

Boundary conditions and renormalized stress-energy tensor on a Poincaré patch of AdS₂

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(Received 26 April 2019; published 17 June 2019)

Quantum field theory on anti-de Sitter spacetime requires the introduction of boundary conditions at its conformal boundary, due essentially to the absence of global hyperbolicity. Here we calculate the renormalized stress-energy tensor $T_{\mu\nu}$ for a scalar field ϕ on the Poincaré patch of AdS₂ and study how it depends on those boundary conditions. We show that, except for the Dirichlet and Neumann cases, the boundary conditions break the maximal AdS invariance. As a result, $\langle\phi^2\rangle$ acquires a space dependence and $\langle T_{\mu\nu}\rangle$ is no longer proportional to the metric. When the physical quantities are expanded in a parameter β which characterizes the boundary conditions (with $\beta = 0$ corresponding to Dirichlet and $\beta = \infty$ corresponding to Neumann), the singularity of the Green’s function is entirely subtracted at zeroth order in β . As a result, the contribution of nontrivial boundary conditions to the stress-energy tensor is free of singular terms.

DOI: 10.1103/PhysRevD.99.125008

I. INTRODUCTION

The construction of quantum field theory on curved backgrounds (solutions of Einstein equations) usually assumes that the spacetime is globally hyperbolic (see, for instance, Ref. [1]). This is a very reasonable assumption since, in this case, a Cauchy surface Σ exists so that the wave problem is well posed and determined by the initial data at Σ [2]. However, it is possible to prescribe a sensible evolution for the wave function even when the spacetime is nonglobally hyperbolic. This prescription was first presented by Wald [3] and amounts to finding the positive self-adjoint extensions of the spatial part of the wave operator. In [4], it was shown that this is the only prescription which is consistent with some very reasonable assumptions—essentially related to causality and energy conservation.

It is well known that the anti-de Sitter spacetime is not globally hyperbolic. At the conformal boundary, information can flow in/out from/to infinity so that no Cauchy surface exists in AdS_n. In particular, in the Poincaré patch PAdS_n given by the metric

$$ds^2 = \frac{l^2}{z^2} \left(-dt^2 + dz^2 + \sum_{i=1}^{n-2} dx_i^2 \right), \quad z \in (0, \infty), \quad (1)$$

the conformal boundary is given by $z = 0$.

Here, we will treat the case of PAdS₂, which will be enough to illustrate our main result: the breaking of AdS invariance and the corresponding extra terms in the stress-energy tensor (which appear in addition to the usual term proportional to the metric).

The vacuum state depends on the choice of the boundary condition to be imposed at $z = 0$ as follows. A solution ϕ of the Klein-Gordon equation,

$$(\nabla^\mu \nabla_\mu - m^2 - \xi R)\phi(x) = 0, \quad (2)$$

can always be expanded in terms a complete set of normalized modes $u_\omega^{(\beta)}(x)$:

$$\phi^{(\beta)}(x) = \sum_\omega \left[a_\omega^{(\beta)} u_\omega^{(\beta)}(x) + a_\omega^{(\beta)\dagger} u_\omega^{(\beta)*}(x) \right], \quad (3)$$

where mode labels were omitted for simplicity. In this equation, $\{u_\omega^{(\beta)}(x)\}$ must satisfy a boundary condition at $z = 0$ identified by the parameter β , which labels possible choices of the self-adjoint extensions (for details, see Sec. II). These modes are eigenfunctions of the Killing vector field $\partial/\partial t$ with eigenvalue $-i\omega$ ($\omega > 0$) and are mutually orthogonal with respect to the scalar product

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$$\langle \phi_1, \phi_2 \rangle = -i \int_0^\infty \phi_1(x) \overleftrightarrow{\partial}_\mu \phi_2(x) [-g_\Sigma(x)]^{1/2} d\Sigma^\mu. \quad (4)$$

The vacuum state is then given by

$$a_\omega^{(\beta)} |0\rangle_\beta = 0 \quad \forall \omega. \quad (5)$$

As the notation suggests, the construction of the vacuum state crucially depends on the boundary condition and this may be chosen arbitrarily, since each choice of β corresponds, in principle, to a legitimate self-adjoint evolution operator for the theory.

