Mean squared displacement in a generalized Lévy walk model

M. Bothe[*]

Institut für Physik, Humboldt Universität zu Berlin, Newtonstraße 15, 12489 Berlin, Germany

F. Sagues

Departament de Química Física, Universitat de Barcelona, 08028 Barcelona, Catalonia, Spain

I. M. Sokolov[†]

Institut für Physik and IRIS Adlershof, Humboldt Universität zu Berlin, Newtonstraße 15, 12489 Berlin, Germany

Lévy walks represent a class of stochastic models (space-time-coupled continuous-time random walks) with applications ranging from the laser cooling to the description of animal motion. The initial model was intended for the description of turbulent dispersion as given by the Richardson law. The existence of this Richardson regime in the original model was recently challenged [T. Albers and G. Radons, Phys. Rev. Lett. **120**, 104501 (2018)]: The mean squared displacement (MSD) in this model diverges, i.e., does not exist, in the regime, where it presumably should reproduce the Richardson law. In the Supplemental Material to that work the authors present (but do not investigate in detail) a generalized model interpolating between the original one and the Drude-like models known to show no divergences. In the present work we give a detailed investigation of the ensemble MSD in this generalized model, show that the behavior of the MSD in this model is the same (up to prefactors) as in the original one in the domains where the MSD in the original model does exist, and investigate the conditions under which the MSD in the generalized model does exist or diverges. Both ordinary and aged situations are considered.

I. INTRODUCTION

A Lévy walk, first introduced in [1], is a space-time-coupled continuous-time random-walk scheme, versatile enough to be applied to many different situations, ranging from laser cooling to animal motion (see [2] for a comprehensive review). A genuine Lévy walk consists of stretches of rectilinear motion. The temporal duration t of a single stretch is a random variable following the probability density function (PDF) $\psi(t)$. In an ordinary Lévy walk the beginning of the observation coincides with the beginning of the first stretch. The situations when the observation starts later correspond to aged walks. Having completed a stretch, the walker changes the direction of motion at random and a new stretch starts; the instant at which this happens will be called a change point. The total observation time after which the position of the walker is recorded does not have to correspond to the end of the last stretch and therefore this last stretch may stay incomplete.

A version of such a Lévy walk commonly used for the modeling of many natural phenomena is given by a motion at a constant speed c so that the displacement in the stretch i is $\mathbf{x}_i = c\tau_i \mathbf{e}_i$, where $\mathbf{e}_i$ is a unit vector of random orientation characterizing the velocity direction during the stretch i of duration τ_i. The special case of the constant speed and fixed

time in a stretch is exactly Pearson's random walk [3] (in the original Pearson question only the position at the end of the complete stretched was of interest). The next more complicated model is the velocity model, in which the speed during a stretch is still assumed constant but the duration of a stretch follows a power law $\psi(t) \simeq t^{-1-\gamma}$. The trajectory of such a walk corresponds to the collection of points in space (corresponding to the times of change points) whose spatial distribution is given by a symmetric Lévy stable law, connected by straight lines. This fact gave the name to the whole process. The variants of this model where the stretches are interrupted by trapping events (rests) at change points arise universally when describing the particles' transport in laminar flows consisting of eddies and jets [4–6].

However, the original model discussed in the very first work [1] contained a discussion of a more complicated case, with the length of complete stretch depending on its duration as $|\mathbf{x}| = ct^\nu$, with some $\nu \geqslant 0$, $\nu \neq 1$ (here the constant c no longer has the meaning of speed). It was claimed that such a model is able to describe, for example, the Richardson law [7] of turbulent dispersion $\langle \mathbf{x}^2 \rangle \propto t^3$ [1,8]. The existence of the Richardson regime in the original model was recently challenged in [9]. The authors carefully investigate this original model and show that the second moment diverges in the domain of parameters γ and ν where it presumably should give rise to the Richardson law.

One may note that the original Lévy walk model may not give the best way for mimicking transport properties in turbulent flows: The Richardson law applies not to a single

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[*]mbothe@physik.hu-berlin.de
[†]igor.sokolov@physik.hu-berlin.de

particle's position but to a distance between two particles in an isotropic and stationary turbulent flow. Lévy-walk-like models proposed for this case, e.g., the ones of Refs. [10,11], differ from the original model and rely on the isotropy, not on the spatial homogeneity of the process. A variant of the model discussed in [11] shows that in a particular regime of motion the process could be multiplicative: If r_i is a distance between the particles corresponding to a change point, then it is r_{i+1}/r_i and not $r_{i+1} - r_i$ that follows a universal distribution. Simulations of turbulent dispersion in a two-dimensional flow show that the statistics of change points is indeed multiplicative [12].

However, the fact, unnoticed until the work in [9], of divergence of the second moment in a model which was essentially invented to prevent divergences of moments is extremely disturbing independently of whether this original model is a good candidate for the description of turbulent dispersion or not. Can this divergence be resolved? Are there similar schemes which would allow for the description of the effects for which the initial scheme was devised? How do they behave?

To answer these questions it is interesting to note that minor changes in the model making it closer to a Drude model for conduction in solids [13] (as proposed in [14,15]) lead to models which do not possess divergences in the Richardson regime. Moreover, in the Supplemental Material to [9] the authors propose a generalized model, which contains an additional parameter, that includes both the original Lévy walk model and its Drude-like variants mentioned above. They show that this model can have a finite mean squared displacement (MSD) in the parameter ranges where one would suspect the Richardson law, but leave the question of how the MSD actually behaves in this new model open.

The present work is therefore devoted to the detailed investigation of the behavior of the mean squared displacement in this general model as a function of time. We consider both ordinary and aged walks and discuss the conditions under which the mean squared displacement does exist (i.e., does not diverge) and investigate in detail the behavior of the MSD in the regimes when it does exist. In the present work we concentrate on the one-dimensional case, since it already shows all necessary richness and since the generalization to higher dimensions is straightforward. The notation (except for sticking to a one-dimensional model) in the present work is the same as in [9]. We note that our method of solution of the problem is very straightforward and differs from the one of Ref. [9] based on the Kubo-like relations.

The further structure of the present work is as follows. In Sec. II we introduce the generalized Lévy walk model and discuss it in some detail. General considerations of ordinary and aged Lévy walks and the notation used in the present work are given in Sec. III. In Sec. IV we consider the ordinary situation. Aged walks are considered in Secs. V–VII: In Sec. V general expressions are given which are then evaluated in the two subsequent sections for special cases of aging times short and long compared to the observation time. The conclusions follow in Sec. VIII. The Appendixes contain information on the use of Tauberian theorems and details of calculations which are too lengthy to be put in the main text.

II. MODEL

The generalized model discussed in the Supplemental Material of [9] starts from the same distribution of the waiting times between the change points, $t = t_i - t_{i-1}$, which follows a power law

$$\psi(t) = \frac{\gamma}{t_0} \frac{1}{(1 + t/t_0)^{\gamma+1}}. \tag{1}$$

Moreover, like in [1], the joint probability density of a stretch's length Δx and duration t in a complete stretch is given by

$$\psi(\Delta x, t) = \tfrac{1}{2}\delta(|\Delta x| - ct^\nu)\psi(t). \tag{2}$$

The models with the same change-point positions still can have different behavior depending on the assumptions about the particle's motion between the change points. In the original model the motion in a single stretch takes place at a constant speed, defined by the stretch duration and by its length:

$$v = \frac{|\Delta x|}{t} = ct^{\nu-1}. \tag{3}$$

The PDF of speeds can be derived from such waiting times and reads

$$
\begin{aligned}
p(v) &= \psi[t(v)]|dt/dv| \\
&= \frac{\gamma}{t_0(\nu-1)} c^{-1/(\nu-1)}[1+(v/c)^{1/(\nu-1)}/t_0]^{-\gamma-1} v^{(2-\nu)/(\nu-1)} \\
&\simeq c^{\gamma/(\nu-1)} t_0^\gamma v^{(1-\gamma-/\nu)(\nu-1)} \quad \text{for } \nu \neq 1.
\end{aligned}
\tag{4}
$$

For $\nu = 1$ the equation above cannot be applied (the derivative $d\tau/dv$ diverges) and one has to take $p(v) = \delta(v - c)$. For $\nu = 0$ (step length independent of time) one should take $v(t) = c\delta(t)$, which will define a jump first, then wait variant of the continuous-time random walk (CTRW).

The unpleasant property of the original model is connected with the properties of single incomplete stretches. The velocities in such stretches in the original model are chosen according to the intended stretch duration, and the speeds assigned to the motion over very long stretches for $\nu > 1$ are very high. Since the stretches whose intended duration is long (e.g., longer than the total measurement time t) often stay incomplete, the MSD is bounded from below by the MSD in the realizations of the walk consisting only of a single stretch of duration not less than t. For $\nu > 1$ this MSD is thus $\langle x^2 \rangle \geqslant A(t) \int_{v(t)}^\infty v^2 t^2 p(v) dv$, where $A(t) > 0$ is the portion of single-stretch trajectories among all trajectories and $v(t) = ct^{\nu-1}$ is the lowest speed in stretches which take more than t to complete. For $\nu \geqslant (\gamma + 2)/2$ the MSD obviously diverges, as correctly stated in [9]. Remarkably, such a divergence rules out the existence of the Richardson regime (which otherwise could appear, e.g., for $\nu = 3/2$ and $\gamma < 1$).

The Drude schemes of Refs. [14,15] assume that within the ith stretch the law of particle motion corresponds to $|\Delta x(t)| = c(t - t_{i-1})^\nu$, where t_{i-1} is the change point preceding the stretch, so the motion proceeds not at a constant speed but accelerates or decelerates. We note that the Drude schemes
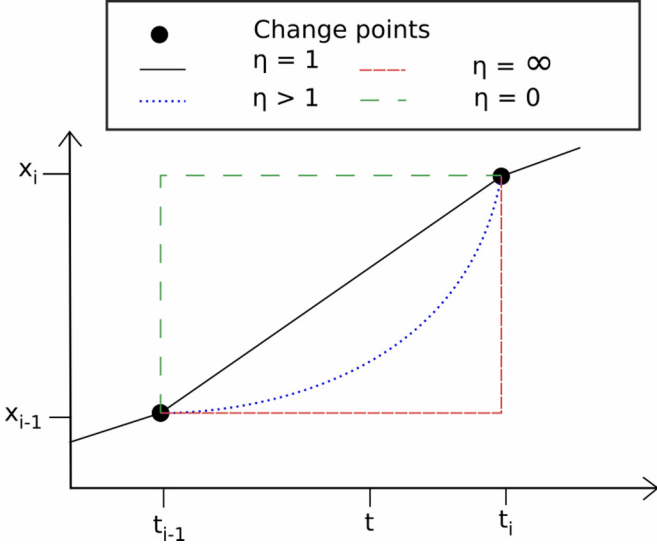
FIG. 1. Comparison of the walker's motion for different values of η: While the positions and times of the change points remain the same, the walker moves at a constant speed for $\eta = 1$ (solid line) and has a nonlinear time dependence of the coordinate for $\eta \neq 1$. In the limits $\eta = 0$ and $\eta = \infty$ the model replicate the time-coupled Lévy flight, where the two limits correspond to the walker jumping first and then waiting or waiting first and then jumping.

of [14,15] differ from the original Drude model [13] by the fact that here the acceleration always shows in the step direction, while in the original model its direction is fixed by the external electric field which introduces a total bias. Such models for Lévy walks were considered in [16,17]. The speed in Drude schemes of [14,15] depends on the time in motion $t' = t - t_{i-1}$ as $v(t') = |\frac{d}{dt'}x(t')| = vct'^{\nu-1}$. The joint probability density of a stretch's length and time in the complete stretch is still given by Eq. (2). However, within an incomplete stretch of duration t' the displacement does not depend on the intended stretch duration and is a deterministic function of the actual time in motion. The discussion above does not apply and the divergences are essentially resolved.

