Vacuum polarization of the quantized massive scalar field in the global monopole spacetime: The renormalized quantum stress energy tensor

Owen Pavel Fernández Piedra*

Departamento de Física, División de Ciencias e Ingenierías, Universidad de Guanajuato, Campus León, Loma del Bosque No. 103, Colonia Lomas del Campestre, CP 37150 León, Guanajuato, México and Grupo de Estudios Avanzados, Universidad de Cienfuegos, Carretera a Rodas, Cuatro Caminos, s/n. C.P 55100, Cienfuegos, Cuba

(Received 22 April 2019; published 14 June 2019)

This paper is devoted to the construction of the renormalized quantum stress energy tensor $\langle T_\mu^\nu \rangle_{\rm ren}$ for a massive scalar field with arbitrary coupling to the gravitational field of a pointlike global monopole, using the Schwinger-DeWitt approximation, up to second order in the inverse mass μ of the field. The given stress energy tensor is constructed by functional differentiation with respect to the metric tensor of the one-loop effective action of a sufficiently massive scalar field, such that the Compton length of the quantum field is much less than the characteristic radius of the curvature of the background geometry. The results are obtained for a general curvature coupling parameter ξ and specified to the more physical cases of minimal and conformal coupling, showing that, in this specific case, the quantum massive scalar field in the global monopole spacetime violates all the pointwise energy conditions.

DOI: 10.1103/PhysRevD.99.125007

I. INTRODUCTION

Quantum field theory in curved spacetime is a well-established branch of modern physics, which has allowed the achievement of novel results since Hawking's discovery of black hole radiation [1]. Within this framework, we consider the quantum dynamics of fields in a gravitational background, considered as a classical external field. That is, all matter fields are considered using the quantum field theory, with the only exception of the external gravitational field that remains satisfying the classical Einstein field equations of general relativity [2,3].

In this context, an important role is played by the quantum stress energy tensor $\langle T_\mu^\nu \rangle$ of the quantum field, which is used as a source in the so-called semiclassical Einstein's equations to take a look at the quantum corrections to the background geometry caused by the quantization of matter fields [3–5]. For this reason, it is very useful to have explicit analytical expressions for the renormalized stress tensor. This quantum stress tensor and the expectation value of the field fluctuation $\langle \varphi^2 \rangle_{\rm ren}$ of a quantum field φ are the main objects to be determined from the quantum field theory in curved spacetime.

The exact determination of $\langle T_\mu^\nu \rangle$ for a generic spacetime is very cumbersome, and some techniques have been developed and applied to this problem, including numerical ones [6–17]. However, for the case of massive fields, there exists a method, called the Schwinger-DeWitt effective action approach, in which one assumes that the vacuum polarization effects can be separated from the particle creation, for masses of fields sufficiently large. This method allows us to obtain approximate analytical expressions for the one-loop quantum effective action as an expansion in the square of the inverse mass of the quantum field. From the effective action, the quantum stress energy tensor can be calculated by functional differentiation with respect to the metric. This approach, based on proper time expansion of the Green's function of the dynamical operator that describes the evolution of the quantum field, can be used to investigate effects like the vacuum polarization of massive fields in curved backgrounds, whenever the Compton's wavelength of the field is less than the characteristic radius of curvature [2,8–16,18].

In a previous paper [19], we took the first step in the investigation of vacuum polarization effects of a quantum massive scalar field with arbitrary coupling to the gravitational field of a pointlike global monopole, using the Schwinger-DeWitt technique to obtain analytical expressions for the field fluctuation $\langle \varphi^2 \rangle_{\rm ren}$ in this background spacetime. We used the simple model, discovered by Barriola and Vilenkin [20], which leads to global monopoles as heavy objects appeared in the early Universe as a result of a phase transition of a self-coupled scalar field triplet whose original global $O(3)$ symmetry is spontaneously broken to $U(1)$. In this systems, the scalar field plays the role of an order parameter which is nonzero outside the monopole's core, where it is concentrated the main part of the monopole's energy.

*opavelfp2006@gmail.com

Previous works that consider quantum fields in global monopole systems include the analysis of massless scalar fields [21–23] and the calculation of the quantum stress energy tensor for a massless spinor field [24,25].

