Finite-power performance of quantum heat engines in linear response

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We investigate the finite-power performance of quantum heat engines working in the linear response regime where the temperature gradient is small. The engine cycles with working substances of ideal harmonic systems consist of two heat transfer and two adiabatic processes, such as the Carnot cycle, Otto cycle, and Brayton cycle. By analyzing the optimal protocol under maximum power we derive the explicitly analytic expression for the irreversible entropy production, which becomes the low dissipation form in the long duration limit. Assuming the engine to be endoreversible, we derive the universal expression for the efficiency at maximum power, which agrees well with that obtained from the phenomenological heat transfer laws holding in the classical thermodynamics. Through appropriate identification of the thermodynamic fluxes and forces that a linear relation connects, we find that the quantum engines under consideration are tightly coupled, and the universality of efficiency at maximum power is confirmed at the linear order in the temperature gradient.

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I. INTRODUCTION

For heat engines working between two heat reservoirs of constant inverse temperatures β_h^r and $\beta_c^r > (\beta_h^r)$ with $(k_B \equiv 1)$ $\beta^r = 1/T^r$, their efficiency is bounded above by the Carnot value $\eta_C = 1 - \beta_h^r/\beta_c^r$ which, however, is achieved in an infinite long time and is of limited practical significance. It is therefore of great necessity to determine the efficiency of engine cycles consisting of finite time state transformations. Under the endoreversible condition, where the irreversibility arises only from the imperfect heat conduction between the system and heat reservoir, Curzon and Ahlborn [1] found that the efficiency of a nonideal Carnot cycle at maximal power is given by

$$\eta^* = \eta_{\rm CA} \equiv 1 - \sqrt{\frac{\beta_h^r}{\beta_c^r}} = 1 - \sqrt{1-\eta_C}, \tag{1}$$

which we call the CA efficiency. The tradeoff between the power and efficiency was subsequently investigated in various engine models working with a system that ranges from nanoscale to macroscale, within different physical frameworks such as endoreversible thermodynamics [2–9], finite time (quantum and classical) thermodynamics [10–15], and irreversible thermodynamics [4,13,16–23]. In the linear response regime where the temperature gradient is small, it has been shown that a universal efficiency

$$\eta^* = \eta_C/2 + \eta_C^2/8 + O(\eta_C^3), \tag{2}$$

exactly the same as one derived by expanding CA efficiency (1) up to the second order, holds for many engines under maximum power under certain conditions. The universal behavior of the optimal efficiency has been found in various classical and quantum heat engines, such as heat engines obeying phenomenological transfer laws [3,4,6], classical particle transport [24], particle transport via Kramers escape [25], quantum optomechanical engines [26], two-level and multilevel quantum engines [7,13,27–35], and quantum-dot engines [36,37].

While the efficiency of physically different systems exhibits a certain universal behavior in certain limits, it varies quite a lot between the lower and upper bounds if these limits are removed. Chen and Yan [3] showed that, for heat engines obeying linear phenomenological heat transfer law, the efficiency at maximum power for the endoreversible engines has the form of

$$\eta^* = \frac{\eta_C}{2 - \gamma_{pl}\eta_C}, \tag{3}$$

where $\gamma_{pl} \equiv (1 + \sqrt{\mathcal{C}_c^{pl}/\mathcal{C}_h^{pl}})^{-1}$, with the heat transfer coefficient $\mathcal{C}_{h,c}^{pl}$ along the hot or cold "isothermal" contact. Alternatively, the low dissipation assumption [38] in which phenomenological heat transfer laws are avoided was introduced to analyze the finite time performance of heat engines. Under the low dissipation condition, the entropy production caused by the dissipation can be assumed to be $\Sigma_\alpha/\tau_\alpha$, where Σ_α is the dissipation constant for the hot ($\alpha = h$) or cold ($\alpha = c$) "isothermal" process. In what follows, the word "isothermal" merely indicates that the system is in contact with a heat reservoir of constant temperature. The change in system entropy along the isothermal contact is then given by [38–40]

$$\Delta S_\alpha = \beta_\alpha^r Q_\alpha - \frac{\Sigma_\alpha}{\tau_\alpha}, \tag{4}$$

with $\alpha = h, c$. Optimizing the power output, $P = (Q_h + Q_c)/(\tau_h + \tau_c)$, with respect to the time allocation τ_c and τ_h, leads to the expression of efficiency at maximum power (3) by replacing γ_{pl} with $\gamma_{ld} \equiv (1 + \beta_h^r\Sigma_c/\beta_c^r\Sigma_h)^{-1}$.

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In the symmetric case when $C_h^{pl} = C_c^{pl}$ or $\Sigma_c = \Sigma_c$, the efficiency at maximum power (3) can be approximated by the universal form as given in Eq. (2). The lower and upper bounds of the efficiency at maximum power are achieved,

$$\eta_-^* = \frac{\eta_C}{2}, \quad \eta_+^* = \frac{\eta_C}{2 - \eta_C}, \tag{5}$$

in the two extremely asymmetric cases, $\Sigma_h/\Sigma_c \to 0$ ($\mathcal{C}_h/\mathcal{C}_c \to 0$) in which the hot isotherm approaches the reversible limit and $\Sigma_h/\Sigma_c \to \infty$ ($\mathcal{C}_h/\mathcal{C}_c \to \infty$) where the cold contact tends to be reversible, respectively. Remarkably, the universal bounds for the optimal efficiency are also found from different engine models under maximum power, for example, a trapped Brownian particle undergoing an Otto cycle [41] or electron transport in a quantum dot [37].

By exploring the finite-power performance, attempts were made to discuss the physical origin [37] of the low dissipation model and to investigate the relationship between the low dissipation and endoreversible models [4–6,8]. Among them, the role of the contact times of the working system with the external heat baths and their symmetries or asymmetries for both heat engines and refrigerators [5,8] were investigated, showing that the equivalence between these two models goes beyond the maximum power regime and also holds in compromise-based figures of merit. The minimally nonlinear irreversible heat engines (refrigerators), in which internal dissipations along the isothermal processes are involved, have been mapped into low dissipation models via appropriate identification of thermodynamic fluxes and forces [23].

