Formation of viscous fingers in regularized Laplacian growth

Oleg Alekseev*

Department of Mathematics and Mechanics, Chebyshev Laboratory, Saint-Petersburg State University, 14th Line, 29b, 199178 Saint-Petersburg, Russia

(Received 5 July 2018; revised manuscript received 18 June 2019; published 22 July 2019)

A systematic analytic treatment of local fluctuations in the regularized Laplacian growth problem is given. The interface dynamics is stabilized by a short-distance cutoff $\hbar$ preventing the cusps production in a finite time. The regularization mechanism results in the violation of the incompressibility condition of the viscous fluid on a microscale in the vicinity of the moving interface, thus producing local fluctuations of pressure. Dissipation of fluctuations with time is described by universal Dyson Brownian motion, which reduces to the complex viscous Burgers equation in the hydrodynamic approximation. Because of the intrinsic instability of the interface dynamics, tiny fluctuations of pressure generate universal complex patterns with well developed fjords and fingers in a long time asymptotic.

DOI: 10.1103/PhysRevE.100.012129

I. INTRODUCTION

A pattern formation in highly unstable, dissipative, nonlinear growth processes still possesses a great challenge in nonequilibrium statistical physics. Although the role of noise on a microscale is believed to be crucial in the pattern formation phenomena [1], this mechanism has never been implemented for a broad class of growth processes known as Laplacian growth [2,3]. The Laplacian growth problem embraces a variety of diffusion driven growth processes typically observed in physical, chemical, and biological systems. The best known examples are the viscous fingering in a Hele-Shaw cell, when a less viscous fluid is injected into a more viscous one in a narrow gap between two plates [2,3], and diffusion-limited aggregation, which is realized by tiny Brownian particles with size $\hbar$ diffusing and sticking to the boundary of the cluster [4,5]. Although it is believed that both these processes are from the same universality class [6], the relation between them is still puzzling [7,8].

A compact formulation of the *idealized* (without a surface tension) Hele-Shaw problem is as follows. Let $D^+(t)$ (where t is time) be a simply connected domain occupied by inviscous fluid or gas. It is surrounded by a viscous fluid, $D^-(t) = \mathbb{C} \setminus D^+(t)$. Both liquids are sandwiched between two parallel close plates. Fluid velocity in $D^-(t)$ obeys the Darcy law, $v = -\nabla p$ (in scaled units), where $p(z, \bar{z})$ is pressure and $z = x + iy$ is a complex coordinate on the plane. Because of incompressibility, $\nabla \cdot v = 0$, then $\nabla^2 p = 0$ in $D^-(t)$, except infinity. Also, $p = 0$ at the interface, $\Gamma(t) = \partial D^+(t)$, between two fluids, if to neglect surface tension. The kinematic identity requires that the normal interface velocity, $v_n(\zeta)$, equals the fluid normal velocity at the interface,

$$v_n(\zeta) = -\partial_n p(\zeta), \quad \zeta \in \Gamma(t), \tag{1}$$

where ∂_n is a normal derivative at the boundary.

The idealized Laplacian growth problem has a convenient formulation in terms of complex variables and analytic functions [9]. Let $z(w, t)$ be a time-dependent conformal map from the complement of the unit disk in the auxiliary w plane to the exterior of the domain $D^+(t)$ in the physical z plane, so that $z(\infty, t) = \infty$, and the conformal radius $r(t) = z'(\infty, t)$ is a positive function of time (see Fig. 1). In terms of the conformal map the interface dynamics (1) can be written as follows:

$$\mathrm{Im}[\overline{\partial_t z(e^{i\phi}, t)}\partial_\phi z(e^{i\phi}, t)] = Q, \tag{2}$$

where Q is a growth rate.

An important feature of this nonlinear partial differential equation is its integrability. Namely, if the initial interface is an algebraic curve of a given order, it will remain so until the solution to Eq. (2) ceased to exist due to the boundary cusps production. Therefore, the idealized Laplacian growth problem is ill defined, because the normal interface velocity diverges in critical points, where the cusps appear [10].

The Mullins-Sekerka instability [11] provides a clue to the cusps production in the Hele-Shaw flow. Let us consider the n-fold perturbation of the circle, $z(e^{i\phi}, t_0) = r(t_0)e^{i\phi} + a(t_0)e^{i(1-n)\phi}$, with $a(t_0) \ll r(t_0)$ at the time instant t_0. Then, the Laplacian growth dynamics (2) leads to the fast growth of the perturbation with time, $a(t) = a_0[r(t)]^{n-1}$, thus evolving in the cusps.

A conventional mechanism stabilizing the interface dynamics is a surface tension. However, it destroys the mathematical structure of the idealized problem and complicates its analytical analysis. Nevertheless, it is believed that the fractal character of patterns does not depend on the particular regularization mechanism at the microscale. This assumption suggests considering other regularization mechanisms, which retain the integrable structure of the idealized problem, e.g., the short-distance regularization at the microscale proposed by the aggregation model [12]. It implies that the area of the growing domains is quantized, so that the growth process is

*teknoanarchy@gmail.com

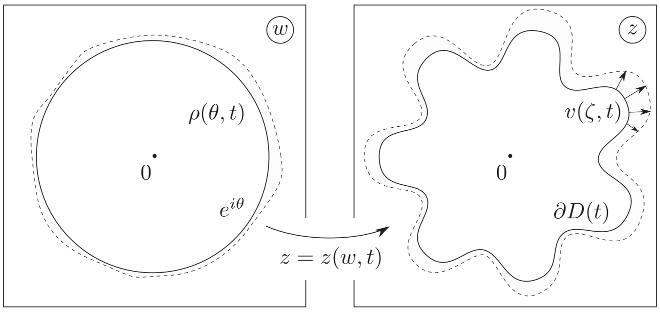

FIG. 1. Time dependent conformal map $z(w,t)$ from the complement of the unit disk in the auxiliary w plane to the exterior of $D(t)$ in the physical z plane. The dashed line in the z plane represents the advance of the interface, $\Gamma(t) \to \Gamma(t+\delta t)$, per time unit δt. The distribution of normal velocities $v_n(\zeta,t) = |z'(e^{i\theta},t)|^{-1}\rho(\theta)$ along $\Gamma(t)$ is determined by the density of eigenvalues $\rho(\theta)$ of the Dyson ensemble.

similar to aggregation of a large number of tiny uncorrelated particles (issued from infinity) at the boundary.

