{(shift)}}{=} v\{f(x + x_a^{(g)}) + g(x + x_a^{(g)})\sigma(t)\} \\ &= v\{f(x + x_a^{(g)}) - g(-x + x_a^{(g)})\sigma(t)\} \\ &\underset{\text{(vect)}}{=} -v\{-f(-x + x_a^{(g)}) + g(x + x_a^{(g)})\sigma(t)\} \\ &\underset{\text{(shift)}}{=} -v\{-f(-x + 2x_a^{(g)}) + g(x)\sigma(t)\}. \end{aligned} \tag{38}$$

Next we continue this chain using the a symmetry of the function $f(x)$:

$$\begin{aligned} &-v\{-f(-x + 2x_a^{(g)}) + g(x)\sigma(t)\} \\ &\underset{\text{(shift)}}{=} -v\{-f(-x + 2x_a^{(g)} + x_a^{(f)}) + g(x - x_a^{(f)})\sigma(t)\} \\ &= -v\{f(x - 2x_a^{(g)} + x_a^{(f)}) + g(x - x_a^{(f)})\sigma(t)\} \\ &\underset{\text{(shift)}}{=} -v\{f(x - 2x_a^{(g)} + 2x_a^{(f)}) + g(x)\sigma(t)\}. \end{aligned} \tag{39}$$

Equating the beginning of the chain [left side of (38)] and its end [right side of (39)] and stressing the antisymmetry of $f(x)$ and $g(x)$ with the subscript, we obtain the following symmetry property:

$$\begin{aligned} &v\{f_a(x) + g_a(x)\sigma(t)\} \\ &\quad = -v\{f_a(x + 2x_a^{(f)} - 2x_a^{(g)}) + g_a(x)\sigma(t)\}. \end{aligned} \tag{40}$$

The meaning of the result is as follows. If the symmetry centers are coincident, $x_a^{(f)} = x_a^{(g)}$, the a-symmetric (in coordinate) function is the total force $F(x,t)$, at which there is no ratchet effect, according to the proof (24). When $x_a^{(f)} \neq x_a^{(g)}$, although each of the functions is a symmetric, but the total function $F(x,t)$ is not, so then, from Eq. (40), the property $v\{F(x,t)\} = 0$ does not follow, and the ratchet effect can exist. Note that since for the functions $f_a(x)$ and $g_a(x)$, one period contains two centers of symmetry shifted by half a period, then in Eq. (40) each of them can be $x_a^{(f)}$ and $x_a^{(g)}$.

VI. ASYMMETRIC DICHOTOMOUS FLUCTUATIONS OF THE FORCE $F(x,t)$

The absence of time dependence in the parameters of a system in certain states or at certain time intervals is a significant fact or an assumption that can simplify calculations of characteristics of this system. The simplification is caused by the fact that there appears the possibility of description, stochastic or deterministic, of time evolution of the system either in terms of transition rates between these states or by involving solutions of equations with time-independent parameters, stitching them at times of transitions between the states. The simplest situation is realized when there are only two such states or intervals—dichotomous model. A dichotomous process is usually understood as a process in which there exist two states, denoted here by the symbols "+" and "–", in which a certain *stochastic* function of time, $R(t)$, takes two values, R_+ or R_-, that is, $R(t) = R_\pm$. The rate constants of transitions between the states, $+ \to -$ and $- \to +$, are designated by γ_+ and γ_-, respectively. Next, we introduce the time asymmetry parameter, $\varepsilon \equiv (\gamma_- - \gamma_+)/(\gamma_- + \gamma_+)$; its zero value means the time symmetry of the dichotomous process. The inverse rate constants of transitions define average lifetimes of the states, $\tau_\pm = \gamma_\pm^{-1}$, which are equal to each other for a time-symmetric process. The time-asymmetry parameter is expressed through the average lifetimes of the states as follows:

$$\varepsilon = \frac{\tau_+ - \tau_-}{\tau_+ + \tau_-}. \tag{41}$$

The average value of $R(t)$ is determined by the obvious ratio

$$\langle R(t)\rangle = \frac{R_+\tau_+ + R_-\tau_-}{\tau_+ + \tau_-} = \frac{R_+ + R_-}{2} + \frac{R_+ - R_-}{2}\varepsilon, \tag{42}$$

which is invariant under the transformation $+ \leftrightarrow -$, $\varepsilon \leftrightarrow -\varepsilon$. For many applications, values $R(t) = R_\pm$ are considered for which this average value is zero, $\langle R(t)\rangle = 0$. Then it is convenient to represent $R(t)$ as

$$R(t) = R[\sigma(t) - \varepsilon], \quad \sigma(t) = \pm 1, \quad \langle \sigma(t)\rangle = \varepsilon, \tag{43}$$

wherein the stochastic function of time $\sigma(t)$ takes values ± 1, and its average value is exactly equal to the asymmetry parameter ε [R is a constant, such as $R_\pm = R(\pm 1 - \varepsilon)$].