We show in this paper that the vacuum $|0\rangle_\beta$ does not respect AdS invariance unless β corresponds to the Dirichlet and Neumann boundary conditions (this fact was already proved for the conformal case by one of the authors in [5], by means of a contradiction argument based on Ref. [6]). This breaking of invariance results in unusual behavior for several physical quantities. For example, the renormalized square field $\langle \phi^2 \rangle_\beta$ turns out to be a function of z notwithstanding the maximal symmetry of the underlying spacetime. Moreover, the renormalized stress-energy tensor $\langle T_{\mu\nu} \rangle_{\text{ren}}$ fails to be proportional to the metric, clearly violating maximal symmetry.

It is known that in the Dirichlet case the two point functions $\langle \varphi(x)\varphi(x') \rangle$ are functions of the geodesic distance $\sigma(x, x')$ [6], which is in line with the fact that the Dirichlet boundary condition respects AdS invariance. By expanding the Green's function in terms of the boundary condition parameter β ,

$$G^{(\beta)}(x, x') = G_0(x, x') + \beta G_1(x, x') + \mathcal{O}(\beta^2), \quad (6)$$

we show that the divergence of $G^{(\beta)}(x, x')$ in the limit $x' \rightarrow x$ is entirely contained in the Dirichlet contribution $G_0(x, x')$, at least for small β . The other terms in (6) come from the purely analytical interaction at the conformal boundary and are finite in the coincidence limit. Since the formal subtraction was already made in Ref. [7] and $\langle \phi^2 \rangle_{\beta=0}$ and $\langle T_{\mu\nu} \rangle_{\beta=0}$ are known, we are able to add the contributions due to β without any further renormalization.

II. BOUNDARY CONDITIONS

In two-dimensions, the Poincaré patch of AdS is given by the metric

$$ds^2 = \frac{1}{z^2} (-dt^2 + dz^2), \quad z > 0, \quad (7)$$

where we set Λ such that $l = 1$ in Eq. (1). The wave equation (2) then becomes

$$\frac{\partial^2 \phi(t, z)}{\partial z^2} - \frac{m_\xi^2}{z^2} \phi(t, z) = \frac{\partial^2 \phi(t, z)}{\partial t^2}, \quad (8)$$

with $m_\xi = m^2 - 2\xi$ (since the scalar curvature in this case is given by $R = -2$).

Note that this equation has the form

$$\frac{\partial^2 \phi(t, z)}{\partial t^2} = -A\phi(t, z), \quad (9)$$

where A is the spatial operator given by (minus) the left-hand side of Eq. (8). Since $z > 0$, it would seem to be reasonable to set $C_0^\infty(0, \infty)$ as the domain of the operator A . However, A would not be self-adjoint (even though it is symmetric) in this case. We must accordingly find the positive self-adjoint extensions of A that generate a sensible dynamics for ϕ .

This problem was thoroughly analyzed in Ref. [8], where the authors study the self-adjoint extensions of the operator

$$A = -\frac{d^2}{dz^2} + \frac{\alpha}{z^2}, \quad (10)$$

with $\alpha \in \mathbb{R}$. If $\alpha \geq 3/4$, A is essentially self-adjoint, i.e., it has a unique self-adjoint extension [9], which corresponds to the Dirichlet boundary condition. This case was analyzed in [7]. If $-1/4 \leq \alpha < 3/4$, there is an infinite number of self-adjoint extensions for A . If $\alpha = -1/4$, only one of these extensions is positive (this corresponds to the Dirichlet boundary condition; all the other choices give rise to bound states). For $-1/4 < \alpha < 3/4$, there is an infinite number of positive self-adjoint extensions and a complete orthonormal system of eigenfunctions can be written in terms of a parameter $\beta \geq 0$ and Bessel functions:

