# Reciprocal conditions in one-dimensional nonlinear wave systems

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We study wave reciprocity in one-dimensional asymmetric systems constructed by multiple nonlinear $\delta$-function scatters embedded within linear scatters. A general reciprocal condition is proposed, in terms of the rotation symmetry between forward and backward transfer matrices. We then derive various resonance conditions, under which all scatterers behave as merging into either a single nonlinear $\delta$ scatter, or a symmetric nonlinear barrier configuration. As such, the reciprocity appears periodically by changing widths of linear constant potentials between neighboring nonlinear $\delta$ scatters. Moreover, the wave reciprocity will not be violated if one replaces the linear constant potential between two $\delta$-nonlinear scatters with any other kind of transparent scatterers.

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## I. INTRODUCTION

The unidirectional wave propagation has long been a crucial topic in controlling energy and information transportation. Recently, different kinds of diodes were proposed in analogy to the familiar electric diodes, including acoustic diodes which can manipulate an acoustic wave with an application in biomedical ultrasound devices [1–3]; optical isolators that can remove undesired light in various optical devices [4–9]; thermal rectifiers [10–19] that permit thermal flux only in one direction with many potential applications such as thermal computers [20–22]; and spin Seebeck diodes which can even manipulate pure spin current [23,24]. Reciprocity is violated in these devices.

Reciprocity, in the Lorentz sense, means the invariance of signals under exchange of source and detector [25,26] and is usually expressed as the symmetry of the Green's function [27] or scattering matrix [25,28]. The reciprocity theorem says that the linear lossless system with time-reversal symmetry is reciprocal [25,28]. Therefore, to realize nonreciprocity in the linear system, time-reversal symmetry needs to be broken [29–31]. For example, magnetic field is applied to magneto-optical materials to realize Faraday rotation and hence the asymmetric propagation of light [29]. An analogy for nonreciprocal acoustic waves in magneto-acoustic materials is also predicted [32]. To achieve nonreciprocity avoiding the physical magnetic field that usually makes devices bulky, one can alternatively introduce materials with absorption [8,9] or time modulation [31] to synthesize an effective magnetic field break the time-reversal symmetry. However, under such conditions although Lorentz reciprocity is broken with the transport coefficient $S_{ij}(B) \neq S_{ji}(B)$, the Onsager reciprocity is still preserved $S_{ij}(B) = S_{ji}(-B)$ under time-reversal of the effective magnetic field $-B$ [33,34] or artificial gauge field. This reciprocity in the sense of time-reversal (including reversing gauge fields) holds in the linear response region.

Nonlinearity is an alternative way to realize nonreciprocity without breaking time-reversal symmetry. For example, acoustic diodes constructed by combining superlattice and nonlinear materials [2], by inserting a point defect near the boundary of the one-dimensional nonlinear lattice [3]; thermal diodes by coupling two distinct nonlinear lattices [18,35], by introducing nonlinear heat radiation in asymmetric holey composites [36] and the static nonreciprocal mechanical system using nonlinear metamaterials [37]. The practicability of nonlinear rectifier devices have also been verified by experiments [2,38,39]. Therefore, introducing nonlinearity in a spatial asymmetric system is an effective way to realize nonreciprocity.

However, nonlinearity, combined with spatial asymmetry, does not necessarily guarantee nonreciprocity. We considered a nonlinear and asymmetric model in the presence of two nonlinear elements [40] embedded within a linear potential. A geometrical resonance condition that can break nonreciprocal propagation was shown in that model. Here, motivated by this previous work [40], we generalize the same notions of symmetric wave propagation but in an asymmetric system with multiple nonlinear elements instead of two. The model is described by the nonlinear Schrödinger equation, which is the governing equation of electron tunneling with many-electron interaction [41,42], or matter wave, Bose-Einstein condensate (BEC), nonlinear optics and is frequently discussed in the literature about asymmetric wave propagation [43–45] as well as solitons [46–49]. Without loss of generality, we investigate the general reciprocity condition based on this model and several special reciprocal cases are provided. The results are also valid for the general wave dynamics described by Helmholtz equation.

The paper is organized as follows. In Sec. II, we review the definition of reciprocity and extend the concept of the scattering matrix for a one-dimensional nonlinear system. A general reciprocity condition in terms of transfer matrix is derived in Sec. III. In Sec. IV, we describe the model with multiple nonlinear $\delta$-function potentials and derive the formal solution of the corresponding wave equation. In Sec. V, we

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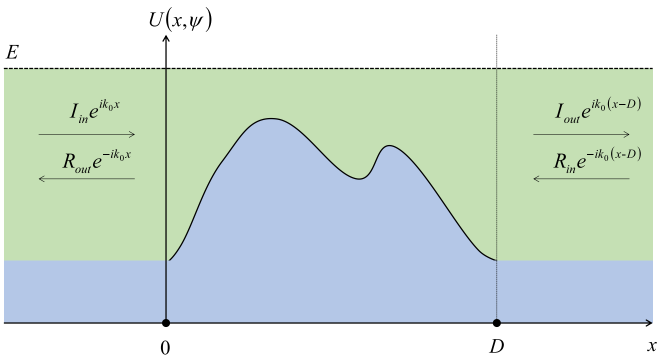


FIG. 1. Diagram of the general model we consider: propagation of wave with single frequency in presence of a scatterer of real nonlinear potential, located at the range from $x = 0$ to $x = D$. $I_{\rm in}$ and $R_{\rm in}$ denotes the amplitude of the incident plane wave from left and right sides of the scatter, respectively. $I_{\rm out}$ and $R_{\rm out}$ denotes the amplitude of the transmitted plane wave to the right and left sides of the scatter, respectively. We consider scattering states with $E > V$.

give four examples that the nonlinear system can also be reciprocal. In Sec. V A, a special reciprocal case named as the global resonance condition is derived from the general reciprocal condition. Under the global resonance condition, two neighboring nonlinear $\delta$-function potentials merge and behave as a single one. Thus the resonance condition in [40] can be generalized to satisfy models with any number of nonlinear $\delta$-function potential. In Secs. V B and V C, we construct two special reciprocal systems that do not satisfy the global resonance condition. Then we argue that they are actually equivalent to systems satisfying the global resonance condition and spatial symmetry, respectively. In Sec. V D, we propose a sandwich-structure linear model and find that it is also transparent. Based on this, we construct another kind of reciprocal model in analogy to the one in Sec. V A. Finally, we briefly discuss the stability of the global resonance condition in Sec. VI.

## II. REVISIT OF RECIPROCAL CONDITION

We consider the one-dimensional system, described by the time-independent nonlinear Schrödinger's equation

$$-\frac{d^2\psi}{dx^2} + U(x, \psi)\psi(x) = E\psi(x), \tag{1}$$

as illustrated in Fig. 1. Note that the governing equation of general waves, such as the mechanical wave and electromagnetic wave, can also be written in a similar form, with replacing $E$ with its quadratic counterpart. The scatter potential in the position $\in [0, D]$ can depend on the wave magnitude as a consequence of the response to nonlinearity. Since we only consider real potentials, the current densities of the system are conserved. The potential outside of the scatterer regime is assumed to be constant.

For the one-dimensional system, reciprocity is defined as the invariance of the transition amplitude while reversing the wave propagating direction. This is usually represented by the symmetry of the scattering matrix for the linear system, which can also be extended to a nonlinear system with some modifications shown below.

