Topology and stochasticity of turbulent magnetic fields

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We present a mathematical formalism for the topology and stochasticity of vector fields based on renormalization group methodology. The concept of a scale-split energy density, $\psi_{l,L} = \mathbf{B}_l \cdot \mathbf{B}_L/2$ for vector field $\mathbf{B}(\mathbf{x}, t)$ renormalized at scales l and L, is introduced in order to quantify the notion of the field topological deformation, topology change, and stochasticity level. In particular, for magnetic fields, it is shown that the evolution of the field topology is directly related to the field-fluid slippage, which has already been linked to magnetic reconnection in previous work. The magnitude and direction of stochastic magnetic fields, shown to be governed, respectively, by the parallel and vertical components of the renormalized induction equation with respect to the magnetic field, can be studied separately by dividing $\psi_{l,L}$ into two $(3+1)$-dimensional scalar fields. The velocity field can be approached in a similar way. Magnetic reconnection can then be defined in terms of the extrema of the L_p norms of these scalar fields. This formulation in fact clarifies different definitions of magnetic reconnection, which vaguely rely on the magnetic field topology, stochasticity, and energy conversion. Our results support the well-founded yet partly overlooked picture in which magnetic reconnection in turbulent fluids occurs on a wide range of scales as a result of nonlinearities at large scales (turbulence inertial range) and nonidealities at small scales (dissipative range). Lagrangian particle trajectories, as well as magnetic field lines, are stochastic in turbulent magnetized media in the limit of small resistivity and viscosity. The magnetic field tends to reduce its stochasticity induced by the turbulent flow by slipping through the fluid, which may accelerate fluid particles. This suggests that reconnection is a relaxation process by which the magnetic field lowers both its topological entanglements induced by turbulence and its energy level.

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I. INTRODUCTION

Newtonian mechanics can be obtained from relativistic mechanics in the limit of $v/c \to 0$, the regime of small velocities v compared to the speed of light c. Quantum mechanics too reduces to classical mechanics as the Planck constant tends to zero $\hbar \to 0$. These examples do not of course exhaust the list of "special" formalisms that emerge from more general theories in physics. The limiting case may be dubbed an ideal regime if it arises as a result of the elimination of nonidealities such as friction or viscosity from the general theory. All this may seem familiar and quite simple, but it is not. Care must be taken in letting specific parameters tend to zero in order to reach an ideal regime. It is true that in recovering Newtonian mechanics from the relativistic theory one basically neglects the higher powers of v/c in a Taylor expansion while equations usually remain well defined with stable solutions. However, this is not a general theme. For example, as viscosity ν tends to zero in a fluid, or correspondingly the Reynolds number Re tends to infinity, the flow becomes very unstable and sensitive to slight perturbations which may lead to the development of turbulence. Indeed, it is well known that the velocity field in a turbulent fluid becomes Hölder singular[1] in the limit $\nu \to 0$ [1,2]. Mathematically, this means that the very concept of taking a derivative, in terms of spatial gradients, would become problematic since the normal derivative of a Hölder-singular function is not well defined and more advanced tools must be employed instead. Analogously, magnetohydrodynamics (MHD) should be treated with care in the limit when viscosity, electrical resistivity, or other such transport coefficients tend to zero. For instance, the magnetic field in a turbulent fluid would become Hölder singular in the limit of vanishing magnetic diffusivity $\eta \to 0$ (see [3–6]). All in all, this suggests that the conventional ideal hydrodynamics (HD) and ideal MHD may only be applied in very special circumstances where flows remain laminar and quantities such as the velocity and magnetic fields are Lipschitz continuous.

Although these concepts are well founded in theoretical physics and formulated with rigorous mathematics, a vast literature in plasma physics as well as astrophysics often appeals to the so-called ideal HD and ideal MHD without a proper examination of their applicability. For example, for a magnetized fluid such as a plasma, it is an easy exercise to show that in the limit of vanishing resistivity, the magnetic field seems to be frozen into the fluid [7]. This principle of

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[1]The complex (or real) valued function g in $\mathcal{R}^n$ is Hölder continuous if two non-negative and real constants C and h exist such that $|g(x) - g(y)| \leqslant C\|x - y\|^h$ for all $x, y \in \text{domain}(g)$. If the Hölder exponent h is equal to unity, then g is Lipschitz continuous. Also g is called Hölder singular if $h < 1$.

magnetic flux freezing is usually applied in the laboratory and astrophysical systems as a common delicacy arising from the ideal MHD (see, e.g., [8,9]). Nevertheless, such environments are usually turbulent, the velocity and magnetic fields singular, and the solutions for the ideal MHD unstable. Application of flux freezing in such systems would involve serious mathematical difficulties.

For Hölder-singular fields, even simple concepts such as field line and particle trajectory, would require careful reconsideration. A mathematical definition for the field lines of a given vector field $\mathbf{F}$ can be given in terms of its integral curves. A field line may be considered as a parametric curve whose tangent vector at any point is parallel to the vector field at that point. Quantitatively, for the vector field $\mathbf{F}(\mathbf{y}) = (F_1(\mathbf{y}), \dots, F_m(\mathbf{y}))$ in $\mathcal{R}^m$, an integral curve (field line) is defined as a solution $\mathbf{y}(\tau) = (y_1(\tau), \dots, y_m(\tau))$ of the ordinary differential equation

$$\frac{d\mathbf{y}(\tau)}{d\tau} = \mathbf{F}(\mathbf{y}(\tau)),$$

with appropriate initial (boundary) conditions. Hence, $\mathbf{y}(\tau)$ is in fact a curve parametrized by τ such that at any point τ_0, $d\mathbf{y}(\tau_0)/d\tau$ is tangent to $\mathbf{F}(\mathbf{y}(\tau_0))$. Mathematically, the existence and uniqueness of integral curves can be determined using Picard's existence (Cauchy-Lipschitz) theorem. A unique solution is guaranteed if $\mathbf{F}(\mathbf{y})$ is Lipschitz continuous. For example, velocity, electric, and magnetic fields in HD and MHD are usually assumed to be continuous, and thus the corresponding integral curves (field lines) are assumed to be well defined. However, in many circumstances, the Lipschitz continuity condition is not satisfied and the uniqueness theorem cannot be applied as a result. What does a magnetic field line mean if the magnetic field is not Lipschitz continuous? The same question can be raised of course for any other vector field.

One example is the velocity field in a turbulent fluid in the limit when viscosity tends to zero $\nu \to 0$ which, as was shown for the first time by Onsager, turns out to be Hölder singular rather than Lipschitz continuous [1,2,5]. This singularity will make all HD equations ill-defined because they contain derivatives of the velocity field (see Sec. II). In fact, similar situations have already been encountered in other fields of physics including quantum electrodynamics (QED) and quantum chromodynamics. Dealing with ultraviolet (UV) infinities, which arise, for example, in calculations of seemingly simple quantities such as the mass or electric charge of particles, has been a major challenge in developing these theories. Regularization and renormalization group (RG) methodologies have been developed to resolve these theoretical difficulties. Today, these well-established formalisms are applicable in many other fields including HD and MHD.

Lagrangian particle trajectories become stochastic in turbulent fluids in the limit when viscosity tends to zero (see, e.g., [10–12]). Under such circumstances, the conventional flux freezing [7] collapses [3–6,13] and instead a distinct stochastic version of it is introduced. This new concept of stochastic flux freezing introduced by Eyink [4] along with stochasticity of field lines [3,5,13] led to deep physical consequences. Magnetic field topology and its evolution with time, for example, would be deeply affected by these new phenomenologies. Topology plays a crucial role in many magnetic phenomena including generation (dynamo action) and reconnection of magnetic fields. Reconnection of stochastic magnetic fields in turbulent fluids, which are ubiquitous in astrophysics and plasma physics, is sometimes defined as a sudden change in magnetic field topology (for a review of magnetic reconnection see, e.g., [14–17]). Magnetic dynamos in fact require persistent magnetic topology change in order to sustain the generation of magnetic fields in stars, galaxies, and accretion disks (for a modern dynamo model applicable to such systems see, e.g., [18] and references therein; for problems involving both generation and reconnection of large-scale fields in accretion disks see [19,20]). In recent years, the problem of magnetic reconnection [6,21–23], as well as a magnetic dynamo [4], in turbulent systems has been approached taking into account the field stochasticity and turbulence. Yet concepts such as topology change and weak and strong stochasticity are widely used without providing concise mathematical definitions. For example, depending on the context, the mathematical notion of stochasticity for a vector field may differ from what a physicist refers to. What an experimentalist can measure is a spatial and temporal *average* field at any point in space-time not the *bare* field. In fact, as discussed before, in many cases the bare field is not well defined at all. If a renormalization method is applied to remove the field singularities, then the stochasticity of the renormalized field is obviously different from the stochasticity of a bare vector field. After all, turbulence itself is a large scale rather than a molecular phenomenon. (For an application of renormalization group methodologies in MHD turbulence see, e.g., [24,25].)

In this paper we give a rigorous mathematical definition of stochasticity level for vector fields in terms of the unit vectors tangent to the renormalized field at different coarse-graining scales. The time-dependent angle between such two unit vectors at a space-time point $(\mathbf{x}, t)$ provides a means to define a local stochasticity level. The average stochasticity level in a volume V can then be obtained using L_p norms. Intuitively, the temporal changes in the stochasticity level indicate topological deformations in the vector field. We use the time derivative of the L_p-average stochasticity level to define topological deformations. These concepts are then applied to turbulent magnetic fields using the coarse-grained induction equation. As an example, we apply the mathematical formalism developed here to the magnetic field evolution in turbulent systems encountered often in astrophysical environments, which may provide plasma physicists and astrophysicists with an alternative statistical approach to study magnetic phenomena.

The detailed plan of this paper is as follows. In Sec. II we first briefly review the problem of Hölder singularities in HD and MHD, and the RG methodologies used to resolve them. We also add a few simple arguments to these known results in order to show that the evolution of magnetic energy is related to the parallel component of the induction equation (with respect to the magnetic field) while the evolution of the field topology is associated with its perpendicular component. Our formulation of topological deformations and stochasticity level for general vector fields is presented in Sec. III. This formalism is then applied to the problem of magnetic reconnection in Sec. IV with a discussion connecting the ideas advanced here to stochastic reconnection and the slippage of magnetic field through the fluid. We discuss our results in Sec. V.

II. RENORMALIZED MHD

In order to give a simple and clear picture of the problem of singularities in HD, let us briefly discuss Onsager's approach to turbulence following the description and notation of Ref. [2], which can be consulted for details (see also [1]). The incompressible Navier-Stokes equation is given by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla_x\mathbf{u} = -\nabla_x p + \nu\nabla^2\mathbf{u}, \quad \nabla_x\cdot\mathbf{u} = 0. \tag{1}$$

In the limit when viscosity tends to zero $\nu \to 0$, the regime of so-called ideal HD, one might expect the energy dissipation to vanish. However, numerous experiments and numerical simulations (see, e.g., [26–28]) have shown that the kinetic energy dissipation rate

$$\epsilon(\mathbf{x},t) = \nu|\nabla_x\mathbf{u}(\mathbf{x},t)|^2, \tag{2}$$

does not vanish as $\nu \to 0$ and has a space average converging $\lim_{\nu\to 0}\langle\epsilon(t)\rangle = \langle\epsilon^*(t)\rangle$ with a nonzero limit $\epsilon^*(t) > 0$. One might argue therefore that the velocity gradients should diverge $\nabla_x\mathbf{u} \to \infty$ in the limit $\nu \to 0$. As mentioned in the Introduction, this is analogous to the UV divergences in quantum field theory (QFT). Such situations may arise, for example, when an integral diverges as a result of contributions from infinitesimal distances. In principle, UV divergences can be removed by invoking an RG methodology. In fact, QED was developed based on renormalization methods to remedy UV infinities. As one goes to smaller scales (higher energies in the language of high-energy physics), higher-order terms in a perturbative formalism (e.g., corresponding to loops in Feynman diagrams in QED) give rise to infinities. Mathematically, this is similar to the inviscid limit of a turbulent fluid: HD equations containing velocity gradients become ill-defined in the limit of $\nu \to 0$. In fact, it was Onsager who showed that such a singularity is necessary to explain the breakdown of the conservation of energy, associated with nonvanishing ϵ in Eq. (2), in the limit of small viscosity in a turbulent fluid.

