PHYSICAL REVIEW D **99,** 125015 (2019)

Perturbative approach to the entanglement entropy and the area law in Fock and polymer quantization

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(Received 29 January 2019; published 26 June 2019)

The area dependence of entanglement entropy of a free scalar field is often understood in terms of coupled harmonic oscillators. In Schrodinger quantization, the Gaussian nature of ground state wave function for these oscillators is sufficient to provide the exact form of the reduced density matrix and its eigenvalues, thus giving the entanglement entropy. However, in polymer quantization the ground state is not Gaussian and the formalism that can provide the exact analytical form of the reduced density matrix is not yet known. In order to address this issue, here we treat the interaction between two coupled harmonic oscillators in the perturbative approach and evaluate the entanglement entropy in Schrodinger and polymer quantization. Contrary to Schrodinger quantization, we show that in the high frequency regime the entanglement entropy decreases for polymer quantization, keeping the ratio of coupling strength to the square of individual oscillator frequency fixed. Furthermore for the free scalar field we validate the area dependence of entanglement entropy in Fock quantization and demonstrate that polymer quantization produces a similar area law.

DOI: 10.1103/PhysRevD.99.125015

I. INTRODUCTION

The fact that one can incorporate thermodynamical attributes to a black hole was first introduced in the seminal work of Bekenstein [1,2]. In these articles and others [3–13] the authors demonstrated that intrinsic entropy $\mathcal{S}_{\rm BH}$ of a black hole should be proportional to the area $\mathcal{A}_h$ of its event horizon $\mathcal{S}_{\rm BH} = \frac{1}{4} M_{\rm Pl}^2 \mathcal{A}_h$, where $M_{\rm Pl}$ is the Planck mass. Then the natural question appeared as to how to connect the concept of quantum states to this entropy of event horizon [14–17] as horizon is not different than any other classical surface with no special local dynamics. To answer this question and to provide a more general realization of the entropy associated to a black hole the authors in [18,19] presented the idea in terms of entanglement entropy. Here it is shown that entanglement entropy of a free scalar field in a certain spatial region is proportional to its area. In these articles the reduced density matrix, essential for estimating the entanglement entropy, is obtained by tracing over the spatial degrees of freedom (d.o.f.) of the ground state density matrix residing inside the considered region.

In the regular formulation of entanglement entropy estimation [19] first the scalar field is partially Fourier transformed with respect to the angular coordinates. The resulting Fourier field Hamiltonian is still dependent on the radial coordinate and it is discretized by assuming a lattice of finite size and interatomic spacing. This discretization transforms the Fourier Hamiltonian to be a collection of coupled harmonic oscillators. The ground state wave function for these coupled harmonic oscillators then provides the corresponding ground state density matrix for the field. Subsequently using the Gaussian nature of this ground state wave function the reduced density matrix and its eigenvalues are obtained, which would produce the entanglement entropy. However this Gaussian nature is a feature specific to the Schrodinger quantization. In polymer quantization [20–22], the quantization method used in loop quantum gravity (LQG) [23–25], the ground state wave functions are expressed in terms of Mathieu functions. Using these polymer wave functions it is still unknown how to obtain the analytic form of reduced density matrix.

In this article we consider a perturbative approach to circumvent these difficulties and obtain the entanglement entropy for free scalar field using Fock and polymer quantization. We treat the interaction between coupled harmonic oscillators in a perturbative manner to get the related ground state and eigenvalues of the reduced density matrix. first we use this procedure to evaluate the entanglement entropy for two coupled harmonic oscillators in Schrodinger and polymer quantization. Then by considering the free scalar field we obtain the area law in Fock quantization. Furthermore, we apply polymer quantization in this formulation and verify that the field theoretic entanglement entropy obeys a similar area law.

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In Sec. II we briefly review the procedures to derive the entanglement entropy in the usual formulation. In this section the detailed description of the considered system is given. Following in Sec. III we recall the perturbative formulation and construct the framework to estimate the entanglement entropy utilizing this technique. In the subsequent sections we use this formulation to obtain entanglement entropy for two coupled harmonic oscillators in Schrodinger and polymer quantization. The following parts include the realization of the area law of entanglement entropy in Fock and polymer quantization utilizing perturbative formulation. We argue about the implications of the obtained results and conclude with the discussion.

II. ENTANGLEMENT ENTROPY AND THE AREA LAW

In the standard derivations of entanglement entropy [18,19,26–33] one considers a system of coupled harmonic oscillators as a basis. In particular, the eigenvalues of the reduced density matrix for two coupled oscillators give the entanglement entropy corresponding to a single oscillator. These eigenvalues are used for a set of coupled harmonic oscillators, which are obtained from the discretized Hamiltonian of a free scalar field, to get the area law of entanglement entropy. In this section we briefly review the key aspects of these procedures and the considered systems, which will also be useful to construct the perturbative formulation.

A. Entanglement entropy for two coupled harmonic oscillators

In order to understand entropy from entanglement at first a system of two coupled harmonic oscillators [19,26] is considered. The two unit mass oscillators are denoted by their position and momentum (x_1, p_1) and (x_2, p_2). The total system can be described by the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}[p_1^2 + p_2^2 + \omega_0^2(x_1^2 + x_2^2) + k_1^2(x_1 - x_2)^2] \\ &= \frac{1}{2}[p_+^2 + \omega_+^2 x_+^2] + \frac{1}{2}[p_-^2 + \omega_-^2 x_-^2], \end{aligned} \tag{1}$$

where the normal coordinates $x_\pm = (x_1 \pm x_2)/\sqrt{2}$, $p_\pm = (p_1 \pm p_2)/\sqrt{2}$ and normal frequencies $\omega_+ = \omega_0$, $\omega_- = (\omega_0^2 + 2k_1^2)^{1/2}$ are defined to make the Hamiltonian decoupled. In decoupled form the ground state wave function becomes simplified and can be expressed in terms of the normal coordinates as

$$\psi_0(x_1, x_2) = \left(\frac{\omega_+\omega_-}{\pi^2}\right)^{\frac{1}{4}} \exp\left[-\frac{\omega_+ x_+^2 + \omega_- x_-^2}{2}\right]. \tag{2}$$

From expression (2) one can find the ground state density matrix to be $\rho(x_1, x_2; x_1', x_2') = \psi_0(x_1, x_2)\psi_0^*(x_1', x_2')$. To discuss the entanglement entropy corresponding to a single oscillator one needs to find its associated reduced density matrix. The reduced density matrix is obtained by tracing out the density matrix with respect to the position d.o.f. of a single oscillator, expressed as

$$\rho^r(x_2, x_2') = \int_{-\infty}^{\infty} dx_1 \psi_0(x_1, x_2)\psi_0^*(x_1, x_2'). \tag{3}$$

The reduced density matrix describes whether the system is in mixed or pure state and the corresponding entanglement entropy is defined as $S_E = -\mathrm{Tr}[\rho^r \ln \rho^r]$. In a suitable basis one can evaluate the entanglement entropy by obtaining the eigenvalues of the reduced density matrix. In particular, for two coupled harmonic oscillators the resulting reduced density matrix from Eq. (3) has eigenvalues

$$\lambda_n = (1 - \xi^2)\xi^{2n} \quad \text{where } \xi = \frac{\sqrt{\omega_-} - \sqrt{\omega_+}}{\sqrt{\omega_-} + \sqrt{\omega_+}}. \tag{4}$$