These studies are important since they contribute towards constituting a bridge between finite time and irreversible thermodynamics [4,7,13]. In these studies, however, specific heat transfer laws were used for classical endoreversible models and a two-level system (obeying Fermi-Dirac statistics) was used for a low dissipation limit. Here, rather than applying a specific model and a given heat transfer law, we consider cyclic heat engines working with harmonic systems and consisting of two adiabatic processes and two heat transfer processes. For these engines, we obtain the expressions for the power output and efficiency in which the time duration is involved via analyzing the dynamics of the engines, with no use of any specific heat transfer laws. Optimization on these engines will be done via three different approaches, namely, the Euler-Lagrange equation, endoreversible assumption, and Onsager linear relation within linear irreversible thermodynamics. We recover many of the well-known expressions of efficiency at maximum power for these engine models, with emphasis on the (dis)similarities of physical origin between the approaches.

In this paper, the expressions of the power and efficiency are derived on the basis of the quantum master equation that describes the time evolution of the systems. When optimized with respect to power output by the scheme based on the Euler-Lagrange equation, these engines can be referred to as a specific example in the low dissipation engine model. We then apply endoreversible thermodynamics to these engines within the linear response regime. We find that the efficiency at maximum power derived from the Euler-Lagrange method is recovered for these endoreversible engines and thus coincides with the result (3) obtained for the classical engines. The Onsager coefficients are calculated via appropriate identification of thermodynamic fluxes and forces, and the universal behavior of the efficiency at maximum power is verified for these engines that are proved to be tightly coupled.

The paper is organized as follows. In Sec. II we optimize with respect to power to determine the corresponding efficiency in the three different approaches: (1) optimization by means of the Euler-Lagrange equation, (2) the endoreversible thermodynamic approach, and (3) the linear irreversible thermodynamic method. Discussions and conclusions are then made in Sec. III.

II. OPTIMIZATION ON CYCLIC QUANTUM HEAT ENGINES

A. Dynamical description of an isothermal process

The cyclic quantum heat engines under consideration, which work between a hot and a cod heat bath and consist of two isothermal and two isentropic adiabatic processes, may be the Carnot cycle, the Brayton cycle, the Otto cycle [42], etc. For these engines, a harmonic system obeying the Bose quantum statistics is employed as the working substance. By assuming the ground state energy to be zero for simplicity of notation, the harmonic system Hamiltonian is described by $\hat{H} = \omega(t)\hat{N} = \omega(t)\hat{a}^\dagger\hat{a}$, where $\hat{N}$ is the number operator and $\hat{a}^\dagger$ ($\hat{a}$) is the bosonic creation (annihilation) operator. All of these consist of the following four consecutive steps.

(1) Along the hot isothermal branch, the system is coupled to the hot reservoir at constant temperature β_h^r in a time period τ_h which starts at $t = t_h^0$ and ends at $t = t_h^f$, and the control variable $\omega(t)$ changes from ω_h^0 to ω_h^f.

(2) The adiabatic expansion is realized by decoupling the system from the hot and cold reservoirs in a time period τ_{hc}. In this step, the working system expands to produce work via changing the control variable $\omega(t)$ from ω_h^f to ω_c^0.

(3) Along the cold isothermal process, the system is coupled to the cold reservoir of constant temperature β_c^r in time duration τ_c, and the control variable $\omega(t)$ varies from ω_c^0 to ω_c^f.

(4) For the adiabatic compression with time duration τ_{ch}, the system is compressed consuming work while isolated from the hot and cold reservoirs.

On this branch, the control variable is explicitly time dependent since it changes back to its initial value ω_h^0 from ω_c^f to close a cycle. Since the entropy of system reads $S = \text{Tr}\hat{\rho}\ln\hat{\rho}$, where the density operator is $\hat{\rho} = \exp(-\beta\hat{H})/\text{Tr}[\exp-(\beta\hat{H})]$, with β being the inverse temperature of the system, it is a function of the parameter $\beta\omega$ only. Therefore, the constancy of entropy S indicates that the population $n = \text{Tr}(\hat{\rho}\hat{N}) = [\exp(\beta\omega) - 1]^{-1}$ is kept constant in these isentropic adiabatic processes. In what follows, the adiabatic expansion and compression of these engines satisfy the sudden limit, in which the time spent on the two adiabatic processes τ_{ch} and τ_{hc} can be negligible compared to the time taken for the isothermal processes, namely, $\tau_{ch} + \tau_{hc} \to 0$. That is, the total cycle time is given by $\tau_{\text{cyc}} \equiv \tau_c + \tau_h$.

Using a dot to denote the differentiation with respect to time t, the time evolution for an operator $\hat{X}$ for the system can be described by the quantum master equation [7,13,29–31]:

$$\dot{\hat{X}} = i[\hat{H}, \hat{X}] + \partial\hat{X}/\partial t + \mathcal{L}_D(\hat{X}), \tag{6}$$

where $\mathcal{L}_D(\hat{X}) = \sum_i k_i(\hat{V}_i^\dagger[\hat{X}, \hat{V}_i] + [\hat{V}_i^\dagger, \hat{X}]\hat{V}_i)$ is Liouville dissipation along the thermalization process. Here $\hat{V}_i^\dagger$ and $\hat{V}_i$ are operators in the Hilbert space of the system and Hermitian conjugates, and k_i are positive coefficients. The quantum version of the first law of thermodynamics $dE = \delta W + \delta Q$ is obtained by substituting $\hat{X} = \hat{H}$ into Eq. (6), $\dot{E} = \dot{W} + \dot{Q} = \langle \partial H/\partial t\rangle + \langle \mathcal{L}_D(\hat{H})\rangle$, with $\dot{W} = \langle \partial H/\partial t\rangle$ and $\dot{Q} = \langle \mathcal{L}_D(\hat{H})\rangle$. We emphasize that such dissipative term $\mathcal{L}_D$, unlike the internal dissipation [13] denoting the degree of nonequilibrium and irreversibility, indicates the heat dissipated into (from) the system due to the thermal interaction with a heat reservoir. Note that along the adiabatic process, in which the working system is decoupled from the hot and cold reservoirs, the Liouville dissipative form $\mathcal{L}_D(\hat{X})$ in Eq. (6) must be vanishing.