One remedy to the problem of singular fields is to use distributions. Mathematically, such an approach involves multiplying equations by a smooth test function and integrating. Physically, this means that we "average" quantities, such as magnetic and velocity fields, over a region in space-time. In fact, what we measure in the laboratory as a magnetic or velocity field at the point $(\mathbf{x}, t)$ is an average over a small volume around $\mathbf{x}$ during a time interval. The reason is that any such a measurement would require a time interval as well as a spatial volume; therefore, a vector field cannot be measured "exactly" at a space-time point. Let us coarse grain, or renormalize, the velocity field $\mathbf{u}(\mathbf{x}, t)$ at a length scale l,

$$\mathbf{u}_l(\mathbf{x},t) = \int_V G_l(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r},t)d^3r, \tag{3}$$

where $G_l(\mathbf{r}) = l^{-3}G(\mathbf{r}/l)$ with $G(\mathbf{r})$ is a smooth, rapidly decaying kernel. In other words, $G \in C_c^\infty(\mathcal{R})$, the space of infinitely differentiable functions with compact support.[2] Without loss of generality, we may assume

$$G(\mathbf{r}) \geqslant 0, \tag{4}$$

$$\lim_{|\mathbf{r}|\to\infty} G(\mathbf{r}) \to 0, \tag{5}$$

$$\int_V d^3r\, G(\mathbf{r}) = 1, \tag{6}$$

$$\int_V d^3r\, \mathbf{r}\, G(\mathbf{r}) = 0, \tag{7}$$

and

$$\int_V d^3r|\mathbf{r}|^2 G(\mathbf{r}) = 1. \tag{8}$$

We may also assume $G(\mathbf{r}) = G(r)$ with $|\mathbf{r}| = r$, i.e., an isotropic kernel, which leads to $\int d^3r\, r_i r_j G(\mathbf{r}) = \delta_{ij}/3$ [2]. Note also that the expression (8) indicates $\int_V d^3r|\mathbf{r}|^2G_l(\mathbf{r}) = l^2$. Intuitively, therefore, the coarse-grained field $\mathbf{u}_l$ represents the average velocity of a fluid parcel of size l at position $\mathbf{x}$. It is basically the low-pass-filtered field, containing only length scales larger than l. A simple change of variable in Eq. (3), as $\mathbf{x}+\mathbf{r} \to \mathbf{r}'$, allows us to write

$$\mathbf{u}_l(\mathbf{x},t) = \int_V G_l(\mathbf{r}'-\mathbf{x})\mathbf{u}(\mathbf{r}',t)d^3r'. \tag{9}$$

Thus all gradients ∇_x would act on G, which also implies that $\mathbf{u}_l$ is now continuous. Without delving into the mathematical details, it is easy to see the implication that the differential equation $d\mathbf{y}(s)/ds = \mathbf{F}(\mathbf{y}(s))$, with $\mathbf{F} = \mathbf{u}_l$, has unique solutions (integral curves) for given boundary conditions. Thus the notion of field line for the renormalized field $\mathbf{u}_l$ is well defined.

Since the space and time derivatives commute with the coarse-graining operator, we can multiply Eq. (1) by the kernel $G_l(r)$ and integrate. This results in the renormalized, or coarse-grained, form of the incompressible Navier-Stokes equation

$$\frac{\partial \mathbf{u}_l}{\partial t} + \nabla_x\cdot(\mathbf{u}\mathbf{u})_l = -\nabla_x p_l + \nu\nabla_x^2\mathbf{u}_l, \quad \nabla_x\cdot\mathbf{u}_l = 0. \tag{10}$$

Applying the Cauchy-Schwarz inequality to $\nabla_x\mathbf{u}$ shows that the renormalized velocity gradient is bounded as long as the total energy remains finite as $\nu \to 0$. Therefore, the ultraviolet divergences (in the form $\nabla_x\mathbf{u} \to \infty$ as $\nu \to 0$, discussed above) are removed: $\nabla_x\mathbf{u}_l < \infty$. Also applying the Cauchy-Schwarz inequality to the viscous term $\nu\nabla_x^2\mathbf{u}_l(\mathbf{x},t) = (-\nu/l)\int d^3r(\nabla G)_l(\mathbf{r})\cdot\nabla_x\mathbf{u}(\mathbf{x}+\mathbf{r},t)$ leads to $|\nu\nabla_x^2\mathbf{u}_l(\mathbf{x},t)| \leqslant \sqrt{(\nu/l)C_l\int d^3r|(\nabla G)_l(\mathbf{r})|^2\,\epsilon(\mathbf{x}+\mathbf{r},t)}$, where $C_l = \int d^3r|(\nabla G)_l(\mathbf{r})|^2$. This expression, at the limit $\nu \to 0$, converges to

$$\left|\nu\nabla_x^2\mathbf{u}_l(\mathbf{x},t)\right| \leqslant \sqrt{(\nu/l)C_l\int d^3r|(\nabla G)_l(\mathbf{r})|^2\,\epsilon^*(\mathbf{x}+\mathbf{r},t)}. \tag{11}$$

[2] A function g is said to have a compact support (set of its arguments for which $g \neq 0$) if $g = 0$ outside a compact set (equivalent to closed and bounded sets in $\mathcal{R}^m$).

Therefore, the renormalized viscous term has an upper bound of order $\nu^{1/2}$ and vanishes for a fixed scale l as $\nu \to 0$. This results in

$$\frac{\partial \mathbf{u}_l}{\partial t} + \nabla_x \cdot (\mathbf{u}\mathbf{u})_l = -\nabla_x p_l, \quad \nabla_x \cdot \mathbf{u}_l = 0 \tag{12}$$

for the inertial range of turbulence, the range of scales l for which the viscous term is negligible. The renormalized Navier-Stokes equation, given by Eq. (10), can be rewritten as

$$\frac{\partial \mathbf{u}_l}{\partial t} + \nabla_x \cdot (\mathbf{u}_l\mathbf{u}_l + \boldsymbol{\tau}_l) = -\nabla_x p_l + \nu\nabla_x^2\mathbf{u}_l, \quad \nabla_x \cdot \mathbf{u}_l = 0, \tag{13}$$

where $\boldsymbol{\tau}_l = (\mathbf{u}\mathbf{u})_l - \mathbf{u}_l\mathbf{u}_l$ is the turbulent stress tensor. Multiplying Eq. (13) by $\mathbf{u}_l$, one can easily obtain the evolution equation for the kinetic energy

$$\partial_t \frac{|\mathbf{u}_l|^2}{2} + \nabla_x \cdot \left[\left(\frac{1}{2}|\mathbf{u}_l|^2 + p_l\right)\mathbf{u}_l + \boldsymbol{\tau}_l \cdot \mathbf{u}_l\right] = -\Pi_l, \tag{14}$$

where

$$\Pi(\mathbf{x}, t) = -\nabla_x \mathbf{u}_l \cdot \boldsymbol{\tau}_l(\mathbf{x}, t) \tag{15}$$

is the energy flux from resolved scales greater than l to unresolved scales less than l (energy cascade). Suppose[3]

$$|\mathbf{u}(\mathbf{x}+\mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)| \leqslant CU\left(\frac{|\mathbf{r}|}{L}\right)^h, \tag{16}$$

for some $C > 0$ with U being the (large-scale) characteristic velocity of the system. Applying this to Eq. (15) leads to

$$\Pi_l(\mathbf{x}, t) = O(l^{3h-1}). \tag{17}$$

Equation (14) indicates that persistent energy decay requires $\int_{\nu\to 0} d^3r\, \Pi_l(\mathbf{x}, t) \neq 0$. Since the scale l is arbitrary, for any fixed l viscosity can be taken arbitrarily small to let the ideal renormalized Eq. (13) be satisfied and then l can be decreased. However, if (16) holds for all space-time points $(\mathbf{x}, t)$ with a Hölder exponent $h > 1/3$, then $\int_{l\to 0} d^3r\, \Pi_l(\mathbf{x}, t) \to 0$. Physics is independent of our eyesight (resolution scale) and thus the energy decay rate cannot depend on our choice of scale l as $l \to 0$. The conclusion is that in the ideal limit $\nu \to 0$, the velocity field becomes Hölder singular with the exponent $h \leqslant 1/3$.

It should be emphasized here that real flows have obviously finite Reynolds numbers (i.e., finite viscosity) although they may be extremely large. In other words, the mathematical condition of vanishing viscosity $\nu \to 0$ is never realized in real classical fluids. The velocity field therefore satisfies the condition (16) with $h = 1$. However, for this to hold in a fluid with finite but very small viscosity, $|\mathbf{r}|$ must be very small. In other words, the velocity field is smooth only below a very small length scale set by viscosity $|\mathbf{r}| < l_d(\nu)$. Nevertheless, in a finite range of scales above this scale, the velocity field will remain nonsmooth and singular. In the presence of such singularities, what does magnetic topology, as well as topology change, mean in terms of magnetic field lines (integral curves)? In fact, similar arguments apply to the magnetic field in the limit of vanishing diffusivity $\eta \to 0$. Similar to the dissipative anomaly associated with velocity field, i.e., nonvanishing energy dissipation rate $\nu|\nabla_x\mathbf{u}|^2$ when $\nu \to 0$, the magnetic field $\mathbf{B}$ in a turbulent fluid is associated with a magnetic dissipative anomaly $\eta|\nabla_x\mathbf{B}|^2 > 0$ when $\eta \to 0$. Therefore, $\nabla_x\mathbf{B} \to \infty$ as $\eta \to 0$. Singular fields, and consequently ill-defined gradients, in MHD obviously have many consequences, one of which is invalidity of magnetic flux freezing in its conventional sense.

[3]Note that this is the Hölder continuity condition; Lipschitz continuity follows only if $h = 1$, while Hölder singularity follows for $h < 1$.

A. Failure of flux freezing

In order to discuss the notion of flux freezing, Maxwell's equations $\nabla \cdot \mathbf{B} = 0$ and $\partial_t\mathbf{B} = -\nabla \times \mathbf{E}$ are usually used along with the equation of motion for electrons simplified in a compact form known as Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{P}, \tag{18}$$

where $\mathbf{P}$ represents any nonideality. For example, it can represent the resistive electric field $\eta\mathbf{J}$, where η is resistivity and $\mathbf{J} = \nabla \times \mathbf{B}$ is the electric current. Combining these, one arrives at the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \mathbf{P}. \tag{19}$$

In the limit of small nonideality $\mathbf{P}$, Eq. (19) is conventionally called the ideal induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{20}$$

Magnetic flux through a smooth and oriented surface $S(t)$ with $S(t=0) = S$ is a Lagrangian variable defined as $\Phi(S, t) = \int_{S(t)} \mathbf{B}(t) d\mathbf{A}$. Its time evolution is given by

$$\frac{d\Phi(S, t)}{dt} = \int_{S(t)} \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B})\right] d\mathbf{A}. \tag{21}$$

Using Ohm's law leads to

$$\frac{d\Phi(S, t)}{dt} = -\oint_{C(t)} \mathbf{P}\, d\mathbf{x}, \tag{22}$$

where $C(t)$ is the boundary of the surface $S(t)$ advected by the flow. The magnetic field will be frozen into the fluid only if $\nabla \times \mathbf{P} = 0$, or in particular $\mathbf{P} = 0$. This result, known as the Alfvén flux-freezing theorem [7], has other derivations too [see Eq. (41) and the Appendix; see also [3] for a detailed discussion].

There are however mathematical and physical difficulties inherent in flux-freezing arguments. If the magnetic field is frozen into an ideal fluid, the flow would entangle the field lines in a very complex way, whereas such patterns have never been observed in magnetized systems including astrophysical bodies. More importantly, one implicit assumption in flux-freezing arguments is the continuity of the velocity and magnetic fields in the limit when the nonideal term $\mathbf{P}$ (e.g., resistivity) tends to zero, in which case the governing equations may become ill-defined. Even in HD, as discussed in the preceding section, it is well known that the solutions of the ideal incompressible Euler equations in turbulent systems should be singular in the limit $\nu \to 0$.