$$u_\omega^{(\beta)}(t, z) = \begin{cases} \sqrt{\frac{\chi}{2}} \frac{J_\chi(\omega z) + \gamma(\beta, \omega) J_{-\chi}(\omega z)}{\sqrt{1 + 2\gamma(\beta, \omega) \cos(\chi\pi) + \gamma^2(\beta, \omega)}} e^{-i\omega t}, & \chi \in (0, 1/2) \cup (1/2, \infty), \\ \frac{1}{\sqrt{\pi\omega}} \frac{\sin(\omega z) + \beta(\omega/\omega_0) \cos(\omega z)}{\sqrt{1 + \beta^2(\omega/\omega_0)^2}} e^{-i\omega t}, & \chi = 1/2, \end{cases} \quad (11)$$

with $\chi = \frac{1}{2}\sqrt{1+4\alpha}$ and

$$\gamma(\beta, \omega) = \beta \frac{\Gamma(1-\chi)}{\Gamma(1+\chi)} \left(\frac{\omega}{2\omega_0} \right)^{2\chi}. \quad (12)$$

Finally, if $\alpha < -1/4$, there are no self-adjoint extensions with spectre bounded below, so we will not consider this case. In terms of χ , the range of interest, $-1/4 < \alpha < 3/4$, corresponds to $0 < \chi < 1$. Notice that $\chi = 1/2$ corresponds to the conformal field with $m = 0$ and $\xi = 0$. The momentum parameter ω_0 in Eq. (12) was introduced to nondimensionalize $\gamma(\beta, \omega)$. This parameter introduces an energy scale to the problem, which entails the breaking of the AdS invariance of the theory.

III. BREAKING OF ADS INVARIANCE

The aim of this section is to present general arguments as to why one should not expect generic boundary conditions to preserve the AdS symmetry of the theory. A more technical discussion is deferred to the next two sections.

The Killing fields on PAdS₂,

$$\begin{aligned} \xi_1 &= \partial_t, \\ \xi_2 &= t\partial_t + z\partial_z, \\ \xi_3 &= (t^2 + z^2)\partial_t + 2tz\partial_z, \end{aligned} \quad (13)$$

generate the infinitesimal transformations

$$\begin{aligned} 1) \quad t &\rightarrow t + \lambda, & 2) \quad t &\rightarrow t + \lambda t, & 3) \quad t &\rightarrow t + \lambda(t^2 + z^2), \\ z &\rightarrow z, & z &\rightarrow z + \lambda z, & z &\rightarrow z + 2\lambda z t, \end{aligned} \quad (14)$$

which clearly preserve the boundary at $z = 0$. In this section we focus on the conformal case, which corresponds to $\chi = 1/2$ in Eq. (11). In this case, the boundary condition determined by the parameter β reads

$$u(t, 0) - \frac{\beta}{\omega_0} \frac{\partial u(t, 0)}{\partial z} = 0, \quad (15)$$

so that the Dirichlet and Neumann boundary conditions correspond to $\beta = 0$ and $\beta = \infty$, respectively.

The derivative term in Eq. (15) transforms at the conformal boundary as

$$1) \frac{\partial}{\partial z} \rightarrow \frac{\partial}{\partial z}, \quad 2) \frac{\partial}{\partial z} \rightarrow (1+\lambda) \frac{\partial}{\partial z}, \quad 3) \frac{\partial}{\partial z} \rightarrow (1+2\lambda t) \frac{\partial}{\partial z}. \quad (16)$$

We immediately see that the last two transformations, when applied to the modes $u_\omega^{(\beta)}(t, z)$, preserve the form of Eq. (15) only for $\beta = 0$ and $\beta = \infty$. We thus see that the AdS symmetry cannot be respected by nontrivial Robin conditions (i.e., those which are neither Dirichlet nor Neumann).