Then the corresponding entanglement entropy [19,26,34] becomes

$$S_E(\xi) = -\sum_n \lambda_n \ln \lambda_n = -\ln(1 - \xi^2) - \frac{\xi^2}{1 - \xi^2} \ln \xi^2. \tag{5}$$

B. Entanglement entropy for *N*-coupled harmonic oscillators

Now it is important to understand the entanglement entropy corresponding to N-coupled harmonic oscillators to get the area law of entanglement entropy for the free scalar field. The general Hamiltonian for the N-coupled harmonic oscillator is

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{j,k=1}^{N} x_j K_{jk} x_k, \tag{6}$$

where the matrix K describes the potential and interaction. The diagonal elements of K give the frequency square of the individual oscillator and symmetric off diagonal elements provide the interaction between two adjacent oscillators. With the help of a suitably chosen orthogonal matrix U this interaction matrix is diagonalized to K_D as $K = U^T K_D U$. The ground state wave function of this N-coupled harmonic oscillator (6) can be expressed as

$$\psi_0(x_1, \ldots, x_N) = \left(\frac{\mathrm{Det}.\Omega}{\pi^N}\right)^{\frac{1}{4}} \exp\left[-\frac{x.\Omega.x}{2}\right], \tag{7}$$

where $\Omega = U^T K_D^{1/2} U$. From this wave function one can obtain the reduced density matrix when the first $\mathtt{m}$ of the total N oscillators are traced out [19]. The reduced density matrix is further evaluated using a general form of the matrix Ω,

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{8}$$

where A is an $\mathbb{m} \times \mathbb{m}$ matrix corresponding to the first $\mathbb{m}$ oscillators, C is a $(N-\mathbb{m}) \times (N-\mathbb{m})$ matrix, and B is a $\mathbb{m} \times (N-\mathbb{m})$ matrix. In terms of a few newly defined quantities $\beta = (1/2)B^T A^{-1} B$ and $\gamma = C - \beta$, the reduced density matrix becomes

$$\rho_{\mathrm{out}}(x, x') \sim \exp\left[-(x.\gamma.x + x'.\gamma.x')/2 + x.\beta.x'\right], \tag{9}$$

where x and x' consist of the $(N-\mathbb{m})$ oscillators after the integration over the first $\mathbb{m}$ d.o.f. $x = V^T \gamma_D^{-1/2} y$ is defined, where $\gamma = V^T \gamma_D V$ such that γ_D is diagonal and V is orthogonal. Then one gets $\rho_{\mathrm{out}}(x, x') \sim \exp[-(y.y + y'.y')/2 + y.\beta'.y']$, where $\beta' = \gamma_D^{-1/2} V \beta V^T \gamma_D^{-1/2}$. Now moving to the basis $z = W^T y$, such that β' is diagonalized as $\beta'_D = W^T \beta' W$, one gets

$$\rho_{\mathrm{out}}(z, z') \sim \prod_{j=\mathbb{m}+1}^{N} \exp\left[-(z_j^2 + z_j'^2)/2 + \beta'_j z_j z'_j\right], \tag{10}$$

where β'_j are the eigenvalues of β'. Then the entanglement entropy [19] corresponding to $(N-\mathbb{m})$ oscillators turns out to be $S = \sum_j S(\xi_j)$, with $S(\xi)$ given by Eq. (5) and $\xi_j^2 = \beta'_j/[1 + (1 - \beta_j'^2)^{1/2}]$.

C. Entanglement entropy for free scalar field and area law

In order to discuss the area law for entanglement entropy, a free massive scalar field $\Phi(\vec{x})$ is considered with mass μ and conjugate momenta $\Pi(\vec{x})$. In Minkowski spacetime the Hamiltonian [35–37] corresponding to the scalar field is

$$H = \frac{1}{2}\int d^3x \left[\Pi^2(\vec{x}) + |\nabla\Phi(\vec{x})|^2 + \mu^2 \Phi^2(\vec{x})\right]. \tag{11}$$

In terms of partial Fourier decomposition the field $\Phi(\vec{x})$ and the conjugate momentum $\Pi(\vec{x})$ are transformed with respect to the angular coordinates as

$$\begin{aligned} \Phi(\vec{x}) &= \sum_{l,m} Z_{lm}(\theta, \phi) \frac{\varphi_{lm}(r)}{r}, \\ \Pi(\vec{x}) &= \sum_{l,m} Z_{lm}(\theta, \phi) \frac{\pi_{lm}(r)}{r}, \end{aligned} \tag{12}$$

where $Z_{lm}(\theta, \phi)$ denotes real spherical harmonics and the Fourier field modes satisfy a commutation relation $[\varphi_{lm}(r), \pi_{l'm'}(r')] = i\delta_{ll'}\delta_{mm'}\delta(r - r')$ among themselves. With this definition of field decomposition from Eq. (12) the Hamiltonian now becomes $H = \sum_{lm} H_{lm}$, where

$$\begin{aligned} H_{lm} = \frac{1}{2}\int_0^\infty dr \Bigg\{ & \pi_{lm}^2(r) + r^2 \left[\frac{\partial}{\partial r}\left(\frac{\varphi_{lm}(r)}{r}\right)\right]^2 \\ & + \left(\frac{l(l+1)}{r^2} + \mu^2\right)\varphi_{lm}^2(r)\Bigg\}. \end{aligned} \tag{13}$$

Next the radial coordinate r is discretized, forming a lattice with interatomic spacing a and size $L = (N+1)a$. The inverse of the spacing a^{-1} signifies the ultraviolet cutoff while the inverse system size L^{-1} denotes infrared cutoff. This discretization makes the Hamiltonian look like a set of coupled harmonic oscillators

$$\begin{aligned} H_{lm} = \frac{1}{2a}\sum_{j=1}^{N} \Bigg[& \pi_{lm,j}^2 + \left(j + \frac{1}{2}\right)^2 \left(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right)^2 \\ & + \left\{\frac{l(l+1)}{j^2} + \mu^2 a^2\right\}\varphi_{lm,j}^2\Bigg], \end{aligned} \tag{14}$$

such that $\varphi_{lm,N+1} = 0$ and $[\varphi_{lm,j}, \pi_{l'm',j'}] = i\delta_{ll'}\delta_{mm'}\delta_{jj'}$. Comparison of this Hamiltonian with the Hamiltonian for N-coupled harmonic oscillators from Eq. (6) gives [26]

$$\begin{aligned} K_{jk} = \frac{1}{j^2}\Bigg[& l(l+1)\delta_{jk} + \frac{9}{4}\delta_{j1}\delta_{k1} + \left(N - \frac{1}{2}\right)^2 \delta_{jN}\delta_{kN} \\ & + (\mu a)^2 j^2 \delta_{jk} + \left\{\left(j + \frac{1}{2}\right)^2 + \left(j - \frac{1}{2}\right)^2\right\}\delta_{jk(j\neq 1,N)}\Bigg] \\ & - \left[\frac{(k+\frac{1}{2})^2}{k(k+1)}\right]\delta_{j,k+1} - \left[\frac{(j+\frac{1}{2})^2}{j(j+1)}\right]\delta_{j,k-1}. \end{aligned} \tag{15}$$

The discretization of the radial coordinate enables one to get a finite expression of matrix K denoting the potential energy and interaction. This in turn would enable one to obtain the entanglement entropy when a finite number $\mathbb{m}$ of spatial points are traced out in total $N+1$ points. Then as one plots the entanglement entropy with respect to $(\mathbb{m} + 1/2)^2$, one gets a straight line, which represents the celebrated area law for entanglement entropy [19]. We present the area curve of entanglement entropy coming from perturbative formulation along with the curve obtained from this usual formulation together in the next section.

