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Why Lévy α-stable distributions lack general closed-form expressions for arbitrary α

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The ubiquitous Lévy α-stable distributions lack general closed-form expressions in terms of elementary functions—Gaussian and Cauchy cases being notable exceptions. To better understand this 80-year-old conundrum, we study the complex analytic continuation $p_\alpha(z)$, $z \in \mathbb{C}$, of the symmetric Lévy α-stable distribution family $p_\alpha(x)$, $x \in \mathbb{R}$, parametrized by $0 < \alpha \leqslant 2$. We first extend known but obscure results, and give a new proof that $p_\alpha(z)$ is holomorphic on the entire complex plane for $1 < \alpha \leqslant 2$, whereas $p_\alpha(z)$ is not even meromorphic on $\mathbb{C}$ for $0 < \alpha < 1$. Next, we unveil the complete complex analytic structure of $p_\alpha(z)$ using domain coloring. Finally, motivated by these insights, we argue that there cannot be closed-form expressions in terms of elementary functions for $p_\alpha(x)$ for general α.

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The ubiquitous Gaussian and Cauchy distributions are given by simple mathematical formulas and are special cases of the Levy α-stable distribution [1], introduced more than 80 years ago. However, even now there is no known general closed-form expression in terms of elementary functions for the full family of Levy α-stable distributions. Here we explain why and argue that there cannot be closed-form expressions in terms of elementary functions for the symmetric Levy α-stable distribution.

The Lévy α-stable distribution [1] plays an important role in areas as distinct as biology (e.g., foraging [2]), chemistry [3], complex systems [4], econophysics [5,6], and social sciences [7,8]. In physics, applications include superdiffusive Lévy walks of particles kinetics [4,9,10], many-body quantum systems [11], intensity distributions in random lasers [12,13], the shape of spectral lines [14], and fluorescence intermittency in colloidal nanocrystals and blinking time distribution of quantum dots [15,16], to cite just a few.

The full family of Lévy α-stable distributions (parametrized by the Lévy index α, the asymmetry β, the scale σ, and the shift μ) is generally defined as ($\alpha \in (0, 2]$)

$$p(x;\alpha,\beta,\sigma,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ \exp[\phi(t)]\exp[-ixt], \quad (1)$$

for $\phi(t) = it\mu - |\sigma t|^\alpha(1 - i\beta\,\mathrm{sgn}[t]\,s_\alpha)$, with $s_\alpha = -\frac{2}{\pi}\ln[|t|]$ ($s_\alpha = \tan[\frac{\pi}{2}\alpha]$) if $\alpha = 1$ ($\alpha \neq 1$), $\mathrm{sgn}(t) = \pm 1$ being the sign function, $\beta \in [-1, 1]$, $\mu \in (-\infty, +\infty)$, and $\sigma > 0$.

Only three special cases can be expressed in closed form, i.e., in terms of a finite number of algebraic operations involving only elementary functions [7,17] (see also below): the Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) are symmetric ($\beta = 0$) and the asymmetric $\beta = 1$ case has an analytic expression for the so-called Lévy distribution ($\alpha = 1/2$). For other parameter values, in principle $p(x;\alpha,\beta,\sigma,\mu)$ can be cast in terms of Fox-H functions [18–20], whose calculation relies on complex integrals of the Mellin-Barnes type. A number of works in the last two decades have addressed the challenging problem of how to write Lévy α-stable distributions in terms of simpler expressions [17,20–23], generally involving approximation schemes and series expansions [24] for particular parameters values. For example, when $\beta = 0$ and $\alpha = 2/M$ (for $M = 1, 2, 3, \ldots$) p is given by an exact finite sum of hypergeometric functions [25] (more generally, for $\alpha = 1/M$ it can be represented by G functions [26,27]).