For an isochoric process, substituting $\hat{V}_i^\dagger = \hat{a}^\dagger$ $(\hat{V}_i = \hat{a})$ and $\hat{X} = \hat{a}^\dagger\hat{a}$ into the master equation (6), the time evolution of the instantaneous mean population $n(t) = \langle \hat{N}(t)\rangle$ at any time t can be obtained:

$$\dot{n}(t) = -\mathcal{C}[n(t) - n^0(t)], \tag{7}$$

where $\mathcal{C} = k_- - k_+$ denotes the relaxation rate of reaching thermal equilibrium for the harmonic system and $n^0(t) \equiv k_+/(k_- - k_+) = 1/[e^{\beta_\alpha^r \omega(t)} - 1]$ is the mean population for the system at local thermal equilibrium with the heat bath. Here the detailed balance $k_+/k_- = e^{-\beta_\alpha^r \omega(t)}$ is assumed to be valid at time t, such that, if the control variable ω is frozen, i.e., $\omega(t) = \omega =$ const, the (global) thermal equilibrium state with $n_\alpha^{\rm eq} = (e^{\beta_\alpha^r \omega} - 1)^{-1}$ is reached in the quasistatic limit [7]. Considering the boundary condition $n^{\rm eq} = n(t \to \infty)$, we solve Eq. (7) to obtain the mean population of the working system at time t:

$$n(t) = n_\alpha^{\rm eq} + \left[n\left(t_\alpha^0\right) - n_\alpha^{\rm eq}\right]e^{-\mathcal{C}_\alpha t}, \tag{8}$$

where $\alpha = h, c$ is used for the hot and cold isothermal processes, respectively. The population n remains constant during the adiabatic process, which implies that

$$n\left(t_h^0\right) = n\left(t_c^f\right), n\left(t_h^f\right) = n\left(t_c^0\right). \tag{9}$$

Substituting the relation (9) into Eq. (8), we find that the difference of the initial and final populations along the hot or cold heat exchange takes the form

$$n\left(t_\alpha^f\right) - n\left(t_\alpha^0\right) = \mp\left(n_h^{\rm eq} - n_c^{\rm eq}\right)\mathcal{F}(\tau_c, \tau_h), \tag{10}$$

where $\mathcal{F}(\tau_c, \tau_h) \equiv \frac{(1-e^{-\mathcal{C}_h\tau_h})(1-e^{-\mathcal{C}_c\tau_c})}{(1-e^{-\mathcal{C}_h\tau_h-\mathcal{C}_c\tau_c})}$ and $0 \leqslant \mathcal{F} \leqslant 1$. Here the plus sign (+) should be applied to the hot isothermal contact, in which the particles are excited to higher energy levels and the mean population is raised to approach its asymptotic (thermal equilibrium) value, and the minus sign (−) is used for the cold isothermal process.

B. Optimizing protocol method

The heat absorbed by the system along such a process of duration of τ_α is obtained as

$$Q_\alpha = \int_{t_\alpha^0}^{t_\alpha^f} \omega(t)\dot{n}(t)dt. \tag{11}$$

Since for each cycle $\langle H(0)\rangle = \langle H(\tau_{\rm cyc})\rangle$ and $W = -Q$, the power output can be determined according to $P = -W/\tau_{\rm cyc} = Q/\tau_{\rm cyc}$. To proceed with an analytical analysis, in the first step we optimize the power output by fixing the variables τ_c and τ_h. In this case, since maximizing the power output is equivalent to maximizing heat or minimizing work, we determine the optimal protocol that maximizes heat via the strategy based on the Euler-Lagrange equation [30,37]. We search for the optimal schedule $n(t)$ and $\dot{n}(t)$, both of which are variables of the control variable $\omega(t)$. Form Eqs. (7) and (11), we can find

$$\beta_\alpha^r Q_\alpha = \int_{t_\alpha^0}^{t_\alpha^f} \mathcal{L}(n, \dot{n})dt, \tag{12}$$

where

$$\mathcal{L} = \dot{n}\ln\left(\frac{\dot{n} + \mathcal{C}_\alpha + \mathcal{C}_\alpha n}{\dot{n} + \mathcal{C}_\alpha n}\right). \tag{13}$$

Integrating the Euler-Lagrange equation gives $\mathcal{L} - \dot{n}\partial\mathcal{L}/\partial\dot{n} = \mathcal{K}_\alpha$, where $\mathcal{K}_\alpha$ is the constant of integration, we obtain

$$\frac{\mathcal{C}_\alpha \dot{n}^2}{(\dot{n} + \mathcal{C}_\alpha n)(\mathcal{C}_\alpha + \dot{n} + \mathcal{C}_\alpha n)} = \mathcal{K}_\alpha. \tag{14}$$

The solution of the quadratic equation for $\dot{n}(t)$ is obtained as

$$\frac{\dot{n}}{\mathcal{C}_\alpha} = \frac{\mathcal{K}_\alpha(1+2n) \mp \sqrt{\mathcal{K}_\alpha}\sqrt{\mathcal{K}_\alpha + 4\mathcal{C}_\alpha n(1+n)}}{2(\mathcal{C}_\alpha - \mathcal{K}_\alpha)}, \tag{15}$$

where the plus sign (+) refers to the upward process with raising quantum level, and the plus sign (−) refers to the downward process [37]. Equation (15), together with Eq. (7), gives rise to the explicit expression for instantaneous mean population:

$$n(t) = n^0(t)\left[1 + \sqrt{\frac{\mathcal{K}_\alpha}{\mathcal{C}_\alpha}\left(1 + \frac{1}{n^0(t)}\right)}\right], \tag{16}$$

where $n^0(t)$ is defined in Eq. (7). When $\mathcal{K}_\alpha = 0$, the system achieves the thermal equilibrium state and $n(t)$ tends to be $n^{\rm eq} = n(t \to \infty)$, thereby implying that $\mathcal{K}_\alpha = 0$ represents the quasistatic limit. If $\mathcal{K}_\alpha \neq 0$, the system evolves in finite time and it is not able to reach the (global) thermal equilibrium state. This means that the mean population cannot reach the thermal equilibrium value, namely, $n < n^{\rm eq}$ $(n > n^{\rm eq})$ for the hot (cold) isothermal process. Therefore, the constant $\mathcal{K}_\alpha$ indicates how far the process that the system undergoes deviates from the quasistatic limit.