Considering the input $\mathbf{a} = [I_{\rm in}, R_{\rm in}]^T$ and output $\mathbf{b} = [R_{\rm out}, I_{\rm out}]^T$, they are related via: $\mathbf{b} = \mathbf{Sa}$. Here $\mathbf{S}$ is a function of $\mathbf{a}$ and $\mathbf{b}$ in analogy to the scattering matrix for the linear system. Note that $\mathbf{S}$ cannot be determined totally by an incident wave due to the existence of multistability [50,51]. In addition, it is difficult to calculate with $\mathbf{S}$ since it operates on $\mathbf{a}$ but its value depends on both $\mathbf{a}$ and $\mathbf{b}$. So we use the transfer matrix $\mathbf{M}$ instead: $\mathcal{L} = \mathbf{M}\mathcal{R}$ where $\mathcal{L} = [I_{\rm in}, R_{\rm out}]^T$ and $\mathcal{R} = [I_{\rm out}, R_{\rm in}]^T$. $\mathbf{S}$ can be represented by $\mathbf{M}$:

$$\mathbf{S} = \frac{1}{\mathbf{M}_{11}}\begin{bmatrix} \mathbf{M}_{21} & 1 \\ 1 & -\mathbf{M}_{12} \end{bmatrix}. \tag{2}$$

Here we applied $\text{Det}(\mathbf{M}) = 1$ as a consequence of time-reversal symmetry and probability current conservation [52]. For the same reason, $\mathbf{S}$ is unitary and symmetric [25].

For the forward incident wave from the left, we have $I_{\rm out} = T$ and $R_{\rm in} = 0$ where $T$ is an arbitrary number. Thus the wave at $x > D$ is $\psi(x) = I_{\rm out}e^{ik_0(x-D)}$ and the scattering matrix denoted as $\mathbf{S}_f$ can be determined according to the Picard uniqueness theorem. Similarly, for a wave propagating backward from the right, we have $I_{\rm in} = 0$ and $R_{\rm out} = T$. The wave at $x < 0$ is $\psi(x) = R_{\rm out}e^{-ik_0x}$ and the scattering matrix is denoted as $\mathbf{S}_b$. Then, the definition of the reciprocity reads

$$\mathbf{S}_{21,f} = \mathbf{S}_{12,b}, \tag{3}$$

where the subscript $f, b$ denotes forward and backward, respectively.

## III. ROTATION SYMMETRY BETWEEN FORWARD AND BACKWARD

Combining the definition of reciprocity $\mathbf{S}_{21,f} = \mathbf{S}_{12,b}$ and the symmetric scattering matrix $\mathbf{S}_{12} = \mathbf{S}_{21}$, we can conclude that for the reciprocal system

$$\mathbf{S}_{21,f} = \mathbf{S}_{21,b} = \mathbf{S}_{12,f} = \mathbf{S}_{12,b}. \tag{4}$$

As $\mathbf{S}$ is unitary, we can derive three independent equations about four entries of $S$:

$$|\mathbf{S}_{11}|^2 + |\mathbf{S}_{21}|^2 = 1, \tag{5a}$$

$$|\mathbf{S}_{12}|^2 + |\mathbf{S}_{22}|^2 = 1, \tag{5b}$$

$$\mathbf{S}_{11}^*\mathbf{S}_{12} + \mathbf{S}_{21}^*\mathbf{S}_{22} = 0. \tag{5c}$$

Comparing Eq. (4) to Eqs. (5a) and (5b), we have $\mathbf{S}_{11,f} = \mathbf{S}_{11,b}e^{i\theta_1}$ and $\mathbf{S}_{22,f} = \mathbf{S}_{22,b}e^{i\theta_2}$ where $\theta_1$ and $\theta_2$ are real numbers. Substituting these into Eq. (5c), we get $\theta_1 = -\theta_2$.

Comparing these arguments with Eq. (3), we can conclude that for the reciprocal system, the transfer matrix for a wave propagating forward and backward are related via a unitary transform

$$\mathbf{M}_f = \Lambda^\dagger\mathbf{M}_b\Lambda, \tag{6}$$

where $\Lambda = \text{diag}[e^{-i\theta}, e^{i\theta}]$ and $\theta$ is an arbitrary real number.

Usually, for a system with nonlinearity or potential varying with position continuously, wave amplitude is not well-defined inside the scatterer. So we introduce a new variable

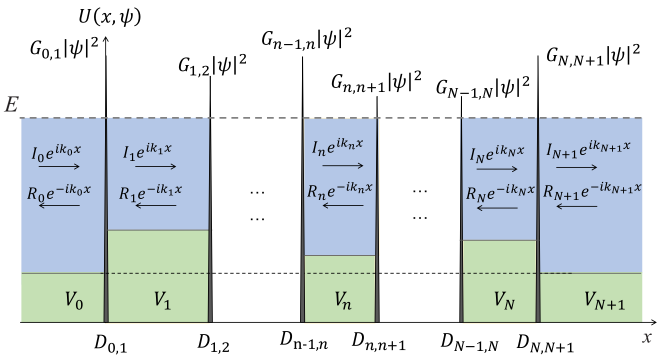


FIG. 2. Diagram of the model with multiple nonlinear $\delta$ potentials. A plane wave of amplitude $I_0$ strikes a nonlinear $\delta$-function potential, giving rise to a reflected wave of amplitude $R_i$ and transmitted wave of amplitude $I_i$ in each region with linear potential $V_i$. We consider scattering states with $E > V$. The backward wave propagation can be similarly described.

$\chi(x) = [\psi(x), \ \ \frac{1}{k(x)}\psi'(x)]^T$ to describe the wave propagation, with factor $k(x) = \sqrt{E - U(x)}$ ensuring that the units of the two elements are the same. At the left end of the scatterer, we have $\chi(0-\epsilon) = \Gamma\mathcal{L}$ where $\Gamma = [1, 1; i, -i]$ and $\epsilon \to 0$. At the right end, $\chi(D+\epsilon) = \Gamma\mathcal{R}$. So, $\chi(x)$ at the two ends of the scatterer can be related $\chi(0-\epsilon) = \Xi\chi(D+\epsilon)$ to the new transfer matrix $\Xi$:

$$\Xi = \tfrac{1}{2}\Gamma\mathbf{M}\Gamma^\dagger. \tag{7}$$

Substituting Eq. (7) into Eq. (6), the reciprocal condition can be derived in terms of $\Xi$:

$$\Xi_f = \Upsilon^\dagger\Xi_b\Upsilon, \tag{8}$$

where $\Upsilon$ is a rotation matrix

$$\Upsilon = \frac{1}{2}\Gamma\Lambda\Gamma^\dagger = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}. \tag{9}$$

Therefore, Eq. (8) can be concluded as the general reciprocal condition for the transfer matrix $\Xi$ of a reciprocal system for a wave propagating forward and backward which are orthogonally similar to each other via a rotation matrix. In the following sections, we will verify various kinds of nonlinear yet reciprocal models based on Eq. (8).

## IV. MODELS AND RESULTS

In the rest of this paper, we examine the reciprocity of the system with $N+1$ nonlinear layers, described by nonlinear Dirac $\delta$-function potentials embedded in a linear system as illustrated by Fig. 2. Similar derivations can be generalized to the discretized coupled-site system or multilayer system.

The potential of the system consists of the nonlinear and linear parts:

$$U(x,\psi) = \sum_{n=0}^{N} G_{n,n+1}\delta(x - D_{n,n+1})|\psi(x)|^2 + V(x), \tag{10}$$

where real numbers $G_{n,n+1}$ denote the nonlinear strength of the $\delta$-function potential located at $x = D_{n,n+1}$. Index $n = 0, 1, \ldots, N$ here, as well as in the following sections. In additions, the linear part is

$$V(x) = \begin{cases} V_0, & x < D_{0,1},\\ V_j, & D_{j-1,j} < x < D_{j,j+1},\\ V_{N+1}, & x > D_{N,N+1},\end{cases} \tag{11}$$

with $j = 1, \ldots, N$ and $V_{N+1} = V_0$. Inside each linear region, the equation is

$$\frac{d^2\psi}{dx^2} + k(x)^2\psi = 0, \tag{12}$$

where wave vector $k(x) = \sqrt{E - V(x)}$ or $k_j = \sqrt{E - V_j}$, $j = 1, 2, \ldots, N$. Note this equation is reminiscent of the Helmholtz equation for general classical waves.