A slightly less widespread terminology supposes a *deterministic* process described by a periodic (with a period τ) function of time $R(t)$ that takes two values R_+ or R_- at time intervals τ_+ and τ_- (with $\tau_+ + \tau_- = \tau$) which is also called dichotomous. The usefulness of such terminology is that the relations (41)–(43) remain valid for the deterministic process as well; the difference is only that the average lifetimes $\tau_\pm$ of the states become their real durations [widths of the steps of the stepwise function $R(t)$ in Fig. 3]. The fact that the functions $R(t)$ or $\sigma(t)$ are now periodic allows one to use, for their characterization, symmetry properties of periodic functions [see Eqs. (20)–(22)]. Not the least of the facts is that for calculating characteristics of systems in question, one can use Fourier components of periodic functions to simplify the model.

Consider the stepwise function shown in Fig. 3. It is easy to see that this function is of s-symmetry type, since it has

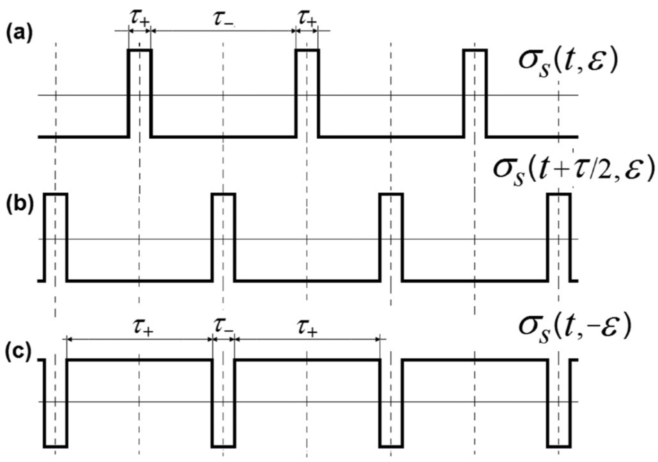


FIG. 3. Symmetry transformations of a stepwise function $\sigma_s(t, \varepsilon)$ describing a deterministic dichotomous process: original function (a), the original function shifted by half a period (b), $\sigma_s(t + \tau/2, \varepsilon)$, and with the sign of ε reversed (c), $\sigma_s(t, -\varepsilon)$, [which is equal to the inverted function $\sigma_s(t + \tau/2, \varepsilon)$; see Eq. (44)].

symmetry axes passing through midpoints of long and short steps. To avoid terminological confusion, we emphasize here that one should distinguish an asymmetry of a dichotomous process, characterized by the parameter ε, and the s symmetry of a periodic function, which describes this process. A stepwise function is always s symmetric (since it contains the mentioned symmetry axes), whereas a dichotomous process is symmetric only at $\varepsilon = 0$. Obviously, in this particular case of $\varepsilon = 0$, the s-symmetric function $\sigma(t)$ is simultaneously *sh* and *a* symmetric, and therefore, it refers to the universal symmetry type (it is u symmetric). Bearing in mind that the parameter ε characterizes symmetry properties of the process itself (the dichotomous process), it is advisable, in description, to explicitly indicate the dependence $\sigma(t)$ on ε, and also to emphasize its s symmetry with the corresponding index: $\sigma(t) = \sigma_s(t, \varepsilon)$. We define the unbiased operation of sign reversal of ε as the replacement $\tau_+ \leftrightarrow \tau_-$ performed in the function $\sigma(t)$ in such a way that its symmetry axes do not undergo any displacements [compare Figs. 3(a) and 3(c)]. Then, one can check directly (see Fig. 3) that for such functions the following symmetry property holds:

$$\sigma_s(t + \tau/2, \varepsilon) = -\sigma_s(t, -\varepsilon), \tag{44}$$

which passes, as already noted above, into u symmetry with $\varepsilon = 0$ [*sh*-symmetry behavior (44) here, $\sigma_s(t + \tau/2, 0) = -\sigma_s(t, 0)$, is a particular case of the arising general u symmetry]. If the origin of the coordinates is chosen coinciding with the position of a symmetry axis of the function $\sigma_s(t, \varepsilon)$, then its Fourier components become

$$\sigma_0(\varepsilon) = \varepsilon, \quad \sigma_j(\varepsilon) \underset{j\neq 0}{=} \frac{2}{\pi j} \sin\frac{\pi j}{2}(1 + \varepsilon). \tag{45}$$

Property (44) in Fourier space is written as $(-1)^j\sigma_j(\varepsilon) = -\sigma_j(-\varepsilon)$. One can verify it by direct substitution of components (45) into this equality. When $\varepsilon = 0$ this property limits possible j values to odd numbers, as it should be for *sh*-symmetric functions.