This breaking of AdS invariance by the boundary conditions affects physical quantities defined by the theory as follows. Let us first analyze the renormalized quantity $\langle \phi^2 \rangle_\beta$. We first notice that, since this quantity is time-independent, we have $\mathcal{L}_{\xi_1} \langle \phi^2 \rangle_\beta = 0$. However, we already know that the second and third transformations in Eq. (16) violate AdS invariance, so that we cannot conclude that $\partial_z \langle \phi^2 \rangle_\beta = 0$. Therefore, $\langle \phi^2 \rangle_\beta$ can now depend on z for nontrivial Robin conditions. In the next section we show that this is indeed the case.

Regarding the stress-energy tensor, it is easy to see that if $\mathcal{L}_{\xi_i} \langle T_\nu^\mu \rangle = 0$ for $i = 1, 2, 3$, then one would have $\partial_t \langle T_\nu^\mu \rangle = 0$, $\partial_z \langle T_\nu^\mu \rangle = 0$, $\langle T_t^t \rangle = \langle T_z^z \rangle$ and $\langle T_t^z \rangle = \langle T_z^t \rangle = 0$, where the last equality follows from time reversal symmetry. However, since ξ_2 and ξ_3 are no longer symmetries for Robin conditions, we can now only guarantee that $\langle T_\nu^\mu \rangle$ is independent of time and that $\langle T_t^t \rangle = \langle T_z^z \rangle = 0$. The other components $\langle T_t^z \rangle$ and $\langle T_z^t \rangle$ might well be different from each other and dependent on z . We show in Sec. V that this is in fact what happens.

IV. RENORMALIZED $\langle \phi^2 \rangle_\beta$

A Green's function can be constructed from Eq. (11) by means of the mode sum

$$G^{(\beta)}(x, x') \sim \int_0^\infty u_\omega^{(\beta)}(t, x) u_\omega^{(\beta)}(t', x') d\omega \quad (17)$$

and a regularization. The integral involving the normal modes in Eq. (11) is impractical to be done analytically. However, an expansion of the Green's function up to first order in β will be already enough to illustrate our main points about the dependence of physical quantities on the boundary conditions.

We will work with the Hadamard's elementary function given by

$$G^{(\beta)}(x, x') = {}_\beta \langle 0 | \{ \phi^{(\beta)}(x), \phi^{(\beta)}(x') \} | 0 \rangle_\beta. \quad (18)$$

Up to first order in β we have

$$\begin{aligned}
G^{(\beta)}(x, x') &= \sqrt{zz'} \int_0^\infty J_\chi(\omega z) J_\chi(\omega z') \cos \omega(t-t') d\omega \\
&+ \frac{\beta}{2^{2\chi}} \frac{\Gamma(1-\chi)}{\Gamma(1+\chi)} \sqrt{zz'} \int_0^\infty \left(\frac{\omega}{\omega_0}\right)^{2\chi} [J_\chi(\omega z) J_{-\chi}(\omega z') + J_{-\chi}(\omega z) J_\chi(\omega z') - 2J_\chi(\omega z) J_\chi(\omega z') \cos \pi\chi] \\
&\times \cos \omega(t-t') d\omega + \mathcal{O}(\beta^2).
\end{aligned} \tag{19}$$

The first integral in Eq. (19) corresponds to the Dirichlet boundary condition ($\beta = 0$). It was shown in [6] that this term respects all the spacetime symmetries since it is a function of the spacetime coordinates only through the geodesic distance $\sigma(x, x')$, and it is clearly divergent in the limit $x \rightarrow x'$. To calculate the value of $\langle \phi^2 \rangle_{\beta=0}$ we can use the Hadamard function for the Dirichlet case which was calculated in [7]. Using this result, the value of $\langle \phi^2 \rangle_{\beta=0}$ after the Hadamard subtraction is given by

$$\langle \phi^2 \rangle_{\beta=0} = \lim_{x' \rightarrow x} \frac{1}{2} (G_{\text{Dirichlet}}(\sigma) - G_{H,\text{singular}}(\sigma)) = \frac{\log 2}{4\pi} - \frac{1}{2\pi} \left[\psi\left(\frac{1}{2} + \chi\right) + \gamma \right], \tag{20}$$

where ψ is the digamma function [10], γ is the Euler-Mascheroni constant and we set the renormalization scale $M = 1$.