III. ENTANGLEMENT ENTROPY IN THE PERTURBATIVE APPROACH

As discussed in the previous section using the Gaussian ground state wave function of coupled harmonic oscillators from Schrodinger quantization Eq. (2), one can easily evaluate the exact form of the reduced density matrix in Eq. (3). However, in polymer quantization, a quantization method used in loop quantum gravity, the ground state wave function is obtained in terms of Mathieu functions. To the best of our knowledge evaluation of the exact analytical form of reduced density matrix is not possible even for two coupled oscillators using these polymer wave functions. This constraint further debars one to obtain the eigenvalues for N-coupled oscillators in polymer quantization and motivates us to take help from the perturbation technique.

In this section we apply perturbation to describe entanglement entropy of coupled harmonic oscillators. We express the Hamiltonian corresponding to the coupled oscillators in terms of a noninteracting free Hamiltonian H_0 and a net interaction term $\lambda H_{\rm int}$ as

$$H = H_0 + \lambda H_{\rm int}. \tag{16}$$

When interaction strength is smaller than the strength of the free Hamiltonian, which is obtained for small λ, one can express the ground state $|\Omega\rangle$ corresponding to the whole system in a perturbative manner as

$$|\Omega\rangle = |0\rangle + \lambda|0^1\rangle + \lambda^2|0^2\rangle + \ldots, \tag{17}$$

where $|0\rangle$ denotes the ground state corresponding to the noninteracting Hamiltonian H_0. On the other hand $|0^1\rangle$ and $|0^2\rangle$ denotes the first and second order perturbative corrections to the noninteracting ground state. From the time independent perturbation theory [38–40] one obtains the first order correction to the ground state as

$$|0^1\rangle = \sum_{n\neq 0} \frac{\langle n|\hat{H}_{\rm int}|0\rangle}{E_0 - E_n}|n\rangle, \tag{18}$$

where E_n denotes the energy of the nth excited state corresponding to the noninteracting Hamiltonian. The second order correction to the ground state is expressed as

$$|0^2\rangle = -\sum_{n\neq 0} \frac{\langle 0|\hat{H}_{\rm int}|0\rangle\langle n|\hat{H}_{\rm int}|0\rangle}{(E_0 - E_n)^2}|n\rangle + \sum_{m\neq 0}\sum_{n\neq 0} \frac{\langle m|\hat{H}_{\rm int}|n\rangle\langle n|\hat{H}_{\rm int}|0\rangle}{(E_0 - E_m)(E_0 - E_n)}|m\rangle. \tag{19}$$

We use these perturbative corrections to obtain the actual ground state up to certain perturbative order in the system of coupled harmonic oscillators. We mention that while discussing polymer quantization [22] we consider only the π-periodic sector for our calculations. In the π-periodic sector, except for the ground state, the even and odd energies become degenerate in high energy regimes. Now as we are interested in the ground state density matrix it is convenient for us to consider the nondegenerate perturbation theory.

A. Entanglement entropy for two coupled harmonic oscillators

We begin with a system of two coupled harmonic oscillators. We recall the Hamiltonian from Eq. (1) and observe that it can be expressed in the form of Eq. (16) with $H_0 = H_1 + H_2$ and $\lambda H_{\rm int} = -k_1^2 x_1 x_2$, where $H_j = [p_j^2 + \omega^2 x_j^2]/2$ and $\omega = (\omega_0^2 + k_1^2)^{1/2}$. Perturbative methods can be applied when k_1^2 is smaller than ω^2, which is always true for nonzero ω_0. Then in this system of two coupled oscillators the correction to the ground state wave function due to first order perturbation would be

$$\lambda|0^1\rangle = A_{nn}|n\rangle_1 \otimes |n\rangle_2 = A_{nn}|n,n\rangle, \tag{20}$$

where in the second compact notation of the wave function the first index corresponds to the first oscillator and the second one corresponds to the second oscillator. Here the operation of $\hat{x}_j$ on the corresponding ground state is given by

$$\hat{x}_j|0\rangle_j = \sum_n C^j_{0n}|n\rangle_j, \tag{21}$$

where in general the most dominating term comes from a single excitation $|n\rangle_j$. Then we get for two coupled oscillators

$$A_{nn} = \frac{k_1^2}{E_{n,n} - E_{0,0}} C^1_{0n} C^2_{0n}. \tag{22}$$

Considering up to first order perturbation, the normalized ground state is $|\Omega\rangle = N^1_{nn}[|0,0\rangle + A_{nn}|n,n\rangle]$, where the normalization factor $N^1_{nn} = (1 + A^2_{nn})^{-1/2}$. The corresponding reduced density matrix for the first oscillator would be

$$\hat{\rho}_1 = {\rm Tr}_2(|\Omega\rangle\langle\Omega|) = (N^1_{nn})^2[|0\rangle\langle 0| + A^2_{nn}|n\rangle\langle n|], \tag{23}$$

where the states now correspond to the first oscillator. This reduced density matrix has eigenvalues $(N^1_{nn})^2$ and $(N^1_{nn}A_{nn})^2$, and it would give the entanglement entropy

$$S^1_E = -(N^1_{nn})^2[\ln(N^1_{nn})^2 + A^2_{nn}\ln(A_{nn}N^1_{nn})^2]. \tag{24}$$

Now we consider second order perturbation and from Eq. (19) we observe that the first quantity would vanish as $\langle 0|\hat{x}_1\hat{x}_2|0\rangle = 0$, when discussing two coupled oscillators. Then second order correction to the ground state can be expressed as

$$\lambda^2|0^2\rangle = A_{nn}k_1^2 \sum_{m\neq 0} \frac{\langle m|\hat{x}_1\hat{x}_2|n,n\rangle}{E_m - E_{0,0}}|m\rangle. \tag{25}$$

We evaluate this quantity explicitly in the Schrodinger quantization and compare the qualitative difference of resulting entanglement entropy with the result obtained from first order perturbation.