The generalized model mixes these two situations and interpolates between the original and Drude-like models. Here the speed in a stretch depends both on the intended stretch duration τ and on the actual time in motion t' according to

$$v(t', \tau) = \eta c \tau^{\nu-\eta} t'^{\eta-1}, \tag{5}$$

so in a complete stretch we always have

$$|\Delta x| = \int_0^\tau v(t', \tau)dt' = c\tau^\nu. \tag{6}$$

The displacement in an incomplete stretch i is given by

$$|\Delta x| = c\tau_i^{\nu-\eta}(t - t_{i-1})^\eta. \tag{7}$$

The original model corresponds to $\eta = 1$ and the Drude-like model to $\eta = \nu$ (see Fig. 1). In the following we recover the result of Ref. [9] that the MSD can only be finite when the condition $\gamma > 2(\nu - \eta)$ is satisfied and show that this condition is sufficient for the existence of the MSD in the ordinary and in the aged case. This implies that the Drude-like model with $\eta = \nu$ never shows divergences. We consider the

ordinary nonaged situation, when the walk is observed from the instant of preparation at $t = 0$ on, and the aged situations, when the system is prepared at $t = 0$ but observed from the time $t_a > 0$ on so that the MSD is $\langle x^2(t) \rangle = \langle [x(t) - x(t_a)]^2 \rangle$. Here the case of short aging times $t \gg t_a \gg t_0$ and the case of long aging times $t_a \gg t \gg t_0$ have to be distinguished. For the case of no aging and for short aging times we find

$$\langle x^2(t) \rangle \propto \begin{cases} t^\gamma & \text{for } \gamma < 1, \, 2\nu < \gamma \\ t^{2\nu} & \text{for } \gamma < 1, \, 2\nu > \gamma \\ t & \text{for } \gamma > 1, \, 2\nu < \gamma \\ t^{2\nu+1-\gamma} & \text{for } \gamma > 1, \, 2\nu > \gamma \end{cases} \tag{8}$$

whereas in the case of long aging times we obtain

$$\langle x^2(t) \rangle \propto \begin{cases} t_a^{\gamma-1}t & \text{for } 2\nu < \gamma \\ t_a^{\gamma-1}t^{2\nu+1-\gamma} & \text{for } \gamma < 2\nu < 1+\gamma \\ t_a^{2\nu-2}t^2 & \text{for } 1+\gamma < 2\nu \end{cases} \tag{9}$$

for $\gamma < 1$ and

$$\langle x^2(t) \rangle \propto \begin{cases} t & \text{for } 2\nu < \gamma \\ t^{2\nu+1-\gamma} & \text{for } \gamma < 2\nu < 1+\gamma \\ t_a^{2\nu-\gamma-1}t^2 & \text{for } 1+\gamma < 2\nu \end{cases} \tag{10}$$

for $\gamma > 1$. These findings are discussed in more detail in Sec. VIII.

We note that all exponents characterizing the temporal evolution of the MSD are independent of the new parameter η, which only appears in the prefactors of the MSD. Therefore, whenever the MSD is finite its behavior coincides with the one found in [9] for the special case $\eta = 1$. This matches our expectations, because changing η leaves the actual change points of the trajectory unaltered. This also means that for suitable values of η this model is indeed able describe the Richardson law.

III. GENERAL CONSIDERATIONS AND NOTATION

Since the notation in our work will be somewhat more complex than in the standard problem, it is important to make it consistent and clear. There will be two kinds of probabilities repeatedly appearing in the theory: the joint probabilities of displacements and times and the probabilities of displacements conditioned on the temporal variable and possibly on something else.

The first ones have in one dimension the dimension $[L]^{-1}[T]^{-1}$ and will be denoted by capital letters. Thus, let $C(x, t)$ be the joint probability density to land between x and $x + dx$ in a last complete step ending between t and $t + dt$ in an ordinary Lévy walk, observed from the instant of preparation. The observation time in an aged walk is counted from the beginning of the observation, i.e., $t = 0$ corresponds to the time t_a from the preparation of the system and all other times are counted accordingly. Thus $F(x, t|t_a)$ is the joint probability density of the displacement in the very first step and time after the beginning of observation at which it is completed, provided the time between the preparation of the system and the beginning of observation is t_a.

The second ones are the (conditional) probability density $p(x|t)$ of displacement at time t, the quantity we are looking
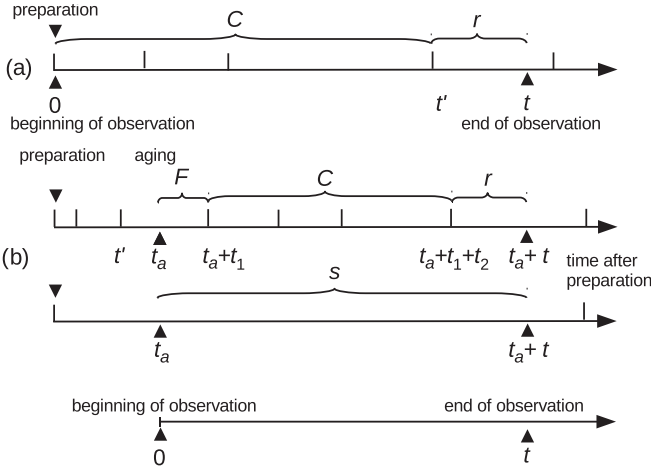
FIG. 2. Times and notation used in ordinary and in aged walks. (a) Ordinary walk starting at preparation at $x = 0$. Here C corresponds to a set of complete stretches (possibly empty) and r to the last, incomplete stretch starting at time t'. (b) Aged walk. The observation starts at $t = 0$, which corresponds to the time t_a after the preparation, as is depicted by the lowest time axis in the figure. The particle's position at the beginning of observation is declared to be the origin. In the aged case two situations are possible. The upper scheme corresponds to the case when the set of change points between the beginning of observation at $t = 0$ and the final observation at t is nonempty. The time t' corresponds to the last change point before the observation starts (it may coincide with the preparation time), the time t_1 to the first change point after the beginning of the observation, and the time t_2 to the last change point before the observation ends (this may coincide with t_1). In addition, F corresponds to the stretch during which the observation started. Other notation is the same as above. The second case corresponds to a situation where there were no change points between $t = 0$ and t. The two cases are mutually exclusive.

for; $r(x|t)$, the conditional probability density of displacement in the last, incomplete, stretch (taking the rest of the observation time from the last change point), provided its duration is t; and $s(x|t, t_a)$, the conditional probability density of the position of the particle x in the case when the first step after the beginning of the observation at $t = 0$ is incomplete at the end of the observation, i.e., at time t. The conditional densities have the dimension $[L]^{-1}$. The choice of letters corresponds to C, complete; r, rest; F, first; and s, single steps. These definitions and the times involved in the integrals below are illustrated in Fig. 2. Then, for the ordinary Lévy walk we have

$$p(x|t) = \int_{-\infty}^{\infty} dx' \int_0^t dt' C(x', t') r(x - x'|t - t'). \quad (11)$$

For the aged Lévy walk we have a more complex structure, namely,

$$p(x|t) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_0^t dt_1 \int_0^{t-t_1} dt_2 F(x_1, t_1|t_a)$$
$$\times C(x_2, t_2) r(x - x_1 - x_2|t - t_1 - t_2) + s(x|t, t_a). \quad (12)$$

In the present work we concentrate on the mean squared displacements in the corresponding walks and note that it

is possible to obtain the MSD without explicitly putting the PDF $p(x|t)$ in the Fourier-Laplace domain, which leads to unnecessarily complicated calculations for the aged case, which we want to treat on the same footing as an ordinary one. To understand the structure we note that the integrals in both Eqs. (11) and (12) have a structure of convolution in their spatial variables. Denoting by $p(k|t)$, $r(k|t)$, $F(k, t|t_a)$, $C(k, t)$, and $s(k|t, t_a)$ the corresponding Fourier transforms and noting that all probability densities p, r, F, C, and s possess mirror symmetry in one dimension or rotational symmetry in higher dimensions, which lets the corresponding first moments in the spatial coordinate vanish, and assuming all of them to possess a finite second spatial moment, we get that the corresponding characteristic functions have the forms

$$p(k|t) = 1 - \tfrac{1}{2}k^2 x_2(t) + o(k^2), \quad (13)$$

$$r(k|t) = r_0(t) - \tfrac{1}{2}k^2 r_2(t) + o(k^2), \quad (14)$$

$$s(k|t, t_a) = s_0(t, t_a) - \tfrac{1}{2}k^2 s_2(t, t_a) + o(k^2), \quad (15)$$

$$C(k, t) = C_0(t) - \tfrac{1}{2}k^2 C_2(t) + o(k^2), \quad (16)$$

$$F(k, t|t_a) = F_0(t|t_a) - \tfrac{1}{2}k^2 F_2(t|t_a) + o(k^2), \quad (17)$$

where $x_2(t) = \langle x^2(t) \rangle$ is the total MSD, $r_2(t) = \int x^2 r(x|t) dx$ is the MSD in the last, incomplete stretch, $s_2(t, t_a) = \int x^2 s(x|t, t_a) dx$ is the same in the case that the last stretch is a single one after the beginning of the observation, $C_2(t) = \int x^2 C(x, t) dx$ is the marginal second moment of displacement in a single complete step, and $F_2(t|t_a) = \int x^2 F(x, t|t_a) dx$ is the same for the first complete step after the beginning of the observation, conditioned on the aging time. The conditions under which the second moments do exist, i.e., the corresponding coefficients in k^2 terms are finite, will follow from the calculations below.

All prefactors of k^2 denoted by small letters have the dimension $[L]^2$ and the ones denoted by capital letters have the dimension $[L]^2[T]^{-1}$. Note that all functions except for p lack normalization on unity; the zero terms of expansions are defined $g_0 = \int g(x|t, t_a) dx \neq 1$, with g being r, s, C, or F.

Passing to the Fourier-Laplace representations of $p(x|t)$ in Eqs. (11) and (12) and taking the corresponding forms of the characteristic functions, we get

$$p(k, s) = C_0(s)r_0(s) - \frac{k^2}{2}[C_0(s)r_2(s) + C_2(s)r_0(s)] + o(k^2) \quad (18)$$

for the ordinary process and

$$p(k|s) = F_0(s|t_a)C_0(s)r_0(s) - \frac{k^2}{2}[F_0(s|t_a)C_0(s)r_2(s)$$
$$+ F_0(s|t_a)C_2(s)r_0(s) + F_2(s|t_a)C_0(s)r_0(s)]$$
$$+ s_0(s, t_a) - \frac{k^2}{2}s_2(s, t_a) + o(k^2) \quad (19)$$

for the aged one, where $r_i(s)$, $C_i(s)$, $F_i(s|t_a)$, and $s_i(s, t_a)$ denote the Laplace transforms of the corresponding functions in their t variable. To obtain the MSD for ordinary walks we will need to get the inverse Laplace transforms of two

combinations $C_0 r_2$ and $C_2 r_0$, and for aged case we will need the inverse transforms of three combinations $F_0(s)C_0(s)r_2(s)$, $F_0(s)C_2(s)r_0(s)$, and $F_2(s)C_0(s)r_0(s)$ and to compute s_2 immediately in the time domain. While the calculation of $C_0(s)$ and $C_2(s)$ may be performed in the Laplace domain, like for the standard model, giving

$$C(k, s) = \frac{1}{1 - \psi(k, s)}, \qquad (20)$$

other integrals have to be first evaluated in the time domain and then transformed to the Laplace domain. After this the combinations $C_0 r_2$ and $C_2 r_0$, as well as $F_0(s)C_0(s)r_2(s)$, $F_0(s)C_2(s)r_0(s)$, and $F_2(s)C_0(s)r_0(s)$, have to be transformed back to the time domain.