In fact, for this important $\beta = 0$ (symmetric) case, we always can promote the rescaling $(x - \mu)/\sigma \to x$ and $\sigma p \to p$ in Eq. (1). Then, for $\beta = 0$, without loss of generality we can set $\sigma = 1$ and $\mu = 0$, obtaining

$$p_\alpha(x) = \frac{1}{\pi}\int_0^\infty \exp[-t^\alpha]\cos[tx]dt. \quad (2)$$

For illustrative plots of $p_\alpha(x)$, see Fig. S1 in the Supplemental Material (SM) [28].

The ubiquity of the Lévy α-stable distributions in stochastic phenomena is due to the generalized central limit theorem (GCLT), a weak form of which states that the distribution of the sum of independent and identically distributed random variables (of possibly infinite variance) converges to $p(x;\alpha,\beta,\sigma,\mu)$ [29]. The standard CLT is the special case $\alpha = 2$ of the GCLT, for which p is a Gaussian. For non-Gaussian cases, $p(x;\alpha < 2,\beta,\sigma,\mu)$ is a "fat tailed" distribution [29] such that for $x \gg 1$ one has an asymptotic power law $p \sim |x|^{-\alpha-1}$.

In many contexts, such as multifractality [19] or if interpreting x as step lengths instead of positions of a walker [2],

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one may have interest in $\Delta_x(\alpha) = \int_0^{+\infty} dx\, x p(x; \alpha, \beta, \sigma, \mu)$, which in the latter case can be thought of as the "single displacement mean." Because of the above-mentioned $x \to \infty$ behavior of p, we have a converging $\Delta_x(\alpha)$ for $\alpha > 1$ and a diverging (marginally diverging; see below) $\Delta_x(\alpha)$ for $\alpha < 1$ ($\alpha = 1$). Hence, this "classification scheme" seems to be uniquely related to Lévy α-state distributions asymptotics, with no further "deeper" analytical feature of p playing any role (but see the interesting α classification in [20]). However, take $p_\alpha(x)$ in Eq. (2). It is well defined for any x in the real line. Nevertheless, depending on α, $p_\alpha(x)$ may diverge if one tries to implement analytic continuation by simply substituting x by $z \in \mathbb{C}$ in Eq. (2). Actually, for $\alpha > 1$ such direct procedure works for any z. In contrast, for $\alpha = 1$ the integral behaves properly, yielding $\pi^{-1}/(1+z^2)$, only if $-1 < \mathrm{Re}(z) < +1$, whereas for $\alpha < 1$ the integral does not converge whenever $\mathrm{Im}(z) \neq 0$. Thus, these three regimes for $p_\alpha(z)$ as a function of α are akin to the convergence of $\Delta_x(\alpha)$ regarding the α intervals. The above simple, yet apparently overlooked, fact suggests that the analytic behavior of p_α could unveil relevant properties of the Lévy α-stable distributions, that are otherwise hard to grasp from the usual Fourier transform representation.

We thus examine the analytic continuation of p_α into the complex plane $\mathbb{C}$. We show the p_α properties become explicitly manifested in $\mathbb{C}$, with its full analytic prescription strongly dependent on α. From the analytic structure of $p_\alpha(z)$, we then are able to discuss certain aspects of the mathematical obstruction, rendering it impossible to find general closed-form expressions for p in terms of elementary functions (e.g., as defined in [30,31]; see next).

We divide our analysis into two parts. First, we classify and characterize the analytic continuation $p_\alpha(z)$ of $p_\alpha(x)$, which can be found in fragmented (and incomplete) form in the literature (see, e.g., [32–34]), but which we derive here in a systematic way, with much simpler proofs. Second, we use the numerical visualization method of domain coloring (Fig. 1) to investigate the full analytic structure of $p_\alpha(z)$, thereby obtaining insight into the notorious difficulty of finding simple general expressions for the Lévy α-stable distributions.

We first recall the following result [7], for which we provide a straightforward proof.