We can solve Eq. (15) via separation of the variables n and t, leading to

$$\mathcal{C}_\alpha t = \mathcal{G}[n(t); \mathcal{K}_\alpha] - \mathcal{G}\left[n\left(t_\alpha^0\right); \mathcal{K}_\alpha\right], \tag{17}$$

where

$$\mathcal{G}[n;\mathcal{K}_\alpha] \equiv -\ln(n+1) + \sqrt{\frac{\mathcal{C}_\alpha}{\mathcal{K}_\alpha}}\ln\{\mathcal{C}_\alpha + 2\mathcal{C}_\alpha n + \sqrt{\mathcal{C}_\alpha[\mathcal{K}_\alpha + 4\mathcal{C}_\alpha n(1+n)]}\} + \frac{1}{2}\ln\frac{-2\mathcal{C}_\alpha(1+n) + \mathcal{K}_\alpha + \sqrt{\mathcal{K}_\alpha[\mathcal{K}_\alpha + 4\mathcal{C}_\alpha n(1+n)]}}{\mathcal{K}_\alpha + 2\mathcal{C}_\alpha n + \sqrt{\mathcal{K}_\alpha[\mathcal{K}_\alpha + 4\mathcal{C}_\alpha n(1+n)]}}. \tag{18}$$

Equation (18) is so complicated that its exact solution can be found by numerical method only. Fortunately, since our analysis is restricted to the weak dissipation case, in which the duration of the process is very long (though it is not infinite), we can adopt a perturbative solution via assumption that $\mathcal{K}_\alpha$ is small. For very small $\mathcal{K}_\alpha$, we expand $\mathcal{G}[n;\mathcal{K}_\alpha]$ with respect to $\sqrt{\mathcal{K}_\alpha}$ via keeping the first order:

$$\mathcal{G}[n;\mathcal{K}_\alpha] = \sqrt{\frac{\mathcal{C}_\alpha}{\mathcal{K}_\alpha}}\ln[\mathcal{C}_\alpha(1 + 2n + 2\sqrt{n+n^2})]. \tag{19}$$

For the process of duration τ_α, we find from Eq. (17) that there is a constraint, $\mathcal{C}_\alpha\tau_\alpha = \mathcal{G}[n(t_\alpha^f);\mathcal{K}_\alpha] - \mathcal{G}[n(t_\alpha^0);\mathcal{K}_\alpha]$, which simplifies to

$$\sqrt{\mathcal{C}_\alpha}\tau_\alpha = \frac{1}{\sqrt{\mathcal{K}_\alpha}}\ln\left[\frac{1 + 2n(t_\alpha^f) + 2\sqrt{n(t_\alpha^f) + n^2(t_\alpha^f)}}{1 + 2n(t_\alpha^0) + 2\sqrt{n(t_\alpha^0) + n^2(t_\alpha^0)}}\right], \tag{20}$$

in the low dissipation limit where the duration τ_α is very long. For given heat transfer process, we use this condition (20) to determine the integration constant $\mathcal{K}_\alpha$. In the linear response regime, where the difference between the temperatures of two heat baths is small, the difference of the equilibrium populations, $\Delta n^{\rm eq} = n_h^{\rm eq} - n_c^{\rm eq}$, must be small. With consideration of Eq. (10), we can obtain the quadratic approximation to the right-hand side of Eq. (20) about $n_c^{\rm eq}$ via making Taylor series expansion, leading to $\sqrt{\mathcal{K}_\alpha\mathcal{C}_\alpha}\tau_\alpha = \Delta n^{\rm eq}/[n_\alpha^{\rm eq}(1+n_\alpha^{\rm eq})]^{1/2}\mathcal{F}(\tau_c,\tau_h) + (\Delta n^{\rm eq})^2(1+2n_\alpha^{\rm eq})/[8n_\alpha^{\rm eq}(1+n_\alpha^{\rm eq})]^{3/2}\mathcal{F}^2(\tau_h,\tau_c) + O[(\Delta n^{\rm eq})^3]$. Since the exponential function $\mathcal{F}(\tau_c,\tau_h)$ decreases (with increasing $\tau_{c,h}$) much more fast than the linear functions toward their maximum value 1, the long (contact) time limit finally leads to

$$\ln\left[\frac{1 + 2n(t_\alpha^f) + 2\sqrt{n(t_\alpha^f) + n^2(t_\alpha^f)}}{1 + 2n(t_\alpha^0) + 2\sqrt{n(t_\alpha^0) + n^2(t_\alpha^0)}}\right] = \Phi, \tag{21}$$

which implies the approximation to Eq. (20):

$$\sqrt{\mathcal{C}_\alpha}\tau_\alpha = \frac{\Phi}{\sqrt{\mathcal{K}_\alpha}}. \tag{22}$$

Here we have defined $\Phi \equiv \Delta n^{\rm eq}/[n_\alpha^{\rm eq}(1+n_\alpha^{\rm eq})]^{1/2} + (\Delta n^{\rm eq})^2(1+2n_\alpha^{\rm eq})/[8n_\alpha^{\rm eq}(1+n_\alpha^{\rm eq})]^{3/2}$.