As in Sec. II A, we use the two-dimensional vector $\chi(x) = [\psi(x) \ \ \frac{1}{k(x)}\psi'(x)]^T$ to represent wave function and its derivation. At the endpoints of each linear region $\chi_{n,r} = \chi|_{x=D_{n,n+1}-\epsilon}$ and $\chi_{n+1,l} = \chi|_{x=D_{n,n+1}+\epsilon}$ where indexes "$l$" and "$r$" denote the left endpoints and right endpoints, respectively. Thus Eq. (12) can be rewritten as $\chi' = ik_j\sigma_y\chi$, where $\sigma_y$ is the Pauli matrix.

So, inside each linear region, we have

$$\chi_{j,l} = \Phi_j\chi_{j,r}, \tag{13}$$

where $\Phi_j$ is

$$\Phi_j = e^{-i\phi_j\sigma_y} = \begin{bmatrix}\cos\phi_j & -\sin\phi_j\\ \sin\phi_j & \cos\phi_j\end{bmatrix}, \tag{14}$$

where $\phi_j = k_j(D_{j,j+1} - D_{j-1,j})$, $j = 1, 2, \ldots, N$. $\Phi_j$ is recognized as the rotation matrix in two-dimensional Euclidean space $\mathbb{R}^2$. So the effect of back-propagating in the $j$th linear potential, without considering the boundary effect, is to rotate both the real and imaginary parts of vector $\chi$ an angle, $\phi_j$. The boundary conditions at $x = D_{n,n+1}$ are

$$\psi|_{D_{n,n+1}-\epsilon} = \psi|_{D_{n,n+1}+\epsilon}, \tag{15}$$

$$\left.\frac{d\psi}{dx}\right|_{D_{n,n+1}-\epsilon}^{D_{n,n+1}+\epsilon} = G_n|\psi(D_{n,n+1})|^2\psi(D_{n,n+1}), \tag{16}$$

where $\epsilon \to 0$.

Rewriting Eqs. (15) and (16) in terms of $\chi_{n,r}$ and $\chi_{n+1,l}$:

$$\chi_{n,r} = \Omega_{n,n+1}\chi_{n+1,l} = \begin{bmatrix}1 & 0\\ -Q_{n,n+1} & P_{n,n+1}\end{bmatrix}\chi_{n+1,l}, \tag{17}$$

where $P_{n,n+1} = \frac{k_{n+1}}{k_n}$ and $Q_{n,n+1} = \frac{G_{n,n+1}}{k_n}|\psi(D_{n,n+1})|^2$ comes from the scattering of the boundary and $\delta$ potential, respectively. By combining Eqs. (13) and (17), we get

$$\chi_{0,r} = \Xi\chi_{N+1,l}, \tag{18}$$

where $\Xi$ is the transfer matrix

$$\Xi = \Omega_{01}\prod_{j=1}^{N}\Phi_j\Omega_{j,j+1}. \tag{19}$$

$\Xi$ describes the relationship of vector $\chi$ at the two endpoints of the scatterer. Each $\Phi_j$ represents the effect of the $j$th linear constant potential alone, while $\Omega_{j,j+1}$ is the scattering of the boundary and $\delta$ potential between the $j$th and $j+1$th linear potential together with the nonlinear $\delta$-function potential located there. Thus we derived a formal solution of Eq. (1) with a potential in the form of Eq. (10), although the nonlinear operator $\Xi$ is determined by the unknown wave function in general.

## V. RESONANCE CONDITIONS FOR RECIPROCITY

Previously, we studied a system consisting of two nonlinear $\delta$-function potentials and a linear interface [40]. Results showed that the system is reciprocal if the linear interface satisfies the resonance condition $\phi = n\pi$. In this section, we will construct more nonlinear models and verify the reciprocity. In Sec. V A, as a generalization of the previous two-nonlinear-spot system [40], we derive a global resonance condition for an arbitrary number of nonlinear spots from the general reciprocal condition Eq. (8), which says that the system is reciprocal if $\phi_j = m_j\pi$ for all linear interfaces $V_j$. We also show that the reciprocity arises from that the two nonlinear spots $G_{j-1,j}$ and $G_{j,j+1}$ at the two ends of linear interface $V_j$ and can be viewed as a single one if $\phi_j = m_j\pi$. Based on this idea, we construct two models that can be viewed as satisfying the global resonance condition and spatial symmetry in Secs. V B and V C, respectively. In Sec. V D, we show that reciprocity will not be violated if the linear constant interface satisfying the resonance condition is replaced by another kind of transparent media.

### A. Global resonance condition

First, we consider the case that $\theta = 0$ in Eq. (8) and reciprocity requires $\Xi_f = \Xi_b$. Comparing this with Eq. (19), the expression of $\Xi$ for our model, we find a sufficient reciprocal condition $\Omega_{f,n,n+1} = \Omega_{b,n,n+1}$, which yields

$$|\psi_f(D_{n,n+1})|^2 = |\psi_b(D_{n,n+1})|^2. \quad (20)$$

As $|\psi_f(D_{N,N+1})| = |\psi_b(D_{0,1})| = |T|$, to realize Eq. (20), we can set

$$|\psi(D_{j-1,j})| = |\psi(D_{j,j+1})|, \quad (21)$$

for $j = 1, 2, \ldots, N$, so that $|\psi_f(D_{n,n+1})| = |\psi_b(D_{n,n+1})| = |T|$. According to Eq. (13):

$$|\psi(D_{j-1,j})| = \left|\psi(D_{j,j+1})\cos\phi_j - \psi'(D_{j,j+1}-\epsilon)\frac{\sin\phi_j}{k_j}\right|. \quad (22)$$

So, to ensure that Eq. (20) holds true, we only need to set $\sin\phi_j = 0$ and $\cos\phi_j = \pm 1$, from which we can obtain the following reciprocal condition:

$$\phi_j = m_j\pi, \quad (23)$$

where $m_j$ are integers. We name Eq. (23) as the global resonance condition since it must be fulfilled for all $j$ from 1 to $N$. Under this condition, $\Phi_j = (-1)^{m_j}\mathbf{1}$. Therefore, the transfer matrix of the subscatter formed by the $j$th interface $V_j$ and nonlinear spots at its two ends is

$$\Omega_{j-1,j}\Phi_j\Omega_{j,j+1} = (-1)^{m_j}\begin{bmatrix} 1 & 0 \\ -Q_{j-1,j+1} & P_{j-1,j+1} \end{bmatrix}, \quad (24)$$

where $P_{j-1,j+1} = \frac{k_{j+1}}{k_{j-1}}$ and $Q_{j-1,j+1} = \frac{G_{j-1,j}+G_{j,j+1}}{k_{j-1}}|\psi(D_{j,j+1})|^2$.

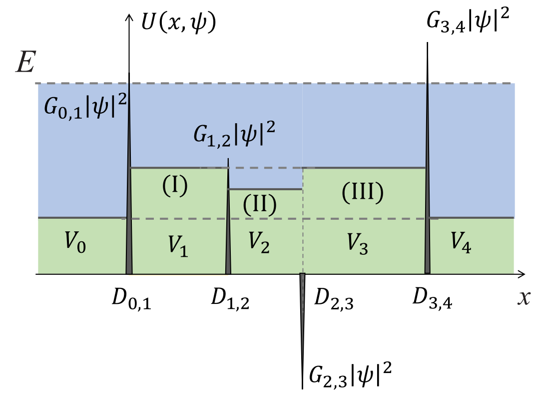


FIG. 3. Diagram of the model considered in Sec.V B. An "inverted" nonlinear $\delta$ potential is inserted in the multiple-nonlinear-layer model. The parameters of this model are $V_0 = V_4$, $V_1 = V_3$, $G_{1,2} + G_{2,3} = 0$, $\phi_2 = m_2\pi$, $\phi_1 + \phi_3 = m_3\pi$, where $m_2$ and $m_3$ are integers.