Suppose that, in the multiplicative-additive form (36), the equality $u(x) = \lambda w(x)$ holds, where $w(x)$ is a sawtooth potential with the widths of its links l and $L - l$. Then its derivative $g(x) = -dw(x)/dx$ is described by a s-symmetric stepwise function, for which the asymmetry parameter $\kappa = 2l/L - 1$ can now be entered as a *spatial* asymmetry parameter as well as the designation $g_s(x, \kappa)$, in full analogy with the description of time dependence for the above deterministic dichotomous process. For the function $g_s(x, \kappa)$, a property similar to the property (44) will be valid:

$$g_s(x + L/2, \kappa) = -g_s(x, -\kappa). \tag{46}$$

By virtue of the proportionality of $f(x)$ and $g(x)$, $f(x) = \lambda g(x)$, the total force will be of the following multiplicative form:

$$F(x, t; \lambda, \kappa, \varepsilon) = g_s(x, \kappa)[\lambda + \sigma_s(t, \varepsilon)]. \tag{47}$$

The average velocity of such a ratchet is an odd functional of $F(x, t; \lambda, \kappa, \varepsilon)$ [see (31)], and therefore of the function $g_s(x, \kappa)$, due to its s symmetry [see also (25)]; hereafter we denote this fact by the index "*odd*". Alternately using the properties (46) and (44), we get the following:

$$\begin{aligned} &v_{\text{odd}}\{g_s(x, \kappa)[\lambda + \sigma_s(t, \varepsilon)]\} \\ &\quad = v_{\text{odd}}\{-g_s(x, -\kappa)[\lambda + \sigma_s(t, \varepsilon)]\} \\ &\quad = v_{\text{odd}}\{g_s(x, \kappa)[\lambda - \sigma_s(t, -\varepsilon)]\}. \end{aligned} \tag{48}$$

At $\varepsilon \neq 0$ the considered odd functional is a sum of two contributions, which are even and odd functionals of $\sigma_s(t, \varepsilon)$ (see the reasoning in Sec. V) and the latter becomes zero at $\varepsilon = 0$. On the other hand, it is easy to verify that, from the relations (48), the average velocity should be turned to zero at $\kappa = 0$, and in the case of $\lambda = 0$ at $\varepsilon = 0$.

Next we will perform similar transformations for a ratchet with a fluctuating force (rocking or forced ratchet), characterized by the function

$$F(x, t; \kappa, \varepsilon) = g_s(x, \kappa) + R_s(t, \varepsilon), \tag{49}$$

with zero-mean fluctuations, $\langle R_s(t, \varepsilon)\rangle = 0$. Since, by (43), $R_s(t, \varepsilon) = R[\sigma_s(t, \varepsilon) - \varepsilon]$, then the property (44) transforms here into a similar property for $R_s(t, \varepsilon)$: $R_s(t + \tau/2, \varepsilon) = -R_s(t, -\varepsilon)$. The average ratchet velocity is an odd functional of $F(x, t; \kappa, \varepsilon)$ due to s symmetry of this function both in x and t. Therefore, the average velocity can be represented as a sum of two contributions: The first one is an odd functional in $g_s(x, \kappa)$ and even in $R_s(t, \varepsilon)$, and the second, in contrast, is an even functional in $g_s(x, \kappa)$ and odd in $R_s(t, \varepsilon)$. Alternately using the properties (46) and (44) for the force (49), we obtain

$$\begin{aligned} &v_{\text{odd}}\{g_s(x, \kappa) + R_s(t, \varepsilon)\} \\ &\quad = v_{\text{odd}}\{-g_s(x, -\kappa) + R_s(t, \varepsilon)\} \\ &\quad = v_{\text{odd}}\{g_s(x, \kappa) - R_s(t, -\varepsilon)\}. \end{aligned} \tag{50}$$

From this relation, it follows that with $\kappa = 0$ only the second contribution, being an even functional in $g_s(x, 0)$ and odd in $R_s(t, \varepsilon)$, is different from zero, and with $\varepsilon = 0$ only the first contribution, which is an even functional in $R_s(t, 0)$ and odd in $g_s(x, \kappa)$. Since from (50) it follows that $v_{\text{odd}}\{g_s(x, \kappa) + R_s(t, \varepsilon)\} =$

$-v_{\rm odd}\{g_s(x,-\kappa)+R_s(t,-\varepsilon)\}$, then the average velocity turns to zero only at $\kappa=\varepsilon=0$.

Now let us analyze relations (48) and (50) to clarify the type of dependences of the functional $v\{F(x,t;\lambda,\kappa,\varepsilon)\}$ on the asymmetry parameters. For this, expand it in small κ and ε:

$$v\{F(x,t;\lambda,\kappa,\varepsilon)\}=A(\lambda)+\kappa B_1(\lambda)+\varepsilon B_2(\lambda)+\kappa\varepsilon C(\lambda)+O(\kappa^2,\varepsilon^2), \tag{51}$$

where the coefficients $A(\lambda)$, $B_1(\lambda)$, $B_2(\lambda)$, and $C(\lambda)$ are functions of ratchet parameters, independent of the asymmetry parameters, and $O(\kappa^2,\varepsilon^2)$ denotes terms of the order κ^2 and ε^2. From (48) it follows that $v\{F(x,t;\lambda,0,\varepsilon)\}=0$, and hence $A(\lambda)=B_2(\lambda)=0$. Since equality $v\{F(x,t;0,\kappa,0)\}=0$ also holds, then $B_1(0)=0$, and therefore, the coefficient $B_1(\lambda)$ itself can be represented as $B_1(\lambda)=\lambda\tilde{B}_1(\lambda)$. Therefore, for a ratchet with the multiplicative force (47), we have the equality

$$v_{\rm odd}\{g_s(x,\kappa)[\lambda+\sigma_s(t,\varepsilon)]\}=\kappa[\lambda\tilde{B}_1(\lambda)+\varepsilon C(\lambda)]+O(\kappa^2,\varepsilon^2). \tag{52}$$