In Ref. [19], we construct various approximations for $\langle\phi^2\rangle$, each one proportional to the coincident limit of the Hadamard-DeWitt coefficient $[a_k]$, starting from the leading term, proportional to $[a_2]$, up to the next-to-next-to-next-to-leading term, that include the coincident limits of coefficients up to $[a_5]$. In terms of the mass μ of the quantum scalar field, the leading approximation leads to $\langle\phi^2\rangle$ proportional to μ^{-2}, whereas the higher-order approximation involves powers μ^{-8}. We also find the trace of the renormalized stress energy tensor for the quantized field in the leading approximation, using the existing relationship between this magnitude, the trace anomaly, and the field fluctuation.

The results obtained in Ref. [19] for the field fluctuations of the quantized massive scalar field in the global monopole background show that taking into account higher-order terms substantially improves the approximation, for which we concluded that, for this spacetime, we need to use the next-to-next-to-next-to-leading term to obtain a good description of the vacuum polarization.

This situation was in contrast with that obtained for other spacetimes with spherical symmetry, such as the one describing a Reissner-Nordstrom black hole, for which previous studies showed that the next-to-leading term, proportional to $[a_3]$, provides a reasonable good approximation [26].

In this paper, we continue the study of vacuum polarization effects in the spacetime of the global monopole. Using the Schwinger-DeWitt approach, we construct an analytic expression of the four-dimensional renormalized quantum effective action for a quantum massive scalar field with arbitrary coupling to a generic spacetime. This expression, which is an expansion in powers of the square of the inverse mass of the quantum field, proportional to the coincident limit of the Hadamard-DeWitt coefficient $[a_3]$, is used to obtain the leading approximation for the renormalized quantum stress energy tensor of the quantum field by functional differentiation with respect to the metric. The general expressions obtained are particularized to the case of a background spacetime corresponding to a pointlike global monopole.

The paper is organized as follows. In Sec. II, we present the line element describing a pointlike global monopole, which will be used as a background to quantize the massive scalar field. In Sec. III, we give a brief description of the Schwinger-DeWitt method to construct the quantum effective action and obtain an analytic expression for this quantity for large mass scalar fields. Section IV is devoted to the construction of the four-dimensional renormalized quantum stress tensor $\langle T_{\mu\nu}\rangle_{\rm ren}$ for a massive scalar field in a general spacetime in terms of the functional derivatives of the coincident limit of the Hadamard-DeWitt coefficients a_3. Explicit analytic results for $\langle T_{\mu\nu}\rangle_{\rm ren}$ in the spacetime of a pointlike global monopoles are presented and discussed in Sec. V, whereas Sec. VI contains our concluding remarks and some perspective about future works on this subject.

II. THE POINTLIKE GLOBAL MONOPOLE SPACETIME

The most simple model which gives rise to global monopoles was constructed by Barriola and Vilenkin in Ref. [20] and starts with the Lagrangian density

$$L = \frac{1}{2}(\partial_\mu \psi^a)(\partial^\mu \psi^a) - \frac{1}{4}\lambda(\psi^a\psi^a - \eta^2)^2, \tag{1}$$

where the parameter η is of the order of 10^{16} GeV for a typical grand unified theory. From the above Lagrangian density and the Einstein equations, we obtain the spherically symmetric solution:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

where $f(r)$, far from the monopole's core, is given by

$$f(r) = 1 - 8\pi\eta^2 - 2m/r, \tag{3}$$

m being the mass parameter. If we neglect this mass term, we obtain the line element that describes the geometry around a pointlike global monopole, which results in

$$ds^2 = -\alpha^2 dt^2 + dr^2/\alpha^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4}$$

where we define the parameter α according to the expression

$$\alpha^2 = 1 - 8\pi\eta^2. \tag{5}$$

Rescaling in the above solution the time and radial variables using $\tau = \alpha t$ and $\rho = \frac{r}{\alpha}$, we arrive to the line element

$$ds^2 = -d\tau^2 + d\rho^2 + \alpha^2\rho^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{6}$$

which shows that this spacetime is characterized by a solid angle deficit, defined as the difference between the solid angle in the flat spacetime 4π and the solid angle in the global monopole spacetime $4\pi\alpha^2$. The parameter $\alpha < 1$ implies a solid angle deficit, whereas $\alpha > 1$ implies a solid angle excess. Taking into account the value of η in (5), we have that the field theory predicts a value for α smaller than unity, which implies a solid angle deficit for the pointlike global monopole spacetime.