The asymptotics of Laplace transforms and the inverse Laplace transforms of functions essentially following power laws are given by Tauberian theorems. These theorems give the asymptotics of a Laplace transform $f(s) = \int_0^\infty f(t)e^{-st}dt$ of a function with the large t behavior $f(t) \simeq t^{\rho-1}L(t)$, where $L(t)$ is slowly varying, in the asymptotic regime $s \to 0$, and the corresponding inverse transforms. In the case where $f(t)$ is nonintegrable due to a divergence at $t \to \infty$, i.e., for $\rho > 0$, Tauberian theorem can be applied directly:

$$f(t) \simeq t^{\rho-1}L(t) \leftrightarrow f(s) \simeq \Gamma(\rho)s^{-\rho}L\left(\frac{1}{s}\right). \qquad (21)$$

In what follows we omit slowly varying functions (i.e., replace them with constants, representing their limiting values L at $t \to \infty$). By this we disregard logarithmic corrections that appear at the points of regime changes. If $f(t)$ is however integrable, i.e., $\rho < 0$, we have to use the more general formula

$$f(s) \simeq \sum_{k=0}^{k_{\max}} \frac{(-1)^k}{k!} I_k^f s^k + (-1)^{k_{\max}+1} L\Gamma(\rho)s^{-\rho}, \qquad (22)$$

where $k_{\max} = \lceil -\rho - 1 \rceil$ is the number of finite repeated integrals of $f(t)$ and I_k^f is the moment integral

$$I_k^f = \int_0^\infty t^k f(t)dt \qquad (23)$$

(see Appendix A for a detailed explanation).

IV. ORDINARY WALK

A. Calculation of $\psi_0(s)$

All properties of the ordinary walk are derived from the waiting-time density $\psi(t)$. The joint probability density of a displacement in a stretch and of the time of stretch is given by

$$\psi(x, t) = \frac{1}{2}\delta(|x| - ct^\nu)\psi(t), \qquad (24)$$

so the Fourier transform of this function in x reads

$$\psi(k, t) = \psi(t) - \frac{k^2}{2}c^2 t^{2\nu}\psi(t) + o(k^2). \qquad (25)$$

Then the Laplace transform in t should be performed. Although the exact Laplace transform of $\psi(t)$ as given by Eq. (1) is possible in quadratures, we are only interested in the asymptotic behavior, which can be found using the Tauberian theorem (22). The forms of the Laplace transform differ for

different relations between the values of γ and ν. We are interested only in the lowest-order terms of s dependence.

1. Condition $\gamma < 1$

For $0 < \gamma < 1$ the function $\psi_0(s) = \psi(s)$ belongs to an integrable class and its Laplace representation reads

$$\psi_0(s) \simeq 1 + \gamma\Gamma(-\gamma)t_0^\gamma s^\gamma = 1 - \Gamma(1 - \gamma)t_0^\gamma s^\gamma. \qquad (26)$$

Keeping t_0 in all calculations and not setting it to unity is reasonable to be able to check the dimension of the ensuing results, especially in the aged case.

2. Condition $\gamma > 1$

The Laplace transform of $\psi(t)$ now has an additional term, due to its first moment being finite:

$$\psi_0(s) \simeq 1 - \tau s - \Gamma(1 - \gamma)t_0^\gamma s^\gamma. \qquad (27)$$

Here τ is defined as

$$\tau = \frac{\gamma}{t_0}\int_0^\infty \frac{t\,dt}{(1 + t/t_0)^{\gamma+1}} = \frac{t_0}{\gamma - 1}. \qquad (28)$$

B. Calculation of $\psi_2(s)$

The marginal second moment of the step distribution is given by

$$\psi_2(t) = \int_{-\infty}^\infty x^2\psi(x, t)dx = c^2 t^{2\nu}\psi(t). \qquad (29)$$

The expressions for $\psi_2(s)$ depend on whether $2\nu < \gamma$ or $2\nu > \gamma$.

1. Condition $2\nu < \gamma$

In this first case the function $\psi_2(t)$ is integrable, $\int_0^\infty \psi_2(t)dt < \infty$, and the expansion of its Laplace transform starts from a constant,

$$\psi_2(s) \simeq \gamma c^2 t_0^{2\nu}\int_0^\infty \frac{x^{2\nu}}{(1 + x)^{\gamma+1}}dx + \gamma\Gamma(2\nu - \gamma)c^2 t_0^\gamma s^{\gamma-2\nu}, \qquad (30)$$

where the integral is given by the dimensionless constant

$$\int_0^\infty \frac{x^{2\nu}}{(1 + x)^{\gamma+1}}dx = B(2\nu + 1, \gamma - 2\nu), \qquad (31)$$

with $B(a, b)$ the Beta function [see Eq. (2.2.4.24) of Ref. [18]].

2. Condition $2\nu > \gamma$

In the second case $\psi_2(t)$ is nonintegrable, the integral $\int_0^\infty \psi_2(t)dt$ diverges, and the asymptotics of its Laplace transform read

$$\psi_2(s) \simeq \gamma\Gamma(2\nu - \gamma)c^2 t_0^\gamma s^{\gamma-2\nu}. \qquad (32)$$

C. Calculation of $C_0(s)$ and $C_2(s)$

For complete steps our generalization does not differ from the original Lévy walk and we can use the general result for

$C(x, t)$ in the Fourier-Laplace domain [19]

$$C(k, s) = \frac{1}{1 - \psi(k, s)}. \tag{33}$$

Expanding $C(k, s)$ for s small (and fixed) and for $k \to 0$, we get

$$C(k, s) \simeq \frac{1}{1 - \psi(s) + (k^2/2)\psi_2(s) + o(k^2)}$$
$$\simeq \frac{1}{1 - \psi(s)} - \frac{k^2}{2} \frac{\psi_2(s)}{[1 - \psi(s)]^2} + o(k^2). \tag{34}$$

Therefore,

$$C_0(s) = \frac{1}{1 - \psi(s)}, \tag{35}$$

$$C_2(s) = \frac{\psi_2(s)}{[1 - \psi(s)]^2}, \tag{36}$$

into which we now have to insert our results for ψ_0 and ψ_2.

1. Conditions $\gamma < 1$ and $2\nu < \gamma$

Here we find, by using Eqs. (26) and (30),

$$C_0(s) \simeq \frac{1}{\Gamma(1 - \gamma)} t_0^{-\gamma} s^{-\gamma}, \tag{37}$$

$$C_2(s) \simeq \gamma \frac{B(2\nu + 1, \gamma - 2\nu)}{\Gamma^2(1 - \gamma)} c^2 t_0^{2\nu - 2\gamma} s^{-2\gamma}. \tag{38}$$

2. Conditions $\gamma < 1$ and $2\nu > \gamma$

In this case we obtain, for C_2,

$$C_2(s) \simeq \gamma \frac{\Gamma(2\nu - \gamma)}{\Gamma^2(1 - \gamma)} c^2 t_0^{-\gamma} s^{-2\nu - \gamma}, \tag{39}$$

while C_0 is the same as in Eq. (37) as long as $\gamma < 1$.

3. Conditions $\gamma > 1$ and $2\nu < \gamma$

Now the first moment of $\psi(t)$ is finite; therefore,

$$C_0(s) \simeq \frac{1}{\tau s}, \tag{40}$$

$$C_2(s) \simeq \gamma B(2\nu + 1, \gamma - 2\nu) c^2 \frac{t_0^{2\nu}}{\tau^2} s^{-2}. \tag{41}$$

4. Conditions $\gamma > 1$ and $2\nu > \gamma$

Again, $C_0(s)$ does not change, and for C_2 we obtain

$$C_2 \simeq \gamma \Gamma(2\nu - \gamma) c^2 \frac{t_0^{\gamma}}{\tau^2} s^{\gamma - 2\nu - 2}. \tag{42}$$

D. Calculation of $r_0(s)$ and $r_2(s)$

The function $r(x|t)$ gives the distribution of the corresponding displacements conditioned on the fact that the total duration of a stretch is longer than t:

$$r(x|t) = \int_t^\infty \frac{1}{2} \delta(|x| - ct^\eta t'^{\nu - \eta}) \psi(t') dt'. \tag{43}$$

This is correctly normalized to the overall probability to stay within a single stretch for a time longer than t,

$$\int r(x|t) dx = \int_t^\infty \psi(t') dt'. \tag{44}$$

Expanding the Fourier transform of $r(x, t)$ for small k,

$$r(k|t) = \int_{-\infty}^\infty dx \, e^{ikx} \int_t^\infty \frac{1}{2} \delta(|x| - ct^\eta t'^{\nu - \eta}) \psi(t') dt'$$
$$= r_0(t) - \frac{k^2}{2} r_2(t) + o(k^2), \tag{45}$$

we find the marginal moments

$$r_0(t) = \frac{1}{(1 + t/t_0)^\gamma}, \tag{46}$$

$$r_2(t) = \gamma c^2 \frac{1}{t_0} t^{2\eta} \int_t^\infty \frac{t'^{2(\nu - \eta)}}{(1 + t'/t_0)^{\gamma + 1}} dt', \tag{47}$$

whose Laplace transforms depend on the relationship between γ, ν, and η.

1. Condition $\gamma < 1$

In this case r_0 is nonintegrable and we find

$$r_0(s) \simeq \Gamma(1 - \gamma) t_0^\gamma s^{\gamma - 1}. \tag{48}$$

For r_2 we can use the asymptotic form of $\psi(t)$, since we are interested in large t,

$$r_2(t) \simeq \gamma c^2 t_0^\gamma t^{2\eta} \int_t^\infty \tau^{2(\nu - \eta) - 1 - \gamma} d\tau, \tag{49}$$

TABLE I. Leading terms of the marginal moments of C and r in the Laplace domain for different parameter ranges.

γ	r_0, C_0	r_2, C_2	
$\gamma < 1$	$C_0(s) \simeq \frac{1}{\Gamma(1-\gamma)} t_0^{-\gamma} s^{-\gamma}$	$C_2(s) \simeq \begin{cases} \gamma \frac{B(2\nu+1, \gamma-2\nu)}{\Gamma^2(1-\gamma)} c^2 t_0^{2\nu-2\gamma} s^{-2\gamma} & \text{for } 2\nu < \gamma \\ \gamma \frac{\Gamma(2\nu-\gamma)}{\Gamma^2(1-\gamma)} c^2 t_0^{-\gamma} s^{-2\nu-\gamma} & \text{for } 2\nu > \gamma \end{cases}$	
$\gamma < 1$	$r_0(s) \simeq \Gamma(1-\gamma) t_0^\gamma s^{\gamma-1}$	$r_2(s) \simeq \gamma \frac{\Gamma(2\nu+1-\gamma)}{\gamma - 2(\nu-\eta)} c^2 t_0^\gamma s^{\gamma-1-2\nu}$	
$\gamma > 1$	$C_0(s) \simeq \frac{1}{\tau s}$	$C_2(s) \simeq \begin{cases} \gamma B(2\nu+1, \gamma-2\nu) c^2 \frac{t_0^{2\nu}}{\tau^2} s^{-2} & \text{for } 2\nu < \gamma \\ \gamma \Gamma(2\nu-\gamma) c^2 \frac{t_0^\gamma}{\tau^2} s^{\gamma-2\nu-2} & \text{for } 2\nu > \gamma \end{cases}$	
$\gamma > 1$	$r_0(s) \simeq \tau$	$r_2(s) \simeq \begin{cases} \gamma \frac{B(2\nu+2, \gamma-2\nu-1)}{2\eta+1} c^2 t_0^{2\nu+1} & \text{for } 2\nu < \gamma - 1 \\ \gamma \frac{\Gamma(2\nu+1-\gamma)}{\gamma-2(\nu-\eta)} c^2 t_0^\gamma s^{\gamma-2\nu-1} & \text{for } 2\nu > \gamma - 1 \end{cases}$	

which is only finite for $\gamma > 2(\nu - \eta)$ and diverges otherwise, meaning that no MSD exists. In the rest of the paper we will concentrate on the case when the second moment converges, where we find

$$r_2(t) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} c^2 t_0^\gamma t^{2\nu - \gamma}. \tag{50}$$

Since this expression is always in the nonintegrable class, we obtain, by using the Tauberian theorem (21),

$$r_2(s) \simeq \gamma \frac{\Gamma(2\nu + 1 - \gamma)}{\gamma - 2(\nu - \eta)} c^2 t_0^\gamma s^{\gamma - 2\nu - 1}. \tag{51}$$

2. Conditions $\gamma > 1$ and $2\nu > \gamma - 1$

For $\gamma > 1$, $r_0(t)$ becomes integrable; therefore, its Laplace transform reads

$$r_0 \simeq \tau + \Gamma(1 - \gamma) t_0^\gamma s^{\gamma - 1}. \tag{52}$$

The function $r_2(t)$ is still nonintegrable and therefore identical to the previous case.