Theorem 1 (The symmetric Lévy α-stable distribution analytic classification). If $p_\alpha(z)$ is the symmetric Lévy stable distribution analytic continuation onto the complex plane $\mathbb{C}$, then

(i) $p_\alpha(z)$ is holomorphic on $\mathbb{C}$ for $1 < \alpha \leqslant 2$;

(ii) $p_\alpha(z)$ is meromorphic on $\mathbb{C}$ for $\alpha = 1$;

(iii) $p_\alpha(z)$ has an essential singularity at $z = 0$, not being meromorphic on $\mathbb{C}$ for $\alpha < 1$.

Proof. A function is holomorphic on an open disk centered at z_0 if its Taylor series at z_0 converges on the disk. Thus, we first calculate the radius of convergence of the Taylor series for $p_\alpha(z)$ at the origin. Differentiating n times Eq. (2) with $x \to z$ and evaluating the result at $z = 0$ we get

$$\left.\frac{d^n p_\alpha}{dz^n}\right|_{z=0} = \frac{1+(-1)^n}{2\pi} \int_0^\infty \exp[-t^\alpha](it)^n\, dt, \tag{3}$$

from which we obtain the Taylor expansion at the origin

$$p_\alpha(z) = \frac{1}{\alpha\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Gamma\left[\frac{1+2n}{\alpha}\right] z^{2n}. \tag{4}$$

The series in Eq. (4) is the complex generalization of a classical result obtained in [33] (see also [35]). To find the radius of convergence we apply the root test. From Stirling's approximation for large n, we obtain up to constant factors

$$\sqrt[n]{\frac{z^{2n}}{(2n)!} \Gamma\left[\frac{1+2n}{\alpha}\right]} \sim z^2 n^{-2+[(1/n+2)/\alpha]}. \tag{5}$$

The condition for infinite radius of convergence is thus $\alpha > 1$. For the case $\alpha > 1$, $p_\alpha(z)$ is an entire function (i.e., holomorphic on $\mathbb{C}$), and claim I follows.

For $\alpha < 1$, the series has zero radius of convergence. Note that the radius of convergence of a power series extends to the nearest singularity, hence there is a singularity at $z = 0$. Since there are only three types of isolated singularities, claim (iii) follows by showing that when $\alpha < 1$, the singularity at $z = 0$ is neither removable nor a pole. Since $p_\alpha(x)$ is continuous on the real line, we rule out a removable singularity. Further, it cannot be a pole of an arbitrary order k because from the series, $z^k\, p_\alpha(z)$ also has a zero radius of convergence.

Lastly, (ii) holds given that $p_1(z) = \pi^{-1}/(1+z^2)$ is the natural analytic continuation of the well-known Cauchy distribution $\pi^{-1}/(1+x^2)$ (for details, see SM). So $p_1(z)$ is meromorphic on $\mathbb{C}$, with a single pair of poles of order 1 on the imaginary axis ($z = \pm i$). ■

The study of $p_\alpha(z)$ for arbitrary $\alpha < 1$ directly from Eq. (2) is not possible. This issue can be overcome as follows.

Theorem 2 (Equivalent expressions to Eq. (2) when α ⩽ 1). For $\alpha \leqslant 1$, let

$$p_\alpha^{(\pm)}(x) = \frac{1}{\pi} \int_0^\infty dt\, \exp[\mp xt] \exp\left[-\cos\left(\alpha\frac{\pi}{2}\right)t^\alpha\right] \times \sin\left[\sin\left(\alpha\frac{\pi}{2}\right)t^\alpha\right], \tag{6}$$

where $p^{(+)}$ ($p^{(-)}$) is a convergent integral if $x \geqslant 0$ ($x \leqslant 0$) and $p^{(+)}(x) = p^{(-)}(-x)$, then

(i) $p_\alpha(x) = p_\alpha^{(+)}(x)$ [$p_\alpha^{(-)}(x)$] for $x > 0$[$x < 0$];

(ii) $p_\alpha^{(\pm)}(0) = \Gamma[1/\alpha]/(\alpha\pi) = p_\alpha(0)$.