Analogous to the velocity field, the magnetic field too can be renormalized to remove its singularities. The renormalized induction equation reads

$$\frac{\partial \mathbf{B}_l}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})_l - \nabla \times \mathbf{P}_l, \tag{23}$$

where we have used the renormalized Ohm law

$$\mathbf{E}_l + (\mathbf{u} \times \mathbf{B})_l = \mathbf{P}_l, \tag{24}$$

which can also be rewritten as

$$\mathbf{E}_l = \mathbf{P}_l + \mathbf{R}_l - \mathbf{u}_l \times \mathbf{B}_l. \tag{25}$$

Therefore, even in the absence of any nonideality $\mathbf{P}$, there is a nonlinear term which is not necessarily negligible,

$$\mathbf{R}_l = -(\mathbf{u} \times \mathbf{B})_l + \mathbf{u}_l \times \mathbf{B}_l \equiv -\mathcal{E}_l. \tag{26}$$

Here the turbulent electromotive force (EMF) $\mathcal{E}_l \equiv -\mathbf{R}_l$ is the motional electric field induced by turbulent eddies of scales smaller than l and it plays a crucial role in magnetic dynamo theories. We find

$$\frac{\partial \mathbf{B}_l}{\partial t} = \nabla \times (\mathbf{u}_l \times \mathbf{B}_l - \mathbf{R}_l - \mathbf{P}_l). \tag{27}$$

Let us assume that the nonideal term $\mathbf{P}$ is negligibly small at larger scales. In other words, we assume that $\mathbf{P}_l$ is negligible in the inertial range of turbulence, which can basically be taken as the definition of the inertial range. Ignoring $\mathbf{P}_l$, Eq. (27) becomes the ideal renormalized induction equation. This form of MHD equations can also be called a coarse-grained, distributional, or weak form. After coarse graining at an arbitrary scale $l > l_d$, where l_d is the viscous damping scale down the inertial range, the ideal MHD equations gain such an additional nonlinear term even for a smooth laminar solution. However, for such solutions the nonlinear term $\mathbf{R}_l$ would vanish rapidly as l tends to zero. A real violation of flux freezing for an ideal MHD solution is one which persists in the limit $l \to 0$ as defined in [3]. Many other similar situations can be found in RG theories.

Employing the renormalization method briefly discussed above, Eyink and Aluie [3] showed that magnetic flux conservation can be violated at an instant of time for an arbitrarily small length scale l in the absence of any nonideality only if at least one of the following necessary conditions is satisfied: (a) nonrectifiability of advected loops, (b) unbounded velocity or magnetic fields, or (c) singular current sheets and vortex sheets that both exist and intersect in sets of large enough dimension. One underlying important fact is that an electrically conducting fluid with a small viscosity and resistivity, or other nonideality represented by the vector field $\mathbf{P}$ in Ohm's law, will be generally turbulent (see, e.g., [1,3,29,30]).

What is the condition for a given magnetic field, satisfying the above renormalized induction equation in the sense of distributions, to be frozen into the fluid? One way to answer this question is through studying the slippage between the magnetic field and the fluid threaded by the field. However, it should be emphasized that flux freezing in its conventional form is meaningless in a turbulent medium, as we will briefly discuss in Sec. II D. Particle (Lagrangian) trajectories are stochastic in a turbulent flow: Which trajectory should a field line be frozen into? Conventional flux freezing is not valid in a turbulent medium since the magnetic field is stochastic in such environments; instead a stochastic version is in demand. In Sec. II D we will briefly discuss the concept of stochastic flux freezing introduced in [4] in the context of spontaneous stochasticity. A mathematical analysis of these concepts is given in the Appendix for self-containment (details can be found in [1,3] and references therein).

B. Field-fluid slippage

Eyink [6] obtained a simple equation that governs the perpendicular slip velocity of magnetic field lines relative to the fluid generalizing the magnetic reconnection theory to turbulent media. Here we only introduce the concept of slip velocity and add a very simple argument to show how the slip-velocity source term is related to the time evolution of the unit vector tangent to the renormalized magnetic field at scale l.

If we denote by $\boldsymbol{\xi}(s; \mathbf{x}, t)$ an arbitrary point on the magnetic field line at time t located at a distance s from a base point $\mathbf{x}$ along the field line, the unit tangent vector to the curve parametrized by s is

$$\frac{d}{ds}\boldsymbol{\xi}(s; \mathbf{x}, t) = \hat{\mathbf{B}}(\boldsymbol{\xi}(s; \mathbf{x}, t), t), \quad \boldsymbol{\xi}(s=0; \mathbf{x}, t) = \mathbf{x}, \tag{28}$$

where $\hat{\mathbf{B}} = \mathbf{B}/|\mathbf{B}|$. On the other hand, the position of a fluid element, which starts at $\mathbf{x}_0$ at time t_0 at a later time t, satisfies the following equation:

$$\frac{d}{dt}\mathbf{x}(t, \mathbf{x}_0, t_0) = \mathbf{u}(\mathbf{x}(t, \mathbf{x}_0, t_0)), \quad \mathbf{x}(t_0, \mathbf{x}_0, t_0) = \mathbf{x}_0. \tag{29}$$

Obviously, if flux freezing holds for a smooth and laminar solution of ideal MHD equations, then we should be able to parametrize both field lines and the trajectories of the fluid elements together using the same function $\boldsymbol{\xi} \equiv \mathbf{x}$. In other words, in that case, we could find a function $s(t, s_0, x_0)$ such that $\boldsymbol{\xi}(s(t; s_0, \mathbf{x}_0); \mathbf{x}(t; \mathbf{x}_0, t_0), t) = \mathbf{x}(t; \boldsymbol{\xi}(s_0; \mathbf{x}_0, t_0), t_0)$. The derivative of this equation reveals that the flux-freezing condition $(d/dt)\boldsymbol{\xi} = \mathbf{u}(\boldsymbol{\xi}, t) \equiv \tilde{\mathbf{u}}$ holds if and only if

$$\dot{s}(t)\hat{\mathbf{B}}(\boldsymbol{\xi}, t) + D_t\boldsymbol{\xi} = \tilde{\mathbf{u}}, \tag{30}$$

where $D_t = \partial_t + \mathbf{u} \cdot \nabla$. To determine $s(t)$ one can use the parallel components to write

$$\dot{s}(t) = (\tilde{\mathbf{u}} - D_t\boldsymbol{\xi}) \cdot \hat{\mathbf{B}} = (\tilde{\mathbf{u}} - D_t\boldsymbol{\xi})_\parallel, \quad s(t_0) = s_0. \tag{31}$$

Therefore, the condition $d\boldsymbol{\xi}/dt = \tilde{\mathbf{u}}$ holds if and only if for all s, $\mathbf{x}$, and t,

$$(D_t\boldsymbol{\xi})_\perp(s; \mathbf{x}, t) - \mathbf{u}_\perp(\boldsymbol{\xi}(s; \mathbf{x}, t), t) = 0. \tag{32}$$

In fact, this is another version of flux freezing since what it really states is that the relative perpendicular velocity (with respect to the field line) between the field line and fluid elements vanishes. Thus when the flux-freezing condition is not satisfied, this relative velocity has a nonzero value which we denote by

$$\Delta\mathbf{w}_\perp(s; \mathbf{x}, t) = (D_t\boldsymbol{\xi} - \tilde{\mathbf{u}})_\perp(s; \mathbf{x}, t). \tag{33}$$

Therefore, the flux-freezing condition translates into $\Delta\mathbf{w}_\perp \equiv 0$. It is easy to show (for details see [6]) that

$$\frac{d\Delta\mathbf{w}_\perp}{ds} - [(\nabla_\xi\hat{\mathbf{B}})^T - (\hat{\mathbf{B}}\hat{\mathbf{B}})\cdot(\nabla_\xi\hat{\mathbf{B}})]\cdot\Delta\mathbf{w}_\perp = -\frac{(\nabla\times\mathbf{P})_\perp}{|\mathbf{B}|}. \tag{34}$$

Hence, assuming that the field remains smooth as $\mathbf{P}\to 0$, one may conclude that flux freezing holds and the field lines move with the fluid elements with no slippage. In fact, the expression (34) indicates that flux freezing holds if $\hat{\mathbf{B}}\times(\nabla\times\mathbf{P}) = 0$. This condition has long been known as the general condition for flux freezing [31]: $(\nabla\times\mathbf{P})_\parallel = 0$. In the next section we will show that this vector has a close relationship with the evolution of the magnitude and direction of magnetic field.

C. Field topology and energy

Here we connect Eyink's [6] slip-velocity source term to magnetic topology and show how the renormalized induction equation gives rise to two equations which govern the topology and energy of the magnetic field. In the inertial range of turbulence, as we discussed before, the nonideal term $\mathbf{P}$ is (by definition) negligible. In this case, the nonlinear term $\mathbf{R}$ will appear instead of $\mathbf{P}$ in the above expressions thus the slip-velocity source term is given by the perpendicular component of (with respect to $\mathbf{B}_l$)[4]

$$\boldsymbol{\Sigma}_l = \frac{\nabla\times\mathbf{R}_l}{B_l}. \tag{35}$$

Similarly, we can define

$$\boldsymbol{\sigma}_l = \frac{\nabla\times\mathbf{P}_l}{B_l}, \tag{36}$$

which will be useful in our later discussions. It is easy to see that the term $\boldsymbol{\Sigma}_l^\perp$ ($\boldsymbol{\sigma}_l^\perp$) is related to the time evolution of the unit vector tangent to the renormalized field $\mathbf{B}_l$. The derivative of the unit vector $\hat{\mathbf{B}} = \mathbf{B}_l/|\mathbf{B}_l|$ is

$$\partial_t\hat{\mathbf{B}}_l = \frac{1}{|\mathbf{B}_l|^2}(|\mathbf{B}_l|\partial_t\mathbf{B}_l - \mathbf{B}_l\partial_t|\mathbf{B}_l|).$$

Noting that $\partial_t|\mathbf{B}_l| \equiv \partial_t(B_l^2)^{1/2} = (\mathbf{B}_l\cdot\partial_t\mathbf{B}_l)(B_l^2)^{-1/2}$, which is $(\mathbf{B}_l/|\mathbf{B}_l|)\cdot\partial_t\mathbf{B}_l = \hat{\mathbf{B}}_l\cdot\partial_t\mathbf{B}_l = (\partial_t\mathbf{B}_l)_\parallel$, we find

$$\partial_t\hat{\mathbf{B}}_l = \frac{1}{|\mathbf{B}_l|}[\partial_t\mathbf{B}_l - \hat{\mathbf{B}}_l(\hat{\mathbf{B}}_l\cdot\partial_t\mathbf{B}_l|)] = \frac{1}{|\mathbf{B}_l|}[\partial_t\mathbf{B}_l - (\partial_t\mathbf{B}_l)_\parallel].$$

Obviously, the terms inside the square brackets are the perpendicular component (with respect to $\mathbf{B}_l$) of the renormalized induction equation

$$\partial_t\hat{\mathbf{B}}_l = \left(\frac{\partial_t\mathbf{B}_l}{B_l}\right)_\perp$$

or

$$\partial_t\hat{\mathbf{B}}_l - \left(\frac{\nabla\times(\mathbf{u}_l\times\mathbf{B}_l)}{B_l}\right)_\perp = -\boldsymbol{\Sigma}_l^\perp - \boldsymbol{\sigma}_l^\perp. \tag{37}$$

[4]Here, for simplicity, we have dropped the minus sign used in [6] in the definition of $\boldsymbol{\Sigma}_l^\perp$.