In this paper we continue to study the short-distance regularization (closely related with diffusion-limited aggregation [4]) of the Hele-Shaw problem initiated in Ref. [8]. Diffusion-limited aggregation is a process where the Brownian particles with size $\hbar$ are issued one by one from infinity and diffuse until they stick to a growing cluster. A naive limit of the vanishing particle size, $\hbar \to 0$, is similar to the ill-defined Hele-Shaw dynamics with zero surface tension. Thus the particle size $\hbar$ serves as a short-distance cutoff curbing singularities of the fluid dynamics.

The short-distance cutoff at the microscale significantly affects the macroscopic interface dynamics. Due to the finite size and irregular shape of particles, taking the vanishing particle size limit of the aggregation model effectively leads to a *compressible* fluid at the microscale in the vicinity of the interface.[1] The violation of the incompressibility condition results in *local* fluctuations of pressure, which are forbidden in the idealized Hele-Show problem. Indeed, because of the continuity and incompressibility conditions small perturbations of the interface, e.g., the n-fold perturbation of the circle, require changing the pressure field in the whole domain occupied by a viscous fluid.

In accordance with common principles of statistical mechanics local microscopic fluctuations of pressure should relax to the equilibrium value. We will show that this process leads to the formation of macroscopic patterns with deep fjords of oil separating fingers of water in a long time asymptotic. These patterns are closely connected to the family of "logarithmic" (and "multicut" [13]) solutions, which are of great importance in the Laplacian growth problem. Namely, the logarithmic solutions remain well defined (under certain constraints) for all positive times and describe the *nonsingular* interface dynamics. Besides, the patterns described by these solutions are found to be in an excellent agreement with some experimental observations [14–17]. This paper elucidates the following question: why are the experimentally observed patterns in the Hele-Shaw cell well described by the logarithmic solutions?

The structure of the paper is straightforward: first, we briefly recall a theory of static fluctuations at the interface (the boundary of the aggregate) in the *regularized* Hele-Shaw problem introduced in Ref. [8]. Then, we consider relaxation of local fluctuations in the pressure and argue that this process can be described by one-dimensional hydrodynamical equations. Afterwards, we study the pattern formation in the regularized Hele-Shaw problem, where the incompressibility condition is violated at the microscale in the vicinity of the interface. We argue that the dissipation of fluctuations leads to the formation of patterns with the well-developed fjords. Finally, we draw our conclusion and indicate some open problems.

II. STATIC FLUCTUATIONS IN LAPLACIAN GROWTH

The short-distance regularization of the Hele-Shaw dynamics suggested by the aggregation model implies that the change of the area of the domain is quantized and equals an integer multiple of the area quanta $\hbar$ [8]. The domain can then be considered as an incompressible aggregate of tiny particles with the size $\hbar$ obeying the Pauli exclusion principle. The growth process is generated by aggregation of a large number, $N \gg 1$, of uncorrelated Brownian particles of the size $\hbar$ issued from infinity and stuck to the interface per time unit [8,18]. Then, statistical properties of the aggregate can be studied within Laughlin's theory of the integer quantum Hall effect [12] (see also Ref. [19]).

Let us consider the joint distribution function of the eigenvalues $\{e^{i\theta_j}\}_{j=1}^N$ of a random unitary $N \times N$ matrix,

$$P(\theta_1,\ldots,\theta_N) = C_{N\beta}\prod_{i<j}\left|2\sin\left(\frac{\theta_i-\theta_j}{2}\right)\right|^\beta, \qquad (3)$$

where $C_{N\beta}$ is a normalization constant and β is a parameter. The density of eigenvalues is given by the sum of one-dimensional delta functions,

$$\rho(\theta) = q\sum_{i=1}^N \delta(\theta-\theta_i), \qquad (4)$$

where $q = \hbar/\delta t$ is a quanta of the growth rate $Q = Nq$, so that $\rho(\theta)$ is normalized as $(2\pi)^{-1}\int_0^{2\pi}\rho(\phi)d\phi = Q$. In the limit $N \to \infty$, the density $\rho(\theta)$ becomes a smooth function of the angle on the unit circle.

In the previous works on stochastic regularized interface dynamics in the Hele-Shaw cell we argued the following *proposition* [8,18] (see also the Appendix A for a brief review).

The normal interface velocity, $v_n(\zeta,t)$, determined by the distribution of aggregated particles per time unit, is connected to the density of eigenvalues (15) of Dyson's circular ensembles on symmetric unitary matrices (i.e., at $\beta = 2$) as follows:

$$v_n(\zeta,t) = |w'(\zeta,t)|\rho(\theta), \quad \zeta = z(e^{i\theta},t). \qquad (5)$$

[1]Note that the incompressibility still holds in the bulk of the oil domain.

In particular, the uniform (most probable) distribution of eigenvalues, $\rho(\theta) = Q$, generates the deterministic Laplacian growth (2) with $v_n(\zeta, t) = Q|w'(\zeta, t)|$.