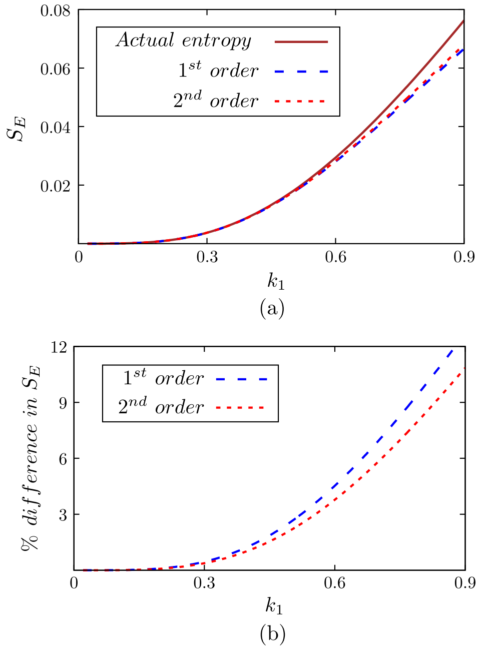
1. Schrodinger quantization

In Schrodinger quantization $C^j_{0n} = \delta_{1,n}/\sqrt{2\omega}$ and $E_n = (n+1/2)\omega$, then we have $\hat{x}_j|0\rangle_j = 1/\sqrt{2\omega}|1\rangle_j$ and $E_{11} - E_{00} = 2\omega$. Then $A_{11} = A = k_1^2/4\omega^2$ and $\lambda|0^1\rangle = k_1^2/4\omega^2|1,1\rangle$. The entanglement entropy would be given by Eq. (24). When the interaction k_1^2 is very small compared to the frequency square ω^2 ($\approx\omega_0^2$, in that case) of the individual oscillator, the expression of entanglement entropy from Eq. (24) can be simplified to

$$S_E^1 \approx -\ln\left[1-\left(\frac{k_1}{2\omega_0}\right)^4\right] - \left(\frac{k_1}{2\omega_0}\right)^4 \ln\left[\left(\frac{k_1}{2\omega_0}\right)^4\right]. \qquad (26)$$

One can observe that in similar conditions exactly the same expression for entanglement entropy is obtained from Eq. (5) using (4). Thus at least for two coupled oscillators, when the interaction is comparatively much lower than the frequency square, the first order perturbation produces reasonable results in accordance with the results from actual formulation. This fact can also be verified from Fig. 1(a). Similarly in Schrodinger quantization the second order correction to the wave function from Eq. (25) becomes

$$\lambda^2|0^2\rangle = A^2[|2,2\rangle + \sqrt{2}(|0,2\rangle + |2,0\rangle)]. \qquad (27)$$



(a)

(b)

FIG. 1. (a) We have plotted the entanglement entropy of two coupled harmonic oscillators with respect to varying k_1 in units of ω_0, from both actual and perturbative formulation. (b) The percentage difference of perturbative entanglement entropies from actual entropy is plotted, with respect to varying k_1 in unit of ω_0.

Then the normalized ground state wave function would be

$$\begin{aligned}|\Omega\rangle = N_2[&|0,0\rangle + A|1,1\rangle \\ &+ A^2\{|2,2\rangle + \sqrt{2}(|0,2\rangle + |2,0\rangle)\}], \end{aligned}\qquad (28)$$

where $N_2 = (1 + A^2 + 5A^4)^{-1/2}$ is the normalization constant. One obtains the reduced density matrix corresponding to the first oscillator as

$$\begin{aligned}\hat{\rho}_1 = N_2^2[&(1+2A^4)|0\rangle\langle 0| + A^2|1\rangle\langle 1| + 3A^4|2\rangle\langle 2| \\ &+ \sqrt{2}A^2(1+A^2)\{|2\rangle\langle 0| + |0\rangle\langle 2|\}]. \end{aligned}\qquad (29)$$

This reduced density matrix has eigenvalues

$$\begin{aligned}&\lambda_1 = N_2^2 A^2 \quad \text{and} \\ &\lambda_{2,3} = \frac{N_2^2}{2}\left[1 + 5A^4 \pm \sqrt{(1+A^2)^2(9A^4 - 2A^2 + 1)}\right], \end{aligned}\qquad (30)$$

which would give the entanglement entropy as $S_E = -\sum_{s=1}^3 \lambda_s \ln\lambda_s$. This entanglement entropy and the entanglement entropy obtained from first order perturbation are plotted with the actual entropy in Fig. 1(a) with respect to a varying coupling between the two oscillators. The percentage difference of the obtained result using perturbative techniques from the actual entanglement entropy is plotted in Fig. 1(b). From these figures we observe that when the coupling is small compared to the individual frequency square of the oscillators, the perturbation method is quite elegant to study entanglement entropy of coupled harmonic oscillators. Furthermore, from these figures we also observe that the results from second order perturbation does not drastically improve compared to first order perturbation. On the other hand as our main objective is to understand the qualitative nature of entanglement entropy from perturbation, it is expected that first order perturbation would be good enough to satisfy our requirement.

2. Polymer quantization

In this part we discuss entanglement entropy for the system of two coupled harmonic oscillators in polymer quantization. Perturbation techniques are used to obtain the entanglement entropy as the wave functions arising from polymer quantization cannot be handled analytically like the Gaussian wave functions. Here we start with a brief overview of the technical aspects of polymer quantization.

Polymer quantization [22] is a background independent quantization procedure arising from LQG. In polymer quantization apart form the Planck constant $\hbar$ a new dimension-full parameter λ is introduced. Here the elementary operators are configuration operator $\hat{x}$ and translation operator $\hat{U}_\lambda \equiv e^{i\hat{\lambda p}}$ and their actions are defined as

$$\hat{x}e^{ipx_k} = i\frac{\partial}{\partial p}e^{ipx_k}, \qquad \hat{U}_\lambda e^{ipx_k} = e^{ip(x_k+\lambda)}. \tag{31}$$

These operators satisfy the basic commutator $[\hat{x}, \hat{U}_\lambda] = -\lambda\hat{U}_\lambda$. Now with the definition of translation operator from Eq. (31) and the inner product

$$\langle x|x'\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dp e^{-ipx}e^{ipx'} = \delta_{x,x'}, \tag{32}$$

it is observed that a momentum operator cannot be defined as the translation operator is not continuous in its parameter. However to describe the kinetic energy part of the Hamiltonian one must have a suitable expression of the momentum operator. In this case the momentum operator should be λ dependent and to be given in terms of the translation operator. One simple definition of the momentum operator as considered in [22] is

$$\hat{p}_\lambda = \frac{1}{2i\lambda}(\hat{U}_\lambda - \hat{U}^\dagger_\lambda). \tag{33}$$

One can then express the eigenvalue equation $\hat{H}\psi = E\psi$, where $H = p^2/2m + m\omega^2x^2/2$ represents the Hamiltonian corresponding to a simple harmonic oscillator with mass m, as

$$\psi''(u) + \left[\sigma - \frac{1}{2g^2}\cos(2u)\right]\psi(u) = 0, \tag{34}$$

which represents a Mathieu equation [41]. Here $\lambda = \lambda_\star$, $u = \lambda_\star p + \pi/2$, $g = m\omega\lambda_\star^2$, and $\sigma = 2E/g\omega - 1/2g^2$. The above differential equation has periodic solutions for σ representing the Mathieu characteristic value functions

$$\begin{aligned} \psi_{2n}(u) &= \pi^{-1/2}ce_n(1/4g^2, u), \qquad \sigma = \mathcal{A}_n(g), \\ \psi_{2n+1}(u) &= \pi^{-1/2}se_{n+1}(1/4g^2, u), \qquad \sigma = \mathcal{B}_n(g). \end{aligned} \tag{35}$$