The entropy flow due to heat exchange, $\beta_\alpha^r Q_\alpha = \int_{t_\alpha^0}^{t_\alpha^f}\beta_\alpha^r\omega(t)\dot{n}dt$, can be expressed as $\beta_\alpha^r Q_\alpha = \int_{n(t_\alpha^0)}^{n(t_\alpha^f)}\beta_\alpha^r\omega(t)dn$. According to Eq. (16), by use of $\beta_\alpha^r\omega = \ln[(\mathcal{K}_\alpha + 2\mathcal{C}_\alpha n + 2\mathcal{C}_\alpha n^2 + \sqrt{\mathcal{K}_\alpha{}^2 + 4n\mathcal{C}_\alpha\mathcal{K}_\alpha + 4n^2\mathcal{C}_\alpha\mathcal{K}_\alpha})/(2\mathcal{C}_\alpha n^2)]$, we can derive the expression of entropy flow as

$$\beta_\alpha^r Q_\alpha = \tilde{S}[n(t_\alpha^f);\mathcal{K}_\alpha] - \tilde{S}[n(t_\alpha^0);\mathcal{K}_\alpha], \tag{23}$$

where

$$\tilde{S}[n;\mathcal{K}_\alpha] = \sqrt{\frac{\mathcal{K}_\alpha}{\mathcal{C}_\alpha}}\ln[\mathcal{C}_\alpha(1+2n) + \sqrt{\mathcal{K}_\alpha\mathcal{C}_\alpha + 4\mathcal{C}_\alpha{}^2 n(1+n)}] + \ln\left[2\mathcal{C}_\alpha(1+n) - \mathcal{K}_\alpha - \sqrt{K_\alpha^2 + 4\mathcal{C}_\alpha\mathcal{K}_\alpha n(1+n)}\right] + n\ln\left[\frac{\mathcal{K}_\alpha + 2\mathcal{C}_\alpha n(1+n) + \sqrt{\mathcal{K}^2 + 4\mathcal{C}_\alpha\mathcal{K}n(1+n)}}{2\mathcal{C}_\alpha n^2}\right]. \tag{24}$$

It follows, making the first-order Taylor expansion of $\tilde{S}[n;\mathcal{K}_\alpha]$ with respect to $\sqrt{\mathcal{K}_\alpha}$, that the entropy flow for the low dissipation process of time duration τ_α becomes

$$\Delta\tilde{S}_\alpha = \Delta S_\alpha - \Delta S_\alpha^{\rm irr}, \tag{25}$$

where

$$\Delta S_\alpha^{\rm irr} = \sqrt{\frac{\mathcal{K}_\alpha}{\mathcal{C}_\alpha}}\ln\left[\frac{1 + 2n(t_\alpha^f) + 2\sqrt{n(t_\alpha^f) + n^2(t_\alpha^f)}}{1 + 2n(t_\alpha^0) + 2\sqrt{n(t_\alpha^0) + n^2(t_\alpha^0)}}\right] \tag{26}$$

denotes the irreversible entropy production, and $\Delta S_\alpha = S(t_\alpha^f) - S(t_\alpha^0)$ is the entropy change, with $S = n^0\ln(1 + 1/n^0) + (1+n^0)$ [43] being the system entropy. In the given process, ΔS_α ($\alpha = c, h$) is a preassigned state variable and $\Delta S_\alpha^{\rm irr}$ is a protocol-dependent quantity. For our engine model, the two isothermal processes linking two adiabatic isentropic processes, there exists the relation $\Delta S \equiv \Delta S_h = -\Delta S_c$. It follows, inserting Eq. (21) into Eq. (26) and using Eq. (22), that the entropy flow due to heat exchange along an isothermal process takes the low dissipation form [38]:

$$\Delta\tilde{S}_\alpha = \Delta S - \frac{\Phi^2}{\mathcal{C}_\alpha\tau_\alpha}, \tag{27}$$

where Φ was defined in Eq. (22). Equation (27) confirms the low dissipation assumption that the irreversible entropy production in an isothermal process is inversely proportional to the time duration. Accordingly, the heat transfers along the hot and cold isothermal contact of the engine cycle are given by

$$Q_h = \frac{\Delta S}{\beta_h^r} - \frac{\Phi^2}{\mathcal{C}_h \beta_h^r \tau_h}, \quad Q_c = \frac{-\Delta S}{\beta_c^r} - \frac{\Phi^2}{\mathcal{C}_c \beta_c^r \tau_c}. \tag{28}$$

Following Refs. [37,38], the optimal efficiency for the engines under maximum power can be expressed by a form similar to Eq. (3):

$$\eta^* = \frac{\eta_c}{2 - \gamma_{el}\eta_C}, \tag{29}$$

where $\gamma_{el} \equiv (1 + \sqrt{\mathcal{C}_h \beta_h^r / \mathcal{C}_c \beta_c^r})^{-1}$. The efficiency at maximum power is thus bounded by Eq. (5) and reaches the upper bound $\eta_C/(2 - \eta_C)$ when $\mathcal{C}_h/\mathcal{C}_c \to 0$ and lower bound $\eta_C/2$ when $\mathcal{C}_h/\mathcal{C}_c \to \infty$, and it is exactly the same as CA efficiency η_{CA} (1) for $\mathcal{C}_h = \mathcal{C}_c$. Note also that the quadratic approximation to η^* about η_C is obtained via Taylor series expansion:

$$\eta^* = \frac{\eta_C}{2} + \frac{\eta_C^2}{4\left(1 + \sqrt{\beta_h^r \mathcal{C}_h / \beta_c^r \mathcal{C}_c}\right)} + O\left(\eta_C^3\right), \tag{30}$$

which reduces to formula (2) when $\beta_h^r \mathcal{C}_h = \beta_c^r \mathcal{C}_c$. In the quasistatic limit ($\tau_\alpha \to \infty$), the irreversible entropy production is vanishing [$\Delta S_\alpha^{\text{irr}} = \Phi^2/(\mathcal{C}_\alpha \tau_\alpha) = 0$]. In this case, the fact that the entropy change after a cycle is zero, namely, $\Delta S = \Delta \tilde{S}_{\text{cyc}} = \beta_h Q_h + \beta_c Q_c = 0$, reproduces the Carnot efficiency $\eta = \eta_C = 1 - \beta_h/\beta_c$. We emphasize that, in deriving the weak dissipation form (27), the condition that the quantum heat engine is required to work in the linear responses was used, as done in low dissipation quantum-dot Carnot engines [37].