The position of eigenvalues $\{e^{i\theta_j}\}_{j=1}^N$ have a clear geometrical interpretation as the points, which provide growth. Indeed, Eq. (5) can be recast in the Loewner-Kufarev equation describing a sequence of conformal maps of subordinal domains parametrized by time t [20,21],

$$\frac{\partial z(w,t)}{\partial t} = wz'(w,t)\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{w+e^{i\phi}}{w-e^{i\phi}}\frac{\rho(\phi)}{|z'(e^{i\phi},t)|^2}, \quad (6)$$

where the function $\rho(\phi)$ is known as Loewner density.[2] In particular, when $\rho(\theta) = |z'(e^{i\phi}, t)|^2 \sum_{j=1}^N \delta(e^{i\theta} - e^{i\theta_j})$ is a sum of normalized Dirac peaks, Eq. (6) generates the well-known Loewner evolution of N curves growing from the points $e^{i\theta_j}$ on the unit circle [20]. Thus, in the limit $N \to \infty$ [so that the density (15) becomes a smooth function on the unit circle], all points of the boundary become the sources of Loewner curves. We also note that, in the case $\rho(\theta) = Q$, Eq. (6) is equivalent to Laplacian growth equation (2).

III. DISSIPATION OF FLUCTUATIONS

Time evolution of the Loewner density. Let us generalize the density (4) in such a way that the distribution of eigenvalues (3) acquires a meaning of the dynamical system which may be in an arbitrary nonequilibrium state changing with time. For this purpose, one can use the following two interpretations of the distribution (3).

(i) The distribution function P is a ground state wave function for the *conservative* Calogero-Sutherland model defined on the unit circle by the Hamiltonian [22]

$$\mathcal{H} = -\sum_{j=1}^N \frac{\partial^2}{\partial\theta_j^2} + \frac{1}{4}\sum_{k\neq j}^N \frac{\beta(\beta/2-1)}{\sin^2(\theta_k-\theta_j)/2}. \quad (7)$$

(ii) The function P is the probability distribution function of N charged Brownian particles, subjected to an electric force $E(\theta_i) = -\partial_{\theta_i} W$ [where $W = -\sum_{i<j}\log|\sin(\theta_i - \theta_j)/2|$ is the potential] and *friction* with strength γ.[3] At temperature β^{-1} during a small "time" interval,[4] $\delta\tau$, the positions of particles change as follows:

$$\langle\delta\theta_i\rangle = qE(\theta_i)\delta\tau, \quad \langle(\delta\theta_i)^2\rangle = 2q\beta^{-1}\delta\tau, \quad (8)$$

and all higher moments are zero. The joint probability density $P(\theta_1, \ldots, \theta_N; \tau)$ then satisfies the Fokker-Planck equation [23],

$$q^{-1}\frac{\partial P}{\partial\tau} = \mathcal{L}P, \quad \mathcal{L} = \sum_i \frac{\partial}{\partial\theta_i}\left(\beta^{-1}\frac{\partial}{\partial\theta_i} + \frac{\partial W}{\partial\theta_i}\right), \quad (9)$$

and its stationary solution is given by (3). Although the transformation $\tilde{P} = P\exp(-W/2)$ recasts the Fokker-Planck operator (9) in the Calogero Hamiltonian (7), the system (9) is *dissipative*.

Hydrodynamical description. In the limit $N \to \infty$ exact collective descriptions of both models, (7) and (9), are known and have the hydrodynamical form closely connected to the Hopf equation [23,24],

$$\partial_\tau u + u\partial_\theta u = 0, \quad (10)$$

where $u = v + i\pi\rho$ (outside the unit disk) is a sum of the density ρ and velocity v operators in the case of Calogero model (7), while in the case of Dyson's model $u = \rho^H - i\rho$, where ρ^H stands for the periodic Hilbert transform of the density,

$$\rho^H(\theta) = \text{P.V.}\int_0^{2\pi}\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')d\theta', \quad (11)$$

where P.V. indicates the principal value. The nonlinearity of the Hopf equation results in the formation of shock waves, which must be regularized either by dispersion (in the Calogero model) or by dissipation (in the Dyson model). In the former case one obtains the *Benjamin-Ono* equation as an effective description of Calogero hydrodynamics in the limit of weak nonlinearity and dispersion [24,25],

$$\partial_\tau u + u\partial_\theta u = -\alpha_{BO}\partial_\theta^2 u^H, \quad (12)$$

where $2\alpha_{BO} = \sqrt{\beta/2} - \sqrt{2/\beta}$. In the latter case one arrives at the *complex Burgers* equation,

$$\partial_\tau u + u\partial_\theta u = \nu\partial_\theta^2 u, \quad (13)$$

with $\nu/q = 1 - \beta/2$. Equation (13) describes the hydrodynamic limit of Dyson's Brownian model as the interplay between nonlinearity and dissipation [23]. Although the Benjamin-Ono term, $\alpha_{BO}\partial_\theta^2 u^H$, is of the same order as the Burgers term, $\nu\partial_\theta^2 u$, it is very different in nature. Namely, the dispersive perturbation results in the real correction, $\delta\omega \sim \alpha_{BO}k|k|$, to the spectrum of linear waves, while the diffusive term leads to the dissipation, $\delta\omega \sim -i\nu k^2$.

The semiclassical limit of the quantum hydrodynamical equations, (12) and (13), is subtle. In particular, the semiclassical limit of the Benjamin-Ono equation (12) requires a shifting $2\alpha_{BO} \to \sqrt{\beta/2}$, so that dispersion survives even for free fermions when $\beta = 2$ [24,26]. Following this result, one can expect that dissipation regularizes shock waves in the complex Burgers equation (13) even when the semiclassical limit of the quantum system with $\beta = 2$ is implemented. Thus below we will refer to ν as to the "effective" quanta of the growth rate.[5]

Dissipation of fluctuations. Solutions to both equations, (12) and (13), can be coupled with the Loewner-Kufarev equation (6), thus generating a nontrivial evolution of the interface in the presence of fluctuations.

In this paper we only consider the dissipation mechanism for regularizing the Hopf equation. The reason is that the Hele-Shaw flow is a highly dissipative process. The relaxation of local fluctuations of pressure in the vicinity of the interface,

[2] More precisely, the Loewner density is $\rho_L(\theta) = \rho(\theta)/|z'(e^{i\theta}, t)|^2$.