Consequently, $\boldsymbol{\Sigma}_l^\perp + \boldsymbol{\sigma}_l^\perp$ is also the source term for the time evolution of $\hat{\mathbf{B}}_l$ which arises from the nonlinear term $\mathbf{R}_l$ and nonideal term $\mathbf{P}_l$. Note that on small scales, i.e., in the dissipative range, $\boldsymbol{\sigma}$ dominates $\boldsymbol{\Sigma}$ in the expression (37), whereas in the inertial range, where $\mathbf{P}$ is negligible, $\boldsymbol{\Sigma}$ is dominant instead. In the inertial range, $\boldsymbol{\Sigma}_l^\perp$ corresponds to the contribution of the turbulent electromotive force $\mathcal{E}_l \equiv -\mathbf{R}_l = (\mathbf{u}\times\mathbf{B})_l - \mathbf{u}_l\times\mathbf{B}_l$ in changing the direction of the magnetic field at scale l. According to the argument following Eq. (35), a nonzero slip-velocity source term $\Sigma_l^\perp \neq 0$ ($\sigma_l^\perp \neq 0$) indicates field-fluid slippage. This in turn means that the conventional flux freezing would not hold in turbulence. In terms of Eq. (37), this is also equivalent to the fact that the turbulent EMF induced by the motions below scale l change the direction of magnetic field at scale l. Note that this argument also implies that the evolution of the field topology, in terms of the temporal changes in its direction $\hat{\mathbf{B}}_l$, is related to the perpendicular component of the induction equation.

Similarly, it is easy to show that the evolution of magnetic energy at scale l is related to the parallel component of the induction equation

$$\frac{\partial B_l}{\partial t} = \left(\frac{\partial\mathbf{B}_l}{\partial t}\right)_\parallel, \quad \frac{\partial}{\partial t}\left(\frac{B_l^2}{2}\right) = B_l\left(\frac{\partial\mathbf{B}_l}{\partial t}\right)_\parallel. \tag{38}$$

Thus one may study magnetic field topology (in terms of $\partial_t\hat{\mathbf{B}}_l$) and magnetic energy (in terms of $\partial_t B_l^2$ or $\partial_t B_l$) separately. In general, the contributions of nonidealities and nonlinearities to the evolution of $\hat{\mathbf{B}}_l$ and B_l can be summarized as

$$(\partial_t\hat{\mathbf{B}}_l)_{\text{non}} = -(\boldsymbol{\Sigma}_l^\perp + \boldsymbol{\sigma}_l^\perp) \tag{39}$$

and

$$\left(\frac{\partial_t B_l}{B_l}\right)_{\text{non}} = \left(\frac{\partial_t B_l^2/2}{B_l^2}\right)_{\text{non}} = -(\Sigma_l^\parallel + \sigma_l^\parallel). \tag{40}$$

Separating the magnitude B_l and direction $\hat{\mathbf{B}}_l$ of the magnetic field $\mathbf{B}_l$, the former governed by the parallel component and the latter by the vertical component (with respect to the magnetic field $\mathbf{B}_l$) of the induction equation, extremely simplifies the study of stochastic fields and shows that $\Sigma^\parallel$ ($\sigma^\parallel$) is related to the temporal changes in the field magnitude (energy) while $\Sigma^\perp$ ($\sigma^\perp$) is related to the temporal changes in the field direction (topology). We will formulate these considerations more rigorously in Sec. III.

D. Spontaneous stochasticity and stochastic flux freezing

Eyink [4] has shown that magnetic flux conservation in turbulent media with small resistivities, or equivalently high magnetic Reynolds numbers, neither holds in the conventional sense nor is entirely broken. Instead, flux freezing holds in a statistical sense associated with the spontaneous stochasticity of Lagrangian particle trajectories. As resistivity tends to zero, the magnetic Reynolds number tends to infinity. If viscosity of the fluid tends to zero simultaneously and the fluid becomes turbulent, Lagrangian trajectories will not be unique anymore. With an infinite number of such trajectories, which trajectory can the magnetic field pick at any point and freeze into?

Even in the absence of any nonideal term in Ohm's law, Eq. (27) indicates that flux freezing would not hold in a turbulent medium. It can be shown (see the Appendix for details) that for particle advection in turbulence the Lagrangian trajectories can remain random in the limit of vanishing conductivity and viscosity. This phenomenon of spontaneous stochasticity [32] resembles spontaneous symmetry breaking in QFT and has been discussed with great details in recent decades (for more details see [4] and references therein).

Expanding $\nabla\times(\mathbf{u}\times\mathbf{B})=\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{B}\nabla\cdot\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{B}+\mathbf{u}\nabla\cdot\mathbf{B}$, one can write the bare induction equation as $D\mathbf{B}/Dt=\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{B}\nabla\cdot\mathbf{u}+\lambda\nabla^2\mathbf{B}$ with Lagrangian derivative $D/Dt\equiv\partial_t+\mathbf{u}\cdot\nabla$. Here we denote magnetic diffusivity by λ. The continuity equation $D\rho/Dt+\rho\nabla\cdot\mathbf{u}=0$ then yields

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right)=\left(\frac{\mathbf{B}}{\rho}\right)\cdot\nabla\mathbf{u}. \tag{41}$$

This is of course another way to state the conventional flux-freezing theorem presuming that MHD equations remain well behaved in the limit $\lambda\to 0$ (i.e., the so-called ideal MHD regime) and the integral curves of $\mathbf{B}/\rho$ are advected with the fluid.[5]

In general, there is no way to find a velocity field $\mathbf{u}^*$ such that the resistive induction equation can be written in the ideal form (20). Nevertheless, it is possible to describe the motion as a stochastic advection. This approach naturally leads to the notion of stochastic flux freezing (see the Appendix; for a more detailed treatment see [4] and references therein). The induction equation can in fact be expanded in a form that resembles a diffusion equation, which is analogous to the problem of particle advection in turbulence. A path-integral formula can then be applied which eventually leads to the important implication that magnetic field is stochastically frozen in and is advected along stochastic Lagrangian trajectories (see the Appendix).

III. MAGNETIC TOPOLOGY AND STOCHASTICITY

In general, a given vector field $\mathbf{B}(\mathbf{x},t)$ would be different at different scales, that is, $\mathbf{B}_l(\mathbf{x},t)$ would generally differ from $\mathbf{B}_L(\mathbf{x},t)$ for $l\neq L$. The scale dependence of $\mathbf{B}(\mathbf{x},t)$ is related to its spatial gradients (if the field is renormalized spatially; otherwise a similar argument would apply to temporal renormalization). Quantitatively, one can see this relationship by writing

$$\frac{\partial\mathbf{B}_l}{\partial l}=\frac{\partial}{\partial l}\int_V G_l(\mathbf{r})\mathbf{B}(\mathbf{x}+\mathbf{r},t)d^3r. \tag{42}$$

This expression may be simplified by changing variables as $\mathbf{r}'=\mathbf{r}/l$ and using the definition $G_l(\mathbf{r})=l^{-3}G(\mathbf{r}/l)$:

$$\begin{aligned} l\frac{\partial\mathbf{B}_l}{\partial l}&=l\int_V d^3r' G(\mathbf{r}')\frac{\partial}{\partial l}\mathbf{B}(\mathbf{x}+l\mathbf{r}',t)\\ &=\int_V d^3r\, G_l(\mathbf{r})\,\mathbf{r}\cdot\nabla\mathbf{B}(\mathbf{x}+\mathbf{r},t)\\ &=\int_V d^3r\, G_l(\mathbf{r})\,(\mathbf{r}+\mathbf{x})\cdot\nabla\mathbf{B}(\mathbf{x}+\mathbf{r},t)\\ &\quad-\int_V d^3r\, G_l(\mathbf{r})\,\mathbf{x}\cdot\nabla\mathbf{B}(\mathbf{x}+\mathbf{r},t). \end{aligned} \tag{43}$$

Therefore, we find

$$l\frac{\partial\mathbf{B}_l}{\partial l}=(\mathbf{x}\cdot\nabla\mathbf{B})_l-\mathbf{x}\cdot\nabla\mathbf{B}_l. \tag{44}$$

Here we have used the fact that $\mathbf{x}_l=\mathbf{x}$, which results from Eqs. (6) and (7). In this paper we are mostly concerned with the behavior of a given vector field in terms of its renormalized components at different scales. However, it should be emphasized that this approach is completely different from the concept of scale separation. One might naively attempt to scale separate the field assuming $L\gg l$ and calling $\mathbf{B}_L\to\mathbf{B}_0$ the large-scale field and $\mathbf{B}_l-\mathbf{B}_L\to\mathbf{b}$ the small-scale field

$$\begin{aligned}\mathbf{B}_l&=\mathbf{B}_L+(\mathbf{B}_l-\mathbf{B}_L)\\ &\overset{?}{\Rightarrow}\mathbf{B}_T=\mathbf{B}_0+\mathbf{b},\end{aligned}$$

with $\mathbf{B}_T$ to be understood as the total field. Nevertheless, scale separation requires $\langle\mathbf{B}_T\rangle=\mathbf{B}_0$ and $\langle\mathbf{b}\rangle=0$, where $\langle\cdot\rangle$ indicates an ensemble averaging. However, in general, $\langle\mathbf{B}_l-\mathbf{B}_L\rangle\neq 0$. In fact, the statistical scale separation method inherently differs from the deterministic approach of renormalization. The latter methodology does not depend on any separation of scales between large-scale mean quantities and their small-scale components. In fact, as mentioned above, we treat the scale l as a variable as it is common in RG analyses (for a discussion on the relation between coarse-graining or renormalizing mean field theory and the filtering approach employed in large-eddy simulation of turbulent flows see [3] and references therein). We will not use this vague but widely used notion in this paper.[6]

Scale-split energy density

We define the scale-split energy density $\psi(\mathbf{x},\mathbf{r};t)$ in terms of the renormalized vector field $\mathbf{B}_l(\mathbf{x},t)$ at scale l and the renormalized field $\mathbf{B}_L(\mathbf{x}+\mathbf{r},t)$ at scale L as[7]

$$\psi_{l,L}(\mathbf{x},\mathbf{r},t)=\tfrac{1}{2}\,\mathbf{B}_l(\mathbf{x},t)\cdot\mathbf{B}_L(\mathbf{x}+\mathbf{r},t). \tag{45}$$

Here we are concerned only with $\psi_{l,L}(\mathbf{x},\mathbf{r}=0,t)\equiv\psi_{l,L}(\mathbf{x},t)$. This quantity is a scale-dependent scalar field and obviously $L=l$ reduces it to the energy density $B_l^2/2$ at scale

[5] Euler potentials α and β are also sometimes used to write the magnetic field in Clebsch notation, $\mathbf{B}=\nabla\alpha\times\nabla\beta$ with vector potential $\mathbf{A}=\alpha\nabla\beta+\nabla\gamma$. The ideal bare induction equation then becomes $\partial_t\mathbf{B}=\nabla\times(\mathbf{u}\times\mathbf{B})+\nabla(D\alpha/Dt)\nabla\beta+\nabla\alpha\times(D\beta/Dt)$. Thus flux freezing, in this formulation, translates into the statement that α and β remain conserved in the flow.

[6] Another analogous method is to decompose a (usually scalar) field to homogeneous and inhomogeneous parts, writing $\phi(\mathbf{x},s)=\phi^*(s)+\phi'(\mathbf{x},t)$. Here $\phi^*(s)=(1/V)\int_V\phi(\mathbf{x},s)d^3x$ is the homogeneous part and $\phi'(\mathbf{x},s)$ is the inhomogeneous part of ϕ. This decomposition is used, for example, in inflationary theories in cosmology for the quantum scalar field $\phi(\mathbf{x},s)$ known as the inflation which drives the cosmic inflation.