C. Endoreversible description

Since the irreversibility of the quantum heat engines under consideration is exclusively coming from the thermal interaction between the system and the heat reservoir, one can apply an endoreversible thermodynamics approach to investigating the performance of these engines. In view of the fact that our analysis is restricted to the linear irreversible region, we approximate the instantaneous mean population $n[\beta_\alpha(t), \omega_\alpha(t)]$ given in Eq. (8) around the (global) thermal equilibrium point $\beta_\alpha(t) = \beta_\alpha^r$ and $\omega_\alpha(t) = \omega_\alpha^f \equiv \omega(t_\alpha^f)$, keeping only the first nonzero term:

$$n(t) = n_{\text{eq}} + \frac{1}{\mathcal{C}_\alpha} n_{\text{eq}}^{\beta_\alpha}\left[\beta_\alpha(t) - \beta_\alpha^r\right] + \frac{1}{\mathcal{C}_\alpha} n_{\text{eq}}^{\omega_\alpha}\left[\omega_\alpha(t) - \omega_\alpha^f\right], \tag{31}$$

where we have introduced $n_{\text{eq}}^{\beta_\alpha} \equiv \mathcal{C}_\alpha \frac{\partial n}{\partial \beta_\alpha}|_{\beta_\alpha(t)=\beta_\alpha^r, \omega_\alpha(t)=\omega_\alpha^f}$ and $n_{\text{eq}}^{\omega_\alpha} \equiv \mathcal{C}_\alpha \frac{\partial n}{\partial \omega_\alpha}|_{\beta_\alpha(t)=\beta_\alpha^r, \omega_\alpha(t)=\omega_\alpha^f}$. Inserting Eq. (31) into Eq. (7) leads to

$$\dot{n}(t) = n_{\text{eq}}^{\beta_\alpha}\left[\beta_\alpha^r - \beta_\alpha(t)\right] + n_{\text{eq}}^{\omega_\alpha}\left[\omega_\alpha^f - \omega_\alpha(t)\right]. \tag{32}$$

Without loss of generality, $\beta_\alpha(t)$ and $\omega_\alpha(t)$ can be written as

$$\beta_\alpha(t) = \beta_\alpha\left(t_\alpha^0\right) + \gamma(t)\left[\beta_\alpha^r - \beta_\alpha\left(t_\alpha^0\right)\right], \tag{33}$$

$$\omega_\alpha(t) = \omega_\alpha\left(t_\alpha^0\right) + g(t)\left[\omega_\alpha^f - \omega_\alpha\left(t_\alpha^0\right)\right], \tag{34}$$

where $\gamma(t)$ and $g(t)$ are the functions of time t only and they must be restricted to the boundary conditions $\gamma(t_\alpha^0) = g(t_\alpha^0) = 0$ and $g(t_\alpha^f) = \gamma(t_\alpha^f \to \infty) = 1$. Returning to the heat exchange along the isothermal contact, we find by combination of Eqs. (11), (33), and (34)

$$Q_\alpha = \int_{t_\alpha^0}^{t_\alpha^f} \left[\omega_\alpha^f + \tilde{g}(t)\left(\omega_\alpha^0 - \omega_\alpha^f\right)\right]\left[n_{\text{eq}}^{\beta_\alpha}\tilde{\gamma}(t)\left(\beta_\alpha^0 - \beta_\alpha^r\right) + n_{\text{eq}}^{\omega_\alpha}\tilde{g}(t)\left(\omega_\alpha^0 - \omega_\alpha^f\right)\right]dt, \tag{35}$$

where $\tilde{\gamma}(t) \equiv 1 - \gamma(t)$, $\tilde{g}(t) \equiv 1 - g(t)$, $\omega_\alpha^0 \equiv \omega(t_\alpha^0)$, and $\beta_\alpha^0 \equiv \beta(t_\alpha^0)$. In the linear response regime where $\Delta\beta/\beta_\alpha^0 \equiv (\beta_\alpha^0 - \beta_\alpha^f)/\beta_\alpha^0 \ll 1$ and thus $\Delta\omega/\omega_\alpha^0 \equiv (\omega_\alpha^0 - \omega_\alpha^f)/\omega_\alpha^0 \ll 1$ [7], Eq. (35) can be approximated by

$$Q_\alpha = \omega_\alpha^f n_{\text{eq}}^{\beta_\alpha}\psi_\alpha\left(\beta_\alpha^0 - \beta_\alpha^f\right)\tau_\alpha + \omega_\alpha^f n_{\text{eq}}^{\omega_\alpha} G_\alpha\left(\omega_\alpha^0 - \omega_\alpha^f\right)\tau_\alpha + O(\Delta^2), \tag{36}$$

where $\Delta^2 = (\Delta\omega)^2 + (\Delta\beta)^2 + \Delta\beta\Delta\omega$, $G_\alpha\tau_\alpha = \int_{t_\alpha^0}^{t_\alpha^f} \tilde{g}(t)dt$, and $\Gamma_\alpha\tau_\alpha = \int_{t_\alpha^0}^{t_\alpha^f} \tilde{\gamma}(t)dt$. For these endoreversible engines, here $\tilde{\gamma}(t)$ and $\tilde{g}(t)$ are assumed to be constant during an isothermal process, since in the linear response regime the differences of both temperature and frequency are very small. The relation $\frac{\partial n(t)}{\partial \beta_\alpha(t)}\beta_\alpha(t) = \frac{\partial n(t)}{\partial \omega_\alpha(t)}\omega_\alpha(t)$ for any instant along the isothermal contact leads to

$$\omega_\alpha^f = \frac{n_{\text{eq}}^{\beta_\alpha}}{n_{\text{eq}}^{\omega_\alpha}}\beta_\alpha^r, \quad \omega_\alpha^0 = \frac{n_{\text{eq}}^{\beta_\alpha}}{n_{\text{eq}}^{\omega_\alpha}}\beta_\alpha^0. \tag{37}$$

Combining Eqs. (37) and (36), the heat exchange between the system and heat bath can finally be written as

$$Q_h = \tilde{\mathcal{C}}_h\left(\beta_h^0 - \beta_h^r\right)\tau_h, \quad Q_c = \tilde{\mathcal{C}}_c\left(\beta_c^r - \beta_c^0\right)\tau_c, \tag{38}$$

where $\tilde{\mathcal{C}}_h \equiv \langle H\rangle_{\text{eq}}^{\beta_h}(\Gamma_h + G_h)$ and $\tilde{\mathcal{C}}_c \equiv \langle H\rangle_{\text{eq}}^{\beta_c}(\Gamma_c + G_c)$ with $\langle H\rangle_{\text{eq}}^{\beta_h} = \omega_h^f n_{\text{eq}}^{\beta_h}$, $\langle H\rangle_{\text{eq}}^{\beta_c} = \omega_c^f n_{\text{eq}}^{\beta_c}$.