[3] Without loss of generality, we will set $\gamma = 1$ below.

[4] The auxiliary time τ is a parameter of the Dyson model. Its relation with the physical time t will be clarified below.

[5] Note that the dimensionless parameter, ν/Q, of the short-distance regularization plays a similar role as the "surface tension parameter," d_0, in the Hele-Shaw problem [3].

described by Eq. (13), can then be considered as a coarse-grained limit of viscous dissipation and/or impact of the third dimension.

In terms of the density, $\rho = -\operatorname{Im} u$, the complex Burgers equation (13) takes the following form:

$$\partial_\tau \rho + \partial_\theta(\rho\rho^H) = \nu\partial_\theta^2\rho. \tag{14}$$

Remarkably, this nonlinear partial differential equation can be reduced to the many-body problem. Indeed, exact analytical formulas for its solutions can be obtained by the method of pole expansion [27,28]. The idea is to consider the density $\rho(\theta,\tau)$ as the meromorphic functions in the complex plane, namely,

$$\rho(\theta,t) = Q - 2\nu \ \mathrm{Re}\sum_{k=1}^{M}\frac{\xi_k(t)}{e^{i\theta}-\xi_k(t)}, \tag{15}$$

where the poles' ξ_k's are located inside the unit disk in the w plane and the normalization, $(2\pi)^{-1}\int_0^{2\pi}\rho(\phi,t)d\phi = Q$, is assumed. By using the ansatz (15) one reduces the transport equation (14) to the system of M coupled ordinary differential equations, which describe motions of the poles' ξ_k's inside the unit circle:

$$\frac{d\xi_k}{\xi_k} = -Q\,d\tau + \nu\sum_{l\neq k}\frac{\xi_k+\xi_l}{\xi_k-\xi_l}d\tau. \tag{16}$$

The exact relation between the "auxiliary" and "physical" times, i.e., the function $\tau(t)$, possesses a nontrivial problem. However, in the large time asymptotic it acquires a particular simple form (see Ref. [29] for details):

$$\frac{d\tau(t)}{dt} \propto \exp\left\{-\frac{2r(t)}{\delta}\right\}, \tag{17}$$

where $r(t)$ and δ are the conformal radius and characteristic scale (e.g., a typical width of fjords) of the domain correspondingly. The exact relation can be obtained from the statistical mechanic point of view, i.e., by coupling regularized Laplacian growth to statistical systems and finding the conditions for certain objects to be martingales. However, this approach requires sophisticated methods of conformal field theories and, therefore, is considered in detail in the separate publication [29].

From Eq. (16) it follows that the poles attract each other in the tangential direction, and repel each other in the radial direction. They tend to form radial lines, eventually coalescing into a single point at the origin (because of the constant drift $-Q$) with a characteristic "lifetime" $\tau^* \sim Q^{-1}$. Thus any initial density relaxes to the uniform distribution $\rho(\theta,t)\to Q$ as $t\to\infty$.

To clarify the physical meaning of the collective coordinates, ξ_k $(k=1,\dots,M)$, we note that stochastic interface dynamics (5), generated by Langevin dynamics of eigenvalues in the w plane (8), admits a separation of scales in a form of a "slow" modulation of "fast" fluctuations of pressure (at a scale of the order of $\hbar$) in the vicinity of the boundary. The hydrodynamical evolution (14) of the Loewner density, $\rho(e^{i\theta},t)$, is obtained by averaging over fast fluctuations, so that ξ_k's parametrize their slow modulations.

Coupling the Burgers hydrodynamics with Laplacian growth. The density of eigenvalues, $\rho(\theta,t)$, determines the interface dynamics in the Hele-Shaw problem (5). The normal interface velocity can be rewritten in terms of the conformal map, $z(w,t)$, as follows:

$$v_n(e^{i\theta},t) = |z'(e^{i\theta},t)|^{-1}\,\mathrm{Im}[\overline{\partial_t z(e^{i\phi},t)}\partial_\theta z(e^{i\theta},t)], \tag{18}$$

where $e^{i\theta} = w(\zeta,t)$ $[\zeta\in\Gamma(t)]$ parametrizes the boundary. Then, from Eqs. (5) and (15), one obtains the following equation of motion of the boundary:

$$\mathrm{Im}[\overline{\partial_t z(e^{i\theta},t)}\partial_\theta z(e^{i\theta},t)] = Q - 2\nu \ \mathrm{Re}\sum_{k=1}^{M}\frac{\xi_k(t)}{e^{i\theta}-\xi_k(t)}, \tag{19}$$

where the functions $\xi_k(t)$ are solutions to Eq. (16).

Formally, Eq. (19) describes the idealized Laplacian growth problem with the sink at ∞ and M *time-dependent* sources located at the points

$$\zeta_k(t) = z(1/\bar{\xi}_k(t),t), \quad k=1,\dots,M. \tag{20}$$

In the case $\zeta_k(t) = \mathrm{const}$, these points have a clear physical interpretation as the positions of oil wells, which *inject* the oil to the reservoir with the rate ν. In our case, the Calogero-type dynamics of ξ_k's (16) results in the motion of the points $\zeta_k(t)$ inside the oil domain.

The emergence of the sources at the points (20) can be misleading, because the only *physical* oil sink is located at infinity [from Eq. (19) it also follows that the total growth rate is Q]. This apparent contradiction has the following resolution. The short-distance regularization effectively leads to a compressible liquid at a microscale due to a finite size and irregular shape of particles. The incompressibility condition holds in the bulk, but is violated in the vicinity of the boundary at a scale of order $\hbar$. Thus tiny perturbations of pressure close to $\partial D(t)$ only formally can be attributed to the oil sources of the idealized Hele-Show flow.