Proof. The demonstration of (i) is given in the SM [28] (see also [34] for a similar, but partial result). (ii) follows from the exact integral $\int_0^\infty dt\, \exp[-c\, t^\alpha] = c^{-\alpha}\, \Gamma[1+1/\alpha]$, $\mathrm{Re}[c] > 0$. ■

Now $p_{\alpha\leqslant 1}(x)$—through the representation in Eq. (6)—is amenable to analytic continuation by just setting $x \to z$. However, we need to consider two separated situations, once $p_\alpha^{(s)}(z)$ converges only for $s\,\mathrm{Re}(z) > 0$ [the exception is $\alpha = 1$, since $p^{(+)}(z)$ and $p^{(-)}(z)$ match perfectly at $\mathrm{Re}[z] = 0$; SM]. For $\alpha < 1$ the whole z-imaginary axis is a branch cut, separating the complex plane into two halves (for a much more involving analytic continuation construction in terms of distinct Riemann surfaces—depending on the explicit $0 < \alpha < 1$—see [32]). From our prescription, one finds that (i) the real part of $p_\alpha(z)$ across $x = 0$ is continuous, but with a discontinuous derivative (see Fig. 2 and the illustrative plots in the SM [28]) and (ii) the imaginary part of $p_\alpha(z)$ is

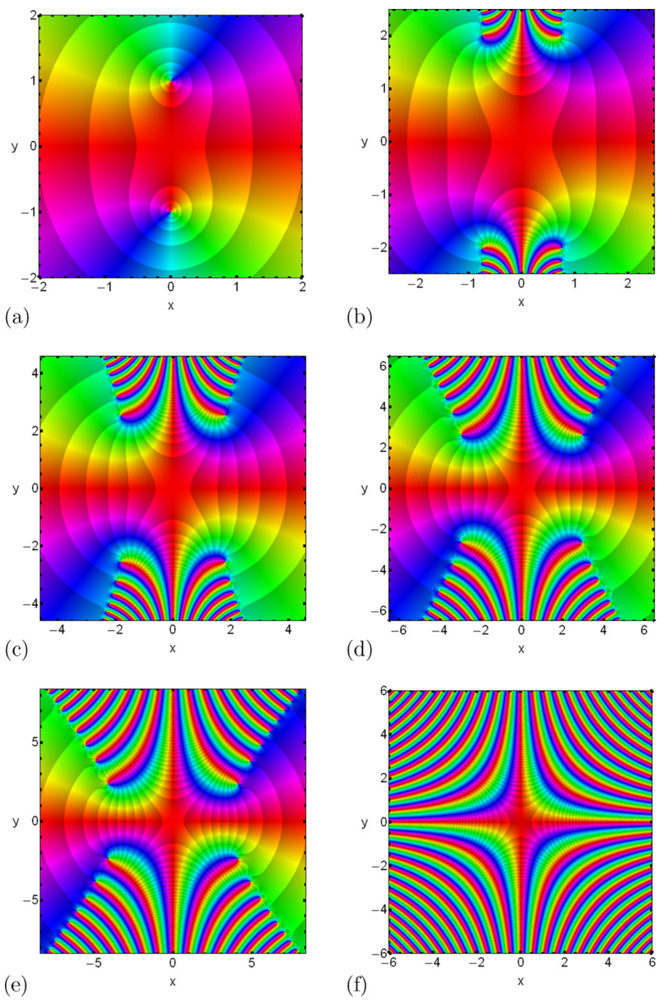

FIG. 1. Domain coloring plots of the analytic continuation $p_\alpha(z)$ of the symmetric Lévy stable distribution, calculated via Eq. (2) for various α: (b) 1.2, (c) 1.4, (d) 1.6, and (e) 1.8. Cases (a) 1.0 and (f) 2.0 are the Cauchy and Gaussian, calculated directly from their formulas. The upper cutoff in the Fourier integral was chosen to be 100 in Eq. (1), but we have verified that the results do not change significantly if the cutoff is increased to 1000 or higher values. Note how as α increases a "palm leaf" structure of zeros sprouts until $\alpha = 2$, where it extends to infinity.