Comparing Eqs. (16) and (23), we find that the transfer matrix of the subscatter is the same as the transfer matrix of the single nonlinear $\delta$ potential with nonlinear coefficient $G_{j-1,j} + G_{j,j+1}$. This indicate that the neighboring nonlinear $\delta$ potential can be viewed as a single one. Moreover, substituting $\Phi_j$ into Eq. (18), we get the transfer matrix of the whole scatter

$$\Xi = (-1)^m \prod_{n=0}^{N} \Omega_{n,n+1} = (-1)^m \begin{bmatrix} 1 & 0 \\ -Q & 1 \end{bmatrix}, \quad (25)$$

where $Q = G|\psi(D_{0,1})|^2/k_0$, $G = \sum_{j=0}^{N} G_{j,j+1}$, $m = \sum_{j=1}^{N} m_j$.

Equation (25) also has the same form as Eqs. (17) and (24). Thus, all nonlinear spots can merge into a single one. It is also known that a single nonlinear spot is not sufficient for nonreciprocity [40]. Specifically, if $G = 0$, all nonlinear potentials are eliminated and the transfer matrix is the identity, which means that the scatterer is then transparent. This is a generalization of Fabry-perot resonance for a linear system [40].

Note that global resonance condition Eq. (23) is a special case of reciprocal condition Eq. (8) since it is derived by first specifying the invariance of wave magnitude at each nonlinear spot in Eq. (20), then further specifying the equality of wave magnitude at every nonlinear spot Eq. (21). So there may exist other cases where the global resonance condition is not fulfilled whereas the system is still reciprocal.

### B. Equivalent systems satisfying global resonance condition

In this section, to investigate other reciprocal conditions, we intentionally design a special reciprocal system that does not satisfy the global resonance condition.

The geometry of a this setup is shown in Fig. 3. The highlight is that $G_{1,2} + G_{2,3} = 0$, $\phi_2 = m_2\pi$, $\phi_1 + \phi_3 = m_3\pi$. As $\phi_2 = m_2\pi$ and $G_{1,2} + G_{2,3} = 0$, two nonlinear potentials $G_{1,2}$, $G_{2,3}$ merge and eliminate so that linear regions (I) and (III) become adjacent. Meanwhile, $\phi_1 + \phi_3 = m_3\pi$. Thus the system is equivalent to one that consists of two nonlinear spots and a linear interface satisfying the global resonance

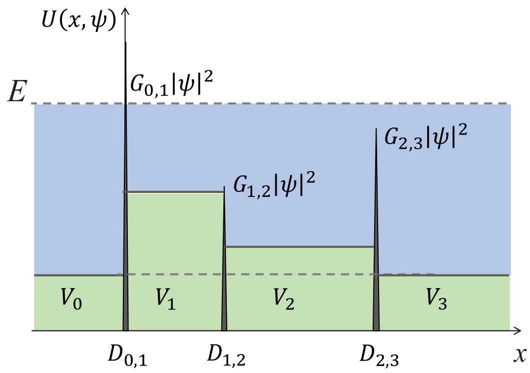


FIG. 4. Diagram of the model considered in Sec. V C. The parameters of this model are $V_0 = V_3$, $G_{1,2} + G_{2,3} = G_{0,1}$, and $\phi_2 = m_2\pi$.

condition. Therefore, the system is reciprocal for very good reasons, even though the global resonance condition Eq. (23) is not fulfilled.

Verification of this reciprocal model is given by exploiting the general condition of reciprocity we just proposed in Eq. (8). As $\phi_2 = m_2\pi$, according to Eq. (14), $\Phi_2 = (-1)^{m_2}\mathbf{1}$ and $\chi_{2,l} = (-1)^{m_2}\chi_{2,r}$. So, the wave magnitude at two ends of the linear potential $V_2$ are the same :$|\psi(D_{1,2})| = |\psi(D_{2,3})|$. The transfer matrix of the subscatterer constructed by linear potential $V_2$ and $\delta$-function potential $G_{1,2}$ and $G_{2,3}$ is

$$\Omega_{1,2}\Phi_2\Omega_{2,3} = (-1)^{m_2}\begin{bmatrix} 1 & 0 \\ -\frac{G_{1,2}+G_{2,3}}{k_1}|\psi(D_{1,2})|^2 & 1 \end{bmatrix} = (-1)^{m_2}\mathbf{1}. \tag{26}$$

Thus, two nonlinear potentials $G_{1,2}$ and $G_{2,3}$ are eliminated, as expected. Namely, the presence of the subscatterer will not affect the wave propagation. According to Eqs. (13), (17), and (26), $\chi_{1,l} = (-1)^{m_2}\Phi_1\Phi_3\chi_{3,r}$. As $\phi_1 + \phi_3 = m_3\pi$,

$$\Phi_1\Phi_3 = e^{-i(\phi_1+\phi_3)\sigma_y} = (-1)^{m_3}\mathbf{1}. \tag{27}$$

Therefore, $\chi_{1,l} = (-1)^{m_2+m_3}\chi_{3,r}$. Thus wave magnitude at two endpoints of the scatterer are the same both for the waves propagating forward and backward: $|\psi(D_{0,1})| = |\psi(D_{3,4})| = |T|$. Substituting Eqs. (26) and (27) into Eq. (19), the transfer matrix $\Xi$ is obtained:

$$\Xi_f = \Xi_b = (-1)^{m_2+m_3}\Omega_{0,1}\Omega_{3,4} = (-1)^{m_2+m_3}\begin{bmatrix} 1 & 0 \\ -\frac{G_{0,1}+G_{3,4}}{k_0}|T|^2 & 1 \end{bmatrix}. \tag{28}$$

So, the reciprocal condition Eq. (8) is fulfilled with $\theta = 0$. Note that this model is different from the global resonance condition. Unlike in Sec. III A, the wave magnitude at $x = D_{1,2}$ and $x = D_{2,3}$ are not invariant in general as $G_{0,1}$ and $G_{3,4}$ take arbitrary values. Namely, Eq. (20) is not fulfilled. But this does not violate reciprocity since the scattering at the two nonlinear spots are eliminated in Eq. (26).

### C. Equivalent to symmetric systems

Now we consider another special reciprocal system as shown in Fig. 4. This model does not fulfill the global resonance condition Eq. (23) either, but is still reciprocal, similar to the previous one.

Nonlinear spots at $x = D_{1,2}$ and $x = D_{2,3}$ merge into a single spot with nonlinear strength equal to $G_{0,1}$ and the system then is equivalent to a symmetric one. It is well known that a symmetric nonlinear model is not sufficient to give rise to nonreciprocity. Thus, this setup is also reciprocal, reasonably.

First, we consider the subscatterer constructed by linear potential $V_2$ and nonlinear $\delta$-function potential $G_{1,2}$ and $G_{2,3}$ as in Sec. V B. Since $\phi_2 = m_2\pi$, $\Phi_2 = (-1)^n\mathbf{1}$ and $|\psi(D_{1,2})| = |\psi(D_{2,3})|$. So, according to Eqs. (13) and (17), the transfer matrix of the subscatterer is

$$\Omega_{1,2}\Phi_2\Omega_{2,3} = (-1)^{m_2}\begin{bmatrix} 1 & 0 \\ -\frac{G_{1,2}+G_{2,3}}{k_1}|\psi(D_{1,2})|^2 & \frac{k_3}{k_1} \end{bmatrix}. \tag{29}$$

Comparing Eqs. (29) and (17), we find that the transfer matrix of the subscatterer is the same as the single $\delta$-function potential with nonlinear strength $G_{1,2} + G_{2,3} = G_{0,1}$. So, the transfer matrix of the model is the same (other than the sign) as another one, which is constructed by two nonlinear identical spots $G_{0,1}$ and a linear interface $V_1$, a surely reciprocal model. Note that we do not need any information about the linear constant interface $V_1$ to get Eq. (29). Therefore, the linear interface can be replaced by another kind of symmetric potential and reciprocity will still not be violated. For example, the interface can be a sandwich structure and the potential can change continuously with position or consists of nonlinearity itself.