3. Conditions $\gamma > 1$ and $2\nu < \gamma - 1$

In this case r_0 does not change but $r_2(s)$ is now integrable and therefore its transform behaves as

$$r_2(s) \simeq I_0^{r_2} - \gamma \frac{\Gamma(2\nu + 1 - \gamma)}{2(\nu - \eta) - \gamma} c^2 t_0^\gamma s^{\gamma - 2\nu - 1}. \tag{53}$$

We evaluate the definite integral $I_0^{r_2}$ by manipulating the area of integration

$$
\begin{aligned}
I_0^{r_2} &= c^2 \int_0^\infty dt\, t^{2\eta} \int_t^\infty dt'\, \psi(t')(t')^{2(\nu - \eta)} \\
&= c^2 \frac{1}{2\eta + 1} \int_0^\infty \psi(t') t'^{2\nu + 1} dt' \\
&= \gamma \frac{1}{2\eta + 1} \int_0^\infty \frac{x^{2\nu + 1} dx}{(1 + x)^{\gamma + 1}} c^2 t_0^{2\nu + 1} \\
&= \gamma \frac{B(2\nu + 2, \gamma - 2\nu - 1)}{2\eta + 1} c^2 t_0^{2\nu + 1},
\end{aligned} \tag{54}
$$

meaning that r_2 is constant in the leading order. Therefore, we find, for $\gamma > 1$ and $2\nu < \gamma - 1$,

$$
\begin{aligned}
r_2 \simeq{}& \gamma \frac{B(2\nu + 2, \gamma - 2\nu - 1)}{2\eta + 1} c^2 t_0^{2\nu + 1} \\
&- \gamma \frac{\Gamma(2\nu + 1 - \gamma)}{2(\nu - \eta) - \gamma} c^2 t_0^\gamma s^{\gamma - 2\nu - 1}.
\end{aligned} \tag{55}
$$

The results so far are summarized in Table I.

E. Mean squared displacement

With these results we can now compute the MSD via the formula

$$\langle x^2(s) \rangle = C_0(s) r_2(s) + C_2(s) r_0(s). \tag{56}$$

1. Conditions $\gamma < 1$ and $2\nu < \gamma$

In the case $2\nu < \gamma$ we have

$$
\begin{aligned}
\langle x^2(s) \rangle \simeq{}& \gamma \bigg[\frac{\Gamma(2\nu + 1 - \gamma)}{\Gamma(1 - \gamma)[\gamma - 2(\nu - \eta)]} c^2 s^{-2\nu - 1} \\
&+ \frac{B(2\nu + 1, \gamma - 2\nu)}{\Gamma(1 - \gamma)} c^2 t_0^{2\nu - \gamma} s^{-\gamma - 1} \bigg], \tag{57}
\end{aligned}
$$

which translates to

$$
\begin{aligned}
\langle x^2(t) \rangle \simeq{}& \gamma \bigg[\frac{\Gamma(2\nu + 1 - \gamma)}{\Gamma(1 - \gamma)[\gamma - 2(\nu - \eta)]\Gamma(2\nu + 1)} c^2 t^{2\nu} \\
&+ \frac{B(2\nu + 1, \gamma - 2\nu)}{\Gamma(1 - \gamma)\Gamma(1 + \gamma)} c^2 t_0^{2\nu - \gamma} t^\gamma \bigg]. \tag{58}
\end{aligned}
$$

This is dominated by the second term since $2\nu < \gamma$, leading to $\langle x^2(t) \rangle \propto t^\gamma$ in this case. This means that for $\gamma < 1$ and $2\nu < \gamma$ the behavior of the walk merges with the one of a CTRW with a fixed step length.

2. Conditions $\gamma < 1$ and $2\nu > \gamma$

In this parameter regime we obtain

$$
\begin{aligned}
\langle x^2(s) \rangle \simeq{}& \gamma c^2 s^{-2\nu - 1} \\
&\times \bigg[\frac{\Gamma(2\nu + 1 - \gamma)}{\Gamma(1 - \gamma)(\gamma - 2(\nu - \eta))} + \frac{\Gamma(2\nu - \gamma)}{\Gamma(1 - \gamma)} \bigg] \\
={}& \gamma \frac{\Gamma(2\nu - \gamma)}{\Gamma(1 - \gamma)} \frac{2\eta}{\gamma - 2(\nu - \eta)} c^2 s^{-2\nu - 1}; \tag{59}
\end{aligned}
$$

therefore, we find in the time domain

$$\langle x^2(t) \rangle \simeq \gamma \frac{\Gamma(2\nu - \gamma)}{\Gamma(2\nu + 1)\Gamma(1 - \gamma)} \frac{2\eta}{2(\nu - \eta) - \gamma} c^2 t^{2\nu}. \tag{60}$$

3. Conditions $\gamma > 1$ and $2\nu < \gamma - 1$

In this case the MSD reads

$$
\begin{aligned}
\langle x^2(s) \rangle ={}& C_0(s) r_2(s) + C_2(s) r_0(s) \\
\simeq{}& \gamma \frac{B(2\nu + 2, \gamma - 2\nu - 1)}{2\eta + 1} c^2 \frac{t_0^{2\nu + 1}}{\tau} \frac{1}{s} \\
&+ \gamma B(2\nu + 1, \gamma - 2\nu) c^2 \frac{t_0^{2\nu}}{\tau} s^{-2}, \tag{61}
\end{aligned}
$$

which is dominated by the second term. Therefore, we find in the time domain in leading order

$$\langle x^2(t) \rangle \simeq \gamma B(2\nu + 1, \gamma - 2\nu) c^2 \frac{t_0^{2\nu}}{\tau} t. \tag{62}$$

4. Conditions $\gamma > 1$ and $\gamma - 1 < 2\nu < \gamma$

Compared to the previous case only r_2 changes; therefore,

$$
\begin{aligned}
\langle x^2(s) \rangle \simeq{}& \gamma \frac{\Gamma(2\nu + 1 - \gamma)}{\gamma - 2(\nu - \eta)} c^2 \frac{t_0^\gamma}{\tau} s^{\gamma - 2 - 2\nu} \\
&+ \gamma B(2\nu + 1, \gamma - 2\nu) c^2 \frac{t_0^{2\nu}}{\tau} s^{-2}. \tag{63}
\end{aligned}
$$

Since $\gamma - 2\nu > 0$ the term quadratic in s is again dominant and the asymptotic behavior in the time domain is identical to the previous case:

$$\langle x^2(t) \rangle \simeq \gamma B(2\nu + 1, \gamma - 2\nu) c^2 \frac{t_0^{2\nu}}{\tau} t. \tag{64}$$

5. Conditions $\gamma > 1$ and $2\nu > \gamma$

Now C_2 is different, giving us

$$\langle x^2(s)\rangle \simeq \gamma \frac{\Gamma(2\nu+1-\gamma)}{\gamma-2(\nu-\eta)}c^2\frac{t_0^\gamma}{\tau}s^{\gamma-2\nu-2}$$

$$+ \gamma\Gamma(2\nu-\gamma)c^2\frac{t_0^\gamma}{\tau}s^{\gamma-2\nu-2}. \qquad (65)$$

This leads to the time dependence

$$\langle x^2(t)\rangle \simeq \gamma\frac{2\eta}{\gamma-2(\nu-\eta)}\frac{\Gamma(2\nu-\gamma)}{\Gamma(2\nu+2-\gamma)}c^2\frac{t_0^\gamma}{\tau}t^{2\nu+1-\gamma}. \quad (66)$$

The results for the ordinary walk under the assumption that the convergence condition $\gamma > 2(\nu - \eta)$ is satisfied can be summarized as follows:

$$\langle x^2(t)\rangle \propto \begin{cases} t^\gamma & \text{for } \gamma < 1, \ 2\nu < \gamma \\ t^{2\nu} & \text{for } \gamma < 1, \ 2\nu > \gamma \\ t & \text{for } \gamma > 1, \ 2\nu < \gamma \\ t^{2\nu+1-\gamma} & \text{for } \gamma > 1, \ 2\nu > \gamma. \end{cases} \quad (67)$$

Thus, in the whole domain of γ there are four regimes with crossovers at $\gamma = 1$ and at $2\nu = \gamma$: For $2\nu < \gamma$ one has $\langle x^2(t)\rangle \propto t^\gamma$ for $\gamma < 1$, crossing over to a faster growth $\langle x^2(t)\rangle \propto t$ for $\gamma > 1$, and for $2\nu > \gamma$ one has universally $\langle x^2(t)\rangle \propto t^{2\nu}$ for $\gamma < 1$, crossing over to a slower growth $\langle x^2(t)\rangle \propto t^{2\nu+1-\gamma}$ for $\gamma > 1$. Note that in an ordinary walk the Richardson regime is possible and is achieved for $\nu = 3/2$ for $\gamma < 1$ and for $\nu = (\gamma + 2)/2$ for $\gamma > 1$, provided $\gamma > 2(\nu - \eta)$. The last restriction corresponds to $\eta > (3 - \gamma)/2$ for $\gamma < 1$ and to $\eta > 1$ for $\gamma > 1$. The original model with $\eta = 1$ indeed does not possess a Richardson regime.

V. AGED WALK: GENERAL EXPRESSIONS

We now consider the functions F and s which are specific for aged walks. The general expression for F reads

$$F(x,t|t_a) = \int_0^{t_a} dt'\psi(t_a+t-t')k(t')\delta(x-c[(t_a+t-t')^\nu$$

$$- (t_a+t-t')^{\nu-\eta}(t_a-t')^\eta]), \qquad (68)$$

where $k(t) = C_0(t)$ is the time-dependent rate of steps. Note that the argument of the δ function is shifted, due to the fact that the distance from the origin x is set to zero at the start of the measurement. The marginal normalization of $F(x,t|t_a)$ is

$$F_0(t|t_a) = \int F(x,t|t_a)dx = \int_0^{t_a}\psi(t_a+t-t')k(t')dt'$$

$$= \psi_1(t|t_a), \qquad (69)$$

where $\psi_1(t|t_a)$ is the forward waiting-time PDF known from the theory of the CTRW [19]. The marginal second moment of F reads

$$F_2(t|t_a) = \int_0^{t_a} dt'c^2\psi(t_a+t-t')k(t')[(t_a+t-t')^\nu$$

$$- (t_a+t-t')^{\nu-\eta}(t_a-t')^\eta]^2. \qquad (70)$$

Additionally, we have to consider the term $s(x|t,t_a)$, which describes the case that both the aging time and the observation time belong to the same stretch,

$$s(x|t,t_a) = \int_0^{t_a} dt'k(t')\int_{t_a+t-t'}^\infty dt''\psi(t'')\delta(x-c[(t'')^{\nu-\eta}$$

$$\times (t_a+t-t')^\eta - (t'')^{\nu-\eta}(t_a-t')^\eta]), \qquad (71)$$

where the inner integral gives the probability that no renewal took place during the time interval between t' and $t_a + t$. The normalization of this function, giving s_0,

$$s_0(t,t_a) = \int_0^{t_a} \Psi(t_a+t-t')k(t')dt', \qquad (72)$$

where $\Psi(t) = \int_t^\infty \psi(t')dt'$ is the survival probability, is not necessary for what follows and will not be calculated. The second moment is given by

$$s_2(t,t_a) = c^2\int_0^{t_a} k(t')\int_{t_a+t-t'}^\infty \psi(t'')[(t'')^{\nu-\eta}(t_a+t-t')^\eta$$

$$- (t'')^{\nu-\eta}(t_a-t')^\eta]^2dt''dt'. \qquad (73)$$

We note that the form of the integrals involved in F_2 and s_2 is very similar. In the following calculation we differentiate between two time regimes, the case of short aging times $t \gg t_a \gg t_0$ and the case of long aging times $t_a \gg t \gg t_0$, which will be discussed separately in the two following sections.