For a ratchet with the fluctuating force (49), the quantity $F(x,t;\lambda,\kappa,\varepsilon)$ and the coefficients in (51) do not depend on λ; therefore the arguments λ can be omitted. Using the property $v\{F(x,t;0,0)\}=0$, we obtain $A=0$, so that

$$v_{\rm odd}\{g_s(x,\kappa)+R_s(t,\varepsilon)\}=\kappa B_1+\varepsilon B_2+O(\kappa^2,\varepsilon^2,\kappa\varepsilon). \tag{53}$$

The main results of this section are the formulas (52) and (53), which show how the presence or absence of spatial and temporal asymmetry can affect the average velocity of pulsating and forced ratchets with a sawtooth coordinate dependence of potential energy and its dichotomous time dependence. For pulsating ratchets, the structure (47) considered here as a dependence of the force on coordinate and time corresponds to fluctuations of the potential in amplitude. Then the formula (52) shows that the ratchet effect is absent in a spatially symmetric system ($\kappa=0$), while, in the presence of spatial asymmetry ($\kappa\neq0$), stopping points become possible with a change in time asymmetry when $\lambda\tilde{B}_1(\lambda)+\varepsilon C(\lambda)=0$. One can say that the presence of such stopping points and the possibility of reversing the motion direction arise due to competition of spatial and temporal asymmetry of the system. The parameter λ characterizes the degree of pulsing of the ratchet in amplitude. Indeed, if we introduce the ratio α of potential energy amplitudes in two states, $U_-(x)=\alpha U_+(x)$, $-1\leqslant\alpha\leqslant1$, then α is associated with λ through the following ratio: $\alpha=(\lambda-1)/(\lambda+1)$. At $\lambda=0$ the potential energy fluctuates in sign [$\alpha=-1$, $U_-(x)=-U_+(x)$], and it follows from Eq. (52) that $v_{\rm odd}\{g_s(x,\kappa)\sigma_s(t,\varepsilon)\}\approx\kappa\,\varepsilon\,C(0)$. This means that the ratchet effect is absent not only at $\kappa=0$, but also at $\varepsilon=0$. Using the concept of the "thermodynamic action," this fact was first established by Tarlie and Astumian [40] for a multiplicative flashing ratchet operating in the overdamped regime with the potential energy fluctuating in sign. The same effect was noted for adiabatic and high-temperature ratchets in Refs. [31–33] and the occurrence of stopping points at $\lambda\ll1$ ($|1+\alpha|\ll1$), $\varepsilon\ll1$ in Ref. [36].

For forced ratchets, the formula (53) shows that the ratchet effect will be absent only when the equalities $\kappa=0$ and $\varepsilon=0$ are fulfilled simultaneously, that is, the effect can exist (unlike pulsating ratchets) in spatially symmetric systems solely due to temporal asymmetry. Reducing the factor $\kappa B_1+\varepsilon B_2$ to zero (that is, appearing ratchet stopping points) can already occur in the first order of smallness in the spatial asymmetry parameter κ. For pulsating ratchets, the factor $\lambda\tilde{B}_1(\lambda)+\varepsilon C(\lambda)$ does not depend on κ, so that the appearance of stopping points can be expected only in the second order of smallness in κ. This explains the fact that the appearance of stopping points is more typical for forced ratchets than for pulsating ratchets.

As for determining the direction of the ratchet velocity between stopping points, it is a complicated problem which is not solvable by a symmetry analysis and usually requires direct calculations. Only in some simple cases similar to those considered in this section, can one introduce parameters of the time and spatial asymmetry owing to the stepwise functional dependence of an applied force on coordinate and time, with different step lengths determining these parameters. Then the structure of the dependence of the ratchet velocity on the asymmetry parameters [given by formulas (52) and (53)] allows one to judge about stopping point locations, the velocity signs between them being determined from other approaches. They suggest calculations of the average velocity [the coefficients $\tilde{B}_1(\lambda)$, $C(\lambda)$, B_1, and B_2 in Eqs. (52) and (53)] or intuitive considerations. Such considerations were carried out in Ref. [33] for dichotomous fluctuations of the force of multiplicative or additive forms [given by Eq. (47) or (49)]. For example, for the potential energy of the multiplicative form (a flashing ratchet) with one minimum and one maximum on its spatial period (the simplest representative here is a sawtooth potential) which undergoes symmetric temporal fluctuations ($\varepsilon=0$, $\lambda\neq0$) in amplitude (pulsating amplitude) or fluctuates asymmetrically in sign ($\varepsilon\neq0$, $\lambda=0$) with the lifetime of the initial potential profile larger than that with the opposite sign, the motion direction is determined by the direction from the minimum to the nearest maximum position ($\kappa\neq0$). If the same single-well potential profile is a contribution to the energy of the additive form (a forced ratchet), the motion direction will be opposite, that is, from the minimum to the farthest maximum position. It is clear that these regularities can be used in mixed cases ($\kappa\neq0$, $\varepsilon\neq0$) when the motion direction follows from the competition of the above factors.