We assume the heat engine to be weak endoreversible [9], which indicates that $\oint \dot{Q}(t)\beta(t) = 0$ [3], which reduces to the conventional endoreversible assumption [1] for fixed system temperature $\beta(t) = \beta$. Physically, this weak endoreversible model, in which the irreversibility arises due to imperfect thermal interaction between the system and the heat reservoirs, can be understood from the context that the working substance relaxes (to the internal equilibrium) much more quickly than the heat exchange along the isothermal heat exchange. It follows, using $\oint \dot{Q}(t)\beta(t) = 0$, that the conventional endoreversible condition

$$\beta_h^0\langle Q_h\rangle + \beta_c^0\langle Q_c\rangle + O(\Delta^2) = 0 \tag{39}$$

holds in the linear response regime. Equation (39) shows that the engine efficiency, $\eta = (Q_h + Q_c)/Q_h$, is given by

$$\eta = 1 - \frac{\beta_h^0}{\beta_c^0}. \tag{40}$$

With consideration of Eqs. (38), (39), and (40), the power output, $P = (Q_h + Q_c)/\tau_{\text{cyc}}$, can be expressed in terms of η and β_h^0:

$$P = \frac{\tilde{\mathcal{C}}_c\tilde{\mathcal{C}}_h\eta\left(\beta_h^0 - \beta_h^r\right)\left[(1-\eta)\beta_c^r - \beta_h^0\right]}{\tilde{\mathcal{C}}_c\left[(1-\eta)\beta_c^r - \beta_h^0\right] + \tilde{\mathcal{C}}_h\left(\beta_h^0 - \beta_h^r\right)(1-\eta)^2}. \tag{41}$$

Setting $\partial P/\partial\beta_h^0 = 0$ and $\partial P/\partial\eta = 0$, we find that the efficiency at maximum power η^* satisfies the expression similar to Eq. (3) or (29), which reads

$$\eta^* = \frac{\eta_C}{2 - \tilde{\gamma}_{qn}\eta_C}, \quad (42)$$

where we have used $\tilde{\gamma}_{qn} = 1/(1 + \sqrt{\tilde{\mathcal{C}}_c/\tilde{\mathcal{C}}_h})$. The efficiency at maximum power only depends on the ratio $\tilde{\mathcal{C}}_c/\tilde{\mathcal{C}}_h$. In the limits $\tilde{\mathcal{C}}_c/\tilde{\mathcal{C}}_h \to 0$ and $\tilde{\mathcal{C}}_c/\tilde{\mathcal{C}}_h \to \infty$, the efficiency at maximum power approaches the upper bound $\eta_+^* = \eta_C/(2 - \eta_C)$ and lower bound $\eta_-^* = \eta_C/2$ [as given in Eq. (5)], respectively. The efficiency at maximum power is expanded with respect to η_C up to the second order, leading to

$$\eta^* = \frac{\eta_C}{2} + \frac{\eta_C^2}{4(1 + \sqrt{\tilde{\mathcal{C}}_c/\tilde{\mathcal{C}}_h})} + O(\eta_C^3). \quad (43)$$

In the symmetric limit when $\tilde{\mathcal{C}}_h = \tilde{\mathcal{C}}_c$, we recover Eq. (2) for a quantum cyclic engine, namely, $\eta^* = \eta_C/2 + \eta_C^2/8 + O(\eta_C^3)$, which thus shares the same universality with the CA efficiency η_{CA} for the linear response regime where the temperature gradient is small.

D. Irreversible thermodynamic analysis

The entropy production rate of the system $\dot{\sigma}$ can be expressed as the sum of the entropy increase rates for the hot and cold reservoir, $\dot{\sigma} = -(\dot{Q}_h\beta_h^r + \dot{Q}_c\beta_c^r)$, where $\dot{Q}_{h,c} = Q_{h,c}/\tau_{\mathrm{cyc}}$ denotes the average heat current along the hot or cold process. In the linear response regime where the temperature difference of the two heat reservoirs is small, we apply the approximation of $\beta_c^0 \simeq \beta_c^r$ to writing the entropy production rate $\dot{\sigma}$ as

$$\dot{\sigma} = (\Gamma_h + G_h)R\beta_c^r\langle H\rangle_{\mathrm{eq}}^{\beta_h}(\Delta\beta - \Delta\beta^r)\frac{\Delta\beta}{\beta_c^r} + (\Gamma_h + G_h)R\langle H\rangle_{\mathrm{eq}}^{\beta_h}(\Delta\beta - \Delta\beta^r)(-\Delta\beta^r), \quad (44)$$

where $\Delta\beta = \beta_h^0 - \beta_c^0$, $\Delta\beta^r = \beta_h^r - \beta_c^r$, and $R = \tau_h/\tau_{\mathrm{cyc}}$. If

$$J_q = \beta_c^r R\langle H\rangle_{\mathrm{eq}}^{\beta_h}(\Gamma_h + G_h)(\Delta\beta - \Delta\beta^r) \quad (45)$$

and

$$J_s = R\langle H\rangle_{\mathrm{eq}}^{\beta_h}(\Gamma_h + G_h)(\Delta\beta - \Delta\beta^r) \quad (46)$$

are identified as the heat and entropy fluxes, respectively, their conjugate affinities are then given by

$$X_q = \Delta\beta/\beta_c^r, \quad X_s = -\Delta\beta^r. \quad (47)$$

We therefore have $\dot{\sigma} = J_qX_q + J_sX_s$ by using Eq. (44). Liner relations between fluxes $J_{q,s}$ and affinities $X_{q,s}$,

$$J_q = L_{qq}X_q + L_{qs}X_s, \quad J_s = L_{sq}X_q + L_{ss}X_s, \quad (48)$$

hold in the linear response regime. Here the Onsager coefficients $L_{\mu,\nu}$ with $\mu,\nu = q,s$ should satisfy the conditions $L_{qs} = L_{sq}$, $L_{qq}, L_{ss} \geqslant 0$, and $L_{qq}L_{ss} \geqslant L_{sq}L_{qs}$, such that the second law of thermodynamics is valid, namely, $\dot{\sigma} \geqslant 0$.