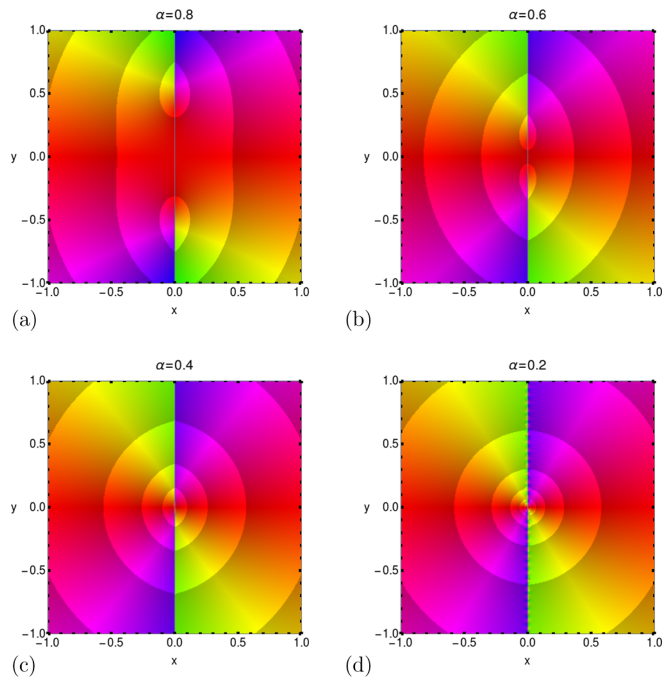

FIG. 2. Domain coloring plots of $p_\alpha(z)$, calculated using Eq. (6) for some values of α. For these values of α, Eq. (7) leads to the exact same graphs.

discontinuous across $x = 0$. The magnitude of all these jumps depends on y.

Based on Theorems 1 and 2 and the above analytic continuation protocol for $\alpha < 1$, we have the following.

Theorem 3 (Only $p_1(z)$ has poles in the complex plane). Assume the above analytic continuation prescription for the case of $\alpha \leqslant 1$. There are no poles for $p_\alpha(z)$ in $\mathbb{C}$ if $\alpha \neq 1$.

Proof. For Re$[z] \geqslant 0$ (Re$[z] \leqslant 0$), straightforwardly $p^{(+)}_{\alpha<1}(z)$ $[p^{(-)}_{\alpha<1}(z)]$ is always finite. This and (i) and (ii) of Theorem 1 conclude the proof. ■

Theorems 1–3 give a full picture of $p_\alpha(z)$. Equation (2) yields an entire function $p_{\alpha>1}(z)$, whereas $p_1(z) = \pi^{-1}/(1 + z^2)$ [see Fig. 1(a)] is meromorphic on $\mathbb{C}$ with two isolated poles. For $\alpha < 1$, the essential singularity at the origin $z = 0$ turns the imaginary axis into a branch cut, with two distinct analytic continuations in each side.

To exemplify the complete characterization of $p_\alpha(z)$ from Theorems 1–3, let $\alpha = 2/M$, $M = 3, 4, 5, \ldots$ (so $\alpha < 1$). The exact $p_\alpha(x)$ is given in terms of a finite sum of hypergeometric functions [25]. From these formula with the substitution $x \to z$, we get ($d_j = 1 + j/M$)