The violation of the incompressibility condition at the microscale allows one to consider *local* fluctuations of pressure, which are forbidden in the idealized Hele-Show problem. Indeed, due to a continuity condition small perturbations of the interface in Eq. (2) result in variations of the pressure field in the viscous fluid. However, because of the incompressibility of oil these variations are nonlocal and require one to change the pressure field in the whole domain occupied by a viscous fluid.

IV. EVOLUTION OF DOMAIN

Evolution of the Schwarz function. The Hele-Shaw problem can be integrated in terms of the Schwarz function [9], which provides an elegant geometrical interpretation of the interface dynamics. The Schwarz function $\mathcal{S}(z)$ for a sufficiently smooth curve Γ drawn on the complex plane is an analytic function in a striplike neighborhood the curve, such that $\bar{z} = \mathcal{S}(z)$ for $z\in\Gamma$ [30]. If the curve $\Gamma(t)$ evolves in time, so does its Schwarz function, $\mathcal{S}(z,t)$. The Schwarz function admits the following decomposition: $\mathcal{S} = \mathcal{S}^+ + \mathcal{S}^-$, where the functions $\mathcal{S}^+ = \sum_{k\geqslant 0}\mathcal{S}_k z^k$ and $\mathcal{S}^- = \sum_{k\geqslant 1}\mathcal{S}_{-k}z^{-k}$, that are regular in D^+ and D^-, respectively.[6]

In terms of the Schwarz function the Laplacian growth equation (19) reads [9]

$$\partial_t \mathcal{S}(z,t) = 2\partial_z \mathcal{W}(z,t), \tag{21}$$

where the complex potential, $\mathcal{W} = -p + i\psi$, is a sum of the negative pressure p and the stream function ψ. In the idealized Laplacian growth the complex potential reads $\mathcal{W}(z,t) = (Q/2)\log w(z,t)$. Since $\partial_z \log w(z,t)$ is analytic everywhere in the oil domain both sides of Eq. (21) are regular in $D^-(t)$. Therefore, idealized Laplacian growth implies that $\mathcal{S}^+(z,t)$ does not vary in time, thus possessing an infinite number of conserved quantities $\mathcal{S}_k = \text{const}$ ($k>0$). From this observation it follows that any algebraic curve of a given order remains so until the solution to Eq. (2) ceased to exist due to the cusps production at the boundary.

The short-distance regularization of the Hele-Shaw problem results in local fluctuations of pressure at the microscale. Because of instability of the interface dynamics the microscopic fluctuations affect the macroscopic evolution of the domain. The *effective* complex potential which drives the interface dynamics (19) reads

$$\mathcal{W}(z,t) = \frac{Q+M\nu}{2}\log w(z,t) - \frac{\nu}{2}\sum_{k=1}^{M}\log\frac{w(z,t)-1/\bar{\xi}_k}{1-w(z,t)1/\xi_k}. \tag{22}$$

The pressure, $p = -\operatorname{Re}\mathcal{W}(z,t)$, satisfies the Laplace equation $\nabla^2 p = \nu\sum_{k=1}\delta^{(2)}(z-\zeta_k)$ with the sources located at the points (20) and the asymptotic behavior $p = -(Q+M\nu)\log|z|$ as $z\to\infty$. Besides $p(z,t)=0$ if $z\in\partial D(t)$. Then, Eq. (21) implies that

$$\partial_t \mathcal{S}^+(z,t) = \nu\sum_{k=1}^{M}\frac{1}{z-\zeta_k(t)}. \tag{23}$$

From this equation one concludes that the function $\mathcal{S}^+$ has branch cuts in the oil domain evolving with time.[7] Integrating both sides of Eq. (23) with respect to time, one obtains the function $\mathcal{S}^+(z,t)$:

$$\mathcal{S}^+(z,t) = \mathcal{S}^+(z,t_0) + \nu\sum_{k=1}^{M}\int_{\zeta_k^0}^{\zeta_k(t)}\frac{\mathcal{P}_k(l)dl}{z-l}, \tag{24}$$

where $\mathcal{P}_k(l)$ are the time-dependent Cauchy densities along the cuts $\gamma_k(t)$ with the end points $\zeta_k^0 = \zeta_k(t_0)$ and $\zeta_k(t)$. These cuts are the trajectories of of the virtual sources $\zeta_k(t)$ in the oil domain. The Cauchy densities, $\mathcal{P}_k(l) = 1/v_k(l)$ are determined by the velocities, $v_k(l) = dl/dt$, of the end points $\zeta_k(t)$.

Let $\zeta_k(t_0)$ parametrize "slow" fluctuations of the pressure in the vicinity of the circle, $\mathcal{S}(z,t_0) = r(t_0)/z$, with the radius $r(t_0)$ at the time instant t_0. Then, the perturbed function $\mathcal{S}^+(z,t_0)$ reads

$$\mathcal{S}^+(z,t_0) = \nu\sum_{k=1}^{M}\frac{1}{z-\zeta_k(t_0)}. \tag{25}$$

It is well known that the idealized Hele-Shaw dynamics (2) of the boundary (25) leads to the formation of cusps at the interface. However, the dissipation of fluctuations, due to the Calogero-type motion of the collective coordinates (16), results in a smooth evolution of the contour and the development of fjords, described by the function (24) (see Fig. 2).

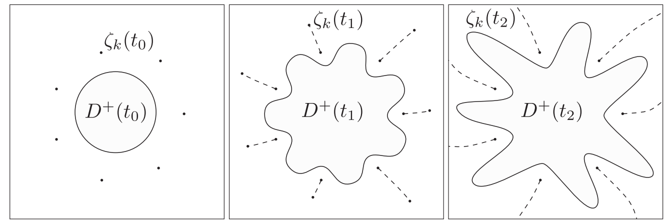

FIG. 2. Dynamical generation of the fjords of oil, which separate fingers of the water droplet, is schematically depicted in three successive instants of time $t_0 < t_1 < t_2$. The center lines of the *dynamically* generated fjords (the dashed curves outside the growing domain) are the branch cuts of $\mathcal{S}^+(z,t)$.