Next, we verify that the general reciprocal condition Eq. (8) is fulfilled. For a wave propagating forward, $\chi_{f,3,l} = [T, iT]^T$. substituting Eq. (29) into Eq. (19), the transfer matrix of the whole scatter is

$$\begin{aligned}\Xi_f &= \Omega_{f,0,1}\Phi_1(\Omega_{f,1,2}\Phi_2\Omega_{f,2,3}) \\ &= (-1)^{m_2}\frac{\cos\phi_1}{k_0}\begin{bmatrix} k_0 & 0 \\ -(\alpha+\beta) & k_0 \end{bmatrix} \\ &\quad + (-1)^{m_2}\frac{\sin\phi_1}{k_1}\begin{bmatrix} \alpha & -k_0 \\ \frac{1}{k_0}(k_1^2 - \alpha\beta) & \beta \end{bmatrix},\end{aligned} \tag{30}$$

where $\alpha = G_{0,1}|\psi_f(D_{1,2})|^2$ and $\beta = G_{0,1}|\psi_f(D_{0,1})|^2$. For a wave propagating backward $\chi_{b,0,r} = [T, -iT]^T$, so $|\psi_b(D_{0,1})| = |\psi_f(D_{1,2})| = |T|$. It can be also verified that $|\psi_b(D_{1,2})| = |\psi_f(D_{0,1})|$, by substituting $\chi_{f,3,l}$ and $\chi_{b,0,r}$ into $\chi_{f,1,l} = \Phi_1(\Omega_{b,1,2}\Phi_2\Omega_{b,2,3})\chi_{f,3,l}$ and $\chi_{b,1,r} = \Phi_1^{-1}\Omega_{b,0,1}^{-1}\chi_{b,0,r}$, respectively. So, the transfer matrix $\Xi_b$ can be obtained just by exchanging the position of $\alpha$ and $\beta$ in Eq. (30):

$$\begin{aligned}\Xi_b &= (-1)^{m_2}\frac{\cos\phi_1}{k_0}\begin{bmatrix} k_0 & 0 \\ -(\alpha+\beta) & k_0 \end{bmatrix} \\ &\quad + (-1)^{m_2}\frac{\sin\phi_1}{k_1}\begin{bmatrix} \beta & -k_0 \\ \frac{1}{k_0}(k_1^2 - \alpha\beta) & \alpha \end{bmatrix}.\end{aligned} \tag{31}$$

Comparing Eqs. (30) and (31), reciprocal condition Eq. (8) is fulfilled with

$$\theta = arctan\left(\frac{k_0(\alpha-\beta)\tan\phi_1}{\left(k_0^2 - k_1^2 + \alpha\beta\right)\tan\phi_1 + k_1(\alpha+\beta)}\right). \tag{32}$$

Recall that $\theta$ is the angle of the rotation matrix in Eq. (9).

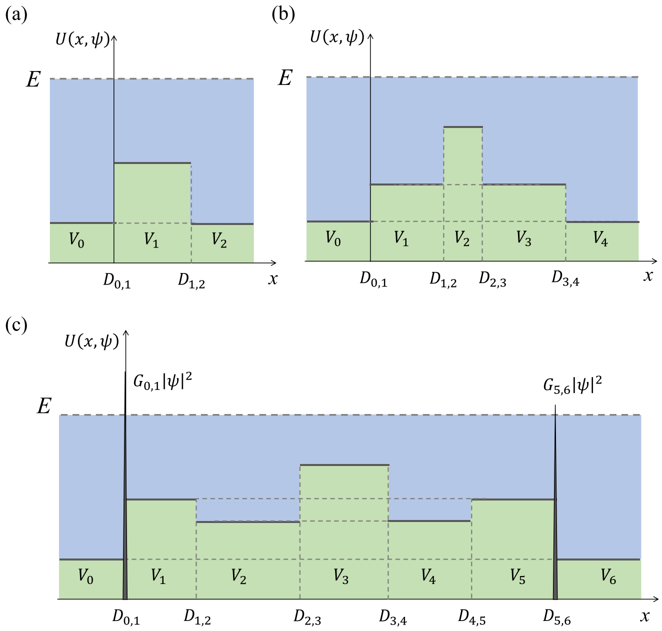


FIG. 5. (a) Diagram of the constant "transparent" scatterer ($V_1$) with $V_0 = V_2$ and $\phi_1 = \pi$. (b) Diagram of the nonconstant "transparent" scatterer ($V_1, V_2, V_3$) with $V_0 = V_4$, $V_1 = V_3$, $k_2/k_1 = k_1/k_0$ and $\phi_1 = \phi_2 = \phi_3 = \pi/2$. (c) Diagram of the model with nonconstant subscatterer sandwiched by two nonlinear $\delta$ functions.

### D. Other reciprocal conditions

In Secs. V B and V C, we constructed two kinds of reciprocal model. Their essences are the same: the transfer matrix of an interface between two neighboring nonlinear spots is a unit matrix, thus the two nonlinear spots can be viewed as a single one. However, to realize this, the linear interface need not be a constant-potential and can be replaced by another one that has a unit transfer matrix. Such an interface can be constructed by a transparent scatter, whose transfer matrix is a rotation $\phi$, and a constant-potential which produces a rotation $n\pi - \phi$.

A well-known example of transparent media is the Fabry-Perot resonance, which we used in the above sections. The transparency of it can be explained by the interference of a wave. As shown in Fig. 5(a), supposing that $\phi_1 = \pi$, $V_1 > V_0 = V_2$, the wave incident comes from the left and it splits into a reflected wave and transmitted wave at $x = 0$. The transmitted wave propagates forward along media $V_1$ and picks a phase $\pi$. Then it is reflected at $x = D_{1,2}$ and picks a phase $\pi$ because of half-wave loss. Finally, the wave reflected at $x = D_{1,2}$ propagates backward and picks a phase $\pi$. Meanwhile, the wave reflected at $x = D_{0,1}$ picks no phase. So, the wave reflected by the boundary at $x = 0$ and $x = D_{1,2}$ have phase difference $3\pi$ and are eliminated with each other, which is why the media are transparent.

In analogy, we construct the model illustrated in Fig. 5(b). $V_0 = V_4, V_1 = V_3, k_1/k_0 = k_2/k_1, \phi_1 = \phi_2 = \phi_3 = \pi/2$. We expect that the wave reflected at $x = 0$ and $x = D_{2,3}$ are eliminated and the wave reflected at $x = D_{1,2}$ and $x = D_{3,4}$ are also eliminated. Different from the common Fabry-Perot resonance, such transparent media are expected to generate a rotation by angle $3\pi/2$ instead of $\pi$ since $\phi_1 + \phi_2 + \phi_3 = 3\pi/2$. So, to construct a reciprocal model, it should be inserted into two nonlinear spots together with another constant media, which generate a rotation by angle $\pi/2$.

For example, we use the nonconstant "transparent" subscatterer to construct the nonlinear model illustrated in Fig. 5(c). $V_0 = V_6$, $V_1 = V_5$, $\phi_q = (\frac{1}{2} + m_q)\pi$ for $q = 2, 3, 4$ and $\phi_1 + \phi_5 = (\frac{1}{2} + m_1)\pi$. The subscatterer formed by $V_2$, $V_3$, and $V_4$ is transparent according to an analysis of the method we introduced above. $V_1$ and $V_2$ are added to fix the phase difference of $\chi$ at two ends of the subscatterer such that the wave function magnitudes at the two nonlinear spots are the same and equals to $|T|$. So, the system is supposed to be reciprocal according to our analysis.