VI. SHORT AGING TIMES $t \gg t_a$

A. Calculation of F_0 and F_2

We are interested in the Laplace transforms of the marginal moments of F_0 and F_2. Since both of them depend on $k(t) = C_0(t)$ whose form we found to be dependent on whether $\gamma < 1$ or $\gamma > 1$, we have to distinguish between these cases.

1. Condition $\gamma < 1$

The function $F_0(t|t_a)$ is equal to the forward-waiting-time PDF $\psi_1(t|t_a)$. For $\gamma < 1$ it is given by the following expression:

$$F_0(t|t_a) = \psi_1(t|t_a) = \frac{\sin\pi\gamma}{\pi}\left(\frac{t_a}{t}\right)^\gamma\frac{1}{t+t_a}. \qquad (74)$$

The expression is normalized to unity, and therefore in the Laplace domain the leading term in F_0 will be 1.

For $t \gg t_a$ the expression in the square brackets in Eq. (70) can be approximated by $t^{2\nu}$ (since $\nu > 0$) and for F_2 we find in this limit

$$F_2(t|t_a) \simeq c^2t^{2\nu}\int_0^{t_a}\psi(t_a+t-t')k(t')dt'. \qquad (75)$$

The integral can again be expressed through the forward waiting time $\psi_1(t|t_a)$. We take the asymptotics of ψ_1 for t large, so

$$F_2(t|t_a) \simeq \frac{\sin\pi\gamma}{\pi}c^2t_a^\gamma t^{2\nu-\gamma-1}. \qquad (76)$$

Here again two situations arise depending on the integrability.

2. Conditions $\gamma < 1$ and $2\nu > \gamma$

In this case F_2 is nonintegrable, so in the Laplace domain

$$F_2(s|t_a) \simeq \Gamma(2\nu - \gamma)\frac{\sin \pi \gamma}{\pi}c^2 t_a^\gamma s^{\gamma - 2\nu}. \tag{77}$$

3. Conditions $\gamma < 1$ and $2\nu < \gamma$

Now F_2 is integrable and the lowest order in its Laplace transform tends to a constant:

$$F_2(s|t_a) \simeq \frac{\sin \pi \gamma}{\pi}c^2 t_a^\gamma \int_0^\infty \frac{t^{2\nu - \gamma}}{t + t_a}dt. \tag{78}$$

The corresponding integral is given by

$$\int_0^\infty \frac{t^{2\nu - \gamma}}{t + t_a}dt = \frac{\pi}{\sin[\pi(2\nu + 1 - \gamma)]}t_a^{2\nu - \gamma} \tag{79}$$

[see Eq. (2.2.5.25) of Ref. [18]], so

$$F_2(s|t_a) \simeq \frac{\sin \pi \gamma}{\sin[\pi(2\nu + 1 - \gamma)]}c^2 t_a^{2\nu}. \tag{80}$$

4. Condition $\gamma > 1$

Now we consider the case $\gamma > 1$. From the preceding section we know that $C_0(s) = \frac{1}{\tau s}$, and therefore

$$k(s) = \frac{1}{\tau s}. \tag{81}$$

With this we can rewrite F_0 as

$$F_0(t|t_a) = \frac{1}{\tau}\int_0^{t_a} \psi(t + y)dy. \tag{82}$$

For $t \to \infty$ it decays as $t^{-\gamma - 1}$ and therefore is of integrable type. To find its lowest-order (constant) term we note that

$$\begin{aligned} F_0 &= \frac{1}{\tau}\int_0^\infty dt \int_0^{t_a} \psi(t + y)dy = \frac{1}{\tau}\int_0^{t_a} dy \int_0^\infty dt\, \psi(t + y) \\ &= \frac{1}{\tau}\int_0^{t_a} dy \int_y^\infty dt\, \psi(t) = \frac{1}{\tau}\int_0^{t_a} \Psi(y)dy, \end{aligned} \tag{83}$$

which tends to unity since $t_a \gg t_0$ and since $\int_0^\infty \Psi(t')dt' = \tau$. The term F_2 for $t \gg t_a$ can again be evaluated by approximating the expression in square brackets in Eq. (70) by t^ν:

$$\begin{aligned} F_2(t|t_a) &\simeq c^2 \frac{1}{\tau}t^{2\nu} \int_0^{t_a} \psi(t + y)dy \\ &= c^2 \frac{t_0^\gamma}{\tau}[(t + t_0)^{-\gamma} - (t + t_a + t_0)^{-\gamma}]t^{2\nu}. \end{aligned} \tag{84}$$

Since the power-law asymptotics of the expression in square brackets is $t^{-\gamma - 1}$, the whole expression

$$F_2(t|t_a) \simeq \gamma c^2 \frac{t_0^\gamma}{\tau}t_a t^{2\nu - \gamma - 1} \tag{85}$$

is of the nonintegrable type for $2\nu > \gamma$ and of integrable type for $2\nu < \gamma$.

5. Conditions $\gamma > 1$ and $2\nu > \gamma$

In this first case $F_2(t|t_a)$ is nonintegrable; therefore, the Laplace transform is

$$F_2(s|t_a) \simeq \gamma\Gamma(2\nu - \gamma)c^2 \frac{t_0^\gamma}{\tau}t_a s^{\gamma - 2\nu}. \tag{86}$$

6. Conditions $\gamma > 1$ and $2\nu < \gamma$

In the second case the Laplace transform of the expression tends to a constant. To evaluate this we write

$$F_2 \simeq \gamma c^2 \frac{t_0^\gamma}{\tau}\int_0^\infty dt\, t^{2\nu}\int_0^{t_a} \frac{1}{(t + t_0 + y)^{\gamma + 1}}dy \tag{87}$$

and interchange the sequence of integrations:

$$\begin{aligned} F_2 &= \gamma c^2 \frac{t_0^\gamma}{\tau}\int_0^{t_a} dy \int_0^\infty \frac{t^{2\nu}}{(t + t_0 + y)^{\gamma + 1}}dt \\ &= \gamma B(2\nu + 1, \gamma - 2\nu)c^2 \frac{t_0^\gamma}{\tau}\int_0^{t_a}(t_0 + y)^{2\nu - \gamma}dy \\ &= \gamma \frac{B(2\nu + 1, \gamma - 2\nu)}{2\nu + 1 - \gamma}c^2 \frac{t_0^\gamma}{\tau}[(t_0 + t_a)^{2\nu + 1 - \gamma} - t_0^{2\nu + 1 - \gamma}] \end{aligned} \tag{88}$$

[in the transition to the second line Eq. (2.2.5.24) of Ref. [18] is used]. The corresponding expression is dominated by the first or by the second term in square brackets, depending on whether $2\nu > \gamma - 1$ or $2\nu < \gamma - 1$.

We summarize our results for $\gamma > 1$ in the following formula:

$$F_2 = \begin{cases} \gamma\frac{B(2\nu + 1, \gamma - 2\nu)}{\gamma - 1 - 2\nu}c^2\frac{t_0^{2\nu + 1}}{\tau} & \text{for } 2\nu < \gamma - 1 \\ \gamma\frac{B(2\nu + 1, \gamma - 2\nu)}{2\nu - \gamma + 1}c^2\frac{t_0^\gamma}{\tau}t_a^{2\nu + 1 - \gamma} & \text{for } \gamma - 1 < 2\nu < \gamma \\ \gamma\Gamma(2\nu - \gamma)c^2\frac{t_0^\gamma}{\tau}t_a s^{\gamma - 2\nu} & \text{for } 2\nu > \gamma. \end{cases} \tag{89}$$

B. Calculation of s_2

We can calculate the second marginal moment of the single step PDF $s_2(t, t_a)$ directly in the time domain. For this we need the stepping rate $k(t) = C_0(t)$, whose behavior depends on whether $\gamma > 1$ or $\gamma < 1$.

1. Condition $\gamma < 1$

In this case we find, by inverse transform of C_0 from Table I,

$$k(t) = \frac{1}{\Gamma(\gamma)\Gamma(1 - \gamma)}t_0^{-\gamma}t^{\gamma - 1} = \frac{\sin \pi \gamma}{\pi}t_0^{-\gamma}t^{\gamma - 1}. \tag{90}$$

Inserting this result into Eq. (73), we find

$$\begin{aligned} s_2(t, t_a) &= \gamma\frac{\sin(\pi\gamma)}{\pi}c^2 \int_0^{t_a}\int_{t_a + t - t'}^\infty [(t_a + t - t')^\eta - (t_a - t')^\eta]^2 \\ &\times (t')^{\gamma - 1}(t'')^{2(\nu - \eta)}\frac{1}{(t_0 + t'')^{\gamma + 1}}dt''dt'. \end{aligned} \tag{91}$$

Just like r_2 in the ordinary case, this integral only converges for $\gamma > 2(\nu - \eta)$, meaning that η again governs the existence

TABLE II. Results for F_0 and F_2 in the Laplace domain as well as s_2 in the time domain for different parameter ranges in the case of weak aging $t \gg t_a$. Dimensionless prefactors are omitted.

γ	F_0	F_2, s_2	
$\gamma < 1$	$F_0(s\|t_a) \simeq 1$	$F_2(s\|t_a) \propto \begin{cases} c^2 t_a^{2\nu} & \text{for } 2\nu < \gamma \\ c^2 t_a^{\gamma} s^{\gamma - 2\nu} & \text{for } 2\nu > \gamma \end{cases}$	
$\gamma < 1$		$s_2(t, t_a) \propto c^2 t_a^{\gamma} t^{2\nu - \gamma}$	
$\gamma > 1$	$F_0(s\|t_a) \simeq 1$	$F_2(s\|t_a) \propto \begin{cases} c^2 t_0^{2\nu} & \text{for } 2\nu < \gamma - 1 \\ c^2 t_0^{\gamma-1} t_a^{2\nu+1-\gamma} & \text{for } \gamma - 1 < 2\nu < \gamma \\ c^2 t_0^{\gamma-1} t_a s^{\gamma - 2\nu} & \text{for } 2\nu > \gamma \end{cases}$	
$\gamma > 1$		$s_2(t, t_a) \simeq c^2 t_0^{\gamma-1} t_a t^{2\nu - \gamma}$	

of the second moment. In the limit $t_0 \ll t''$ we can write

$$s_2(t, t_a) \simeq \gamma \frac{\sin(\pi\gamma)}{\pi} \frac{1}{\gamma - 2(\nu - \eta)} c^2 \int_0^{t_a} (t_a + t - t')^{2(\nu-\eta)-\gamma}$$
$$\times [(t_a + t - t')^{\eta} - (t_a - t')^{\eta}]^2 (t')^{\gamma-1} dt'. \quad (92)$$

The expression in square brackets is again approximated by $t^{2\eta}$ and $(t_a + t - t')$ by its value at t, so

$$s_2(t, t_a) \propto c^2 t_a^{\gamma} t^{2\nu - \gamma}. \quad (93)$$

2. Condition $\gamma > 1$

In this regime we have $k(t) = \frac{1}{\tau}$ again. Substituting this into Eq. (73) and approximating the term in square brackets again results in

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} c^2 \frac{t_0^{\gamma}}{\tau} t^{2\eta}$$
$$\times \int_0^{t_a} (t_a + t - t')^{2(\nu-\eta)-\gamma} dt', \quad (94)$$

which gives us

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} c^2 \frac{t_0^{\gamma}}{\tau} t_a t^{2\nu - \gamma}. \quad (95)$$

The results so far are summarized in Table II, where we used the fact that $\tau \propto t_0$ [see Eq. (28)].