From the line element (4), we see that the time component of the metric tensor, $g_{tt} = -\alpha^2$, is constant. In the Newtonian approximation, one can write the Newtonian gravitational potential $\Phi(r)$ in terms of this metric component as $\Phi = -\frac{1}{2}(1 + g_{tt})$, which turns out to be constant if

g_{tt} is constant. Hence, the Newtonian gravitational force, which is the gradient of this gravitational potential, turns out to be null in these approximations. This fact can be understood if we take into account that the gravitational mass $M(r)$ of the monopole is divergent as $r \to \infty$. From $g_{tt} = -(1 - \frac{2M(r)}{r}) = -\alpha^2$, we find $M(r) = 4\pi\eta^2 r$, and the Newtonian gravitational potential $\Phi(r) \sim \frac{M(r)}{r}$ is constant. Therefore, the pointlike global monopole exerts no gravitational force on the matter around it, apart from the tiny gravitational effect due to the core.

However, the geometry around the global monopole has nonvanishing curvature, and we can expect gravitational effects around it. For example, although the global monopole exerts no Newtonian gravitational force on nearby matter, it gives an enormous tidal acceleration $a \approx \frac{M(r)}{r^3} \approx \frac{1}{r^2}$, proportional to the inverse of the square of the distance from the monopole's core, a fact that was considered in Ref. [27] in a cosmological context to obtain an upper bound on the number density of them in the Universe, resulting in at most one global monopole in the local group of galaxies. In contrast with this result, in Ref. [28], the authors show, using numerical simulations, that the real upper boundary is smaller by many orders than that derived by Hiscock in Ref. [27], finding a scaling solution which corresponds to a few global monopoles per horizon volume.

III. THE RENORMALIZED ONE-LOOP EFFECTIVE ACTION

In this section, we construct the one-loop effective action for a massive scalar field with mass μ and arbitrary coupling to a generic gravitational background with metric tensor $g_{\mu\nu}$, using the Schwinger-DeWitt approach. Details for the results presented in this section can be found in Refs. [8,9,11,12,14,16,29].

The nonminimally coupled massive scalar field satisfies the Klein-Gordon equation

$$(-\Box + \mu^2 + \xi R)\phi = 0, \tag{7}$$

where ξ is the coupling constant and R is the Ricci scalar.

The one-loop effective action $S^{(1)}$ is related with the Feynman Green's function $G^F(x, x')$ of the Klein-Gordon operator in (7) by the expression

$$S^{(1)} = -\frac{i}{2} \,\mathrm{Tr}\, \ln G^F. \tag{8}$$

In the following, we use the Schwinger-DeWitt proper-time formalism, which assumes that $G^F(x, x')$ is given by

$$G^F(x, x') = \frac{i\Delta^{1/2}}{(4\pi)^2} \int_0^\infty ids \frac{1}{(is)^2} \exp\left[-i\mu^2 s + \frac{i\sigma(x, x')}{2s}\right] \times A(x, x'; is), \tag{9}$$

where

$$A(x, x'; is) = \sum_{k=0}^{\infty} (is)^k a_k(x, x'), \tag{10}$$

s is the proper time, and the biscalars $a_k(x, x')$ are called Hadamard-DeWitt coefficients. Also, $\Delta(x, x')$ is the Van Vleck–Morette determinant, and the biscalar $\sigma(x, x')$ represents one-half of the geodetic distance between the spacetime points x and x'.

In four dimensions, the first three terms in (9), respectively proportional to a_k with $k = 0$, 1, 2, are divergent. Then, defining the regularized biscalar $A_{\mathrm{reg}}(x, x'; is)$ as

$$A_{\mathrm{reg}}(x, x'; is) = A(x, x'; is) - \sum_{k=0}^{2} a_k(x, x')(is)^k, \tag{11}$$

we can put, in Eq. (9), $A_{\mathrm{reg}}(x, x'; is)$ instead of $A(x, x'; is)$, which gives finally the regularized four-dimensional Green's function $G^F_{\mathrm{reg}}(x, x')$ as

$$G^F_{\mathrm{reg}}(x, x') = \frac{i\Delta^{1/2}}{(4\pi)^2} \int_0^\infty ids \frac{1}{(is)^2} \exp\left[-i\mu^2 s + \frac{i\sigma(x, x')}{2s}\right] \times \sum_{k=3}^{N} a_k(x, x')(is)^k. \tag{12}$$