To gain a better understanding of solutions (24), let us consider the uniform motion of the end points $\zeta_k(t)$ toward infinity, i.e., $v_k(t) = v_k$. This process generates the Schwarz function with logarithmic branch cuts in the oil domain:

$$\mathcal{S}^+(z,t) = \sum_{k=1}^{M}c_k\log\frac{z-\zeta_k(t)}{z-\zeta_k^0}, \qquad t>t_0, \tag{26}$$

where $c_k = \nu/v_k$. In particular, in the limit $\zeta_k(t)\to\infty$ as $t\to\infty$ solutions (32) reduce[8] to the well known logarithmic solutions of the idealized Hele-Shaw problem. Then, the logarithmic terms in (32) have a clear geometric interpretation of the fjords of oil with parallel walls separating the fingers of water. The stagnation points of the fjords are located at $\zeta_k^0 - \bar{c}_k\log 2$ and their widths are $\pi|c_k|$. The direction of the fjord is given by $\arg\bar{c}_k$ (for more details about logarithmic solutions, see Ref. [31]).

Below, we explain how to construct explicit formulas for conformal maps, $z(e^{i\phi},t)$, knowing the singularities of the Schwarz function, $\mathcal{S}^+(z,t)$, in the oil domain (24).

V. SOLUTION SCHEME

In order to obtain the Hele-Shaw dynamics, one should recover the conformal map $z(w,t)$ given the function $\mathcal{S}^+(z,t)$. It can be formally determined by integrating both sides of the relation $z(w,t) = \bar{\mathcal{S}}(\bar{z}(1/w,t),t)$ with the kernel $(u-w)^{-1}$ over the unit circle provided that $|w|>1$:

$$z(w,t) = rw + u_0 + \oint_{|u|=1}\frac{\bar{\mathcal{S}}^+(\bar{z}(1/u,t),t)}{w-u}\frac{du}{2\pi i}. \tag{27}$$

[6]The coefficients $\mathcal{S}_{\pm k} = \mp\int_{D^{\mp}} z^{\mp k}d^2z$ are the external and internal harmonic moments of the domain correspondingly.

[7]In the idealized Laplacian growth one has $\partial_t\mathcal{S}^+ = 0$. Thus all singularities of $\mathcal{S}^+$ are frozen in time.

[8]It is implicitly assumed that $\sum_{k=1}^{M}c_k = 0$, so that $\mathcal{S}^+$ has no branching at infinity.

Then, by substituting (24) in the right hand side of Eq. (27), taking the integral over $|u| = 1$, and changing the integration variable, $l = \bar{z}(1/\xi, t)$, one obtains [13]

$$z(e^{i\phi}, t) = r\, e^{i\phi} + u_0 - \sum_{k=1}^{M} \int_{\tilde{\gamma}_k(t)} \frac{\bar{\mathcal{P}}(1/\xi(t), t)}{\xi - e^{i\phi}} \frac{d\xi}{2\pi i}. \quad (28)$$

To clarify, it is convenient to consider the growth process, Eqs. (19) and (16), in a discrete framework, so that the integrals in (24) are replaced by the sums over the points along the cuts. For example, the change of the Schwarz function (23) reads

$$\mathcal{S}^+(z, t_i) - \mathcal{S}^+(z, t_{i-1}) = \nu \Delta t \sum_{k=1}^{M} \frac{1}{z - \zeta_k(t_i)}. \quad (29)$$

The interface dynamics described by this equation is similar to the idealized Hele-Show dynamics with N oil sources at the points $\zeta_k(t_i)$ and a sink at infinity. Equation (29) implies that singularities of $\zeta_k(t_j)$ $(0 \leqslant j < i)$ of the function $\mathcal{S}^+(z, t_{i-1})$ are integrals of motion. Then, the dynamics of poles, $\xi_k^j(t)$, of the conformal map is governed by the following system of equations:

$$\zeta_k(t_j) = z(1/\bar{\xi}_k^j(t_i), t_i) = \text{const} \quad (j < i), \quad (30)$$

and $2\pi t_i = (1/2i) \oint_{|w|=1} \overline{z(w, t_i)} z'(w, t_i) dw$ is the area of the domain at time t_i. The positions of the remaining N poles, $\xi_k^i(t_i)$, are determined by Eq. (16). Thus the system of coupled equations, (19) and (16), is self-consistent.

Contrary to the idealized Laplacian growth (2), the evolution of the interface, (19) and (16), *does not preserve* the number of singular points: each time step, Δt, the function $z'(w, t_i)$ develops N new poles[9] at the points $\xi_k(t_i) \equiv \xi_k^i(t_i)$ determined by the Calogero-type dynamics (16). In order to prove this statement, we recast Eq. (29) in the form

$$\Delta \mathcal{S}^+(z, t_i) = \sum_{k=1}^{M} \frac{\nu}{v_k(t_{i-1})} \log \frac{z - \zeta_k(t_{i-1})}{z - \zeta_k(t_i)} + O((\Delta t)^2), \quad (31)$$

where $v_k(t_i)$ is the velocity of $\zeta_k^i(t_i)$. Therefore, the branch cuts of the Schwarz function can be written as a sum of logarithmic terms with tips at the points $\zeta_k(t_i)$:

$$\mathcal{S}^+(z, t_i) = \mathcal{S}^+(z, t_0) + \sum_{j=0}^{i} \sum_{k=1}^{M} c_{k,j} \log[z - \zeta_k(t_j)], \quad (32)$$

where the coefficients $c_{k,j}$ read

$$c_{k,j} = \nu \big[v_k^{-1}(t_j) - v_k^{-1}(t_{j-1}) \big]. \quad (33)$$

The one-to-one correspondence between singularities of $\mathcal{S}^+(z, t)$ and $z(w, t)$ allows one to determine the conformal map. Namely, if the Schwarz function has a logarithmic singularity at the point ζ_k with the coefficient c_k, then the conformal map also has the logarithmic singularity at the point $\xi_k = 1/\overline{w(\zeta_k, t)}$ inside the unit disk with the coefficient $\bar{c}_k$ [9]. Therefore, Eq. (32) allows one to obtain the following expression for the conformal map:

$$z(w, t_i) = rw + \sum_{k=1}^{N} \sum_{j=0}^{i} \bar{c}_{k,j} \log \big[w - \xi_k^j(t_j) \big], \quad (34)$$

where $\xi_k^j(t_i)$ is a position of the jth point of the cut $\tilde{\gamma}_k(t_i)$ of the map (28).