For $(n = 0, 1, \ldots)$, ce_n and se_n represent the elliptic cosine and sine functions, where for even n they are π periodic and for odd n they are π-antiperiodic functions. The corresponding energy eigenvalues are given by

$$\begin{aligned} E_{2n} &= \frac{\omega}{4g}[2g^2\mathcal{A}_n(g) + 1], \\ E_{2n+1} &= \frac{\omega}{4g}[2g^2\mathcal{B}_{n+1}(g) + 1]. \end{aligned} \tag{36}$$

Using the asymptotic expansions of the Mathieu characteristic value functions $\mathcal{A}_n(g)$ and $\mathcal{B}_n(g)$ one can get in small g limit, i.e., when $g \ll 1$,

$$\frac{E_{2n}}{\omega} \approx \frac{E_{2n+1}}{\omega} = \left(n + \frac{1}{2}\right) - \frac{(2n+1)^2 + 1}{16}g + \mathcal{O}(g^2). \tag{37}$$

On the other hand in high g regimes, i.e., when $g \gg 1$, one gets

$$\frac{E_0}{\omega} = \frac{1}{4g} + \mathcal{O}\left(\frac{1}{g^3}\right), \qquad \frac{E_{2n-1}}{\omega} \approx \frac{E_{2n}}{\omega} = \frac{n^2 g}{2} + \mathcal{O}\left(\frac{1}{g}\right). \tag{38}$$

From the asymptotic expression (37) we observe that when $g \ll 1$ the energy levels corresponding to the π-periodic and π-antiperiodic sectors become degenerate among themselves. On the other hand from (38) we observe that for $g \gg 1$ the energy levels within the separate π-periodic and π-antiperiodic sectors become degenerate. We consider only the π-periodic sector of the wave functions, containing the nondegenerate ground state, to discuss the corresponding entanglement entropy. We mention that the asymptotic expressions of the energy eigenvalues from Eqs. (37) and (38) are also utilized in [42–44] to observe the Unruh and Hawking effect for polymer observer. One can also look into [45–48] where polymer quantization is used in different systems to study particle creation.

Entanglement entropy in polymer quantization: In this part we evaluate the perturbative corrections to the ground state in polymer quantization, which basically requires the estimation of A_{nn}. The operation $\hat{x}_j|0\rangle_j$ is already discussed in Eq. (21) and in polymer quantization C^j_{0n} are given by

$$C^j_{0n} = {}_j\langle n|\hat{x}_j|0\rangle_j = \lambda_\star\int_0^{2\pi}\psi^j_n\frac{\partial}{\partial u^j}\psi^j_0 du^j, \tag{39}$$

where $u^j = \lambda_\star p^j + \pi/2$ and $\lambda_\star$ is the polymer length scale; see [22]. There are infinite numbers of nonzero C^j_{0n} in polymer quantization, where as in Schrodinger quantization there is only one $C^j_{0n} = \delta_{1,n}/\sqrt{2\omega}$. In order to compute polymer corrections we only consider the first and most dominating nonzero C^j_{0n}, which is C^j_{03}. In the small $g = \omega\lambda_\star^2$ limit, i.e., when $g \ll 1$, these coefficients are given by

$$C^j_{03} = C_{03} = \frac{i}{\sqrt{2\omega}}\left[1 - \frac{3}{4}g + \mathcal{O}(g^2)\right], \tag{40}$$

and the corresponding energy correction is given by

$$\Delta E_{30} = E_3 - E_0 = \omega\left[1 - \frac{g}{2} + \mathcal{O}(g^2)\right]. \tag{41}$$

The expression of A_{nn}, now obtained as A_{33}, is changed and using Eq. (22) becomes

$$A_{33} = A \approx -\frac{k_1^2}{4\omega^2(1-g/2)}\left(1-\frac{3}{2}g\right)$$
$$\approx -\frac{k_1^2}{4\omega^2}(1-g). \tag{42}$$

One can observe from this expression of A that as one takes $g \to 0$, one gets back the result from Schrodinger quantization. We note that the sign of A does not affect the end result as the entanglement entropy is obtained using A^2. In the ultraviolet limit when $g \gg 1$, one has the expressions

$$C_{03}^j = C_{03} = i\sqrt{\frac{g}{2\omega}}\left[\frac{1}{4g^2} + \mathcal{O}\left(\frac{1}{g^6}\right)\right], \tag{43}$$

and

$$\Delta E_{30} = \omega\left[2g + \mathcal{O}\left(\frac{1}{g^3}\right)\right]. \tag{44}$$

Then the expression of A_{33} is

$$A_{33} = A \approx -\left(\frac{k_1}{2\omega}\right)^2 \frac{1}{2(2g)^4}. \tag{45}$$

We know that the reduced density matrix in first order perturbation has eigenvalues $\lambda_1 = N_1^2 = (1+A^2)^{-1}$ and $\lambda_2 = (N_1 A)^2$. Then for fixed (k_1/ω) as we take $g \to \infty$, we observe that $\lambda_1 \to 1$ and $\lambda_2 \to 0$, because in this limit $A \to 0$. Then the entanglement entropy evaluated from these eigenvalues vanishes, providing a very new feature in the ultraviolet regime of energy in polymer quantization.

We now intend to express this result in a more general fashion without using asymptotic forms of the Mathieu functions. For this we need some numerical help. First the expression of general energy difference of our concern in polymer quantization is

$$E_{33}^0 - E_{00}^0 = \omega g[\mathcal{B}_2(g) - \mathcal{A}_0(g)], \tag{46}$$

where $\mathcal{A}_n(g)$ and $\mathcal{B}_n(g)$ are the Mathieu characteristic value functions corresponding to even and odd Mathieu functions. The expressions of C_{03} are obtained from Eq. (39) with wave functions represented in terms of Mathieu functions. Then we have

$$A = A_{33} = \frac{k_1^2}{\omega g[\mathcal{B}_2(g) - \mathcal{A}_0(g)]} C_{03}^1 C_{03}^2, \tag{47}$$