C. Mean squared displacement

We are now ready to calculate the MSD in the weakly aged case. Recall our earlier result

$$\langle x^2 \rangle(s|t_a) = F_0(s|t_a) C_0(s) r_2(s) + F_0(s|t_a) C_2(s) r_0(s)$$
$$+ F_2(s|t_a) C_0(s) r_0(s) + s_2(s, t_a). \quad (96)$$

We can now write the first three terms in the Laplace domain using the results from Tables I and II and transform them back into the time domain. The last term in the time domain is already known. The calculation results in different asymptotics depending on γ.

1. Condition $\gamma < 1$

In this regime the terms $F_0 C_0 r_2$ and $F_0 C_2 r_0$ reproduce the result for the nonaged walks. The term $F_2 C_0 r_0(s) \propto t_a^{\gamma} s^{-2\nu-1}$ translates into $F_2 C_0 r_0(t) \propto c^2 t^{2\nu} (t_a/t)^{\gamma}$ and is subdominant for $t \gg t_a$ for $2\nu > \gamma$. For $2\nu < \gamma$ this term tends to const $\times$

$c^2 t_a^{2\nu} s^{-1}$, i.e., is a constant proportional to $c^2 t_a^{2\nu}$ in the time domain, and is again subdominant with respect to the previous ones. The term s_2 has the same asymptotics as the previous one in the first case, $s_2 \propto c^2 t^{2\nu} (t_a/t)^{\gamma}$, and therefore is also subdominant.

2. Condition $\gamma > 1$

For this regime the contributions $F_0 C_0 r_2$ and $F_0 C_2 r_0$ give the same behavior as in the nonaged case, proportional to $t^{2\nu+1-\gamma}$ for $2\nu > \gamma$ or proportional to t in the opposite case. The contribution $F_2 C_0 r_0(s)$ corresponds to $s^{\gamma-2\nu-1}$ and translates to either $t^{2\nu-\gamma}$ for $2\nu > \gamma$ or a constant for $2\nu < \gamma$ and is always subdominant. The contribution of s_2 is always subdominant as well.

In conclusion, we find that the behavior for short aging times reproduces the behavior of the ordinary walk up to prefactors, as one might expect, and has the same range of convergence, namely, the second moment exists for $\gamma > 2(\nu - \eta)$.

VII. LONG AGING TIMES $t_a \gg t$

A. Calculation of F_0 and F_2

Here again the cases $\gamma < 1$ and $\gamma > 1$ have to be distinguished.

1. Condition $\gamma < 1$

In this domain we can reuse the previous result in Eq. (74), but we now expand it for $t_a \gg t$,

$$F_0(t|t_a) = \frac{\sin \pi\gamma}{\pi} \left(\frac{t_a}{t}\right)^{\gamma} \frac{1}{t + t_a} \simeq \frac{\sin \pi\gamma}{\pi} t_a^{\gamma-1} t^{-\gamma}, \quad (97)$$

so we get in the Laplace domain

$$F_0(s|t_a) \simeq \frac{\sin \pi\gamma}{\pi} \Gamma(1 - \gamma) t_a^{\gamma-1} s^{\gamma-1}. \quad (98)$$

For $F_2(t|t_a)$ we can use our result for $k(t)$, Eq. (90), and insert it into (70):

$$F_2(t|t_a) = \gamma \frac{\sin(\pi\gamma)}{\pi} c^2 \int_0^{t_a} (t_a + t - t')^{2(\nu-\eta)}$$
$$\times [(t_a + t - t')^{\eta} - (t_a - t')^{\eta}]^2$$
$$\times \frac{(t')^{\gamma-1}}{(t_0 + t_a + t - t')^{1+\gamma}} dt'. \quad (99)$$

Neglecting t_0 in the expression in the last line, we get

$$F_2(t|t_a) \simeq \gamma \frac{\sin(\pi \gamma)}{\pi} c^2 t_a^{2\nu-2} \int_0^{t_a} \left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^{2(\nu-\eta)-\gamma-1}$$
$$\times \left[\left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^{\eta} - \left(1 - \frac{t'}{t_a}\right)^{\eta}\right]^2 \left(\frac{t'}{t_a}\right)^{\gamma-1} dt'.$$
(100)

We introduce the dimensionless variables $z = 1 - \frac{t'}{t_a}$ and $y = \frac{t}{t_a}$ and rewrite the integral:

$$F_2(t|t_a) = \gamma \frac{\sin(\pi \gamma)}{\pi} c^2 t_a^{2\nu-1} \int_0^1 (z+y)^{2(\nu-\eta)-1-\gamma}$$
$$\times [(z+y)^{\eta} - z^{\eta}]^2 (1-z)^{\gamma-1} dz.$$
(101)

Since we are going to encounter integrals of this type several times, we will calculate them generally. The general form

$$I_{a,b,c}(y) = \int_0^1 (z+y)^a [(z+y)^c - z^c]^2 (1-z)^b dz$$
(102)

can be expressed in terms of Gauss hypergeometric functions, leading to the following asymptotic behavior for $y \to 0$:

$$I_{a,b,c}(y) \simeq \begin{cases} C(a,c) y^{1+a+2c} & \text{for } a + 2c < 1 \\ B(1+b, a+2c-1) y^2 c^2 & \text{for } a + 2c > 1. \end{cases}$$
(103)

A detailed derivation and the bounds for the constant $C(a,c)$ are given in Appendix B.

The behavior of F_2 follows with the substitutions $a = 2(\nu-\eta)-\gamma-1$, $b = \gamma - 1$, and $c = \eta$. Omitting dimensionless constants, we obtain two distinct regimes in the limit $t_a \gg t$, depending on the relation between ν and γ:

$$F_2(t|t_a) \propto \begin{cases} c^2 t_a^{2\nu-3} t^2 & \text{for } 2\nu > \gamma + 2 \\ c^2 t_a^{\gamma-1} t^{2\nu-\gamma} & \text{for } 2\nu < \gamma + 2. \end{cases}$$
(104)

Since both of these cases belong to the nonintegrable class, we obtain in the Laplace domain

$$F_2(s|t_a) \propto \begin{cases} c^2 t_a^{2\nu-3} s^{-3} & \text{for } 2\nu > \gamma + 2 \\ c^2 t_a^{\gamma-1} s^{\gamma-2\nu-1} & \text{for } 2\nu < \gamma + 2. \end{cases}$$
(105)

2. Condition $\gamma > 1$

By substituting $k(t') = \tau^{-1}$ we find

$$F_0 = \int_0^{t_a} \psi(t_a + t - t') \frac{1}{\tau} dt' \to \frac{1}{\tau} \Psi(t) \simeq \frac{t_0^{\gamma}}{\tau} t^{-\gamma}. \quad (106)$$

Since $\gamma > 1$, the term F_0 is of the integrable type, and therefore

$$F_0(s|t_a) \simeq \text{const.} \quad (107)$$

Now we turn to F_2. Starting from Eq. (70) we find

$$F_2(t|t_a) \simeq \gamma c^2 \frac{t_0^{\gamma}}{\tau} t_a^{2\nu-\gamma-1} \int_0^{t_a} \left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^{2(\nu-\eta)-\gamma-1}$$
$$\times \left[\left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^{\eta} - \left(1 - \frac{t'}{t_a}\right)^{\eta}\right]^2 dt'. \quad (108)$$

The calculation is similar to the one in the case $\gamma < 1$. With Eq. (103) for $I_{2(\nu-\eta)-\gamma-1,0,\eta}(t/t_a)$ we obtain

$$F_2(t|t_a)$$
$$\simeq \gamma c^2 \frac{t_0^{\gamma}}{\tau} \begin{cases} \eta^2 B(1, 2\nu-\gamma-2) t_a^{2\nu-\gamma-2} t^2 & \text{for } 2\nu > \gamma + 2 \\ C t^{2\nu-\gamma} & \text{for } 2\nu < \gamma + 2. \end{cases}$$
(109)

The case $2\nu > \gamma + 2$ still belongs in the nonintegrable class and therefore transforms into

$$F_2(s|t_a) \simeq 2\gamma \eta^2 B(1, 2\nu-\gamma-2) c^2 \frac{t_0^{\gamma}}{\tau} t_a^{2\nu-\gamma-2} s^{-3}; \quad (110)$$

however, for $2\nu < \gamma + 2$ we have to distinguish between $\gamma - 1 < 2\nu < \gamma + 2$, where the F_2 is nonintegrable, and $2\nu < \gamma - 1$, where it is integrable. Therefore,

$$F_2(s|t_a) \simeq \gamma c^2 \frac{t_0^{\gamma}}{\tau} \begin{cases} \eta^2 B(1, 2\nu-\gamma-2) t_a^{2\nu-\gamma-2} s^{-3} & \text{for } 2\nu > \gamma + 2 \\ C\Gamma(2\nu+1-\gamma) s^{\gamma-2\nu-1} & \text{for } \gamma + 2 > 2\nu > \gamma - 1 \\ \text{const} \times t_0^{2\nu+1-\gamma} & \text{for } 2\nu < \gamma - 1. \end{cases}$$
(111)

B. Calculation of s_2

1. Condition $\gamma < 1$

The calculations for s_2 from (73) are very similar to that for F_2 case and yield

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu-\eta)} \frac{\sin(\pi \gamma)}{\pi} c^2 \begin{cases} 2\eta^2 B(2\nu-\gamma-1, \gamma) t_a^{2\nu-2} t^2 & \text{for } 2\nu > \gamma + 1 \\ C t_a^{\gamma-1} t^{2\nu+1-\gamma} & \text{for } 2\nu < \gamma + 1. \end{cases}$$
(112)

TABLE III. Results for F_0 and F_2 in the Laplace domain as well as s_2 in the time domain for different parameter ranges in the case of long aging times $t_a \gg t \gg t_0$. Dimensionless prefactors are omitted.

γ	F_0	F_2, s_2	
$\gamma < 1$	$F_0(s\|t_a) \propto t_a^{\gamma-1}s^{\gamma-1}$	$F_2(s\|t_a) \propto c^2 \begin{cases} t_a^{2\nu-3}s^{-3} & \text{for } 2\nu > \gamma+2 \\ t_a^{\gamma-1}s^{\gamma-2\nu-1} & \text{for } 2\nu < \gamma+2 \end{cases}$	
$\gamma < 1$		$s_2(t, t_a) \propto c^2 \begin{cases} t_a^{2\nu-2}t^2 & \text{for } 2\nu > \gamma+1 \\ t_a^{\gamma-1}t^{2\nu+1-\gamma} & \text{for } 2\nu < \gamma+1 \end{cases}$	
$\gamma > 1$	$F_0(s\|t_a) \propto 1$	$F_2(s\|t_a) \propto c^2 \begin{cases} t_0^{\gamma-1}t_a^{2\nu-\gamma-2}s^{-3} & \text{for } 2\nu > \gamma+2 \\ t_0^{\gamma-1}s^{\gamma-2\nu-1} & \text{for } \gamma+2 > 2\nu > \gamma-1 \\ t_0^{2\nu} & \text{for } 2\nu < \gamma-1 \end{cases}$	
$\gamma > 1$		$s_2(t, t_a) \propto c^2 \begin{cases} t_0^{\gamma-1}t_a^{2\nu-\gamma-1}t^2 & \text{for } 2\nu > \gamma+1 \\ t_0^{\gamma-1}t^{2\nu+1-\gamma} & \text{for } 2\nu < \gamma+1 \end{cases}$	

2. Condition $\gamma > 1$

In this case we have

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} c^2 \frac{t_0^{\gamma}}{\tau} \int_0^{t_a} [(t_a + t - t')^{\eta} - (t_a - t')^{\eta}]^2 (t_a + t - t')^{2(\nu-\eta)-\gamma} dt'$$

$$= \gamma \frac{1}{\gamma - 2(\nu - \eta)} I_{2(\nu-\eta)-\gamma, 0, \eta} c^2 \frac{t_0^{\gamma}}{\tau} (t_a)^{2\nu+1-\gamma} \left(\frac{t}{t_a}\right). \tag{113}$$

Using Eq. (103) again we find

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} c^2 \frac{t_0^{\gamma}}{\tau} \begin{cases} \eta^2 B(2\nu - \gamma - 1, 1) t_a^{2\nu-\gamma-1} t^2 & \text{for } 2\nu > \gamma+1 \\ C t^{2\nu+1-\gamma} & \text{for } 2\nu < \gamma+1. \end{cases} \tag{114}$$

The corresponding results for the case of long aging times are summarized in Table III.