[7] All the definitions and arguments presented here can be obviously applied to any other vector field including the velocity field. In this paper, however, we are primarily concerned with the magnetic field.

l. In general, $\psi_{l,L}$ provides us with pointwise information about the angle between the coarse-grained fields on different scales and also their magnitude. Note that $\psi_{l,L}(\mathbf{x},t)$ is in fact a renormalized space-correlation function

$$\psi_{l,L}(\mathbf{x},t) = \frac{1}{2}\int_V d^3r \int_V d^3r' G_l(\mathbf{r}) G_L(\mathbf{r}') \times \mathbf{B}(\mathbf{x}+\mathbf{r},t)\cdot\mathbf{B}(\mathbf{x}+\mathbf{r}',t).$$

Therefore, this scalar field is also an indicator of the self-alignment of the field $\mathbf{B}$ at any point $(\mathbf{x},t)$, that is to say, it depends on the angle between $\mathbf{B}(\mathbf{x}+\mathbf{r},t)$ and $\mathbf{B}(\mathbf{x}+\mathbf{r}',t)$ for any $\mathbf{r}$ and $\mathbf{r}'$ in the neighborhood of $\mathbf{x}$ (since G is a rapidly decaying function). It also proves more convenient to consider the direction and magnitude of the field separately by writing $\psi(\mathbf{x},t) = \phi(\mathbf{x},t)\chi(\mathbf{x},t)$ with two scalar fields

$$\phi_{l,L}(\mathbf{x},t) = \begin{cases} \hat{\mathbf{B}}_l(\mathbf{x},t)\cdot\hat{\mathbf{B}}_L(\mathbf{x},t) & \text{for } B_L \neq 0,\ B_l \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

and

$$\chi_{l,L}(\mathbf{x},t) = \tfrac{1}{2} B_l(\mathbf{x},t) B_L(\mathbf{x},t). \tag{47}$$

Therefore, the energy field $\chi_{l,L}$ is associated with the magnitude of the field at different scales whereas the topology field $\phi_{l,L}$ is associated with its direction. Physically, we are interested in volume-averaged quantities in a given spatial volume V and on a given range of scales $[l,L]$. (For simplicity, we will drop the indices l, L hereafter.) Averaging can be done using L_p norms, with $p=2$ corresponding to rms averaging. Mathematically, the scalar field $\phi(\mathbf{x},t)$ is basically the cosine of the angle between two coarse-grained components of the magnetic field $\mathbf{B}(\mathbf{x},t)$ at different scales l and L at point $(\mathbf{x},t)$, i.e., $\phi = \cos\theta = \hat{\mathbf{B}}_l\cdot\hat{\mathbf{B}}_L$. Thus the term inside the parentheses in Eq. (46), after rms averaging, is a measure of the average deviation of $\hat{\mathbf{B}}_l$ from $\hat{\mathbf{B}}_L$, i.e., $[1-\cos\theta]_{\text{rms}}/2$. In other words, it is a measure of self-entanglement of the vector field, which is why we call it the topology field. The term $\chi_{l,L}$ is twice the geometric mean of the energy densities at scales l and L, $\chi = 2\sqrt{U_l U_L}$, where $U_l = B_l^2/2$ and similarly $U_L = B_L^2/2$. Hence its squared rms average is proportional to the volume average of energy densities $\chi^2_{\text{rms}} = 4\langle U_l U_L\rangle_V$, which is the reason why we call it the energy field. In short, the scalar field ϕ is related to the field topology whereas χ is related to the field energy density. Also note that at magnetic nulls, $\phi = 0$ by definition.

To make our arguments more tractable and quantitative, let us begin with a few definitions. For any $t\in[t_0, t_0+T]$ and $\mathbf{x}\in V$ and in the range of scales $[l,L]$, we give the following definitions.

Definition 1. The vector field $\mathbf{B}(\mathbf{x},t)$ is scale independent if $l'\partial_{l'}\mathbf{B}_{l'}(\mathbf{x},t) = (\mathbf{x}\cdot\nabla\mathbf{B})_{l'} - \mathbf{x}\cdot\nabla\mathbf{B}_{l'} \equiv 0$ for all $l'\in[l,L]$. Otherwise, $\mathbf{B}(\mathbf{x},t)$ is scale dependent.

Definition 2. The scale-dependent vector field $\mathbf{B}(\mathbf{x},t)$ is stochastic if $-1\leqslant\phi(\mathbf{x},t)\leqslant 1$ is a stochastic variable.[8]

Definition 3. The self-entanglement (of order p) of the vector field $\mathbf{B}(\mathbf{x},t)$ is

$$S_p(t) = \frac{1}{2}\|1-\phi(\mathbf{x},t)\|_p \equiv \frac{1}{2}\left[\int_V \left|1-\phi(\mathbf{x},t)\right|^p \frac{d^3x}{V}\right]^{1/p}. \tag{48}$$

Obviously, $S_p(t)\approx 0$ corresponds to a nonstochastic field, while $S_p(t)\approx 1$ indicates a strongly tangled field (between the scales l and L and in volume V). Hence, a smooth field is one for which $S_\infty(t) = 0$.[9] For a stochastic field, the self-entanglement level would be a measure of its stochasticity.

Definition 4. The stochasticity level (of order p) of a stochastic field $\mathbf{B}(\mathbf{x},t)$ is its self-entanglement.

Definition 5. The topological deformation (of order p) of a stochastic (nonstochastic) field $\mathbf{B}(\mathbf{x},t)$ is the rate of change of its pth power of the stochasticity level (self-entanglement) with time:[10]

$$T_p(t) = \frac{\partial S_p(t)}{\partial t} = \frac{S_p^{1-p}(t)}{2^p}\int_V (\phi-1)\frac{\partial\phi}{\partial t}|\phi-1|^{p-2}\frac{d^3x}{V}. \tag{49}$$

Note that $T_p\equiv 0$ generally indicates a stationary stochasticity rather than nonstochasticity. We will give a more precise definition of weak stationarity in terms of time series in Sec. IV.

We emphasize in passing the distinction between topology change and topological deformation. Under a smooth deformation of a given stochastic field renormalized at scale l, such that close points on its integral curves (field lines) remain close to each other, the topology will not change; there will be only deformation. This is why a doughnut is topologically "equivalent" to a coffee cup since each of these objects can be obtained from the other one by a smooth deformation that keeps initially close points (on one object) close to each other (on the second object).[11] The topology would change, however, if one tears the object while deforming it; thus a reconnecting magnetic field undergoes topology change. Of course such a topology change will in general translate into a change in stochasticity level too, i.e., $T_p\neq 0$. Nevertheless, a change in stochasticity, i.e., $T_p\neq 0$, by itself does not

[8]A stochastic (random) variable is a variable whose numerical values are determined based on the outcomes of a random process such as tossing a die or flipping a coin. Mathematically, a random variable $Y: O\to M$ is a measurable function from a set of possible outcomes O to a measurable space M.

[9]The L_p-norm of $\mathbf{f}:\mathcal{R}^m\to\mathcal{R}^m$ is the mapping $\mathbf{f}\to\|\mathbf{f}\|_p = [\int_V|\mathbf{f}(\mathbf{x})|^p(d^mx/V)]^{1/p}$. For $p=2$, $\|f\|_2 = f_{\text{rms}}$ is the root-mean-square value of $\mathbf{f}$. For $p\leqslant q$, $\|\mathbf{f}\|_p\leqslant\|\mathbf{f}\|_q$. Also $\|\mathbf{f}\|_\infty = \lim_{p\to\infty}\|\mathbf{f}\|_p = |\mathbf{f}|_{\max}$.

[10]In this paper we work in an Eulerian system for simplicity and focus on points in space rather than following fluid particles. Otherwise, all time derivatives should be replaced with Lagrangian derivatives.

[11]Mathematically, such a deformation can be represented by a map f from one topological space (e.g., a doughnut) to another (e.g., coffee cup) with an inverse f^{-1} representing the backward deformation. To keep the close points close to each other f and f^{-1} need to be continuous functions. Such a bijective continuous map f with continuous inverse f^{-1} is called a homeomorphism between topological spaces.

necessarily represent a topology change; rather it generally indicates a topological deformation. Also note that stochasticity level and topological deformation, as defined above, are relative concepts in the sense that they are scale dependent.

Definition 6. The (pth-order) cross energy of the vector field $\mathbf{B}$ is

$$E_p(t) = \|\chi\|_p. \tag{50}$$

Its time derivative represents the dissipation rate

$$D_p(t) = \frac{\partial E_p(t)}{\partial t} = E_p^{1-p}(t) \int_V \chi \frac{\partial \chi}{\partial t} |\chi|^{p-2} \frac{d^3x}{V}. \tag{51}$$

This concept of field dissipation, rather than field energy dissipation, will prove useful as a bookkeeping device when we discuss magnetic reconnection in Sec. IV.

Throughout this paper, we will take $p = 2$. Thus, the stochasticity level S_2, topological deformation T_2, cross energy E_2, and dissipation D_2 are given by

$$S_2(t) = \frac{1}{2}(1-\phi)_{\text{rms}}, \tag{52}$$

$$T_2(t) = \frac{1}{4S_2(t)} \int_V (\phi - 1) \frac{\partial \phi}{\partial t} \frac{d^3x}{V}, \tag{53}$$

$$E_2(t) = \chi_{\text{rms}}, \tag{54}$$

and

$$D_2(t) = \frac{1}{E_2(t)} \int_V \chi \, \partial_t \chi \frac{d^3x}{V}. \tag{55}$$

Locally, at any arbitrary point $\mathbf{x} \in V$, the time derivative of the scalar field $\phi(\mathbf{x}, t)$ is given by

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \left[\frac{\partial_t \mathbf{B}_l}{B_l} \cdot (\mathbb{I} - \hat{\mathbf{B}}_l \hat{\mathbf{B}}_l)\right] \cdot \hat{\mathbf{B}}_L + \left[\frac{\partial_t \mathbf{B}_L}{B_L} \cdot (\mathbb{I} - \hat{\mathbf{B}}_L \hat{\mathbf{B}}_L)\right] \cdot \hat{\mathbf{B}}_l \\ &= \hat{\mathbf{B}}_L \cdot \left(\frac{\partial_t \mathbf{B}_l}{B_l}\right)_{\perp \mathbf{B}_l} + \hat{\mathbf{B}}_l \cdot \left(\frac{\partial_t \mathbf{B}_L}{B_L}\right)_{\perp \mathbf{B}_L}.\end{aligned} \tag{56}$$

Here $(\,)_{\perp \mathbf{B}}$ represents the perpendicular component with respect to $\mathbf{B}$. Hence, the time evolution of S_2^2 is given by

$$\begin{aligned}T_2(t) = \frac{1}{4S_2} \int_V [\hat{\mathbf{B}}_l \cdot \hat{\mathbf{B}}_L - 1] \Bigg[\hat{\mathbf{B}}_L \cdot \left(\frac{\partial_t \mathbf{B}_l}{B_l}\right)_{\perp \mathbf{B}_l} \\ + \hat{\mathbf{B}}_l \cdot \left(\frac{\partial_t \mathbf{B}_L}{B_L}\right)_{\perp \mathbf{B}_L} \Bigg] \frac{d^3x}{V}.\end{aligned} \tag{57}$$

The time evolution of the scalar field $\chi(\mathbf{x}, t)$ can be similarly obtained,

$$\frac{\partial \chi}{\partial t} = \frac{1}{2} B_l B_L \left[\left(\frac{\partial_t \mathbf{B}_L}{B_L}\right)_{\parallel \mathbf{B}_L} + \left(\frac{\partial_t \mathbf{B}_l}{B_l}\right)_{\parallel \mathbf{B}_l} \right]. \tag{58}$$

Here $(\,)_{\parallel \mathbf{B}}$ represents the parallel component with respect to $\mathbf{B}$. Hence, for the time evolution of cross energy $E_2(t)$ we find

$$\begin{aligned}D_2(t) &= \frac{1}{4E_2} \int_V \left[B_l^2 \partial_t (B_L^2/2) + B_L^2 \partial_t (B_l^2/2) \right] \frac{d^3x}{V} \\ &= \frac{1}{E_2} \int_V \left(\frac{B_l B_L}{2}\right)^2 \left[\left(\frac{\partial_t \mathbf{B}_L}{B_L}\right)_{\parallel \mathbf{B}_L} + \left(\frac{\partial_t \mathbf{B}_l}{B_l}\right)_{\parallel \mathbf{B}_l} \right] \frac{d^3x}{V},\end{aligned} \tag{59}$$

which is obviously related to the temporal changes in energy densities $B_l^2/2$ and $B_L^2/2$ at scales l and L. This is in turn related to the parallel component of the renormalized induction equation at these scales, as we discussed previously.