The geometrical interpretation of solutions (34) is straightforward: we put many identical (logarithmical) fjords along the centerlines. Thus, instead of the "U" shaped fjord with parallel walls, one obtains the so-called "V" shaped fjords with nonzero opening angles (see also Refs. [31,32] for a similar construction). We summarize as follows.

(a) The branch cuts of the Schwarz function, $\gamma_k(t)$, i.e., the trajectories of ζ_k's inside the oil domain, are the center lines of the *dynamically generated* fjords of oil separating the fingers of water (see Fig. 2). Because of the interaction (16), the center lines are typically curved.

(b) The walls of the fjords are not parallel, but have nonzero opening angles determined by the velocities of the branch points, $\zeta_k(t)$, of the Schwarz function.

However, the quantitative analysis of solutions requires one to establish an exact relation between the "auxiliary" and physical times, i.e., the function $\tau(t)$. We address this problem in a separate publication [29].

VI. CONCLUSION AND DISCUSSION

In this work we considered a stochastic regularization (suggested by the aggregation model similar to diffusion-limited aggregation) of the Hele-Shaw problem. The growth of the droplet is generated by aggregation of a large number, $N \gg 1$, of tiny uncorrelated particles with the size $\hbar$. Although the naive limit of the vanishing particle size, $\hbar \to 0$, is widely believed to be similar to the ill-defined Hele-Shaw problem, we argued that the short-distance cutoff changes the interface dynamics *qualitatively*. Namely, the regularized interface dynamics does not result in the formation of needlelike fingers,[10] but rather to the tip splitting and side branching, thus generating the fjords of oil separating the fingers of water. Solutions (28) are well defined in the limit $\hbar \to 0$ and do not lead to the cusps production. Let us briefly summarize the main results.

(i) The short distance regularization allows one to study *analytically* random distributions of the attached particles (per time unit), which have a clear geometrical interpretation as static fluctuations at the interface. Statistics of the fluctuations is described by Dyson's circular ensemble (3) (see Ref. [19] and the AppendixA for a brief review).

(ii) Due to discreteness of particles, the short-distance regularization effectively leads to a *compressible* viscous liquid (oil) at a microscale. The incompressibility condition holds in the bulk of the oil, but is violated in the vicinity of the boundary at a scale of order $\hbar$. The violation of the incompressibility condition at the microscale allows one to introduce *local* fluctuations of the pressure in the vicinity of

[9] In the limit $\Delta t \to 0$ the generation of poles results in the evolution of the branch cuts of $z'(w, t)$.

[10] A curvature of the tip of the needle is controlled by $\hbar$ and becomes infinite in the limit $\hbar \to 0$.

the interface, which are forbidden in the idealized Hele-Show problem. Assuming that the dissipation of fluctuations with time is described by Dyson's Brownian motion (8), we obtain a set of coupled partial differential equations, (19) and (16), which describe the interface dynamics in the presence of local fluctuations.

(iii) The formation of cusps is forbidden. The effect of cutoff is to create side-branch structures on the interface.[11] Thus the long-time solutions (28) for small and positive $\hbar$ differ from that for zero $\hbar$ fundamentally in that secondary structures develop. In particular, the interface remains smooth during the whole evolution and the complex irregular shapes with the well-developed fjords and fingers are observed at large times.

The geometric features of the solutions (28) are in an qualitative agreement with some experimental observations [14,15]. Namely, solutions (28) describe the formation of fjords of oil left behind the moving fronts with nonparallel walls. Besides, the walls of the fjords are not always straight, but more often curved. The dynamics of poles (16) shows a tendency of small fjords to coalesce and form the larger ones.

Viscous fingering patterns are extremely complex, because the fingers repeatedly split and branch to form new fingers and fjords. Thus it is valuable that the regularized dynamics of the interface can be described from the point of view of *weakly nonequilibrium* statistical mechanics. Namely, in this work we argued that relaxation of *local* fluctuations of the pressure in the vicinity of the interface results in the side branching and tip splitting of viscous fingers. The complexity of patterns in the large time asymptotic is then explained by the instability of the interface dynamics with respect to tiny fluctuations of pressure close to equilibrium.

To conclude, let us also mention some open problems and further directions.

Although the proposed model captures the main features of the experimentally observed patterns, the quantitative analysis of the model requires one to couple properly the dynamics of poles (16) to the Laplacian growth equation (19), i.e., to determine the function $\tau(t)$. There are two ways to proceed. First, one can fine-tune the pole dynamics in such a way that the fjord's evolution, described by Eqs. (19) and (16), reproduces the experimental observations [16]. The second possibility requires a further investigation of the mathematical structure underlying the regularized growth process. Namely, the interface dynamics, (19) and (16), can be considered as a classical limit of stochastic process, because the points ξ_k, which parametrize the "fast" fluctuations at the interface, are random variables. One can put strong constraints on possible forms of the pole's dynamics [and the function $\tau(t)$] by studying martingales[12] of stochastic growth process. We address this problem in detail in the separate publication [29].

Once the dynamics of poles (16) is coupled with the Laplacian growth equation (19) properly, one can study *analytically* the process of the fjord's formation. In particular, the proposed theory promises to explain remarkable geometrical features of the fjords observed in experiments [16]. Afterwards, it becomes possible to address the long standing problems in the wedge geometry by novel methods, and to revise the pattern selection problems without surface tension.