Let us now take into account the contribution of the β -dependent term in Eq. (19). In order to do this we take the limit $t' \rightarrow t$ and make use of the formula [11]

$$\int_0^\infty x^{-s} J_\mu(ax) J_\nu(bx) dx = 2^{-s} b^\nu a^{s-\nu-1} \times \frac{\Gamma((\mu + \nu - s + 1)/2)}{\Gamma(\nu + 1) \Gamma((\mu - \nu + s + 1)/2)} \times {}_2F_1\left(\frac{\nu - \mu - s + 1}{2}, \frac{\nu + \mu - s + 1}{2}; \nu + 1; \frac{b^2}{a^2}\right), \tag{21}$$

with $0 < b < a$. Eq. (6) then becomes (by considering $z' < z$)

$$\begin{aligned}
G^{(\beta)}(z, z') &= G_{\text{Dirichlet}}(\sigma) + \frac{\beta}{\omega_0^{2\chi}} \frac{\Gamma(1-\chi)}{\Gamma(1+\chi)} \sqrt{zz'} \left[z'^{-\chi} z^{-\chi-1} \frac{\Gamma(\frac{1+2\chi}{2})}{\Gamma(1-\chi)\Gamma(\frac{1}{2})} {}_2F_1\left(\frac{1}{2}, \frac{1+2\chi}{2}; 1-\chi; \frac{z'^2}{z^2}\right) \right. \\
&+ z'^\chi z^{-3\chi-1} \frac{\Gamma(\frac{1+2\chi}{2})}{\Gamma(1+\chi)\Gamma(\frac{1-4\chi}{2})} {}_2F_1\left(\frac{1+4\chi}{2}, \frac{1+2\chi}{2}; 1+\chi; \frac{z'^2}{z^2}\right) \\
&\left. - 2z'^\chi z^{-3\chi-1} \frac{\Gamma(\frac{1+4\chi}{2})}{\Gamma(1+\chi)\Gamma(\frac{1-2\chi}{2})} {}_2F_1\left(\frac{1+2\chi}{2}, \frac{1+4\chi}{2}; 1+\chi; \frac{z'^2}{z^2}\right) \cos \pi\chi \right] + \mathcal{O}(\beta^2).
\end{aligned} \tag{22}$$

Making the use of the identity [10]

$$\begin{aligned}
{}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-1)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1, 1-z) \\
&+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b; c-a-b+1, 1-z),
\end{aligned} \tag{23}$$

we find that the dependence of the Green's function on the boundary condition is given by (up to first order in β and in the limit $z' \rightarrow z$)

$$G_1(z) = \frac{1}{(\omega_0 z)^{2\chi}} \left(1 + \chi \frac{\epsilon}{z}\right) \times \frac{\pi 4^{1-\chi} \sin^2(\pi\chi) \csc(4\pi\chi) \Gamma(1-\chi)}{\Gamma(\frac{1}{2}-2\chi) \Gamma(\frac{1}{2}-\chi) \Gamma(\chi+1)^2}, \tag{24}$$

with $z' = z - \epsilon$. Note that the above expression is clearly nondivergent in the coincidence limit. As a result, the Hadamard subtraction on the full Green's function is given by

$$\langle \phi^2 \rangle_\beta = \frac{\log 2}{4\pi} - \frac{1}{2\pi} \left[\psi\left(\frac{1}{2} + \chi\right) + \gamma \right] + \frac{\beta}{2(\omega_0 z)^{2\chi}} \frac{\pi 4^{1-\chi} \sin^2(\pi\chi) \csc(4\pi\chi) \Gamma(1-\chi)}{\Gamma(\frac{1}{2}-2\chi) \Gamma(\frac{1}{2}-\chi) \Gamma(\chi+1)^2} + \mathcal{O}(\beta^2). \tag{25}$$

This explicitly shows that nontrivial boundary conditions ($\beta > 0$) break the invariance of the theory. The AdS invariant result is recovered only when $z \rightarrow \infty$.