C. Mean squared displacement

With these results we can now compute the MSD in the strongly aged case. Using Tables I and III, we can write the asymptotic behavior of the combinations $F_0C_0r_2$, $F_0C_2r_0$, and $F_2C_0r_0$ in the Laplace domain. The inverse transforms are then performed using Tauberian theorems. The corresponding results for $\gamma < 1$ and for $\gamma > 1$ are summarized in Tables IV and V. Therefore, for considerably aged walks we have the following conditions.

1. Condition $\gamma < 1$

For this case we have

$$\langle x^2(t) \rangle \propto \begin{cases} t_a^{\gamma-1}t & \text{for } 2\nu < \gamma \\ t_a^{\gamma-1}t^{2\nu+1-\gamma} & \text{for } \gamma < 2\nu < 1+\gamma \\ t_a^{2\nu-2}t^2 & \text{for } 1+\gamma < 2\nu. \end{cases} \tag{115}$$

Obviously, no Richardson superballistic regime $\langle x^2 \rangle \propto t^3$ is possible in the aged walk. This fact by itself does not make the model a poor candidate for the description of turbulent dispersion, since the Richardson law in turbulence corresponds to starting the observation of the interparticle distance

TABLE IV. Asymptotic behavior of the contributions to the MSD for $\gamma < 1$ in the limit $t_a \gg t \gg t_0$. All dimensionless prefactors are omitted. The dominant terms are highlighted in boldface.

Parameters	$F_2C_0r_0$	$F_0C_0r_2$	$F_0C_2r_0$	s_2
$2\nu < \gamma$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$\boldsymbol{c^2 t_0^{2\nu-\gamma}t_a^{\gamma-1}t}$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$
$\gamma < 2\nu < 1+\gamma$	$\boldsymbol{c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}}$	$\boldsymbol{c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}}$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$\boldsymbol{c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}}$
$1+\gamma < 2\nu < 2+\gamma$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$\boldsymbol{c^2 t_a^{2\nu-2}t^2}$
$2+\gamma < 2\nu$	$c^2 t_a^{2\nu-3}t^3$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1}t^{2\nu+1-\gamma}$	$\boldsymbol{c^2 t_a^{2\nu-2}t^2}$

TABLE V. Asymptotic behavior of the contributions to the MSD for $\gamma > 1$ in the limit $t_a \gg t \gg t_0$. All dimensionless prefactors are omitted. The dominant terms are highlighted in boldface.

Parameters	$F_2C_0r_0$	$F_0C_0r_2$	$F_0C_2r_0$	s_2
$2\nu < \gamma - 1$	$c^2 t_0^{2\nu}$	$c^2 t_0^{2\nu}$	$\mathbf{c^2 t_0^{2\nu-1}t}$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$
$\gamma - 1 < 2\nu < \gamma$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$\mathbf{c^2 t_0^{2\nu-1}t}$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$
$\gamma < 2\nu < 1+\gamma$	$\mathbf{c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}}$	$\mathbf{c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}}$	$\mathbf{c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}}$	$\mathbf{c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}}$
$1+\gamma < 2\nu < 2+\gamma$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$\mathbf{c^2 t_0^{\gamma-1}t_a^{2\nu-\gamma-1}t^2}$
$2+\gamma < 2\nu$	$c^2 t_0^{\gamma-1}t_a^{2\nu-\gamma-2}t^3$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2 t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$\mathbf{c^2 t_0^{\gamma-1}t_a^{2\nu-\gamma-1}t^2}$

immediately after introducing the tracers into the stationary flow, i.e., to an ordinary nonaged case.

2. Condition $\gamma > 1$

For this case we have

$$\langle x^2(t) \rangle \propto \begin{cases} t & \text{for } 2\nu < \gamma \\ t^{2\nu+1-\gamma} & \text{for } \gamma < 2\nu < 1+\gamma \\ t_a^{2\nu-\gamma-1}t^2 & \text{for } 1+\gamma < 2\nu. \end{cases} \quad (116)$$
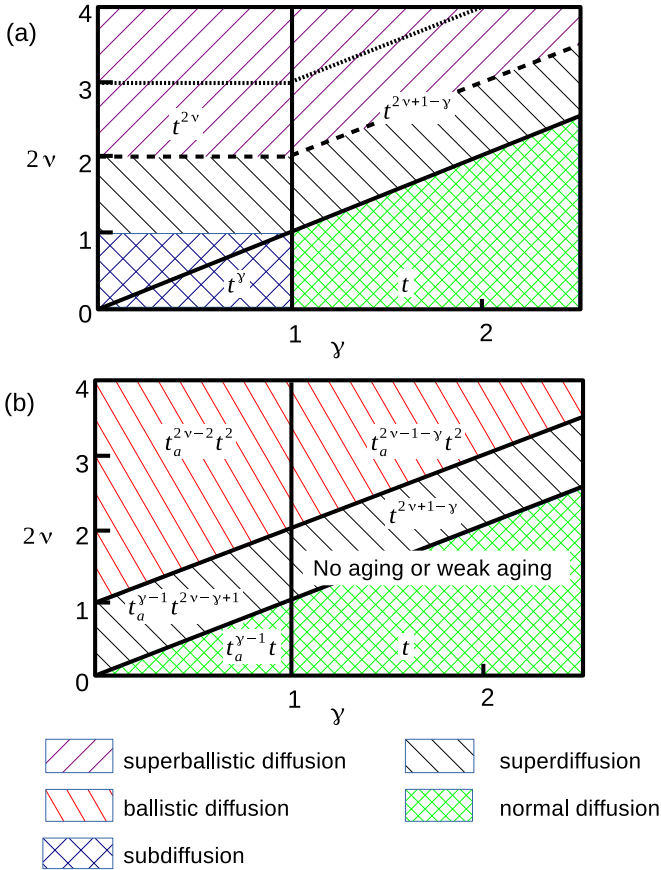

FIG. 3. (a) Map of behaviors of the ensemble average $\langle x^2(t) \rangle$ in the ordinary Lévy walk. The thick solid lines differentiate distinct regimes in the calculations that result in different time exponents. The different hatchings indicate the type of diffusion for this regime in this region of parameter space as shown in the picture. Ballistic behavior is found on the dashed line and the dotted line corresponds to the Richardson law. (b) Same as (a) but for the aged walk in the limit of long aging times.

Therefore, in the regime where the ordinary walk shows normal diffusion or enhanced diffusion there are no or weak changes due to aging (differing only by a prefactor). In the regime where the ordinary walk is superballistic, it ages to a ballistic one. This finding complies with the fact that a ballistic walk with $\nu = 1$ shows only weak aging, i.e., again the ballistic aged behavior [20,21].

Summarizing the findings for the case of considerably aged walks, we state that for the parameter range where the ordinary walk showed subdiffusion we now find regular diffusion but with a prefactor that decays with growing aging times. There is no place for the Richardson regime in the considerably aged case. Again, the second moment exists only for $\gamma > 2(\nu - \eta)$, and if it exists the value of η only enters the prefactors, but does not change the power-law dependences of the MSD on all times involved.

We moreover note that the double time-ensemble average $\langle\langle x^2(t) \rangle_T \rangle_E$, also discussed in Ref. [9], whose calculation involves an additional integration over the time $\langle\langle x^2(t) \rangle_T \rangle_E \simeq (T - t)^{-1} \int_0^{T-t} \langle x^2(t|t_a) \rangle dt_a$, shows the same behavior as the aged walk if the measurement time t is associated with the time lag in the double average and the aging time t_a is changed for the data acquisition time T.

VIII. CONCLUSION

We considered a generalized Lévy walk model interpolating between the original Lévy-walk model of [1] (characterized by two parameters γ and ν) and Drude-like models of [14,15], as proposed in the Supplemental Material of Ref. [9]. The model contains an additional parameter η which allows for such an interpolation: The case $\eta = 1$ corresponds to the original Lévy walk, the case $\eta = \nu$ to a Drude-like model. We showed that the mean squared displacement in such a walk is finite for the sets of parameters obeying $\gamma > 2(\nu - \eta)$ and diverges otherwise. The temporal evolution of the MSD follows the same patterns for all values of η for which it does not diverge; the particular value of η enters only the prefactors. In particular, Drude models never show divergences and possess the Richardson regime in the ordinary case.

While ordinary walks show a plentitude of regimes ranging from subdiffusive to superballistic behavior, the considerably aged walks only show the behavior ranging from diffusive to ballistic. The results of the calculations are best summed up in Fig. 3.

ACKNOWLEDGMENTS

APPENDIX A: TAUBERIAN THEOREMS

In the present work we continuously employed direct and inverse Laplace transforms of functions asymptotically following power laws. The asymptotic forms of Laplace transforms of functions which at long times behave as $f(t) = t^{\rho-1}L(t)$, where $L(t)$ is a slowly varying function, in the asymptotic regime $s \to 0$, and back transforms may be obtained by use of Tauberian theorems. We assume that a Laplace transform $f(s) = \int_0^t f(t)e^{-st}dt$ exists, i.e., the function $f(t)$ does not possess a strong divergence at 0. All functions $f(t)$ appearing in our work are non-negative. For such functions Laplace transforms are monotonically decaying functions of s. Depending on the behavior of $f(t)$ at infinity, two cases should be considered: The function $f(t)$ might be integrable on $[0, \infty)$, so that $\int_0^\infty f(t)dt = I_0^f < \infty$, or this integral may diverge. The first case corresponds to $\rho < 0$ and the second one to $\rho > 0$ [the case $\rho = 0$ may belong to the either class, depending on the concrete form of $L(t)$].

In the second case a Tauberian theorem may be applied immediately, stating that if $f(t)$ is a regularly varying function, i.e., when its Laplace transform is given by

$$f(t) \simeq t^{\rho-1}L(t) \leftrightarrow f(s) \simeq \Gamma(\rho)s^{-\rho}L\left(\frac{1}{s}\right) \quad \text{(A1)}$$

for $\rho \geqslant 0$. As in the main text, all slowly varying functions will be omitted (i.e., replaced with constants L).

We note that for $\rho < 0$, Eq. (A1) gives $f(s)$ as a growing function of s and therefore is wrong. In this case let us consider the function

$$S(t) = \int_t^\infty f(t')dt'. \quad \text{(A2)}$$

The integrability of $f(t)$ means that $S(t)$ is well defined and that $I_0^f = \int_0^\infty f(t')dt' = S(0)$ is finite. The function $S(t)$ has the power-law asymptotics

$$S(t) \simeq -\frac{Lt^\rho}{\rho} \quad \text{(A3)}$$

and if this is no longer integrable (i.e., for $\rho > -1$) can be transformed via the Tauberian theorem so that

$$S(s) \simeq -L\frac{\Gamma(\rho+1)}{\rho}s^{-(\rho+1)} = -L\Gamma(\rho)s^{-(\rho+1)}, \quad \text{(A4)}$$

where in the last equality the identity $\Gamma(x+1) = x\Gamma(x)$ was used. Noting that $f(t) = -\frac{d}{dt}S(t)$ and using the Laplace representation of the derivative, we get

$$f(s) = S(t=0) - sS(s) = I_0^f - L\Gamma(\rho)s^{-\rho}. \quad \text{(A5)}$$

The direct application of the Tauberian theorem would give us a correct form of the singular term (up to a sign), but not the regular one.