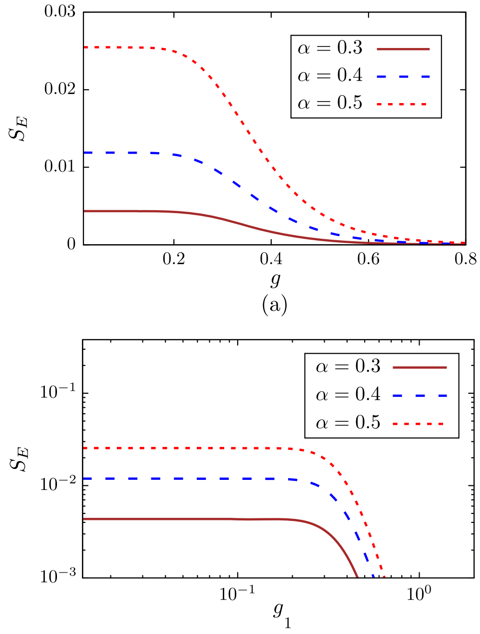
and the corresponding eigenvalues are

$$\lambda_1 = N_1^2 = \left[1 + \left\{\frac{(\alpha C_{03}/\lambda_\star)^2}{\mathcal{B}_2(g) - \mathcal{A}_0(g)}\right\}^2\right]^{-1}, \tag{48}$$

and

$$\lambda_2 = \left\{N_1 \frac{(\alpha C_{03}/\lambda_\star)^2}{\mathcal{B}_2(g) - \mathcal{A}_0(g)}\right\}^2, \tag{49}$$

where $\alpha = k_1/\omega$. We have plotted this entanglement entropy coming from first order perturbation for different fixed values of α with varying polymer parameter g, in Fig. 2. Then the change of g signifies the change in harmonic oscillator frequency ω for the fixed ratio k_1/ω and fixed polymer length scale $\lambda_\star$. In these plots the fixed ratio (k_1/ω) is considered to be less than 1. It implies that the interaction strength is less than 1 and permits the application of perturbation theory even for high g regimes. We observe that at high frequency regime as g increases the entanglement entropy decreases and becomes very low at large g; see Fig. 2. This situation was not present in Schrodinger quantization as there the entanglement entropy is a function of $A = k_1^2/4\omega^2$ and its value is fixed for fixed $\alpha = k_1/\omega$. We note that same phenomena can also be observed in polymer quantization using second order perturbation with very little quantitative difference. For a unit mass harmonic oscillator one can interpret the inverse square root of its frequency to be a length scale characteristic of the harmonic oscillator. Now as the frequency increases this length decreases and even reaches polymer length scale $\lambda_\star$ when g becomes very high. One then interprets the above deviation of entanglement entropy in polymer quantization from usual quantization, as a result of the physics in very high energy or in a very small length scale addressed by polymer quantization.



(a)

(b)

FIG. 2. (a) The entanglement entropy of two coupled harmonic oscillators using perturbative formulation in polymer quantization, with varying g keeping $\alpha = k_1/\omega$ fixed. (b) The log-log plot of the entanglement entropy in polymer quantization, with respect to varying g keeping $\alpha = k_1/\omega$ fixed.

B. Area law for free scalar field

In this part we use perturbation technique to evaluate the entanglement entropy corresponding to a massive free scalar field described by the Hamiltonian (11). We first provide a prescription for the eigenvalues of the reduced density matrix corresponding to N-coupled oscillators. By considering only first order perturbation, one can express the ground state wave function of N weakly coupled harmonic oscillators as

$$|\Omega\rangle_N = |000...\rangle + A_1|nn0...\rangle + \cdots + A_{N-1}|...0nn\rangle, \quad (50)$$

where $A_1, \ldots, A_{N-1}$ are coefficients of the first order perturbative correction to wave function, and they are functions of the individual frequency and interaction between different oscillators. In both Fock and polymer quantization the state $|00\ldots n_j n_{j+1}\ldots\rangle$ is obtained from $\hat{x}_j\hat{x}_{j+1}|00\ldots\rangle = C^j_{0n}C^{j+1}_{0n}|00\ldots n_j n_{j+1}\ldots\rangle$, where we have omitted the sum on n as the most dominating contribution comes from a single term. We first include an appropriate normalization factor to the wave function from Eq. (50) and calculate the corresponding density matrix and its eigenvalues for a successively increasing number of coupled harmonic oscillators. Then the eigenvalues, corresponding to the reduced density matrix after tracing over $\mathbb{m}$ d.o.f. out of total N-coupled harmonic oscillators, can be found by guessing from the consecutive eigenvalue evaluation, as

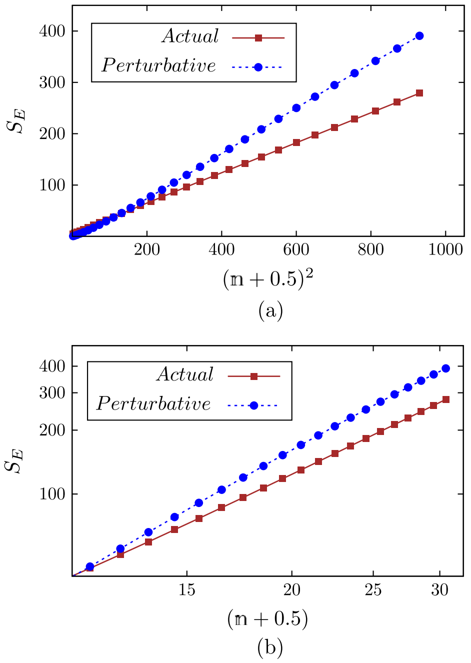
$$\lambda^{\mathbb{m}}_s = N_c^2 A^2_{\mathbb{m}},$$

$$\frac{N_c^2}{2}\left[\sum_{\substack{j=0\\ j\neq\mathbb{m}}}^{N-1} A_j^2 \pm \left\{\left(\sum_{\substack{j=0\\ j\neq\mathbb{m}}}^{N-1} A_j^2\right)^2 - 4\left(\sum_{\substack{j=1\\ j\neq\mathbb{m}}}^{\mathbb{m}-1} A_j^2\right)\left(\sum_{\substack{j=\mathbb{m}+1\\ j\neq\mathbb{m}}}^{N-1} A_j^2\right)\right\}^{\frac{1}{2}}\right] \quad (51)$$

where $A_0 = 1$ and subscript s denotes different eigenvalues that are three in number for any particular reduction $\mathbb{m}$. $N_c = (\sum_{j=0}^{N-1} A_j^2)^{-1/2}$ is the normalization factor corresponding to the perturbed ground state. As discussed earlier the entanglement entropy corresponding to these eigenvalues would be $S^{\mathbb{m}}_E = -\sum_{s=1}^{3}\lambda^{\mathbb{m}}_s \ln\lambda^{\mathbb{m}}_s$.

1. Area law in Fock quantization

In order to obtain the area law of entanglement entropy in Fock quantization we first consider the discrete Hamiltonian, formed out of a partially Fourier transformed field Hamiltonian in a lattice of finite size, from Eq. (14). With the help of Eq. (15) one can get frequency ω_j of the jth oscillator and coupling k_j^2 between the jth and $(j+1)$th oscillator. Now according to Eq. (22) we want to find the expressions of the coefficients of first order perturbation $A_j = k_j^2/\{2\sqrt{\omega_j\omega_{j+1}}(\omega_j + \omega_{j+1})\}$, which are used in Eq. (51) to get the eigenvalues. We note that the perturbative coefficients A_j are, in principle, functions of l and m and sums over these quantities are taken to evaluate the entropy, but for brevity we omitted their index from the notation. In Fock quantization they are evaluated using $C^j_{0n} = \delta_{1,n}/\sqrt{2\omega_j}$ and $E^j_n = (n + 1/2)\omega_j$. We have numerically computed the entanglement entropy using the obtained eigenvalues. In Fig. 3 the entanglement entropy from perturbative and actual formulation is presented for a massless free scalar field. The entanglement entropy in actual formulation is obtained nonperturbatively, utilizing the Gaussian nature of the ground state wave functions. We have used the results from Eq. (10) and the potential from Eq. (15) to evaluate the entanglement entropy in actual formulation [19]. Figure 3 shows that first order perturbation is sufficient to provide an area law for entanglement entropy. We note that the slope from this area curve (~0.42) is different than the one obtained from actual formulation (~0.29). One can notice from Eqs. (14) and (15) that the ratio of the frequency square to interaction strength decreases when j increases as we consider a larger system size. Thus perturbation theory becomes less effective and the results obtained from first order perturbation deviate more from