In conclusion of this section, we note that the results formulated here did not use the time reversal operation, and, therefore, are valid in the general case, taking into account inertial dynamics. The type of dependence of the average velocity on parameters of spatial and temporal asymmetry both for flashing ratchets pulsating in amplitude and for rocking ratchets has been found in Ref. [33], based on the solutions obtained within the high-temperature approximation for stochastic dichotomous fluctuations (that is, for a particular regime of ratchet operation). In this section, the results (52) and (53) are obtained without using these restrictions based only on the analysis of general symmetry properties.

VII. DISCUSSION AND CONCLUSIONS

The main results of this paper can be divided into three categories. To the first one, we attribute the derivation of a compact analytical representation for the average particle velocity as a series in the inverse friction coefficient. A solution of the Smoluchowski equation as an expansion in the inverse friction coefficient was obtained in [33]. Then such representation was used in [31,34–36,41] to obtain analytical expressions for the average velocity of high-temperature (low-energy) ratchets, in which it is sufficient to use only the first expansion terms. In this work, due to the periodicity of an applied force $F(x,t)$ in both spatial and temporal variables, a compact exact analytical expression for the average ratchet velocity has been obtained, which also turned out to be convenient for an effective analysis of symmetry properties of this expression. Therefore, the second category of the results of this paper refers to a visual proof of hidden symmetry of this expression [see Eq. (15)], reflecting the symmetry of summation indices of the applied force harmonics with respect to their numbering from left to right and from right to left. Appendix B gives a similar proof for the particular case of the hidden symmetry, which arises for forced ratchets [see Eqs. (B2) and (B10)]. These proofs should be regarded as alternative and quite simple with respect to the cumbersome proofs of hidden symmetries found in [37].

To the third category of results, we attribute finding various symmetry conditions imposed on the function $F(x,t)$ at which either there is no ratchet effect or the average velocity is an even (odd) functional $v\{F(x,t)\}$. In order to structure and simplify the presentation of these results, in Sec. III we give the main symmetry types of periodic functions—shift, symmetric, and antisymmetric types, as well as the universal type, which is important in the functioning of ratchets and combines the first three types. This section is well illustrated by examples of potential reliefs and applied forces corresponding to them, as well as by piecewise linear temporal dependencies of fluctuations of these reliefs that relate to the above symmetry types (we chose the examples which are either actual for real systems or models but can clearly demonstrate the features "hidden" in each of the types). For clarity, we use a constructional method to get periodic functions with the listed types of symmetries, which applies transformations (compression, shift, reflection) of one arbitrary pattern. In this section, we also give proofs of some important, but not quite obvious at first glance, properties of periodic functions [such as, for example, (i) the sufficiency of two of the three main types of symmetries for the appearance of the universal type, or (ii) the fact that, for symmetric or antisymmetric functions, there always exist two symmetry axes or two symmetry centers at a period, shifted by half a period relative to each other].

In Sec. IV we analyze the forces $F(x,t)$ of an arbitrary structure in which one of the arguments, x or t, is assumed to be fixed, and the symmetry of dependence on the second belongs to the main symmetry types of periodic functions [see relations (20) and (22)]. To reveal certain properties of the functional $v\{F(x,t)\}$, we depart from the symmetry type of $F(x,t)$ and use one of the three symmetry properties, (12), (13), and (15). The first two of them, vector and shift symmetries, are general properties; they are not related to solutions (6)–(9) of the Smoluchowski equation describing the overdamped regime, and are also valid when taking into account inertial effects. Therefore, the results obtained from the use of arbitrary types of symmetries in the spatial variable and the shift symmetry in the temporal variable are also of a general nature. The results that imply performing the time reversal operation and, therefore, use the hidden symmetry of Cubero and Renzoni (15), turn out to be valid only in the overdamped regime. Such a "distribution" of results by the regimes of ratchet functioning leads to a number of interesting properties. Among them, for example, is the absence of the ratchet effect for an inertialess Brownian particle with a multiplicative structure of its potential energy, the time dependence of which is described by a function of the universal symmetry type, and the removal of this symmetry restriction by inertia effects for all ratchets except for those in which the coordinate dependence of nanoparticle potential energy refers to symmetric or antisymmetric functions.

The simplest realization of the universal type of symmetry in ratchets is a symmetric dichotomous process of changing potential energy with time at which the energy fluctuates in sign so that in each of the states a particle remains for the same time. If we confine ourselves to adiabatically fast fluctuations, at which each of the average lifetimes of states, τ_+ and τ_-, exceeds the characteristic diffusion time $\tau_D = L^2/D$ on a spatial period L of potential energy changes, then the average ratchet velocity is independent of the time asymmetry parameter $\varepsilon = (\tau_+ - \tau_-)/(\tau_+ + \tau_-)$ and inversely proportional only to the sum of times τ_+ and τ_- [31]. Therefore, for an adiabatic ratchet, a dichotomous process can always be only symmetric, and since the description of a stochastic and deterministic ratchet is the same in the adiabatic approximation, the time dependence of potential energy can be described by a function of time of the universal type of symmetry. This explains the conclusion of Ref. [34], that the prohibition of the ratchet effect at dichotomous adiabatically fast fluctuations of potential energy of an inertialess nanoparticle in sign is annulled when switching to the inertial regime, but only if the potential profile is not symmetric or antisymmetric. In this article, we prove that such inertia-induced "permission" of ratchet motion is of a general nature. The authors of Refs. [24,42] observed the presence of the ratchet effect for an electron flashing ratchet with the potential energy fluctuating in sign. Since inertial effects are inherent to electron motion, this observation is in agreement with the conclusions of this paper, although it can also be explained by the three-dimensional motion of electrons.