The regularized Laplacian growth problem, (19) and (16), describes the interface dynamics resulted from the relaxation of local fluctuations of the pressure (in the vicinity of the interface) to the equilibrium value. However, fluctuations repeatedly appear during the whole growth process. Therefore, the next important step is to study the regularized Laplacian growth model obtained by driving the Laplacian growth equation (19) with a compound random process producing fluctuations with time.[13]

ACKNOWLEDGMENT

The work is supported by Native towns, a social investment program of PJSC Gazprom Neft.

APPENDIX: STATIC FLUCTUATIONS IN LAPLACIAN GROWTH

In this section we briefly describe the relation between the distribution of eigenvalues in Dyson's circular ensemble and stochastic interface dynamics in the Hele-Shaw cell. Let us consider the growth process generated by aggregation of a large number, $N \gg 1$, of uncorrelated Brownian particles of size $\hbar$ issued from infinity and stuck to the interface per time unit. Let us partition the boundary into $K \gg 1$ tiny segments of the size $\sqrt{\hbar}$. Then, the N issued uncorrelated particles are to be distributed into K bins of the boundary, such that the particles stuck to the same bin form a column. The statistical weight (probability) of the possible outcome of this growth step is given by the multinomial formula [8],

$$P(\{k_i\}) = N! \prod_{i=1}^{K} \frac{p_i^{k_i}}{k_i!}, \tag{A1}$$

where k_i is the number of Brownian particles attached to the ith bin of the boundary[14] and p_i is the probability of attachment. By definition, the probability of attachment is equal to the harmonic measure of the boundary and, therefore, can be expressed in terms of the conformal map, $w(z)$, from the exterior of the growing domain, D, in the physical z plane to the complement of the unit disk in the auxiliary w plane:

$$p_i = |w'(\zeta_i)|\sqrt{\hbar}, \tag{A2}$$

where $\zeta_i \in \partial D$.

In the Stirling approximation, $N \gg 1$, statistical weights of randomly attached Brownian particles can be recast in the

[11]Similar phenomena is expected in the Hele-Shaw problem in the presence of surface tension. The short-distance regularization, however, allows one to study the branching and splitting processes analytically.

[12]Roughly speaking, the martingales are the stochastic processes whose expectation values are constant in time [33].

[13]A similar idea was previously implemented to describe the Loewner evolution with random branching curves [34].

[14]Note that $\sum_{i=1}^{K} k_i = N$.

following form [18]:

$$P(\{k_i\}) \propto \exp\left\{-\sum_{i=1}^{K} k_i \log \frac{k_i}{N p_i}\right\}. \tag{A3}$$

The variation of (A3) shows that P is maximal when the number of Brownian particles, $k_i^* = N p_i$, attached to the ith bin of the interface is proportional to the harmonic measure of the boundary (A2). This maximum is exponentially sharp when $\hbar \to 0$ so all fluctuations around k_i^* are suppressed. Hence k_i^* is a classical trajectory for the stochastic process, which describes deterministic Laplacian growth. Indeed, the normal interface velocity $v_n(\zeta_i)$ is related to the numbers k_i in the following way:

$$v_n(\zeta_i) = \sqrt{\hbar} k_i/\delta t. \tag{A4}$$

Taking into account the identity, $v_n(\zeta) = \mathrm{Im}[\partial_t z(e^{i\phi}, t)\, \partial_\phi z(e^{i\phi}, t)]/|\partial_\phi z(e^{i\phi}, t)|$, one obtains the Laplacian growth equation

$$\mathrm{Im}[\partial_t z(e^{i\phi}, t)\partial_\phi z(e^{i\phi}, t)] = Q, \tag{A5}$$

which was intensely studied earlier [1–3]. Thus it turned out possible to derive the Laplacian growth equation directly from variational calculus based on elementary combinatorics.

The next step is to take a continuum limit of (A3). In this limit $\sqrt{\hbar} k_i \to 0$ and $\delta t \to 0$, so that the normal interface velocity (A4) remains fixed. The key observation is that the exponent of (A3) can be rewritten in a remarkably simple way (see Refs. [8,18] for details):

$$P[v_n] \propto e^{q^{-2} \oint_{\partial D} \oint_{\partial D} v_n(\zeta) \log|w(\zeta) - w(\zeta')| v_n(\zeta')|d\zeta||d\zeta'|}, \tag{A6}$$

where $q = \hbar/\delta t$ is the quanta of the growth rate $Q = Nq$ and $w = w(z)$ is the conformal map from the exterior of the growing domain in the physical z plane to the exterior of the unit disk in the mathematical w plane (see Fig. 1). Finally, introducing the density $\rho(\theta)$ by (5) and recasting the contour integrals in (A6) into the integrals along the corresponding angles in the w plane one obtains

$$P[\rho] \propto e^{\int_0^{2\pi} \int_0^{2\pi} \rho(\theta) \log|e^{i\theta} - e^{i\theta'}| \rho(\theta') d\theta\, d\theta'}. \tag{A7}$$

The expression in the exponent of (A7) has a clear electrostatic interpretation, namely, it is the electrostatic energy of self-interacting charge induced on the unit circle with density $\rho(\theta)$ kept at zero potential. Thus the entropy (A1) was transformed to electrostatic energy, so that the probability of a single growth step takes the form of the Gibbs-Boltzmann distribution (implying that probability $P_i \propto e^{-E_i}$, where E_i is the energy of the ith state).

The function $\rho(\theta)$ can be considered as a macroscopic density function of N point charges on the unit circle in the limit $N \to \infty$ (15). Therefore, in the discrete approximation the probability distribution function (A7) takes the form of joint probability distribution function of eigenvalues of a random unitary $N \times N$ matrix (3) with $\beta = 2$.

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