Of particular interest is the conformal case ($\chi = 1/2$), which can be analytically solved. In fact, the contribution to the Green's function coming from nonzero values of β is given by

$$G^{(\beta)}(z, z') - G_{\text{Dirichlet}}(\sigma) = \frac{\sqrt{zz'}}{\pi} \int_0^\infty \left\{ \frac{1}{(1 + \tilde{\beta}^2 \omega^2)} \left[\frac{\sin(\omega z)}{\sqrt{\omega z}} + \frac{\tilde{\beta} \omega \cos(\omega z)}{\sqrt{\omega z}} \right] \left[\frac{\sin(\omega z')}{\sqrt{\omega z'}} + \frac{\tilde{\beta} \omega \cos(\omega z')}{\sqrt{\omega z'}} \right] - \frac{\sin(\omega z) \sin(\omega z')}{\omega} \right\} d\omega, \quad (26)$$

with $\tilde{\beta} = \beta/\omega_0$. This integral can be calculated exactly, and is given by [12]

$$G^{(\beta)}(z, z') - G_{\text{Dirichlet}}(\sigma) = -\frac{e^{\frac{z+z'}{\beta}} \text{Ei}\left(-\frac{z+z'}{\beta}\right)}{\pi}, \quad (27)$$

where $\text{Ei}(x)$ is the exponential integral function. By using the asymptotic expansion $\text{Ei}(-x) \sim -\frac{e^{-x}}{x}$ we have

$$\begin{aligned} \langle \phi^2 \rangle_\beta &= \frac{\log 2}{4\pi} - \frac{e^{\frac{2z}{\beta}} \text{Ei}\left(-\frac{2z}{\beta}\right)}{2\pi} \\ &= \frac{\log 2}{4\pi} + \frac{1}{2\pi\omega_0 z} \beta + \mathcal{O}(\beta^2) \end{aligned} \quad (28)$$

in the limit of $z' \rightarrow z$. It is easy to see that this exactly agrees with Eq. (25) in the limit $\chi \rightarrow 1/2$. Once again, this β dependence shows that the invariance is broken for nontrivial boundary conditions.

V. RENORMALIZED STRESS-ENERGY TENSOR

To calculate the renormalized stress-energy tensor we use the results of Ref. [13], where the authors found that, in two dimensions,

$$\begin{aligned} \langle T_{\mu\nu} \rangle_{\text{ren}} &= \frac{1}{2\pi} \left[-w_{\mu\nu} + \frac{1}{2}(1 - 2\xi)w_{;\mu\nu} + \frac{1}{2} \left(2\xi - \frac{1}{2} \right) g_{\mu\nu} \square w \right. \\ &\quad \left. + \xi R_{\mu\nu} w - g_{\mu\nu} v_1 \right] + \Theta_{\mu\nu}, \end{aligned} \quad (29)$$

where

$$\begin{aligned} w &= \lim_{x' \rightarrow x} W(x, x') = \lim_{x' \rightarrow x} -2\pi(G^+(x, x') - G_{\text{H,sing}}^+(\sigma)), \\ w_{\mu\nu} &= \lim_{x' \rightarrow x} W(x, x')_{;\mu\nu}, \\ v_1 &= -\frac{1}{2}m^2 - \frac{1}{2} \left(\xi - \frac{1}{6} \right) R, \\ \Theta_{\mu\nu} &= \frac{\ln M^2}{4\pi} \left[-\frac{1}{2}m^2 g_{\mu\nu} \right]. \end{aligned} \quad (30)$$

The term in v_1 is responsible for the trace anomaly and $\Theta_{\mu\nu}$ is a conserved quantity depending on the renormalization

scale M . Since we want to compare the results for Dirichlet and other Robin boundary conditions, we just set $M = 1$.