The above considerations are general and can be applied to any vector field. Let us now concentrate on magnetic field $\mathbf{B}(\mathbf{x}, t)$ which satisfies the renormalized induction equation (27). Time evolution of ϕ, given by Eq. (56), and the renormalized induction equation (27) can be used to write

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \hat{\mathbf{B}}_L \cdot \left(\frac{\nabla \times (\mathbf{u}_l \times \mathbf{B}_l)}{B_l} - \boldsymbol{\Sigma}_l - \boldsymbol{\sigma}_l \right)_{\perp \mathbf{B}_l} \\ &\quad + \hat{\mathbf{B}}_l \cdot \left(\frac{\nabla \times (\mathbf{u}_L \times \mathbf{B}_L)}{B_L} - \boldsymbol{\Sigma}_L - \boldsymbol{\sigma}_L \right)_{\perp \mathbf{B}_L}.\end{aligned} \tag{60}$$

Therefore, the time evolution of ϕ driven by the nonlinearities and nonidealities, rather than the turbulent flow, is

$$\begin{aligned}\left(\frac{\partial \phi}{\partial t}\right)_{\text{non}} &= - \underbrace{\hat{\mathbf{B}}_L \cdot (\boldsymbol{\Sigma}_l^\perp + \boldsymbol{\sigma}_l^\perp)}_{\text{field-fluid slippage of } \mathbf{B}_l \text{ along } \mathbf{B}_L} \\ &\quad - \underbrace{\hat{\mathbf{B}}_l \cdot (\boldsymbol{\Sigma}_L^\perp + \boldsymbol{\sigma}_L^\perp)}_{\text{field-fluid slippage of } \mathbf{B}_L \text{ along } \mathbf{B}_l}.\end{aligned}$$

In the inertial range (dissipative range),[12] terms proportional to $\boldsymbol{\Sigma}$ ($\boldsymbol{\sigma}$) would dominate. Note that the slip-velocity source terms, defined by Eqs. (35) and (36), have appeared again here at two scales.

The term χ can be treated similarly. To get $\partial_t \chi$, we can rewrite Eq. (58) using the renormalized induction equations for $\mathbf{B}_l$ and $\mathbf{B}_L$. The part that depends on the nonlinearity $\mathbf{R}$ and nonideality $\mathbf{P}$ is given by

$$\left(\frac{\partial_t \chi}{\chi}\right)_{\text{non}} = -(\Sigma_L^\parallel + \Sigma_l^\parallel + \sigma_L^\parallel + \sigma_l^\parallel). \tag{61}$$

The implication is that the magnetic energy conversion is related to $\Sigma^\parallel$ and $\sigma^\parallel$. Therefore, the local time evolution of ψ driven by nonlinear terms is given by

$$\begin{aligned}\left(\frac{\partial \psi(\mathbf{x}, t)}{\partial t}\right)_{\text{non}} &\equiv \frac{1}{2} \chi(\mathbf{x}, t) \left(\frac{\partial \phi(\mathbf{x}, t)}{\partial t}\right)_{\text{non}} \\ &\quad + \frac{1}{2} \phi(\mathbf{x}, t) \left(\frac{\partial \chi(\mathbf{x}, t)}{\partial t}\right)_{\text{non}}\end{aligned}$$

or

$$\begin{aligned}2\left(\frac{\partial \psi}{\partial t}\right)_{\text{non}} &= -2\chi \underbrace{(\hat{\mathbf{B}}_L \cdot (\boldsymbol{\Sigma}_l^\perp + \boldsymbol{\sigma}_l^\perp) + \hat{\mathbf{B}}_l \cdot (\boldsymbol{\Sigma}_L^\perp + \boldsymbol{\sigma}_L^\perp))}_{\text{field-fluid slippage}} \\ &\quad - \psi \underbrace{(\Sigma_L^\parallel + \Sigma_l^\parallel + \sigma_L^\parallel + \sigma_l^\parallel)}_{\text{magnetic dissipation}}.\end{aligned} \tag{62}$$

One ubiquitous magnetic process, which can be used as an example to apply the formulation developed so far, is magnetic reconnection to be discussed in the following section.

[12]For simplicity, we assume a magnetic Prandtl number of unity throughout this paper, that is, the viscosity is assumed to be equal to the magnetic diffusivity $\nu/\eta = 1$.

IV. PHYSICAL IMPLICATIONS: SLIPPAGE AND RECONNECTION

In this section we apply the formalism developed in the previous sections to the problem of the slippage of magnetic field through the fluid and the closely related concept of magnetic reconnection. In a more detailed treatment, the formalism developed so far should be applied to the velocity field as well. We postpone such a detailed approach to a future work and present a general picture here in terms of magnetic field evolution only. Some definitions of magnetic reconnection require the magnetic field to evolve without flux freezing [33]. Some others include a topology change in magnetic field [21], although the notion of topology change mostly remains vague in such definitions. Yet others include also a conversion of magnetic energy into heat or kinetic energy of fluid particles (see, e.g., [34]; for a more detailed discussion of reconnection definitions see [14,17]). Axford [35] argued that the change of magnetic connections between fluid particles can be taken as the basal definition of reconnection [6]. As the turbulent flow tangles an initially smooth magnetic field, its stochasticity level $S_p(t)$ increases with time. Magnetic field lines, however, are not easily bent and they will resist bending and tangling by means of tension forces. At some point, the field would slip through the fluid to reduce its stochasticity level (self-entanglement).

Equation (57) can be expanded using the renormalized induction equation as

$$T_2(t) = \frac{1}{4S_2}\int_V \frac{d^3x}{V} \underbrace{[\hat{\mathbf{B}}_l\cdot\hat{\mathbf{B}}_L - 1]}_{\text{self-entanglement (stochasticity)}} \Bigg[\underbrace{\left(\frac{\hat{\mathbf{B}}_L}{B_l}\cdot\nabla\times(\mathbf{u}_l\times\mathbf{B}_l)_{\perp\mathbf{B}_l} + \frac{\hat{\mathbf{B}}_l}{B_L}\cdot\nabla\times(\mathbf{u}_L\times\mathbf{B}_L)_{\perp\mathbf{B}_L}\right)}_{\text{turbulence (flow)}}$$
$$ - \underbrace{(\hat{\mathbf{B}}_L\cdot\mathbf{\Sigma}_l^{\perp} + \hat{\mathbf{B}}_l\cdot\mathbf{\Sigma}_L^{\perp} + \hat{\mathbf{B}}_L\cdot\boldsymbol{\sigma}_l^{\perp} + \hat{\mathbf{B}}_l\cdot\boldsymbol{\sigma}_L^{\perp})}_{\text{slippage (reconnection)}}\Bigg], \tag{63}$$

where we have also recovered the terms related to nonideality **P**. The term inside the first set of square brackets in Eq. (63) acts as a weight function $w(\mathbf{x},t) = \hat{\mathbf{B}}_l\cdot\hat{\mathbf{B}}_L - 1$, which represents the local field stochasticity (or self-entanglement for nonstochastic fields). The smoother the field is (i.e., more aligned $\hat{\mathbf{B}}_l$ and $\hat{\mathbf{B}}_L$), the smaller the value of this weight function would be. As the turbulent flow tangles an initially smooth magnetic field (leading to large deviations between $\hat{\mathbf{B}}_l$ and $\hat{\mathbf{B}}_L$), whose effect is represented by the terms inside the first set of large parentheses in the second set of square brackets, the stochasticity level increases, $T_2 = \partial_t S_2 \geqslant 0$. However, such a tangled magnetic field would interact with the flow and resist more tangling and bending. Magnetic field lines can slip through the fluid, an effect already known to be related to $\Sigma^{\perp}\neq 0$ or $\sigma^{\perp}\neq 0$, whose effect is represented by the terms in the second set of parentheses inside the large square brackets. This can lead to a sudden motion of the field lines relative to the fluid quickly decreasing the stochasticity level $T_2 = \partial_t S_2 \leqslant 0$. Therefore, at some point between these two stages, $T_2 = \partial_t S_2 = 0$. Note that this should be interpreted as a general trend, which might be disrupted by small fluctuations induced by turbulence and intermittency.

Sudden field-fluid slippage, motion of the field relative to the fluid, will generally lead to particle acceleration by extracting energy from magnetic field: $D_2 = \partial_t E_2 \leqslant 0$. As the slippage peaks and the stochasticity level reaches its maximum $T_2 = \partial_t S_2 = 0$, the magnetic field dissipation also ramps up. The evolution of the cross energy, as discussed before, is related to the parallel component of the induction equation at any scale. We also have

$$D_2(t) = \frac{1}{E_2}\int_V \left(\frac{B_lB_L}{2}\right)^2\left[\frac{\partial_t(B_L^2/2)}{B_L^2} + \frac{\partial_t(B_l^2/2)}{B_l^2}\right]\frac{d^3x}{V}$$
$$= \frac{1}{E_2}\int_V \left(\frac{B_lB_L}{2}\right)^2\Bigg[\underbrace{\left(\frac{\nabla\times(\mathbf{u}_l\times\mathbf{B}_l)_{\parallel\mathbf{B}_l}}{B_l} + \frac{\nabla\times(\mathbf{u}_L\times\mathbf{B}_L)_{\parallel\mathbf{B}_L}}{B_L}\right)}_{\text{magnetic-velocity field interaction}} - \underbrace{(\Sigma_l^{\parallel} + \sigma_l^{\parallel} + \Sigma_L^{\parallel} + \sigma_L^{\parallel})}_{\text{magnetic dissipation}}\Bigg]\frac{d^3x}{V}. \tag{64}$$

Since $\mathbf{B}_L$ is obtained by averaging (coarse graining) the magnetic field over the length scale L, it is in fact a weighted sum of fine-grained magnetic fields. Indeed, $\mathbf{B}_L(\mathbf{x},t) = \int_V G_L(\mathbf{r})\mathbf{B}(\mathbf{x}+\mathbf{r},t)d^3r$ can be thought of as the expected value of **B** with respect to the probability distribution function $G_L(r)$. Therefore, on average, less aligned fine-grained fields at smaller scales in a volume V would mean a weaker $\int_V B_L^2/2$ (because of local cancelations) especially if $\int_V B_l^2/2$ decreases as well. In other words, we expect that $T_2 = \partial_t S_2 \geqslant 0$ and $\int_V \partial_t(B_l^2/2) \leqslant 0$ lead to $\int_V \partial_t(B_L^2/2) \leqslant 0$. Using the first equation in Eq. (64), it is easy to see that $D_2 = \partial_t E_2 \leqslant 0$. In an analogous manner, $T_2 = \partial_t S_2 \leqslant 0$ and $\int_V \partial_t B_l^2 \geqslant 0$ would lead to $\int_V \partial_t(B_L^2/2) \geqslant 0$; therefore $D_2 = \partial_t E_2 \geqslant 0$. During magnetic reconnection at scale l, the kinetic energy of accelerating particles has to be extracted from the available magnetic energy. Apart from small fluctuations and deviations from the global trend, at the peak of reconnection, E_2 reaches a minimum whereas S_2 reaches a maximum. Therefore, magnetic reconnection may be defined as a field-fluid slippage in which (a) the stochasticity level increases toward a maximum, (b) the magnetic energy is maximally dissipated, and (c) there are magnetic nulls at which $B_l = 0$ or $B_L = 0$,

$$\left\{\frac{\partial S_2}{\partial t} = 0, \quad \frac{\partial^2 S_2}{\partial t^2} \leqslant 0; \quad \frac{\partial E_2}{\partial t} = 0, \quad \frac{\partial^2 E_2}{\partial t^2} \geqslant 0\right\}, \tag{65}$$

or equivalently $\{T_2 = D_2 = 0; \partial_t T_2 \leqslant 0, \partial_t D_2 \geqslant 0\}$. As the field lines disconnect and reconnect, the magnetic field dissipates and stochasticity increases; hence as the reconnection peaks, so do the relative field dissipation and stochasticity level. Note that the scale l (as well as L taken as the system size here) is arbitrary and the arguments and conditions discussed above are applied at any scale in the renormalization range: Reconnection occurs on all scales.