$$p_{2/M}(z) = \Gamma\left[\frac{M}{2}\right]\frac{M}{2\pi}\frac{1}{C}\sum_{j=1}^{M-1}\Gamma\left[d_j - \frac{1}{2}\right]\frac{\prod_{i=1,i\neq j}^{i=M-1}\Gamma[d_i - d_j]}{(M^M z^2/4)^{(1/2)+(j/M)}} \times {}_1F_{M-2}\left[\begin{matrix}(d_j - 1/2)\\ c^{(j)}\end{matrix}; \frac{4(-1)^{M-1}}{M^M z^2}\right], \tag{7}$$

for $C = \prod_{i=1}^{i=M-1}\Gamma[d_i - 1/2]$, ${}_1F_{M-2}$ the generalized hypergeometric functions (GHFs)${}_pF_q$, and $\mathbf{c}^{(j)} = (d_j - \frac{1}{M}, d_j - \frac{2}{M}, \ldots, d_j - \frac{j-1}{M}, d_j - \frac{j+1}{M}, d_j - \frac{j+2}{M}, \ldots, d_j - \frac{M-1}{M})$. ${}_pF_q(-;w)$ with w in $\mathbb{C}$ and $p \leqslant q$ has an irregular singularity at $w = \infty$, otherwise being holomorphic (see, e.g., [36]). Since in Eq. (7) we have for ${}_1F_{M-1}$ that $w \propto 1/z^2$, from Eq. (7) we directly identify an essential singularity at $z = 0$ and no poles for $p_{\alpha=2/M<1}(z)$. Also, plots of Eq. (7) in the complex plane (see below) display discontinuities across the imaginary axes. These facts are clearly in agreement with some of our previous general results.

Now, we turn to numerical calculations so as to unveil the complete analytic structure on $\mathbb{C}$. We use domain coloring (using *Mathematica*): (i) the colors show the argument of $p_\alpha(z)$, with red being zero and cycling through rainbow colors back to 2π; thus blue indicates negative $p_\alpha(z)$ and green and purple indicate positive and negative imaginary values, respectively; (ii) shading shows the absolute value of $p_\alpha(z)$, with the gradient from lighter to darker signifying increasing absolute values; (iii) discontinuities in shading indicate a doubling of absolute value; (iv) zeros (poles) of $p_\alpha(z)$ appear as points around which the color cycles through the rainbow colors in counterclockwise (clockwise) direction. For further details, see the SM. Importantly, essential singularities appear as points in whose proximity the function becomes wildly oscillatory (e.g., Fig. S7 in the SM for the function $z \mapsto \exp(1/z)$, with an essential singularity at $z = 0$). Picard's great theorem guarantees that in a punctured neighborhood of the essential singularity the function assumes every complex value except possibly one, infinitely often.

Figure 1 displays the domain coloring plot analysis of $p_\alpha(z)$ (for some $1 \leqslant \alpha \leqslant 2$). Figures 1(a) and 1(f) show the Cauchy ($\alpha = 1$) and Gaussian ($\alpha = 2$) distributions. The Gaussian is a saddle on the complex plane, with a saddle point at the origin. For the Cauchy, the pair of poles at $z = \pm i$ can be seen on the imaginary axis. Since $p_{1<\alpha<2}(z)$ is holomorphic and closed-form expressions are known only for rational α's (see below), we have performed numerical integration of Eq. (2). Figures 1(b)–1(e) show the representative cases $\alpha = 1.2, 1.4, 1.6, 1.8$. As α grows from 1 to 2, a series of zeros (in the form of a "palm leaf" pattern) approaches the real axis and begins to extend toward infinity in both x positive and negative directions.

The behavior changes drastically for $\alpha < 1$. As previously mentioned, instead of Eq. (2) we must use Eq. (6), observing the distinct expressions for the two halves of $\mathbb{C}$. In Fig. 2 we show plots for some values of $\alpha < 1$. Once the colors represent the argument of $p_\alpha(z)$, one can see that $p_\alpha(z)$ display discontinuities across $\mathrm{Re}[z] = 0$ (the branch cuts are needed because the true analytic continuation for $\alpha < 1$ is a Riemann surface with multiple branches [32]). All the qualitative results in Fig. 2 corroborate Theorems 1–3.