If $S(t)$ is still integrable, we consider the function $P(t) = \int_t^\infty S(t')dt'$, whose power-law asymptotics for $t \to \infty$ is

$$P(t) \simeq \frac{Lt^{\rho+1}}{\rho(\rho+1)} \quad \text{(A6)}$$

and whose connection to $f(t)$ is given by $f(t) = \frac{d^2}{dt^2}S(t)$. For $-2 < \rho$ the function $P(t)$ is not integrable and the application of the Tauberian theorem gives

$$P(s) = L\frac{\Gamma(\rho+2)}{\rho(\rho+1)}s^{-\rho-2} = L\Gamma(\rho)s^{-\rho-2}. \quad \text{(A7)}$$

Using the Laplace representation for the second derivative, we get

$$f(s) = -sP(t=0) - P'(t=0) + s^2P(s). \quad \text{(A8)}$$

The value of $P'(t=0)$ is $-S(t=0) = -I_0^f$. The value $P(t=0)$ is given by the integral

$$P(t=0) = \int_0^\infty dt \int_t^\infty f(t')dt'. \quad \text{(A9)}$$

Changing the sequence of integrations in t and t', we get

$$P(t=0) = \int_0^\infty dt' f(t') \int_0^{t'} dt = \int_0^\infty t'f(t')dt'. \quad \text{(A10)}$$

Since $f(t)$ decays with t faster than t^{-2}, the integral converges and will be denoted by I_1^f. Therefore, we have

$$f(s) = I_0^f - sI_1^f + L\Gamma(\rho)s^{-\rho}. \quad \text{(A11)}$$

For $\rho < -2$ the procedure has to be repeated again for the function being the integral of $P(t)$, etc. The general result is

$$f(s) = \sum_{k=0}^{k_{\max}} \frac{(-1)^k}{k!} I_k^f s^k + L\Gamma(\rho)s^{-\rho}, \quad \text{(A12)}$$

with $k_{\max}$ being the whole part of $-\rho$ and I_k^f being the moment integral

$$I_k^f = \int_0^\infty t^k f(t)dt. \quad \text{(A13)}$$

In the main text we never have to use more than the first three terms of this expansion.

APPENDIX B: ESTIMATES FOR THE INTEGRAL $I_{a,b,c}(y)$

We are interested in the integral

$$\begin{aligned}
I_{a,b,c}(y) &= \int_0^1 (1-z)^b[(z+y)^c - z^c]^2(z+y)^a dz \\
&= \int_0^1 (1-z)^b[(z+y)^{a+2c} - 2(z+y)^{a+c}z^c \\
&\quad + (z+y)^a z^{2c}]dz
\end{aligned} \quad \text{(B1)}$$

in the limit of small $y = \frac{t}{t_a} \ll 1$ for the parameter ranges $c > 0$, $b > -1$, and $a \in \mathbb{R}$.

To evaluate it we use Euler's integral representation for the Gauss hypergeometric function [22] for $\Re c' > \Re b' > 0$,

$$\begin{aligned}
&{}_2F_1(a', b'; c'; x) \\
&= \frac{1}{B(b', c'-b')} \int_0^1 z^{b'-1}(1-z)^{c'-b'-1}(1-zx)^{-a'}. \quad \text{(B2)}
\end{aligned}$$

As the existence condition $1 + b > 0$ is always satisfied for all three terms in (B1), we can write the integral as

$$
I_{a,b,c}(y) = y^a \left[y^{2c} B(1, 1+b) \, _2F_1\left(-a-2c, 1; 2+b; -\frac{1}{y}\right) - 2y^c B(1+c, 1+b) \, _2F_1\left(-a-c, 1+c; 2+b+c; -\frac{1}{y}\right) \right.
$$
$$
\left. + B(1+2c, 1+b) \, _2F_1\left(-a, 1+2c; 2+b+2c; -\frac{1}{y}\right) \right] dz, \tag{B3}
$$

with $B(x, y)$ being the Beta function. Although the integral can be expressed in terms of three Gauss hypergeometric functions, its investigation is somewhat tricky, since the asymptotic regimes appear as a subleading terms in a sum of three large contributions whose leading terms cancel. First, to avoid evaluating hypergeometric functions at $-\infty$, we make use of the Pfaff transformations

$$
_2F_1(a', b'; c'; z) = (1 - z)^{-b'} \, _2F_1\left(b', c' - a'; c; \frac{z}{z-1}\right), \tag{B4}
$$

$$
_2F_1(a', b'; c'; z) = (1 - z)^{-a'} \, _2F_1\left(a', c' - b'; c; \frac{z}{z-1}\right). \tag{B5}
$$

These two different forms will be useful in different domains of parameters. Under the transformations, the argument of the corresponding functions on the right-hand side, equal to $\frac{1}{1+y}$, will tend to 1.

Applying the Pfaff transformation (B4) to the integrals in Eq. (B3) we find

$$
I_{a,b,c}(y) = y^{1+a+2c} \left[(1+y)^{-1} B(1, 1+b) \, _2F_1\left(1, 2+a+b+2c; 2+b; \frac{1}{1+y}\right) \right.
$$
$$
- 2(1+y)^{-1-c} B(1+c, 1+b) \, _2F_1\left(1+c, 2+a+b+2c; 2+b+c; \frac{1}{1+y}\right)
$$
$$
\left. + (1+y)^{-1-2c} B(1+2c, 1+b) \, _2F_1\left(1+2c, 2+a+b+2c; 2+b+2c; \frac{1}{1+y}\right) \right].
$$

We now use the Euler integral representation (B2) again, but switch between a' and b',

$$
_2F_1(a', b'; c'; x) = \frac{1}{B(a', c' - a')} \int_0^1 z^{a'-1}(1-z)^{c'-a'-1}(1 - zx)^{-b'},
$$

for $\Re c' > \Re a' > 0$. Note that the existence condition for the integrals is the same as before, $b + 1 > 0$, which is satisfied for all relevant cases in this paper, so we can write

$$
I_{a,b,c}(y) = y^{1+a+2c} \int_0^1 \left[(1+y)^{-1}(1-z)^b \left(1 - \frac{z}{1+y}\right)^{-2-a-b-2c} - 2(1+y)^{-1-c} z^c (1-z)^b \left(1 - \frac{z}{1+y}\right)^{-2-a-b-2c} \right.
$$
$$
\left. + (1+y)^{-1-2c} z^{2c} (1-z)^b \left(1 - \frac{z}{1+y}\right)^{-2-a-b-2c} \right] dz.
$$

The integrals of each of the three contributions in square brackets would diverge for $y \to 0$, but the integral of the whole sum is convergent for $a + 2c < 1$ since for $y \to 0$ the integrand tends to

$$
(1 - 2z^c + z^{2c})(1-z)^{-2-a-2c} = (1 - z^c)^2 (1-z)^{-2-a-2c}
$$

and the integral

$$
C(a, c) = \int_0^1 (1 - z^c)^2 (1-z)^{-2-a-2c} dz
$$

of this expression converges in the range $a + 2c < 1$ (to prove the convergence it is enough to expand the first term in vicinity of $z = 1$). This integral cannot be expressed in terms of "simple" functions, but the (loose) bounds for it follow easily.

Let us find two constants $B > A > 0$ such that for all $0 < z < 1$,

$$
A(1 - z) < 1 - z^c < B(1 - z).
$$

To do so consider the function

$$
f(z) = \frac{1 - z^c}{1 - z},
$$

with $f(0) = 1$ and with its limiting value at $z \to 1$ given by the l'Hôpital's rule $\lim_{z \to 1} = c$. Therefore, the limit of the function at 1 is larger than its value at 0 when $c > 1$ and smaller than this value when $c < 1$. For $c = 1$ this function equals unity identically.

Now we consider $c \neq 1$ and proceed to show that the function $f(z)$ is monotonically growing for $c > 1$ and monotonically decaying for $c < 1$. To show this it is enough to show that its derivative on [0,1] does not vanish. The derivative of the corresponding function is

$$f'(z) = \frac{1 - z^c + cz^c - cz^{c-1}}{(1-z)^2}$$

and can only vanish when the numerator, $g(z) = 1 - z^c + cz^c - cz^{c-1}$, vanishes somewhere at $0 \leqslant z < 1$. Vanishing of the numerator at $z = 1$ does not pose a problem since $f'(z)$ diverges and tends to $(c-1)(1-z)^{-2}$ for $z = 1$, being positive in the vicinity of $z = 1$ for $c > 1$ and negative for $c < 1$ due to the fact that the denominator vanishes even faster. Now we show that this function never changes its sign on $0 \leqslant z < 1$. Calculating the derivative

$$g'(z) = -c(c-1)z^{c-2} + c(c-1)z^{c-1} = -c(c-1)z^{c-2}(1-z),$$

we see that it is strictly positive for all $z < 1$ for $c < 1$ and strictly negative for $c > 1$. Therefore, the bounds for the function $f(z)$ are given by its limiting values of 1 at $z = 0$ and c at $z = 1$. Therefore, we have $A = \min(1, c^2)$ and $B = \max(1, c^2)$. Since for $1 > a + 2c$,

$$\int_0^1 (1-z)^{-a-2c} dz = \frac{1}{1 - a - 2c}$$

we get

$$\frac{\min(1, c^2)}{1 - a - 2c} \leqslant C \leqslant \frac{\max(1, c^2)}{1 - a - 2c}. \tag{B6}$$

Therefore, for $a + 2c < 1$ we have, for y small,

$$I_{a,b,c}(y) \simeq Cy^{1+a+2c}, \tag{B7}$$

where the bounds for the constant C are given by Eq. (B6).

For the opposite case $a + 2c > 1$ we have to use the other Pfaff transformation, Eq. (B5), resulting in

$$\begin{aligned} I_{a,b,c}(y) = (1+y)^a [&(1+y)^{2c} B(1, 1+b) \,_2F_1(-a-2c, 1+b; 2+b; 1/(1+y)) \\ &- 2(1+y)^c B(1+c, 1+b) \,_2F_1(-a-c, 1+b; 2+b+c; 1/(1+y)) \\ &+ B(1+2c, 1+b) \,_2F_1(-a, 1+b; 2+b+2c; 1/(1+y))]. \end{aligned} \tag{B8}$$

We now use the integral representation (B2) again and find

$$\begin{aligned} I_{a,b,c}(y) = (1+y)^a \int_0^1 \Bigg[&(1+y)^{2c} z^b \left(1 - \frac{z}{1+y}\right)^{a+2c} - 2(1+y)^c z^b (1-z)^c \left(1 - \frac{z}{1+y}\right)^{a+c} \\ &+ z^b (1-z)^{2c} \left(1 - \frac{z}{1+y}\right)^a \Bigg] dz. \end{aligned} \tag{B9}$$

Now we expand the expression in each term of the integrand up to the second order in y using the fact that

$$(1+y)^\alpha \simeq 1 + \alpha y + \frac{\alpha(\alpha-1)}{2} y^2$$

and

$$\left(1 - \frac{z}{1+y}\right)^\alpha \simeq (1-z)^\alpha + \alpha(1-z)^{\alpha-1} zy + \frac{1}{2}[\alpha(\alpha-1)(1-z)^{\alpha-2} z^2 - 2\alpha(1-z)^{\alpha-1} z] y^2,$$

and using the definition of the Beta function

$$B(a, b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz,$$

we find

$$I_{a,b,c}(y) \simeq c^2 y^2 [B(1+b, 1+a+2c) + 2B(2+b, a+2c) + B(3+b, a+2c-1)].$$

The first two orders in y have canceled, so the leading term goes as y^2. From the argument of the last Beta function it is clear that the result only holds for $a + 2c > 1$, i.e., exactly in the parameter range where Eq. (B7) ceases to be applicable, and that the exponents are continuous at $a + 2c = 1$. Rewriting the Beta functions as $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ and (repeatedly) using the identity $\Gamma(x+1) = x\Gamma(x)$, we find a compact representation of the sum of the three Beta functions, namely,

$$I_{a,b,c}(y) \simeq c^2 y^2 B(1+b, a+2c-1). \tag{B10}$$

In conclusion, we have

$$I_{a,b,c}(y) \simeq \begin{cases} C(a,c)y^{1+a+2c} & \text{for } a+2c < 1 \\ c^2 B(1+b, a+2c-1)y^2 & \text{for } a+2c > 1, \end{cases}$$

which is Eq. (103) of the main text, with the bounds on a constant $C(a,c)$ given by Eq. (B6).

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