It can be verified that the model illustrated in Fig. 5(c) satisfies the general reciprocal condition Eq. (8). Substituting $\phi_q = (\frac{1}{2} + m_q)\pi$ for $q = 2, 3, 4$ into Eq. (14), $\Phi_q$ is

$$\Phi_q = (-1)^{m_q}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{33}$$

As $k_2/k_1 = k_3/k_2 = k_3/k_4 = k_4/k_5$,

$$\Omega_{1,2} = \Omega_{2,3} = \Omega_{3,4}^{-1} = \Omega_{4,5}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & k_2/k_1 \end{bmatrix}. \tag{34}$$

Substituting Eqs. (33) and (34) into Eq. (19), the transfer matrix of the subscatterer formed by $V_2$, $V_3$, and $V_4$ is

$$\Omega_{1,2}\prod_{q=2}^{4}\Phi_q\Omega_{q,q+1} = (-1)^{\sum_{q=2}^{4} m_q}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{35}$$

According to Eqs. (13) and (17),

$$\begin{aligned}\chi_{1,l} &= \prod_{q=1}^{4}\Phi_q\Omega_{q,q+1}\Phi_5\chi_{5,r} \\ &= (-1)^{\sum_{q=2}^{4} m_q}\begin{bmatrix} \sin(\phi_1+\phi_5) & \cos(\phi_1+\phi_5) \\ -\cos(\phi_1+\phi_5) & \sin(\phi_1+\phi_5) \end{bmatrix}\chi_{5,r} \\ &= (-1)^{\sum_{q=1}^{4} m_q}\chi_{5,r}. \end{aligned} \tag{36}$$

So, the wave magnitude at nonlinear spots are the same: $|\psi(D_{0,1})| = |\psi(D_{5,6})| = |T|$, both for a wave propagating forward and backward. The transfer matrix of the system is

$$\Xi_f = \Xi_b = \begin{bmatrix} 1 & 0 \\ -\frac{|T|^2}{k_0}(G_{0,1} + G_{5,6}) & 1 \end{bmatrix}. \tag{37}$$

So reciprocal condition Eq. (8) is fulfilled.

## VI. STABILITY ANALYSIS

In this section, we analyze the stability of the reciprocal condition. The dynamical stability in the conventional sense is the stability against small perturbations of the initial condition, which has been thoroughly discussed in previous works [48,49] for a wave packet propagating in a similar model. Here we study the stability of the reciprocal conditions we proposed, namely, whether the reciprocal models in Sec. V remain approximately reciprocal under small perturbation. Since the general derivation of stability is complicated for our model, we examine several special cases as examples. We will show that the global resonance condition is "approximately reciprocal" if it is slightly violated, either by perturbing the value of the linear constant potential in Sec. VI A or replacing the constant linear potential with a slowly varying one in Sec. VI B. But other reciprocal conditions could be unstable against the perturbation of the nonlinear coefficient as discussed in Sec. VI C.

### A. Perturbation of $\phi_s$

We reconsider the model in Fig. 2 which satisfies the global resonance condition $\phi_n = m_n\pi$, for $n = 1, 2, \dots, N$. For stability analysis, we consider a small deviation from the global resonance condition by changing one linear potential labeled by $n = s$, which changes $\phi_s$ slightly to $\phi_s'$ : $\phi_s' = (1+\epsilon)\phi_s$ with $\epsilon \ll 1$. Then according to Eq.(19), the transfer matrix is

$$\Xi' = (-1)^m \begin{bmatrix} 1 & 0 \\ -Q_l & k_s/k_0 \end{bmatrix} e^{-i\sigma_y\epsilon\phi_s} \begin{bmatrix} 1 & 0 \\ -Q_r & k_0/k_s \end{bmatrix}, \quad (38)$$

where $Q_l = G_l|\psi(D_{0,1})|^2/k_0$ and $Q_r = G_r|\psi(D_{N,N+1})|^2/k_s$, $G_l = \sum_{j=0}^{s-1} G_{j,j+1}$, $G_r = \sum_{j=s}^{N} G_{j,j+1}$. As $\epsilon$ is small, substituting $e^{-i\sigma_y\epsilon\phi_s} = 1 - i\sigma_y\epsilon\phi_s + O(\epsilon^2)$ into Eq. (38), we can derive

$$\Xi' = (-1)^m \begin{bmatrix} 1 & 0 \\ -(Q_l + k_sQ_r/k_0) & 1 \end{bmatrix} + (-1)^m\epsilon\phi_s \begin{bmatrix} Q_r & -k_0/k_s \\ k_s/k_0 - Q_lQ_r & k_0Q_l/k_s \end{bmatrix} + O(\epsilon^2). \quad (39)$$

For a wave propagating forward, $\chi_{f,N+1,l} = [T, iT]^T$ and $Q_r = G_r|T|^2/k_s$. Substituting Eq. (39) into $\chi_{f,0,r} = \Xi'\chi_{f,N+1,l}$, we can derive $Q_l = G_l(1 + 2\epsilon\phi_s G_r|T|^2/k_s)|T|^2/k_0 + O(\epsilon^2)$. Similarly, for a wave propagating backward, $Q_l = G_l|T|^2/k_0$ and $Q_r = G_r(1 + 2\epsilon\phi_s G_l|T|^2/k_s)|T|^2/k_s + O(\epsilon^2)$. By ignoring higher-order terms, the transfer matrices for a wave propagating forward and backward are identical:

$$\Xi_f' \cong \Xi_b' \cong \Xi + (-1)^m\epsilon\phi_s \begin{bmatrix} \frac{G_r}{k_s}|T|^2 & -\frac{k_0}{k_s} \\ \frac{k_s}{k_0} - \frac{3G_lG_r}{k_0k_s}|T|^4 & \frac{G_l}{k_s}|T|^2 \end{bmatrix}, \quad (40)$$

where $\Xi$ is the transfer matrix of the original reciprocal system: $\Xi = (-1)^m[1, 0; -(G_l + G_r)|T|^2/k_0, 1]$. So, the system can still be viewed as reciprocal if the perturbation is sufficiently small. To verify this, we define the quantity $\Delta A = |\mathbf{S}_{21,f} - \mathbf{S}_{12,b}|^2/4$ and $\overline{A} = |\mathbf{S}_{21,f} + \mathbf{S}_{12,b}|^2/4$. If $\Delta A = 0$, the system is reciprocal. If $\Delta A/\overline{A} \ll 1$, we reasonably consider the system "approximately reciprocal." For small $\epsilon$ case, the difference between forward transfer matrix and backward transfer matrix is proportional to $\epsilon$, then $\Delta A$ is proportional to $\epsilon^2$. As shown in Fig. 6, for small $\epsilon$ though $\overline{A}$ changes obviously, $\Delta A/\overline{A}$ remains small compared with $\epsilon^2$.

### B. Spatially varying linear potential

For the global resonance condition in the above section, we consider only the constant linear potential. We now consider a spatially varying symmetric linear potential to check the stability of the reciprocal condition. The model would satisfy a condition similar to the resonance condition: $\int_{D_{s-1,s}}^{D_{s,s+1}} k(x)dx = m_s\pi$ and $k_jd_j = m_j\pi$ for $j \neq s$. For mathematical simplicity, we assume that $k(x) = k_s + A\cos[\epsilon m_s\pi(x - D_c)/d_s]$ for $D_{s-1,s} < x < D_{s,s+1}$, where $k_s$ is to be determined from given $m_s$ and $d_s$ and $D_c = (D_{s-1,s} + D_{s,s+1})/2$. Then ignoring small terms in order $\epsilon^2$, the solution of $\frac{d^2}{dx^2}\psi + k(x)^2\psi = 0$ is $\psi(x) = \frac{1}{\sqrt{k(x)}}(C_1e^{i\int k(x)dx} + C_2e^{-i\int k(x)dx})$. And $\frac{1}{k(x)}\frac{d}{dx}\psi(x) = -\frac{1}{2k(x)^2}\frac{dk}{dx}\psi(x) + \frac{1}{\sqrt{k(x)}}(iC_1e^{i\int k(x)dx} - iC_2e^{-i\int k(x)dx})$. Similar to the case for linear constant potential, the vector $\chi$ at $x = D_{s-1,s}$ and $x = D_{s,s+1}$ can be related: $\chi_{s,l} = \Phi_s'\chi_{s,r}$ where $\Phi_s' = (-1)^{m_s}\mathbf{1} + O(\epsilon^2)$ is identical to $\Phi_s$ of the original reciprocal system. So, the transfer matrix is $\Xi' = (-1)^m[1, 0; -Q, 1] + O(\epsilon^2)$, which is identical to the transfer matrix of the original system in Eq. (25) and the system is approximately reciprocal. The simulation results shown in Fig. 7 agree with these arguments.