We start by calculating T_{tt} . It follows from the discussion above that

$$\begin{aligned} \langle T_{tt} \rangle_{\text{ren}} &= \frac{1}{2\pi} \left[-w_{tt} + \frac{1}{2}(1 - 2\xi)w_{;tt} \right. \\ &\quad \left. - \frac{1}{2z^2} \left(2\xi - \frac{1}{2} \right) \square w + \frac{\xi w}{z^2} + \frac{v_1}{z^2} \right]. \end{aligned} \quad (31)$$

The result is known for the Dirichlet case ($\beta = 0$) and is given by [7]

$$\begin{aligned} \langle T_{tt} \rangle_{\beta=0} &= -\frac{1}{8\pi z^2} \left[\left(-2\chi^2 - 4\xi + \frac{1}{2} \right) \left(\psi \left(\frac{1}{2} + \chi \right) \right. \right. \\ &\quad \left. \left. + \gamma - \frac{\ln 2}{2} \right) + \chi^2 + \frac{1}{12} \right]. \end{aligned} \quad (32)$$

We note that for this case the quantity w in Eq. (31) is constant since the Dirichlet case is AdS invariant. For nonzero values of β this is no longer true since w is then a function of z [see Eq. (22)]. It follows from the wave equation that the $t't'$ derivative of w can be related to its $z'z'$ derivative by

$$\frac{\partial^2 G_1(x, x')}{\partial t'^2} = \frac{\partial^2 G_1(z, z')}{\partial z'^2} + \frac{(1 - 4\chi^2)}{4z'^2} G_1(z, z'). \quad (33)$$

This leads to a renormalized component of T_{tt} of the form

$$\begin{aligned} \langle T_{tt} \rangle_\beta - \langle T_{tt} \rangle_{\beta=0} &= \lim_{z' \rightarrow z} \left\{ -\frac{\partial^2 G_1(z, z')}{\partial z'^2} - \frac{(1 - 4\chi^2)}{4z'^2} G_1(z, z') \right. \\ &\quad \left. - \frac{1}{z'} \frac{\partial G_1(z, z')}{\partial z'} \right\} + \frac{1}{2z} (1 - 2\xi) \frac{\partial G_1(z, z)}{\partial z} \\ &\quad - \frac{1}{2} (2\xi - 1/2) \frac{\partial^2 G_1(z, z)}{\partial z^2} + \frac{\xi}{z^2} G_1(z, z). \end{aligned} \quad (34)$$

As a result, we obtain

$$\langle T_{tt} \rangle_\beta = \langle T_{tt} \rangle_{\beta=0} + \frac{\beta}{\omega_0^{2\chi} z^{2+2\chi}} \frac{\pi^2 2^{-2\chi-1} (2\chi-1) (8\xi(\chi+1) - 2\chi-1) \sin(\pi\chi) \csc(4\pi\chi)}{\chi^2 (\chi+1) \Gamma(\frac{1}{2}-2\chi) \Gamma(-\chi-\frac{1}{2}) \Gamma(\chi)^3} + \mathcal{O}(\beta^2). \quad (35)$$

We also note that for $\chi = 1/2$, i.e., in the conformal case, the first order correction is zero.

Applying the same arguments to calculate $\langle T_{zz} \rangle_\beta$ we find that

$$\begin{aligned} \langle T_{zz} \rangle_\beta = & -\frac{1}{8\pi z^2} \left[\left(-2\chi^2 - 4\xi + \frac{1}{2} \right) \left(\psi \left(\frac{1}{2} + \chi \right) + \gamma - \frac{\ln 2}{2} \right) + \chi^2 + \frac{1}{12} \right] \\ & + \frac{\beta}{\omega_0^{2\chi} z^{2+2\chi}} \frac{4^{-4\chi-1} (2\chi-1) (8\xi(\chi+1) - 2\chi-1) \sin(\pi\chi) \Gamma(-\chi-1) \Gamma(4\chi)}{\chi \Gamma(\chi)^3} + \mathcal{O}(\beta^2). \end{aligned} \quad (36)$$

The fact that the stress-energy tensor is no longer proportional to the metric for nonzero values of β is clearly a manifestation of the loss of AdS invariance of the theory.