Among other interesting properties of adiabatic ratchets, we note that the average velocity of a ratchet with a spatially antisymmetric potential profile, undergoing adiabatically slow changes, does not depend on the "trajectory" of this change (type of time dependence) and is determined only by the initial and final states (shapes) of this profile [30]. A comparative analysis of adiabatically slow and fast regimes of motion showed that, for them, the ratchet average velocity will be an even and odd functional of potential energy, respectively [31].

Note that there are many ratchet systems the description of which rejects the overdamped approximation and requires taking into account inertial effects [1]. They are, for example, molecular shuttles with the motion direction dependent on the particle mass [43], underdamped rocked ratchets which are capable of rectifying the ac input signal more efficiently than in the overdamped regime [44], and inertial ratchets driven by nonlinear velocity-dependent friction forces [45]. But it is one thing just to reveal the removal of symmetry restrictions in case of rejection of the overdamped approximation, and another to find the symmetry properties of inertial systems that require an analysis of solutions of the more complicated Kramers equation. Thus, the discussion of the symmetry properties of inertial ratchets is beyond the scope of this article.

The results of an analysis of various combinations of possible symmetry types, in spatial and temporal variables, of the multiplicative force $F(x,t) = g(x)\sigma(t)$ is generalized in Table I. Section V deals with the symmetry properties of the additive-multiplicative force $F(x,t) = f(x) + g(x)\sigma(t)$, which is often chosen in describing ratchet systems. Depending on symmetry of the time dependence $\sigma(t)$, the average velocity can be an even or an odd functional of $f(x)$ and $g(x)$. The basic properties of ratchets with such $F(x,t)$ are contained in relations (37), with the help of which one can answer the question whether directional motion will arise under various additional conditions imposed on the functions $f(x)$ and $g(x)$. For example, if these two functions are chosen a symmetric, then the ratchet effect will occur only if the symmetry centers of $f(x)$ and $g(x)$ do not coincide.

The results obtained for asymmetric dichotomous fluctuations of nanoparticle potential energy, widely used in the ratchet theory, are presented in Sec. VI. A sawtooth potential profile described by functions $u(x) = \lambda w(x)$ in Eq. (36) begets a stepwise form of an applied force $f(x) = \lambda g(x)$ for which it is easy to introduce the spatial asymmetry parameter κ. This parameter is defined similar to the temporal asymmetry parameter ε when discussing the types of time dependencies $\sigma(t)$; this similarity makes efficient presentation and analysis of ratchet properties, including by analogy. The main result of Sec. VI is the conclusion that the average velocity of pulsating and forced ratchets depends differently on the asymmetry parameters κ and ε [see relations (52) and (53)]. This result is consistent with the similar result of Ref. [33], obtained, however, in the framework of the high-temperature approximation for stochastic dichotomous fluctuations of potential energy, as well as with the results of Ref. [36] on the consequences of competition of spatial and temporal asymmetry of the potential energy.

In conclusion, we note that the symmetry properties of various ratchet systems found in this paper do not pretend to be comprehensive. Rather, they provide the key to describing possible regimes which arise from combinations of symmetry properties of potential reliefs and their fluctuations in time, by using basic symmetry transformations.

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APPENDIX A: REDUCING THE EXPRESSION (9) TO ZERO FOR A TIME-INDEPENDENT POTENTIAL

In the absence of fluctuations $F(x,t) = f(x)$, $F_{qj} = f_q\delta_{j,0}$, so in relations (4)–(8) all indices j_l are equal to zero, and, taking into account the equality $a_{q,0} = i/(Dk_q)$, iteration procedure (7) yields the following representation of Fourier components of the distribution function:

$$\rho_q = L^{-1}\delta_{q,0} - \frac{i\beta}{k_q}(1-\delta_{q,0})L^{-1}\left[f_q + \sum_{n=2}^{\infty}(i\beta)^{n-1} \times \sum_{q_2\ldots q_n(\neq 0)} \frac{f_{q+q_2}f_{-q_2+q_3}\cdots f_{-q_{n-1}+q_n}f_{-q_n}}{k_{q_2}\cdots k_{q_n}}\right], \quad \text{(A1)}$$

where $\beta = (D\zeta)^{-1} = (k_BT)^{-1}$. On the other hand, it is obvious from Eq. (2) that a stationary solution $[\partial\rho(x,t)/\partial t = 0]$ of the Smoluchowski equation for a particle in a time-independent periodic potential $u(x)$ with a force $f(x) = -du(x)/dx$ means no current $[J(x,t) = 0]$, and the distribution function itself is equilibrium,

$$\rho(x) = \exp[-\beta u(x)] \bigg/ \int_0^L dx \exp[-\beta u(x)]. \quad \text{(A2)}$$