Finally, we discuss the extreme difficulty of finding simple exact expressions for $p_\alpha(x)$. A nice overview about closed solutions is presented in [22]. In particular, for $\beta = 0$ and $\alpha = 2r/k$ (with r, k positive integers and either $0 < 2r/k < 1$ or $1 < 2r/k < 2$), Ref. [22] shows that $p_\alpha(x)$ is given as a sum of $N = \max(2r, k)$ GHFs (Eq. (7) corresponds to $r = 1$ and $k = M$ [25]). Certainly, to be able to write $p_\alpha(x)$ in terms of special functions is a great advantage. First, because $p_\alpha(x)$ will naturally display the general properties—like symmetries, asymptotics, small argument behavior, etc.—of the associated special functions. Second, because often there exist efficient algorithms to numerically compute the special functions series or integral representations. But why are there solutions only for $\alpha = 1, 2$ in terms of elementary functions and why only for rational α are there solutions in terms of GHFs (or G functions [27])? To make clear what we mean by elementary functions, we consider the following definition [30,31]: an elementary function is a function of one variable that can be expressed in terms of a finite number of algebraic operations (of the real or complex field) and a finite number of compositions of the following functions: powers and roots, exponential and logarithmic functions, trigonometric and hyperbolic functions, and their inverses. For example, elliptic integrals are not elementary, whereas the derivatives of elementary functions are themselves elementary.

We start with $1 \leqslant \alpha \leqslant 2$. From the domain coloring plots, $\alpha = 1$ and $\alpha = 2$ display a rather similar pattern in the whole complex plane [Figs. 1(a) and 1(f)]. Nonetheless, this contrasts with the plots for $1 < \alpha < 2$ [Figs. 1(b)–1(e)], in which distinct structures qualitatively akin (but of course, not quantitatively, as we have numerically tested) to either Cauchy-like or Gaussian-like shapes, are developed in the different regions of $\mathbb{C}$. Hypothetically, if $p_{1<\alpha<2}(z)$ were to be written as an elementary function in the full complex plane, and furthermore in certain regions to have the form of $1/(A + z^\mu)^\gamma$ (away from the isolated points $z^\mu = -A$) and in others $\exp[-Bz^\nu]$, this would require the exclusive series derived from Eq. (2) to agree with these two expressions, which cannot be the case due to their distinct series expansions. To have a simple expression, in fact, we would need a single function $f_\alpha(z)$ interpolating between the Gaussian and Cauchy expressions, moreover strongly depending on the exact location of z in the complex plane—a requirement too restrictive mathematically [37] and not met here.

We have already mentioned that for a rational $1 < \alpha = 2r/k < 2$ [22], $p_\alpha(x)$ [and therefore $p_\alpha(z)$] can be written as a sum of $N = 2r$ GHFs. Thus, these N distinct power series, each encoded in one of the GHFs [and in principle obtained from proper series expansions of the integrand in Eq. (2); see, e.g., the method in [25]], contrary to $\alpha = 1, 2$ could handle the different qualitative behavior of $p_\alpha(z)$ seen in $\mathbb{C}$. Extending the analysis, for irrational α's, successive continued fraction approximations would lead to an infinite N, obviously with no gain over the direct series equivalent to Eq. (2), not constituting a *bona fide* closed solution. In fact, we can understand such results for $\alpha > 1$ considering $x \to z = |z| \exp[i\theta]$ in Eq. (2) (for convergence reasons, we address the case of $0 \leqslant \theta < \pi/4$, enough for our purposes here). By properly deforming the integral Eq. (2) in the complex plane, we find $p_\alpha(z) = \int_{\mathcal{C}} d\tau \, \exp[-\tau^\alpha] \cos[|z\,\tau|]$ for $\mathcal{C}$ the line $\{\rho \, \exp[-i\theta] | 0 \leqslant \rho < \infty\}$. So, from the multivalued function $\exp[-\tau^\alpha]$ in $\mathbb{C}$, unless α is rational there is infinite branching and hence conceivably no solutions, say, in terms of a finite sum of GHFs. But for rational $\alpha = 2r/k > 1$, each one of the $N = 2r$ GHFs resulting in the exact $p_{\alpha=2r/k>1}(z)$ [22] could be ascribed to distinct Riemann surfaces of our $2r$-valued integrand in the complex plane.