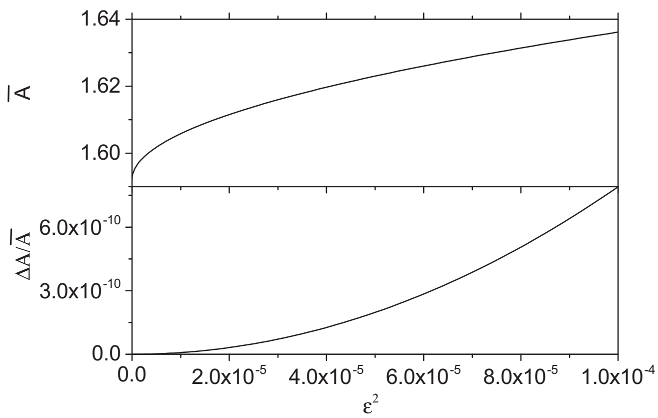


FIG. 6. $\overline{A}$ and $\Delta A/\overline{A}$ for system under perturbation: $\phi_s' = (1+\epsilon)\phi_s$ with $\epsilon$ varying between 0 and 0.01. The model consists of two nonlinear $\delta$ spots: $G_{0,1} = 1$ and $G_{1,2} = 0.9$. $k_1 = 1$ $k_0 = 1$ and $T = 0.9$.

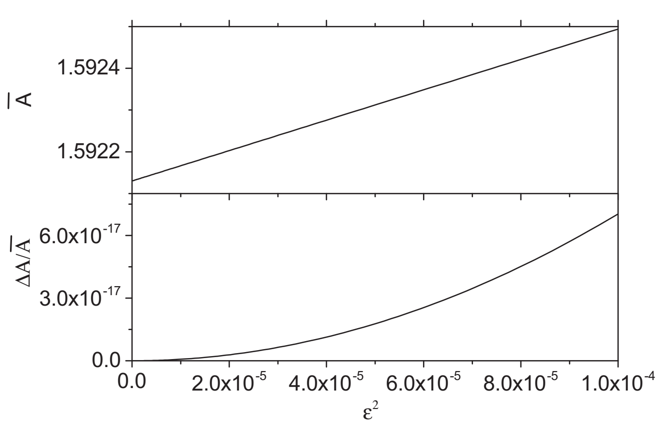


FIG. 7. $\overline{A}$ and $\Delta A/\overline{A}$ for the slowly varying potential $k(x) = k_s + A\cos[\epsilon m_s\pi(x - D_c)/d_s]$ with $\epsilon$ varying between 0 and 0.01. The model consists of two nonlinear $\delta$-spots: $G_{0,1} = 1$ at $D_{0,1} = 0$ and $G_{1,2} = 0.9$ at $D_{1,2} = 4\pi$ and a linear interface with $A = 0.4$ and $\int_0^{4\pi} k(x)dx = 4\pi$. $k_0 = 1$ and $T = 0.9$.

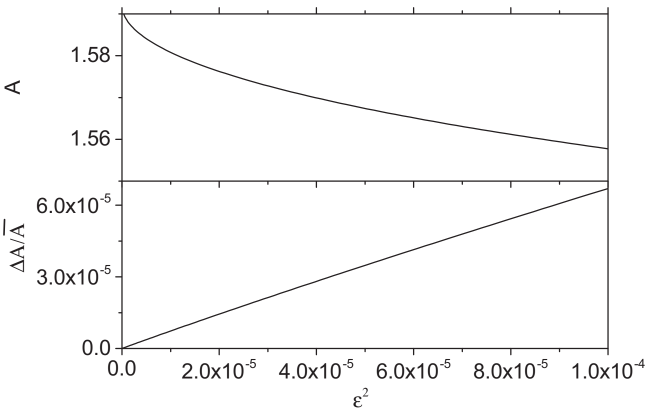


FIG. 8. $\overline{A}$ and $\Delta A/\overline{A}$ under perturbation: $G_{1,2}+G_{2,3}=\epsilon(G_{0,1}+G_{3,4})$ with $\epsilon$ varying between 0 and 0.01. The model consists of four nonlinear $\delta$ spots: $G_{0,1}=1$, $D_{3,4}=0$ and $G_{1,2}$ and $G_{2,3}$ are to be determined. $k_j=1$ for all $j$. $\phi_1=\pi/3$ and $\phi_3=2\pi/3$. $T=0.9$.

### C. Perturbation of nonlinear coefficient

Changing the nonlinear coefficient does not violate the global resonance condition, but it could violate other reciprocal conditions. As an example, we examine the system in Sec. V B and the condition for the nonlinear potential is slightly violated $G_{1,2}+G_{2,3}=\epsilon(G_{0,1}+G_{3,4})$. So, for the subscatter formed by $G_{1,2}, V_2$, and $G_{2,3}$, we have $\Omega_{1,2}\Phi_2\Omega_{2,3}=(-1)^{m_2}[1,0;-\epsilon Q_{1,3},1]$ where $Q_{1,3}=(G_{0,1}+G_{3,4})|\psi(D_{1,2})|^2/k_1$. The trace of the transfer matrix $\Xi'=\Omega_{0,1}\prod_{n=1}^{3}\Phi_n\Omega_{n,n+1}$ is

$$Tr(\Xi')=2(-1)^{m_2+m_3}+(-1)^{m_2}\epsilon B\frac{G_{0,1}+G_{3,4}}{k_1}\times|\psi(D_{1,2})|^2+O(\epsilon^2), \quad (41)$$

where $B=\sin(m_3\pi)+\sin\phi_1\sin\phi_3(G_{0,1}+G_{3,4})|T|^2/k_1)$. As $G_{0,1}$ and $G_{3,4}$ can take arbitrary value, which makes the forward and backward wave scattering different, $|\psi_f(D_{1,2})|^2\neq|\psi_b(D_{1,2})|^2$ in general. So, $Tr(\Xi'_f)\neq Tr(\Xi'_b)$ and thus the eigenvalues of $\Xi'_f$ and $\Xi'_b$ are different and the reciprocal condition Eq. (8) does not hold anymore, even if we ignore small terms in order $\epsilon^2$. As shown in Fig. 8, $\Delta A$ is proportional to $\epsilon^2$, which also indicates that the model is not approximately reciprocal.