Therefore, the average current (the average velocity of directed motion) is zero, so that from Eq. (4) at $j = 0$ the equality follows:

$$\sum_{q_1} f_{q_1}\rho_{-q_1} = 0. \quad \text{(A3)}$$

Substituting the Fourier component of the distribution function (A1) with $q = q_1$ into Eq. (A3) and taking into account that $f_0 = 0$ and $\sum_{q\neq 0}|f_q|^2/q = 0$ (the latter is obvious from the replacement $q \to -q$), we obtain the sum rule:

$$\sum_{q_1\ldots q_n(\neq 0)} \frac{f_{q_1}f_{-q_1+q_2}f_{-q_2+q_3}\cdots f_{-q_{n-1}+q_n}f_{-q_n}}{k_{q_1}\cdots k_{q_n}} = 0, \quad \text{(A4)}$$

which is used to write Eq. (11).

APPENDIX B: HIDDEN SYMMETRY OF FORCED RATCHETS

For forced ratchets, characterized by an additive structure of the force

$$F(x,t) = g(x) + R(t), \quad \text{(B1)}$$

the following property of hidden symmetry was given in Ref. [11],

$$v\{g(x) + R(t)\} = v\{g(-x) + R(t)\}, \quad \text{(B2)}$$

the quite cumbersome proof of which is given in the supplemental materials to Ref. [37]. In the end of Sec. II, we demonstrated that this property is equivalent to the oddness of the functional $v\{F(x,t)\}$ with respect to $F(x,t)$; that is, the

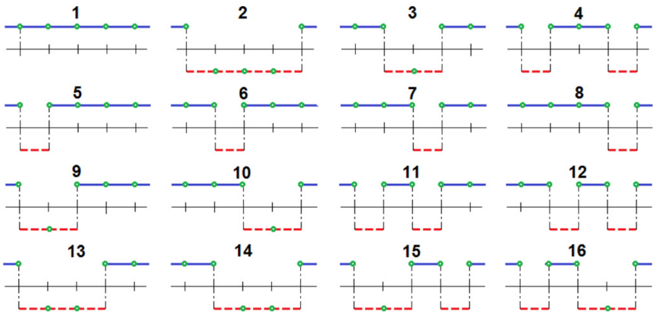

FIG. 4. Diagrams that schematically depict summands in Eq. (B6) (renumbered over the diagrams). The marks on the thin horizontal lines indicate five values of numbers l of the summation indices $q_l,\ j_l,\ l=1,\dots,5$. The bold solid and dashed lines (blue and red in the online version) between the mesh points l and $l+1$ correspond to $g_{-q_l+q_{l+1}}$ and $R_{-j_l+j_{l+1}}$. The circles on these lines (green in the online version) located above and under the points l denote the contributions a_{q_l,j_l}.

following equality is fulfilled:

$$v_{\rm even}\{g(x)+R(t)\}=0. \tag{B3}$$

Taking into account Eq. (9), equality (B3) is equivalent to the property

$$R^{(2l-1)}\{g(x)+R(t)\}=0,\quad l=1,\ 2,\cdots. \tag{B4}$$

Let us demonstrate the mechanism for converting this functional to zero.

The substitution of the Fourier component of force (B1),

$$F_p=F_{qj}=g_q\delta_{j,0}+R_j\delta_{q,0}, \tag{B5}$$

into expression (9) leads to the additional summation over various combinations of products g_q and R_j. We will go over these combinations for a case $n=5$ which is still relatively simple, but already contains certain symmetric combinations, the sums of which make zero contributions to the functional. In this regard, the case $n=1$ is trivial due to the properties $g_0=0$ and $R_0=0$ imposed on a ratchet, and the cases $n=3$ and $n\geqslant 7$ can be considered similarly to the case $n=5$.

We will write the functional (9) with an additive force at $n=5$ in the following form:

$$\begin{aligned}
&R^{(5)}\{g(x)+R(t)\}\\
&\quad=\sum_{\substack{q_1\dots q_5\\ j_2,j_3,j_4}} g_{q_1}a_{q_1,0}\big(g_{-q_1+q_2}\delta_{j_2,0}+R_{j_2}\delta_{q_1,q_2}\big)a_{q_2,j_2}\\
&\qquad\times\big(g_{-q_2+q_3}\delta_{j_2,j_3}+R_{-j_2+j_3}\delta_{q_2,q_3}\big)a_{q_3,j_3}\\
&\qquad\times\big(g_{-q_3+q_4}\delta_{j_3,j_4}+R_{-j_3+j_4}\delta_{q_3,q_4}\big)a_{q_4,j_4}\\
&\qquad\times\big(g_{-q_4+q_5}\delta_{j_4,0}+R_{-j_4}\delta_{q_4,q_5}\big)a_{q_5,0}g_{-q_5},
\end{aligned} \tag{B6}$$

which uses the identity $a_{0,j}=0$ [see Eq. (5)]. The multiplying of the terms in the four parentheses gives 16 contributions, shown schematically in Fig. 4. Contribution 1 contains only the product of g_q and corresponds to the absence of fluctuations, $R(t)=0$, and both forced and pulsating ratchets are characterized by zero average velocity (see Appendix A). For odd n values, there exists an additional symmetry, according to which the expression (11) reverses its sign when changing the summation variables $q_l\to -q_{n+1-l}$ (due to an odd number of factors k_l).