For $\alpha < 1$, Eq. (4) nonconvergence is due to the stretched exponential characteristic function ϕ in Eq. (1), which displays a strong singularity (a logarithmic branch point) at the origin. This of course dictates the possibility of analytic solutions for $p_{\alpha<1}(z)$. The reasoning becomes more explicit from Theorem 2 [and the standard analytic continuation of setting $x \to z$ in Eq. (6)]. First, $p_{\alpha<1}(z)$ as a straightforward elementary function could not handle a branch cut comprising the entire imaginary axis (e.g., $\arctan[z]$ has as branch cut $|y| \geqslant 1$, but which is not the whole y axis). The integrand involving α in Eq. (6) can be easily manipulated, yielding terms of the form $F = \exp[-e^{\pm i\alpha(\pi/2)} t^\alpha]$, where $0 \leqslant t < \infty$, or yet $F = \exp[-(e^{\pm i(\pi/2)} t)^\alpha]$. By deforming the integration path into the complex plane through the variable change $\tau = e^{\pm i(\pi/2)} t$, we get $F = \exp[-\alpha \, \ln(\tau)]$. Finally, the rest of the analysis is quite similar to that already done for the case of $\alpha > 1$. The only difference is that now, for rational $\alpha = 2r/k < 1$, the sum of the $N = k$ GHFs given $p_{\alpha=2r/k<1}(z)$ [22] is associated with the Riemann surfaces of the k-valued integrand in $\mathbb{C}$.

Note that the problem of when the Fourier transform of an elementary function is itself elementary is more difficult than may at first appear [38]. Although our general findings bring new light to the study of Lévy α-stable distributions, they are not entirely surprising. One can classify the p_α's in terms of H functions [20] and even express the distributions in terms of specific integrals [39,40] related to fractional differential equations (FDEs) [27,41]. These results establish an important link between Lévy distributions and FDEs.

Finally, observe that, intuitively, we expect an elementary function to have power series coefficients that are themselves elementary in the degree n of the monomials z^n. Recall that the gamma function $\Gamma(x)$ assumes simple values only for positive integer and positive half-integer x. Although proofs are lacking, it is not inconceivable that for other values of x, the value of the gamma function $\Gamma(x)$ cannot be expressed in terms of elementary functions of x. Now observe that the coefficients in the formal power series z^n in Eq. (4) are gamma functions of quantities that have a factor $(1+2n)/\alpha$. Hence, excepting $\alpha = 2$ and the finite number of cases $\alpha = 1/m$ for positive integer m, the Taylor coefficients of $p_\alpha(x)$ are not elementary functions of the degree n of the monomials z^n. For the countable number of cases $\alpha = 1/m$ where $m = 2, 3, \ldots$, let $M = 2m$ and $z = x$ in Eq. (7), then $p_{1/m}(x)$ is represented by GHFs, and hence is nonelementary. Note that although the formal power series Eq. (4) converges for $z \neq 0$ only for $\alpha \geqslant 1$, the quantity $\Gamma(1/\alpha)$ appears also in Eq. (7), which has a positive radius of convergence. We thus conclude with the following.

Conjecture 1 (Nonexistence of closed-form expressions for $p_\alpha(x)$). Except for $\alpha = 1, 2$, $p_\alpha(x)$ cannot be given by a general closed-form expression in terms of elementary functions.

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