Comparing Eqs. (45) and (46) to Eq. (48), and using Eq. (47), we obtain the Onsager coefficients $L_{\mu\nu}$ ($\mu,\nu = q,s$) as

$$L_{qq} = R\langle H_{\mathrm{eq}}^{\beta_h}\rangle(\Gamma_h + G_h)(\beta_c^r)^2, L_{qs} = R\langle H_{\mathrm{eq}}^{\beta_h}\rangle(\Gamma_h + G_h)\beta_c^r, \quad (49)$$

$$L_{sq} = R\langle H_{\mathrm{eq}}^{\beta_h}\rangle(\Gamma_h + G_h)\beta_c^r, L_{ss} = R\langle H_{\mathrm{eq}}^{\beta_h}\rangle(\Gamma_h + G_h), \quad (50)$$

which fulfill the Onsager reciprocity $L_{qs} = L_{sq}$ and confirm that $\dot{\sigma} \geqslant 0$. Since the coupling strength fulfills $\widetilde{q} \equiv L_{qs}/\sqrt{L_{ss}L_{qq}} = 1$ [16], the models under consideration are proved to be tightly coupled. The power output, $P = \dot{Q}_h\eta = J_q\eta$, can be expressed in terms of the Onsager coefficients: $P = J_q\frac{\Delta\beta}{\bar{\beta}^r} = (L_{qq}X_q + L_{qs}X_s)X_q$. Using the condition $\partial P/\partial X_e = 0$, we obtain the corresponding efficiency η^*:

$$\eta^* = -\frac{\Delta\beta^r}{2\bar{\beta}^r} = \frac{\eta_C}{2} + O(\eta_C^2). \quad (51)$$

It shows that the efficiency at maximum power, when accurate to the first order of η_C, attains the upper bound η_{CA}, namely, $\eta^* = \eta_{\mathrm{CA}} + O(\eta_C^2)$, thereby supporting an argument in favor of our approach.

III. DISCUSSIONS AND CONCLUSIONS

A natural extension of the present model based on a multilevel system is to use a spin-1/2 (two-level) system described by Fermi-Dirac statistics as the working substance for the engines. Consider an engine with working system composed of a spin-1/2 system. The Hamiltonian can be parametrized by $\hat{H} = \omega(t)a_s^\dagger\hat{a}_s$, where $a_s^\dagger$ and $\hat{a}_s$ are the spin creation and annihilation operators [13,30], respectively, and they satisfy the anticommutation relation $[a_s, a_s^\dagger]_+ = 1$. Into Eq. (6) we insert $\hat{V}_i^\dagger = \hat{a}_s{}^\dagger$ ($\hat{V}_i = \hat{a}_s$) and $\hat{X} = \hat{a}_s^\dagger\hat{a}_s$, and we obtain the time evolution of the instantaneous mean polarization $\dot{n}_s(t) = -\mathcal{C}_s[n_s(t) - n_s^0(t)]$. Here the heat conductivity between the spin system and the heat bath is given by $\mathcal{C}_s = k_- + k_+$, and the mean polarization of the system at local equilibrium takes the form $n_s^0 = (k_+ - k_-)/[2(k_- + k_+)] = -1/2\tanh[\beta\omega(t)/2]$, where the transition rates fulfill the detailed balance $k_-/k_+ = e^{\beta\omega}$. Given control variable $\omega(t) = \omega = \mathrm{const}$, the system in the hot ($\alpha = h$) or cold ($\alpha = c$) isothermal contact would relax to global thermal equilibrium with the mean polarization $n_{s,\alpha} = -1/2\tanh(\beta_\alpha^r\omega/2)$. Employing the same approach as was adopted for determining the optimal protocol under maximum power, one can easily obtain the low dissipation form of irreversible entropy production as given in Eq. (20), but with the parameter Φ that merely depends on $\Delta n_s^{\mathrm{eq}} = n_{s,h}^{\mathrm{eq}} - n_{s,c}^{\mathrm{eq}}$. Within the framework of endoreversible or irreversible thermodynamics, we investigated the finite time performance of the engine that works with a harmonic system, without using the concrete expression of mean population n (system energy $\langle H\rangle$). Therefore, for spin-1/2 engines, the expressions of the efficiency at maximum power remain the same as the corresponding ones given by Eqs. (42) and (51).

To conclude, we have examined the finite time performance of quantum heat engines with emphasis on the universal behavior of efficiency at maximum power in certain limits. The power and efficiency were evaluated via the dynamic analysis of the engines, based on three different approaches.

First, we optimized the engines (subject to finite time cycle duration) with respect to the power output via the strategy of optimization based on the Euler-Lagrange equation. The results obtained in this way show that the engine operating in the optimal protocol can be mapped into the low dissipation engine model, and thus the efficiency at maximum power recovers that obtained from the low dissipation case. Endoreversible thermodynamic description was applied to the optimization of power output for these engines in linear responses. The expression for efficiency at maximum power is in nice agreement with that obtained via classical thermodynamics, where the phenomenological heat transfer laws were used. We finally formulated the power and efficiency in the form of linear irreversible thermodynamics, and showed that these cyclic engines satisfy the tight-coupling condition, thereby confirming the universality of efficiency at maximum power for the engines in the linear response regime.

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