The summations in the other contributions, from 2 to 16, containing both g_q and R_j are considerably simplified by the presence of delta symbols in expression (B5). They lead to the fact that the solid lines in the diagrams (not separated by dashed lines) correspond to the same indices j_l, and vice versa, the dashed lines (not separated by solid lines) correspond to the same indices q_l. Due to this, for example, the contribution 2 can be written as

$$\sum_{q_1;\,j_2,j_3,j_4}|g_{q_1}|^2a_{q_1,0}^2a_{q_1,j_2}a_{q_1,j_3}a_{q_1,j_4}R_{j_2}R_{-j_2+j_3}R_{-j_3+j_4}R_{-j_4}. \tag{B7}$$

Its conversion to zero is a result of the sign reversal at the replacement $q_1\to -q_1$, which is a particular case of the transformation $q_l\to -q_{n+1-l}$ with $l=n=1$. The fact that the contributions corresponding to the symmetric diagrams 3 and 4 are zero is proved in a similar way.

Diagrams 5–8 contain only one dashed line connecting adjacent points, which corresponds to the R_0 value. The equality to zero of this quantity [imposed on forced ratchets with $\langle R(t)\rangle=0$] causes zero contributions from diagrams 5–8.

We turn to the consideration of asymmetric diagrams 9–16. For example, the contribution of diagram 9 is determined by the expression

$$\begin{aligned}
&\sum_{q_3,\,q_4,\,q_5;j_2} a_{q_3,0}^2a_{q_3,j_2}a_{q_4,0}a_{q_5,0}g_{q_3}g_{-q_3+q_4}g_{-q_4+q_5}g_{-q_5}|R_{j_2}|^2\\
&\quad=-\sum_{q_3,\,q_4,\,q_5;j_2} a_{q_3,0}a_{q_4,0}a_{q_5,j_2}a_{q_5,0}^2g_{-q_5}g_{-q_4+q_5}\\
&\qquad\times g_{-q_3+q_4}g_{q_3}|R_{j_2}|^2,
\end{aligned} \tag{B8}$$

in which the equalities $q_1=q_2=q_3$ and $j_3=j_4=0$ are already used due to the presence of the corresponding delta symbols, and the replacement of indices $q_3\to -q_5,\ q_4\to -q_4,\ q_5\to -q_3$ is used in writing the right side of Eq. (B8). Expressions in the left and right sides of (B8) differ not only by sign, but also by indices in the factors $a_{q,j}$. Therefore, the contribution of diagram 9 itself is not zero. But if we do similar transformations with the contribution of diagram 10, which is symmetric to 9 in the index number space q_l, j_l, $l=1,\dots,5$,

$$\begin{aligned}
&\sum_{q_1,\,q_2,\,q_3;j_4} a_{q_1,0}a_{q_2,0}a_{q_3,j_4}a_{q_3,0}^2g_{q_1}g_{-q_1+q_2}g_{-q_2+q_3}g_{-q_3}|R_{j_4}|^2\\
&\quad=-\sum_{q_1,\,q_2,\,q_3;j_4} a_{q_3,0}a_{q_2,0}a_{q_1,0}^2a_{q_1,j_4}g_{-q_3}g_{-q_2+q_3}\\
&\qquad\times g_{-q_1+q_2}g_{q_1}|R_{j_4}|^2,
\end{aligned} \tag{B9}$$

and change summation variables in (B9) as $q_1\to q_3,\ q_2\to q_4,\ q_3\to q_5,\ j_4\to j_2$, then we can verify the following: The left side of (B9) is the right side of (B8), and the right side of (B9) is the left side of (B8). This means that the sum of (B8) and (B9) is equal to the same sum, taken with the opposite sign, that is, is equal to zero. One can verify that the sums of the contributions of symmetric diagrams 11 and 12, 13 and 14, and 15 and 16 also vanish in exactly the same way.

Thus, it is demonstrated that expression (B6) is equal to zero, just like the sum of contributions with odd n. The vanishing of even functionals (B3) proves the symmetry property (B2), which is a hidden symmetry, in addition to (15), valid for forced ratchets. Note that this hidden symmetry is also a consequence of the symmetry in numbering of summation indices of harmonics arising in the Fourier representation of solutions of the Smoluchowski equation with potential energy periodic in coordinate and time. If we apply two symmetry properties (15) and (B2) simultaneously,

$$v\{g(x)+R(t)\} \underset{\text{(C-R)}}{=} v\{g(-x)+R(-t)\} = v\{g(x)+R(-t)\}, \tag{B10}$$

we get an additional property of hidden symmetry of forced ratchets, which is the invariance with respect to time reversal in $R(t)$.

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