It should be emphasized that the relationship between the magnetic stochasticity level $S_p(t)$ and cross energy $E_p(t)$ discussed above is to be interpreted as a global pattern. After all, this is a statistical approach and we should expect small deviations from the general pattern. The chaotic motions and small fluctuations, inherent features of turbulence, will certainly affect the relationship between the stochasticity and cross energy during short-time intervals. Apart from the problem of intermittency, which we have ignored here, even the volume-averaged quantities such as $S_p(t)$ and $E_p(t)$ can generally suffer from turbulent fluctuations. Also strong magnetic energy dissipation, e.g., in regions where the magnetic field is efficiently annihilated, may affect the relationship between $S_2(t)$ and $E_2(t)$. This would require then a consideration of kinetic energy too. In fact, because of the interplay between magnetic and velocity fields, this formalism should also be applied to the velocity field. The relationship between magnetic and kinetic topology changes is beyond the scope of the present work.

This is all theory so far. In practice, i.e., in experiments and numerical simulations, it may be difficult or impossible to work with monstrous expressions such as Eq. (63) and their derivatives. We can obtain coarse-grained fields $\mathbf{B}_l$ and $\mathbf{B}_L$, in a volume V and on a range of scales $[l, L]$. It is straightforward then to find ϕ and χ and spatially average them to obtain S_2 and E_2 at any given time. These time-dependent functions may be obtained as a discrete set of values measured at different times with fluctuations rather than a smooth graph. This requires a time series analysis to be briefly discussed in Sec. IV. In any case, reconnection would correspond to a time interval $\Delta t = t - t_0$ during which S_2 reaches its maximum while E_2 reaches its minimum value: $\partial_t T_2 \leqslant 0$ and $\partial_t D_2 \geqslant 0$. At a later time $t = t_0 + \tau$, when T_2 and D_2 change sign, the conditions $\partial_t T_2 \leqslant 0$ and $\partial_t D_2 \geqslant 0$ change to $\partial_t T_2 \geqslant 0$ and $\partial_t D_2 \leqslant 0$. This can be used to define a reconnection rate τ^{-1}.

Definition 7. The reconnection intensity (or field-fluid slippage intensity) in time τ, during which $\partial_t^2 S_p \leqslant 0$ and $\partial_t^2 E_p \geqslant 0$, is

$$I_p(\tau) = \left| \int_{t_0}^{t_0+\tau} T_p(t) dt \right| = |S_p(t_0 + \tau) - S_p(t_0)|. \qquad (66)$$

Note that generally field-fluid slippage may or may not be associated with magnetic null points. If it is, and the above conditions hold, magnetic field lines disconnect and reconnect again, and therefore close points on the field lines will not generally remain close to each other as the field lines disconnect. Hence magnetic reconnection is field-fluid slippage in which magnetic energy is reduced, magnetic connectivity breaks apart, and topology changes. Topological deformation T_p then also indicates topology change.

Note that $\boldsymbol{\sigma}_l^{\perp}$ dominates at the dissipative range where $\boldsymbol{\Sigma}_l^{\perp}$ is negligible. Thus, field-fluid slippage at small scales is driven by the nonidealities. On the other hand, $\boldsymbol{\Sigma}_l^{\perp}$ dominates in the inertial range where $\boldsymbol{\sigma}_l^{\perp}$ is negligible, which indicates that field-fluid slippage is driven by the nonlinearities at larger scales. As an order of magnitude estimate, $|\mathbf{R}_l| \sim |\delta \mathbf{u}(l) \times \delta \mathbf{B}(l)|$, where $\delta \mathbf{u}(l)$ and $\delta \mathbf{B}(l)$ are, respectively, the velocity and magnetic field increments across distance l. Therefore, the vector field $\boldsymbol{\Sigma}_l = (\nabla \times \mathbf{R}_l)/B_l$ is of order of

$$\Sigma_l \sim \left| \frac{\delta \mathbf{u}(l)}{l} \times \frac{\delta \mathbf{B}(l)}{B_l} \right| \lesssim \frac{\delta u(l)}{l} \frac{\delta B(l)}{B_l}. \qquad (67)$$

In the inertial range of turbulence, Kolmogorov scaling [36] leads to $\delta u(l)/l \sim l^{-2/3}$. In the next section we will see that this is related to Richardson two-particle diffusion and stochastic reconnection. Note that in the dissipative range of turbulence, Kolmogorov scaling yields $\delta u/l \sim \nu^{-1/2}\epsilon^{1/2}$ where ν is viscosity and ϵ the energy dissipation rate. In fact, $|\nabla \times \mathbf{R}_l|$ increases as we go to smaller scales in the inertial range and finally approaches the nonideal term $|\nabla \times \mathbf{P}|$ at the turbulence microscale l_d:

$$\frac{\partial}{\partial l} |\nabla \times \mathbf{R}_l| \leqslant 0 \quad \text{(inertial range).}$$

Therefore, $|\nabla \times \mathbf{R}_l| \gg |\nabla \times \mathbf{P}_l|$ for $l \gg l_d$ in the inertial range (see [3,6]). One may expect to find larger values for $|\nabla \times \mathbf{R}_l|$ at smaller scales in the inertial range than the larger scales. However, any such small effect would be magnified by Richardson diffusion at larger scales. We find connections, therefore, to the stochastic model of reconnection proposed by Lazarian and Vishniac [21].

In fact, Lazarian and Vishniac [21] showed that a stochastic magnetic field would enhance magnetic reconnection because stochasticity causes more efficient fluid diffusion from the reconnection region. In other words, the stochastic reconnection model implies the existence of negative feedback. They also pointed out that in a turbulent medium "if the magnetic field is too smooth, reconnection speeds decrease and the field becomes more tangled. If the field is extremely chaotic, reconnection speeds increase, making the field smoother." This argument is intimately connected to the picture we have advanced in this section in terms of stochasticity $S_p(t)$, cross energy $E_p(t)$, and their time derivatives. In fact, the original derivation of stochastic reconnection was based on a simple geometric picture in which the field lines have some small-scale wandering. In this context, weak stochasticity means that on any given scale the typical angle by which field lines differ from their neighbors is very small $\theta \ll 1$. If one makes this argument mathematically more precise, using renormalization methodology rather than scale separation invoked by Lazarian and Vishniac, one recovers a picture analogous to the one advanced in this section based on $\theta = \cos^{-1} \phi$. A detailed numerical evaluation of these theoretical predictions is beyond the scope of the present work.

In passing, note that the completely different derivation of the stochastic reconnection model based on Richardson diffusion by Eyink *et al.* [22] is even more indicative of the indeterministic behavior of the magnetic field. In fact, Eyink's [6] general reconnection elaborately connects stochastic reconnection to field-fluid slippage in a precise mathematical treatment, which was very briefly touched on Sec. II B. Generalization of stochastic reconnection to viscous media by Jafari *et al.* [23] too relies on the stochasticity and lack of preserved identity over time for magnetic field lines even in the dissipative range.

Topological time series and regression

One way to study the magnetic field evolution in complex systems, e.g., in astrophysical objects, and in particular magnetic reconnection is to use time series analysis. The values of the stochasticity level $S_p(t)$ measured at different times can be considered as a discrete set of random variables instead of taking $S_p(t)$ as a continuous function of time. This approach naturally leads to a time series analysis to be discussed in the next section. Consider the stochasticity level as a stochastic process

$$\{S_t = S_p(t), t \in T\}, \tag{68}$$

where T is an arbitrary set called an index. If observations are made at times $t = 1, 2, \ldots$, or generally if $T \subseteq Z$, the above set defines a time series. The mean and covariance can be defined in the conventional way

$$\mu_S(t) = E[S_t], \quad \gamma_S(t) = \text{Cov}(S_r, S_\tau),$$

where $E[S_t]$ is the expectation value and $\text{Cov}(S_r, S_\tau)$ is the covariance. Similarly autocovariance $\gamma_S(h)$ and autocorrelation $\rho_S(t)$ are defined as

$$\gamma_S(t) = \text{Cov}(S_{t+h}, S_t), \quad \rho_S(t) = \frac{\gamma_S(h)}{\gamma_S(0)}$$

for any h (called the lag).

Definition 8. The stochasticity (self-entanglement) of a given magnetic field is weakly stationary if the following conditions for all $t, r, \tau \in Z$ hold:

$$\text{Var}(S_t) < \infty,$$
$$\mu_S(t) = \mu = \text{const},$$
$$\gamma_S(r, \tau) = \gamma_S(r + t, \tau + t).$$

For nonstationary stochastic (entangled) magnetic fields, instead of time derivatives, we may use the backshift $\mathcal{B}$ and difference $\mathcal{L}$ operators

$$\mathcal{L}S_t = S_t - S_{t-1} = (1 - \mathcal{B})S_t, \tag{69}$$

with $\mathcal{B}S_t = S_{t-1}$, $\mathcal{B}^n S_t = S_{t-n}$, and $\mathcal{L}^n \equiv \mathcal{L}(\mathcal{L}^{n-1})$. Note that $T_t := \mathcal{L}S_t$ is the discrete analog of $T_p = \partial_t S_p$.

Trend and seasonality can also be defined in conventional ways. For example, we can decompose the stochasticity in terms of a long-run trend T_t, which is a slowly changing function; a seasonal component P_t, which is a periodic function; and a stationary time series R_t, which is called the residual component:

$$S_t = \mathscr{T}_t + \mathscr{P}_t + \mathscr{R}_t. \tag{70}$$

One main goal usually is to find the deterministic components T_t and P_t in such a way that the residual component can be approximated as a stationary time series.

These are all well known in time series analyses and there are standard methods to describe the emerging pattern, explain how the past values affect the future values, and forecast future values. For the magnetic fields in which we are interested here, the residual component is expected to arise because of turbulence, while the seasonality may be the result of a dynamo action (e.g., solar cycles). The abrupt changes in stochasticity may be related to magnetic reconnection (Sec. IV), which can be studied using (univariate) temporal event detection methodologies. Regression analyses can also be applied to the stochasticity level in order to gain deeper insight into the magnetic topology in turbulent media. For example, simple autoregressive models of order k [called AR(k) models] may be applied to the time series $\{S_t, t \in Z\}$ to estimate S_t at a given time as a linear function of its earlier values $S_t = \beta_0 + \beta_1 S_{t-1} + \cdots + \beta_n S_{n-k} + w_t$, with parameters $\beta_i, 0 \leqslant i \leqslant n - k$, and white noise w_t. Such a detailed approach in which incompressible, homogeneous MHD numerical simulations can be used to evaluate stochasticity level $S_2(t)$ as a time series to infer local field reversals (topology changes) and a global Sweet-Parker-type [37,38] reconnection event is deferred to future work.

V. CONCLUSION

The quantitative description of the topology and stochasticity of turbulent magnetic fields, presented in this paper, shows that field-fluid slippage, magnetic reconnection, stochasticity level, and magnetic topological deformation are all intimately related and should be studied using a renormalized (coarse-grained) version of MHD equations. Renormalization of magnetic and velocity fields in turbulent media resolves the issue of the singularity of these fields, which otherwise would make the notion of field lines ill-defined since the integral curves of Hölder-singular vector fields are not uniquely defined. As shown in this paper, the magnetic topology change is related to the perpendicular component of the induction equation, whereas the evolution of the magnitude of the field, which is related to magnetic energy, is governed by the parallel component of the induction equation. In the presence of turbulence and stochasticity, a renormalized version of the induction equation should be used. This in turn allows the topology and energy of a given magnetic field to be studied separately, introducing the notion of the scale-split energy density $\psi_{l,L} = \mathbf{B}_l \cdot \mathbf{B}_L/2$ for the vector field $\mathbf{B}(\mathbf{x}, t)$ renormalized at scales l and L. In general, replacing the notion of scale separation in turbulence, which is convenient but mathematically vague, with the robust concept of renormalization (coarse graining) yields a unified and systematic formulation of the evolution of turbulent vector fields. In a more general treatment, in magnetized fluids, this formalism should be applied to both magnetic and velocity fields.