(a)

(b)

FIG. 3. (a) The plot of entanglement entropy with respect to $(\mathbb{m}+0.5)^2$ for a free massless scalar field in a lattice of total size $N = 60$. Here $\mathbb{m}$ is the number of position d.o.f. traced out of the ground state of the scalar field. This plot shows the area law for entanglement entropy in both actual and perturbative formulation. Here the ratio of the slopes from perturbative entropy and actual entropy is ~1.85. (b) This figure provides the log-log plot of entanglement entropy vs $(\mathbb{m}+0.5)$ in both actual and perturbative formulation. The slope of the curve from actual formulation is about ~1.9 where the slope from perturbative formulation is ~2.1. We have used first order perturbation to evaluate the entanglement entropy.

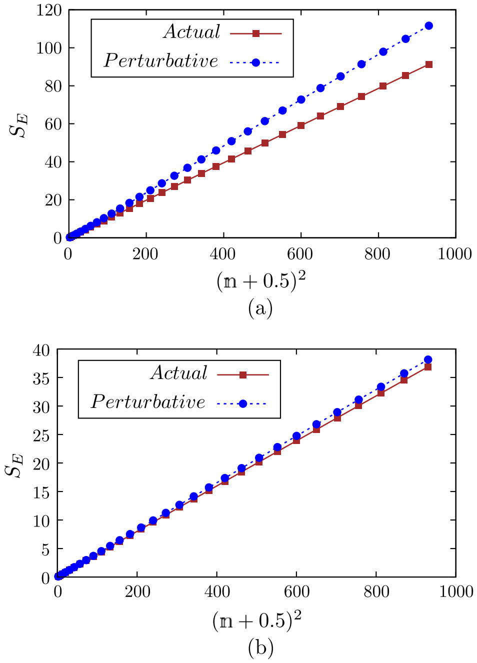


FIG. 4. (a) The plot of entanglement entropy of a massive free scalar field with respect to $(\mathfrak{m}+0.5)^2$ for $N=60$. Here $\mu a=2$ and the ratio of the slopes from perturbative entropy and actual entropy is ∼1.22. (b) The entanglement entropy vs $(\mathfrak{m}+0.5)^2$ plot for massive free scalar field with $N=60$. Here $\mu a=4$ and the ratio of the slopes from perturbative entropy and actual entropy is ∼1.04. In both cases only first order perturbation is used.

the actual nonperturbative result for large j. We also note that as second order perturbation is employed in this formulation the area curve is quantitatively very little improved and gets closer to the area curve from actual formulation. Now as we consider a field with increasing mass, the ratio of the frequency square and coupling strength from Eq. (15) increases and the perturbative formulation becomes more effective in describing the original system.

In addition to the massless case we have also considered a massive scalar field with discretized Hamiltonian from Eq. (14). We have taken different values for the parameter μa and for each value obtained the area law in both the actual and first order perturbation. Then the massless case becomes a special case when this parameter μa is taken to be 0. In particular, we have observed that as we increase the value of this parameter the perturbation becomes stronger and the slopes of the two area curves corresponding to actual and perturbative formulation get closer. The plots corresponding to this feature are shown in Fig. 4.

2. Area law in polymer quantization

We take the Fourier Hamiltonian density for a massive free scalar field according to Eq. (14). We note that the field modes $\varphi_{lm,j}$ and their conjugate momenta $\pi_{lm,j}$ are dimensionless here. To consistently introduce polymer quantization in this formulation we make the transformations $\tilde{\pi}_{lm,j} \equiv \pi_{lm,j}/\sqrt{a}$ and $\tilde{\varphi}_{lm,j} \equiv \sqrt{a}\varphi_{lm,j}$ such that the new field mode and the conjugate momentum become dimension full. Then the Fourier Hamiltonian density from Eq. (14) takes the form

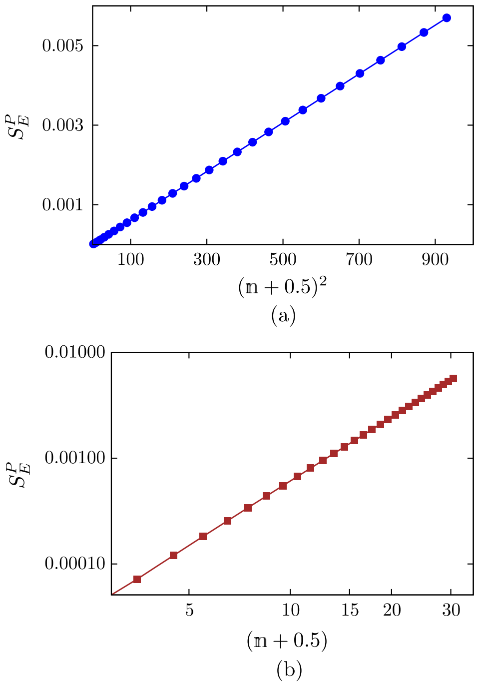
$$H_{lm} = \frac{1}{2}\sum_{j=1}^{N}\left[\tilde{\pi}_{lm,j}^2 + \left(\frac{j+\frac{1}{2}}{a}\right)^2\left(\frac{\tilde{\varphi}_{lm,j}}{j} - \frac{\tilde{\varphi}_{lm,j+1}}{j+1}\right)^2 + \left\{\frac{l(l+1)}{a^2 j^2} + \mu^2\right\}\tilde{\varphi}_{lm,j}^2\right]. \tag{52}$$

This Hamiltonian also describes a system of N-coupled harmonic oscillators given by Eq. (6). In polymer quantization a new dimension full parameter $\lambda_\star$ is introduced with dimension $(\text{length})^{1/2}$, inverse of the dimension of momentum. Here the basic variables are taken to be $\tilde{\varphi}_{lm,j}$ and $U_{lm,j}^{\lambda_\star} = \exp\{i\lambda_\star \tilde{\pi}_{lm,j}\}$ with Poisson bracket $\{\tilde{\varphi}_{lm,j}, U_{lm,j}^{\lambda_\star}\} = i\lambda_\star U_{lm,j}^{\lambda_\star}$. From the above system of coupled harmonic oscillators we observe that for a general jth oscillator the frequency is

$$\omega_{lm,j} = \frac{1}{aj}\left[l(l+1) + \left\{\left(j+\frac{1}{2}\right)^2 + \left(j-\frac{1}{2}\right)^2\right\}\Bigg|_{j\neq 1,N} + \mu^2 a^2 j^2 + \frac{9}{4}\delta_{j1} + \left(N-\frac{1}{2}\right)^2\delta_{jN}\right]^{\frac{1}{2}} = \frac{\Omega_{lm,j}}{a}. \tag{53}$$

Now we want to get the expressions of perturbative coefficients A_j used in Eq. (51). They are constructed using expressions of C_{0n}^j and E_n^j in polymer quantization from Eq. (39) and (46), which are further given by the dimensionless polymer parameter $g_{lm,j} = \omega_{lm,j}\lambda_\star^2 = \Omega_{lm,j} l_\star / a$. We also note that the interatomic distance a and the polymer length scale $l_\star = \lambda_\star^2$ both have the same nature and should have the same order as they signify the ultraviolet cutoff. Then we take their ratio $\gamma = l_\star/a$ to be unity, which adds further simplification to the evaluation of entanglement entropy. We have plotted the entanglement entropy from first order perturbative formulation in Fig. 5 considering massless free scalar fields in polymer quantum field theory. In these figures we observe that the area law is valid in polymer quantization too. However the corresponding slope is now very low compared to results from Fock quantization. One can also get the area law in polymer quantization for massive free scalar field with a further decreased slope.