We note that $\langle T_{zz} \rangle_\beta$ tends to $\langle T_{zz} \rangle_{\beta=0}$ as $\chi \rightarrow 1/2$, as was the case for T_{tt} . This is expected on the grounds of Refs. [14, 15]. In [14], the authors found the stress-energy tensor in the presence of a single plate which splits the Minkowski spacetime into two disjoint regions. There it was shown that $\langle T_{\mu\nu} \rangle_{\text{ren}}$ depends on the (Robin) boundary condition at the plate, except for the conformal case. In [15], it was shown that the renormalized stress-energy tensor for conformal fields in conformally flat spacetimes is given by

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = : \langle T_{\mu\nu} \rangle : + t_{\mu\nu}, \quad (37)$$

where $: \langle T_{\mu\nu} \rangle :$ is the normal ordering operator—which is zero by the results of Ref. [14]—and $t_{\mu\nu}$ is a purely geometrical quantity. In our case $t_{\mu\nu}$ must be of the form

$$t_{\mu\nu} = \frac{1}{24\pi} g_{\mu\nu}, \quad (38)$$

due to trace anomaly. It can be easily checked that this is indeed the case in Eq. (32). We note that, since $|0\rangle_\beta$ is still invariant under time reversal, $\langle T_{tz} \rangle_\beta = \langle T_{zt} \rangle_\beta = 0$.

Finally, we must check whether our stress-energy tensor is conserved or not. This is one of Wald's axioms [16] on the construction of $\langle T_{\mu\nu} \rangle$. It is easily seen that the nontrivial component of the divergence of $\langle T_{\mu\nu} \rangle$ is given by

$$\nabla_\mu \langle T_t^\mu \rangle_\beta = -\frac{[\langle T_t^t \rangle_\beta + \langle T_z^z \rangle_\beta]}{z} + \frac{\partial \langle T_z^z \rangle_\beta}{\partial z} \quad (39)$$

and it follows from Eqs. (35) and (36) that, in fact,

$$\nabla_\mu \langle T_\nu^\mu \rangle_\beta = 0 + \mathcal{O}(\beta^2). \quad (40)$$

VI. CONCLUSIONS

It is usually assumed that quantum field theory on anti-de Sitter spacetime respects its maximal symmetry. This is due to the fact that the vacua constructed from Dirichlet and Neumann boundary conditions respect AdS invariance. However, the theory says nothing about which boundary condition should be chosen when solving the wave equation. We have shown that if a nontrivial generalized Robin boundary condition (i.e., one which is neither Dirichlet nor Neumann) is used, the maximal AdS symmetry of the theory is broken. This manifests itself in an unexpected behavior of physical quantities like the stress-energy tensor.

The good news is that the Hadamard decomposition, being essentially geometric, is contained in the symmetric part of the Green's function. This is exactly what we found. The divergence of the Green's function is entirely subtracted at zeroth order, so that the analytic interaction of the field with the conformal boundary does not lead to any further divergences.

However, there are some strong physical restrictions that must still be respected, regardless the choice of boundary conditions: the divergencelessness of $\langle T_{\mu\nu} \rangle$, which states conservation of energy, and trace anomaly, which is purely geometrical in principle. We have shown that this in fact happens.

ACKNOWLEDGMENTS

J. P. M. Pitelli and V. S. Barroso are grateful to Professor R. M. Wald and G. Satishchandran for enlightening discussions and also thank the Enrico Fermi Institute for the kind hospitality. J. P. M. Pitelli thanks Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) (Grant No. 2018/01558-9). V. S. Barroso thanks FAPESP (Grant No. 2018/09575-0). Finally we all thank FAPESP (Grant No. 2013/09357-9).

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