We have shown in particular that magnetic reconnection can be thought of as a relaxation process which allows the magnetic field tangled by the turbulent flow to untangle itself by slipping through the fluid. Previous work has shown that this field-fluid slippage is related to magnetic reconnection. In our formulation, this tangling and untangling translate into the increase and decrease in stochasticity level which is used to quantify the topological deformations. In this respect, magnetic reconnection corresponds to a maximum in the stochasticity level $S_p(t)$ or equivalently a change in the sign of topological deformation $T_p(t)$. This formalism can be readily applied to the turbulent velocity field as well, connecting magnetic stochasticity and topology to their kinetic counterparts. We also emphasize in passing that the predicted relationship between stochasticity and cross energy should be interpreted statistically in terms of a global trend since small deviations and fluctuations will certainly plague the evolution of stochasticity and cross energy. We have also briefly touched on the idea that the stochasticity level and topological deformation introduced here can be studied in the context of time series and regression analyses. Such a consideration would be different from conventional statistical methods currently used to study turbulent magnetic fields since such an approach relies on the self-entanglement of a stochastic field relative to two different scales in a renormalized version. A detailed numerical treatment of the formalism presented here is postponed to future work.

Finally, we point out that our results are in favor of stochastic reconnection, proposed by Lazarian and Vishniac [21], whose connections with spontaneous stochasticity and Richardson diffusion were explored by Eyink *et al.* [5,22]. Also our formalism is closely related to the general reconnection theory of [6], which introduced the slip-velocity source term Σ, extensively used in the present paper.

APPENDIX

Here we briefly discuss the concept of spontaneous stochasticity [32] and stochastic flux freezing [4]. One way to understand the fact that even in the absence of any nonideal term in Ohm's law, flux freezing would not hold in turbulence is through considering particle advection. Following the notation of Eyink [4], we write

$$\frac{d}{dt}\tilde{\mathbf{x}}(t) = \tilde{\mathbf{u}}^{\nu}(\tilde{\mathbf{x}}(t), t) + \sqrt{2\kappa}\,\tilde{\boldsymbol{\eta}}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{A1}$$

where the advecting velocity is perturbed by a Gaussian white noise $\tilde{\boldsymbol{\eta}}(t)$. Here κ is a constant and velocity field is smooth at scales smaller than the viscous scale $l < l_\nu$ with viscosity ν. Note that the velocity realization $\mathbf{u}^\nu$ is nonrandom, which can be used to write the transition probability for a fluid element

$$G_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x}_f, t_f|\tilde{\mathbf{x}}_0, t_0) = \int_{\mathbf{x}(t_0)=\mathbf{x}_0} \mathcal{D}\mathbf{x}\,\delta^3(\mathbf{x}_f - \mathbf{x}(t_f)) \times \exp\left(-\frac{1}{4\kappa}\int_{t_0}^{t} d\tau|\dot{\mathbf{x}}(\tau) - \mathbf{u}^\nu(\mathbf{x}(\tau),\tau)|^2\right).$$

This path-integral formulation resembles the Feynman path integral, related to the Schrödinger equation, in quantum mechanics. Here and in other similar contexts, $D\mathbf{x}$ denotes integration over all paths $\mathbf{x}$. Similarly, the above formulation can be used to solve the advection-diffusion equation

$$\partial_t\theta + \mathbf{u}^\nu\cdot\nabla\theta = \kappa\nabla^2\theta, \tag{A2}$$

where κ denotes the molecular diffusivity now. The solution is obtained using the Feynman-Kac formula

$$\theta(\mathbf{x}, t) = \int d^3x_0\theta(\mathbf{x}_0, t_0)G_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0). \tag{A3}$$

In a more explicit form, for $t_0 < t$, we can write

$$\theta(\mathbf{x}, t) = \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a}\,\theta(\mathbf{a}_0, t_0) \times \exp\left(-\frac{1}{4\kappa}\int_{t_0}^{t} d\tau|\dot{\mathbf{a}}(\tau) - \mathbf{u}^\nu(\mathbf{a}(\tau),\tau)|^2\right), \tag{A4}$$

which corresponds to solving backward in time the stochastic equation

$$\frac{d}{dt}\tilde{\mathbf{a}}(t) = \tilde{\mathbf{u}}^{\nu}(\tilde{\mathbf{a}}(t), t) + \sqrt{2\kappa}\,\tilde{\boldsymbol{\eta}}(t) \quad \text{for } t < \tau < t_0. \tag{A5}$$

One might naively assume that as the molecular diffusivity tends to zero $\kappa \to 0$, the transition probability collapses to a δ function

$$G_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) \to \delta^3(\mathbf{x}_f - \mathbf{x}(t_f)), \tag{A6}$$

where $\mathbf{x}(t)$ solves $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$. However, this argument would break down if at the same time $\nu \to 0$ too and $\mathbf{u}^\nu \to \mathbf{u}$ for a nonsmooth and singular velocity field $\mathbf{u}$. This means that as the molecular diffusivity and viscosity simultaneously tend to zero $\kappa, \nu \to 0$, we get

$$G_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0) \to G_{\mathbf{u}}(\mathbf{x}_f, t_f|\mathbf{x}_0, t_0). \tag{A7}$$

This is a remarkable result: The Lagrangian trajectories can remain random in the limit $\kappa, \nu \to 0$. This phenomenon of spontaneous stochasticity [32] resembles spontaneous symmetry breaking in QFT and has been discussed in great detail in recent decades (for more details see [4] and references therein).

One can apply a mathematical treatment analogous to the one presented above to magnetic field evolution in a turbulent fluid. As we showed in Eq. (41), expanding $\nabla\times(\mathbf{u}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{B}\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B} + \mathbf{u}\nabla\cdot\mathbf{B}$, one can write the bare induction equation as $D\mathbf{B}/Dt = \mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{B}\nabla\cdot\mathbf{u} + \lambda\nabla^2\mathbf{B}$ with Lagrangian derivative $D/Dt \equiv \partial_t + \mathbf{u}\cdot\nabla$. Here we denote magnetic diffusivity by λ. The continuity equation $D\rho/Dt + \rho\nabla\cdot\mathbf{u} = 0$ then yields

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\right)\cdot\nabla\mathbf{u},$$

which is, as discussed in the main text [Eq. (41)], another way to represent the magnetic flux-freezing theorem presuming that MHD equations remain well behaved in the limit $\lambda \to 0$ and the integral curves of $\mathbf{B}/\rho$ are advected with the fluid. In general, there is no way to find a velocity field $\mathbf{u}^*$ such that the resistive induction equation can be written in the ideal form (20). However, one still can describe the motion as a stochastic advection. Here we briefly discuss this approach, which naturally leads to the notion of stochastic flux freezing (for a detailed treatment see [4] and references therein). The starting point is to note that the induction equation can be expanded in a form that resembles the diffusion equation (A2):

$$\frac{\partial\mathbf{B}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} - \mathbf{B}(\nabla\cdot\mathbf{u}) + \lambda\nabla^2\mathbf{B}. \tag{A8}$$

The path-integral formula given by Eq. (A4) can be applied to (A8). The solution of the induction equation (A8) is

$$\mathbf{B}(\mathbf{x},t) = \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a}\,\mathbf{B}[\mathbf{a}(t_0)]\cdot\mathcal{J}(\mathbf{a},t) \times \exp\left(-\frac{1}{4\lambda}\int_{t_0}^{t} d\tau|\dot{\mathbf{a}}(\tau)-\mathbf{u}^{\nu}(\mathbf{a}(\tau),\tau)|^2\right), \quad \text{(A9)}$$

which is a sum over histories $\mathbf{a}(t)$. Here $\mathbf{B}$ is taken as a row vector in three dimensions and $\mathcal{J}(\mathbf{a},t)$ is a 3×3 matrix which satisfies

$$\frac{d}{d\tau}\mathcal{J}(\mathbf{a},t) = \mathcal{J}(\mathbf{a},t)\nabla_x\mathbf{u}(\mathbf{a}(\tau),\tau) - \mathcal{J}(\mathbf{a},t)(\nabla_x\cdot\mathbf{u})(\mathbf{a}(\tau),\tau), \quad \text{(A10)}$$

with $\mathcal{J}(\mathbf{a},t_0)=\mathcal{I}$ (3×3 identity tensor). Similar to the scalar diffusion equation, with the condition $\mathbf{a}(t)=\mathbf{x}$ as the initial point, the path-integral trajectories correspond to the solution of the following stochastic equation integrated backward in time from $\tau = t$ to $\tau = t_0$:

$$\frac{d}{d\tau}\tilde{\mathbf{a}}(\tau) = \mathbf{u}(\tilde{\mathbf{a}}(\tau),\tau) + \sqrt{2\lambda}\tilde{\boldsymbol{\eta}}(\tau), \quad \tilde{\mathbf{a}}(t)=\mathbf{x}. \quad \text{(A11)}$$

Of course, Eq. (A11) can also be solved in the usual way, that is, forward in time from $\tau = t_0$ to $\tau = t$. To do so, exactly the same trajectories can be obtained if one considers only those particles with initial locations selected to arrive at $\mathbf{x}$ at time t for a given white noise $\tilde{\boldsymbol{\eta}}(t)$. Such a group of time histories $\tilde{\mathbf{x}}(\tau)$ solves the equation

$$\frac{d}{d\tau}\tilde{\mathbf{x}}(\mathbf{a},\tau) = \mathbf{u}(\tilde{\mathbf{x}}(\mathbf{a},\tau),\tau) + \sqrt{2\lambda}\tilde{\boldsymbol{\eta}}(\tau), \quad \tilde{\mathbf{x}}(\mathbf{a},t_0)=\mathbf{a}, \quad \text{(A12)}$$

with $\tau > t_0$. Here the inverse mapping $\tilde{\mathbf{a}}(\mathbf{x},\tau)$ to $\tilde{\mathbf{x}}(\mathbf{a},\tau)$ fixes the starting point by $\mathbf{a} = \tilde{\mathbf{a}}(\mathbf{x},t)$. Applying the operator ∇_a to (A12) shows that a solution for Eq. (A10) is given by

$$\tilde{\mathcal{J}}(\mathbf{a},t) = \frac{1}{\det[\nabla_a\tilde{\mathbf{x}}(\mathbf{a},t)]}\nabla_a\tilde{\mathbf{x}}(\mathbf{a},t). \quad \text{(A13)}$$

As a result, we can write the path integral given by Eq. (A9) in the following form:

$$\mathbf{B}(\mathbf{x},t) = \left\langle \frac{1}{\det[\nabla_a\tilde{\mathbf{x}}(\mathbf{a},t)]}\mathbf{B}_0(\mathbf{a})\cdot\nabla_a\tilde{\mathbf{x}}(\mathbf{a},t)\Big|_{\tilde{\mathbf{a}}(\mathbf{x},t)}\right\rangle. \quad \text{(A14)}$$

Here $\langle\cdot\rangle$ indicates the average over realizations of the random white-noise process $\tilde{\boldsymbol{\eta}}(t)$ used in Eq. (A12). We call the expression (A14) the Eyink-Lundquist formula, which is the stochastic generalization of the standard Lundquist formula obtained by Eyink [4]. The determinant can be interpreted as the ratio of initial and final mass densities:

$$\det[\nabla_a\tilde{\mathbf{x}}(\mathbf{a},t)] = \frac{\rho_0(\mathbf{a})}{\tilde{\rho}(\tilde{\mathbf{x}}(\mathbf{a},t),t)}. \quad \text{(A15)}$$

The important implication is that the vector field $\tilde{\mathbf{B}}/\tilde{\rho}$ is stochastically frozen in and is advected along stochastic Lagrangian trajectories, where $\tilde{\mathbf{B}}$ is given by the term inside the angle brackets in Eq. (A14), that is,

$$\tilde{\mathbf{B}}(\mathbf{x},t) = \frac{1}{\det[\nabla_a\tilde{\mathbf{x}}(\mathbf{a},t)]}\mathbf{B}_0(\mathbf{a})\cdot\nabla_a\tilde{\mathbf{x}}(\mathbf{a},t)\Big|_{\tilde{\mathbf{a}}(\mathbf{x},t)}. \quad \text{(A16)}$$

Magnetic flux freezing in turbulent fluids should be understood only in this stochastic form (for details and numerical evaluations see [4,5]). Note that Eq. (A14) is the stochastic version of Eq. (41).

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