## VII. CONCLUSION

In the present work, we investigate reciprocal conditions for a system consisting of multiple nonlinear layers separated from each other by a linear potential. The geometric (global) resonance condition in [40] is generalized: when the width of linear region $d_j$ meets resonant condition $k_jd_j=m_j\pi$, all nonlinear spots merge and behave as a single nonlinear spot and so that nonreciprocity vanishes. As an extension of this idea, two reciprocal systems not satisfying the global resonance condition are proposed. The resonance condition can be also generalized for a system where the linear potential between two nonlinear spots is transparent but does not need not to be constant. Our results indicate that Lorentz reciprocity will not be violated in a nonlinear system as well as some kind of symmetry exits, such as the invariance of wave magnitude at nonlinear spots while reversing the propagation direction in Secs. V A and V D. So, to realize nonreciprocity in a nonlinear asymmetric system, such symmetry must be broken. This suggests further research about the general theory of what kind of symmetry can ensure reciprocity for a nonlinear system, independent of the exact potential form. Moreover, another interesting topic deserving further investigation is the reciprocity of the wave packet or soliton propagating in this sand-witch structure. Since the superposition principle does not hold anymore in the nonlinear system, the reciprocity of the wave packet is not a trivial generalization of a single frequency wave in our work and whether reciprocity still exists is an open question. Instead of assuming the linear potential to be constant and ignoring scattering outside the media, one can also consider other configurations such as a similar model with periodic boundary condition [53] or a harmonic trap potential [47,54]. For two such kinds of models, the wave packet is expected to be scattered repeatedly and reciprocity in such a model deserves further investigation.

## ACKNOWLEDGMENTS

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[1] C. Liu, Z. Du, Z. Sun, H. Gao, and X. Guo, Phys. Rev. Appl. **3**, 064014 (2015).
[2] B. Liang, X. S. Guo, J. Tu, D. Zhang, and J. C. Cheng, Nat. Mater. **9**, 989 (2010).
[3] N. Boechler, G. Theocharis, and C. Daraio, Nat. Mater. **10**, 665 (2011).
[4] E. A. Kittlaus, N. T. Otterstrom, P. Kharel, S. Gertler, and P. T. Rakich, Nat. Photon. **12**, 613 (2018).
[5] K. Gallo and G. Assanto, Appl. Phys. Lett. **79**, 314 (2001).
[6] X. Hu, Z. Li, J. Zhang, H. Yang, Q. Gong, and X. Zhang, Adv. Funct. Mater. **21**, 1803 (2011).
[7] M. D. Tocci, M. J. Bloemer, M. Scalora, J. P. Dowling, and C. M. Bowden, Appl. Phys. Lett. **66**, 2324 (1995).
[8] N. Peng, X. Li, and W. She, Opt. Express **22**, 17546 (2014).
[9] S. Longhi, Opt. Lett. **40**, 1278 (2015).
[10] M. Terraneo, M. Peyrard, and G. Casati, Phys. Rev. Lett. **88**, 094302 (2002).
[11] B. Li, J. Lan, and L. Wang, Phys. Rev. Lett. **95**, 104302 (2005).
[12] B. Hu, L. Yang, and Y. Zhang, Phys. Rev. Lett. **97**, 124302 (2006).

[13] C. R. Otey, W. T. Lau, and S. Fan, Phys. Rev. Lett. **104**, 154301 (2010).
[14] D. Segal and A. Nitzan, Phys. Rev. Lett. **94**, 034301 (2005).
[15] W. Kobayashi, Y. Teraoka, and I. Terasaki, Appl. Phys. Lett. **95**, 171905 (2009).
[16] H. Tian, D. Xie, Y. Yang, T. L. Ren, G. Zhang, Y. F. Wang, C. J. Zhou, P. G. Peng, L. G. Wang, and L. T. Liu, Sci. Rep. **2**, 523 (2012).
[17] C. W. Chang, D. Okawa, A. Majumdar, and A. Zettl, Science **314**, 1121 (2006).
[18] B. Li, L. Wang, and G. Casati, Phys. Rev. Lett. **93**, 184301 (2004).
[19] J. Ren and J.-X. Zhu, Phys. Rev. B **87**, 241412(R) (2013).
[20] N. Li, J. Ren, L. Wang, G. Zhang, P. Hänggi, and B. Li, Rev. Mod. Phys. **84**, 1045 (2012).
[21] L. Wang and B. Li, Phys. Rev. Lett. **99**, 177208 (2007).
[22] L. Wang and B. Li, Phys. Rev. Lett. **101**, 267203 (2008).
[23] J. Ren, Phys. Rev. B **88**, 220406(R) (2013).
[24] J. Ren and J. X. Zhu, Phys. Rev. B **88**, 094427 (2013).
[25] A. A. Maznev, A. G. Every, and O. B. Wright, Wave Motion **50**, 776 (2013).
[26] L. Ranzani and J. Aumentado, New J. Phys. **16**, 103027 (2014).
[27] P. T. Leung and K. Young, Phys. Rev. A **81**, 032107 (2010).
[28] D. Jalas, A. Petrov, M. Eich, W. Freude, S. Fan, Z. Yu, R. Baets, M. Popovic, A. Melloni, J. D. Joannopoulos, M. Vanwolleghem, C. R. Doerr, and H. Renner, Nat. Photon. **7**, 579 (2013).
[29] L. Deák, L. Bottyán, T. Fülöp, G. Kertész, D. L. Nagy, R. Rüffer, H. Spiering, F. Tanczikó, and G. Vankó, Phys. Rev. Lett. **109**, 237402 (2012).
[30] L. Deák and T. Fulop, Ann. Phys. (NY) **327**, 1050 (2011).
[31] T. T. Koutserimpas and R. Fleury, Phys. Rev. Lett. **120**, 087401 (2018).
[32] B. Lüthi, *Physical Acoustics in the Solid State* (Springer, Berlin, 2005).
[33] L. Onsager, Phys. Rev. **37**, 405 (1931).
[34] L. Onsager, Phys. Rev. **38**, 2265 (1931).
[35] G. Casati, Chaos: An Interdisciplinary Journal of Nonlinear Science **15**, 015120 (2005).
[36] W. Zhu, G. Wu, H. Chen, and J. Ren, Front. Energy Res. **6**, 9 (2018).
[37] C. Coulais, D. Sounas, and A. Alù, Nature **542**, 461 (2014).
[38] L. Fan, J. Wang, L. T. Varghese, H. Shen, B. Niu, Y. Xuan, A. M. Weiner, and M. Qi, Science **335**, 447 (2012).
[39] A. Rosario Hamann, C. Müller, M. Jerger, M. Zanner, J. Combes, M. Pletyukhov, M. Weides, T. M. Stace, and A. Fedorov, Phys. Rev. Lett. **121**, 123601 (2018).
[40] G. Wu, Y. Long, and J. Ren, Phys. Rev. B **97**, 205423 (2018).
[41] B. A. Malomed and M. Y. Azbel, Phys. Rev. B **47**, 10402 (1993).
[42] C. Presilla, G. Jona-Lasinio, and F. Capasso, Phys. Rev. B **43**, 5200 (1991).
[43] J. D'Ambroise, P. G. Kevrekidis, and S. Lepri, J. Phys. A: Math. Theor. **45**, 444012 (2012).
[44] N. Li and J. Ren, Sci. Rep. **4**, 6228 (2014).
[45] S. Lepri and G. Casati, Phys. Rev. Lett. **106**, 164101 (2011).
[46] J. L. Helm, T. P. Billam, and S. A. Gardiner, Phys. Rev. A **85**, 053621 (2012).
[47] H. Sakaguchi and B. A. Malomed, New J. Phys. **18**, 025020 (2016).
[48] T. Mayteevarunyoo, B. A. Malomed, and G. Dong, Phys. Rev. A **78**, 053601 (2008).
[49] A. Shapira, N. Voloch-Bloch, B. A. Malomed, and A. Arie, J. Opt. Soc. Am. B **28**, 1481 (2011).
[50] W. Chen and D. L. Mills, Phys. Rev. B **36**, 6269 (1987).
[51] E. Lidorikis, K. Busch, Q. Li, C. T. Chan, and, C. M. Soukoulis, Physica D **113**, 346365 (1998).
[52] L. L. Snchez-Soto *et al.*, Phys. Rep. **513**, 191 (2012).
[53] X.-F. Zhou, S.-L. Zhang, Z.-W. Zhou, B. A. Malomed, and H. Pu, Phys. Rev. A **85**, 023603 (2012).
[54] E. Shamriz and B. A. Malomed, Phys. Rev. E **98**, 052203 (2018).