Implication of the result: From [26] we get to understand that the slope of the area curve for entanglement entropy can be different due to many reasons, such as different discretization procedures, inclusion of mass, or taking excited states instead of the ground state. We mention here a consistency check to understand whether this result



(a)

(b)

FIG. 5. (a) The entanglement entropy of a massless free scalar field with respect to $(\mathbb{m}+0.5)^2$ using first order perturbation in polymer quantization. This plot shows the area law in polymer quantization with slope 6.12×10^{-6}. (b) The log-log plot of the entanglement entropy vs $(\mathbb{m}+0.5)$ in perturbative formulation using polymer quantization. The total lattice size is taken to be $N=60$ and the obtained slope is 2.002.

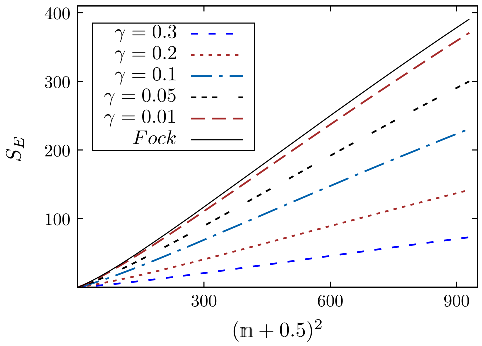


FIG. 6. The plot of entanglement entropy in polymer quantization for different values of $\gamma = l_\star/a$.

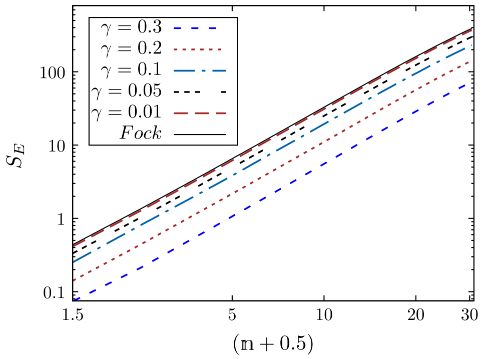


FIG. 7. The log-log plot of entanglement entropy in polymer quantization for different values of $\gamma = l_\star/a$.

from polymer quantization is a plausible one or not. It is noted in [22] that in low energy regimes polymer quantization reproduces the results from usual Fock quantization. Now in this formulation of entanglement entropy evaluation we observe that one direct influence of polymer quantization over Fock is dictated by the factor $(l_\star/a)$. When this factor is unity the system is completely interpreted in terms of polymer quantization. On the other hand when the value of this quantity decreases the value of the dimensionless polymer frequency $g_{lm,j}$ decreases and the system becomes more and more Fock-like as the lower energy regimes of polymer quantization tend to contribute to the description of the system. We have plotted the entanglement entropy for different values of this factor and we observed that as the value decreases the area curve of entanglement entropy from polymer quantization approaches the one from Fock quantization; see Figs. 6 and 7. Thus the very low slope of the entanglement entropy can be described as a feature coming from the disentangling nature of polymer quantization at high energy regimes. We note that massive scalar fields also show disentangling nature and lower the slope of the area curve [49–51].

The entanglement entropy of the free scalar field in polymer quantization gives rise to another question, which relates to the corrections to the area law as predicted by quantum gravity [52–55]. In this manner we note that the slope of the entanglement entropy vs $(\mathbb{m}+0.5)$ curve in the log-log plot is 2.002, which automatically discards any possible departure from the area law. This area dependence of entanglement entropy in polymer quantization is enthralling in its own right since it validates the generality of the area law in quantizations other than Fock.

IV. DISCUSSION

In usual formulation, procurement of the area curve for entanglement entropy [3–9] is simplified using the mathematical structure of Gaussian ground state wave function from Schrodinger quantization. However, not all quantization procedures provide this Gaussian nature of ground state and polymer quantization is one of them. We note that though entanglement entropy for two coupled harmonic oscillators is specifically evaluated for polymer quantization in [56], the framework to obtain entanglement entropy for a large number of coupled oscillators is not provided; thus one cannot obtain the area law. In this work we have treated the interaction between coupled harmonic oscillators in a perturbative manner. Our procedure is different than the ones discussed in [49–51,57], where the eigenvalues of the reduced density matrix and momentum space entanglement are estimated using perturbation. For two coupled harmonic oscillators we noticed a disentangling nature from polymer quantization at the high frequency regime. We observed that in Schrodinger quantization the entanglement entropy is unchanged while in polymer quantization it decreases at high oscillator frequencies, keeping the ratio of interaction strength to frequency square fixed. We showed that in our formulation, by considering the free scalar field, one obtains the area law of entanglement entropy for Fock quantization. As the mass of the scalar field increases the individual oscillator frequency increases; thus perturbation strength increases and the obtained area curve approaches the area curve from usual formulation. Furthermore, we showed that in polymer quantization also this formulation provides a similar area law, but with a very decreased slope. We inferred that this decrease of slope is due to the disentangling nature of polymer quantization at higher energies. We further noticed that as the effect of polymer quantization becomes smaller, by lowering the value of the ratio of polymer length scale $l_\star$ to interatomic distance a, the area curve from polymer quantization using first order perturbation tends to approach the area curve from Fock quantization. This phenomenon is not quite surprising as in the limit $l_\star/a \to 0$, the physical result from polymer quantization would converge to the result obtained from the standard Fock quantization. The disentangling nature of polymer quantization is very intriguing in its own right as it is known that usual quantization loses its predictability in trans-Planckian energy regimes [58,59]. We note that this disentangling phenomenon in polymer quantization is analogous to the suppression of propagation at large energies. We mention that there are other derivations to obtain the area law and harvest entanglement entropy for the scalar field [60–76] and it would be interesting to see whether an exact form of the entanglement entropy can be found using these derivations in polymer quantization. In conclusion we address that our formulation opens up an avenue to understand entanglement entropy in terms of perturbative corrections.

ACKNOWLEDGMENTS

We thank Golam Mortuza Hossain, Narayan Banerjee, Ritesh K. Singh, and Ananda Dasgupta for discussions. We also thank Sumanta Chakraborty, Abhishek Majhi, and Sudipta Saha for useful suggestions. S. B. thanks Indian Institute of Science Education and Research Kolkata (IISER Kolkata) for supporting this work through a doctoral fellowship.

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