Substituting (12) in (8) and taking into account the definition of the trace and the logarithm of an operator given, for example, in Ref. [3], we have for the renormalized one-loop effective action

$$S^{(1)}_{\mathrm{ren}} = \lim_{x' \to x} \int d^4x \sqrt{-g} \frac{\Delta^{1/2}}{32\pi^2} \int_0^\infty ids \frac{1}{(is)^3} \times \exp\left[-i\mu^2 s + \frac{i\sigma(x, x')}{2s}\right] \sum_{k=3}^{N} a_k(x, x')(is)^k. \tag{13}$$

The limiting processes in the above equation gives

$$S^{(1)}_{\mathrm{ren}} = \frac{1}{32\pi^2} \int d^4x \sqrt{-g} \int_0^\infty ids \frac{1}{(is)^3} \exp[-i\mu^2 s] \times \sum_{k=3}^{N} [a_k](is)^k, \tag{14}$$

where $a_k = \lim_{x' \to x} a_k(x, x')$ are the coincidence limits of the Hadamard-DeWitt biscalars and the upper sum limit N gives the order of the Schwinger-DeWitt approximation in $S^{(1)}_{\mathrm{reg}}$. Integrating over s in (14) by making the substitution $\mu^2 \to \mu^2 - i\varepsilon$ ($\varepsilon > 0$) [29], we obtain for the renormalized one-loop effective action of the massive scalar field the result

$$S^{(1)}_{\rm ren} = \frac{1}{32\pi^2}\int d^4x\sqrt{-g}\sum_{k=3}^{N}\frac{\Gamma(k-2)}{(\mu^2)^{k-2}}[a_k], \tag{15}$$

which, using the properties of the Gamma function, can be written as

$$S^{(1)}_{\rm ren} = \frac{1}{32\pi^2}\int d^4x\sqrt{-g}\sum_{k=3}^{N}\frac{(k-3)!}{(\mu^2)^{k-2}}[a_k]. \tag{16}$$

We expect that if the Compton length associated with the field, λ_c, is less than the characteristic radius of the curvature of the background geometry, L, then a reasonable approximation to $S^{(1)}_{\rm ren}$ is given by the leading term in (16), proportional to the inverse of the squared field's mass:

$$S^{(1)}_{\rm ren} = \frac{1}{32\pi^2\mu^2}\int d^4x\sqrt{-g}[a_3]. \tag{17}$$

The inclusion of higher-order terms in the above expansion will be always well motivated, in order to obtain a value for one-loop effective action close to the exact value of this quantity. However, in the rest of the paper, our aim is to take into account only the leading term in (16), which implies the calculation of the coincidence limit of the Hadamard-DeWitt coefficient $[a_3]$.

As we see from (17), the main task for the calculation of the one-loop effective action in the Schwinger-DeWitt approximation is the determination, up to the order of $k=3$, of the coincidence limit of the Hadamard-DeWitt biscalar $a_k(x,x')$, that satisfies the recurrence equation

$$\sigma^{;i}a_{k;i} + ka_k - \Delta^{-1/2}\Box(\Delta^{1/2}a_{k-1}) + \xi R a_{k-1} = 0, \tag{18}$$

with the boundary condition $a_0(x,x')=1$.

The results for this coefficient up to the order of $k=5$ can be found, for example, in Refs. [2,9,19,26,30,31]. Using the transport equation approach of Ottewill and Wardell [31], we can obtain easily general expressions for the coincidence limit of the Hadamard-DeWitt coefficients a_3, by solving the transport equations given in Ref. [31] using the software package xAct for Wolfram *Mathematica* [32]. The result is

$$\begin{aligned}[a_3] = {} & \frac{1}{15120}(584R_{\alpha\beta}R^{\alpha}{}_{\gamma}R^{\beta\gamma} - 654RR_{\alpha\beta}R^{\alpha\beta} + 99R^3 + 456R_{\alpha\beta}R_{\gamma\delta}R^{\alpha\gamma\beta\delta} + 72RR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \\ & - 80R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\delta\epsilon\rho} + 51R_{;\alpha}R^{;\alpha} - 12R_{\alpha\gamma;\beta}R^{\alpha\beta;\gamma} - 6R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma} + 27R_{\alpha\beta\gamma\delta;\epsilon}R^{\alpha\beta\gamma\delta;\epsilon} \\ & + 84R\Box R + 36R_{;\alpha\beta}R^{\alpha\beta} - 24R_{\alpha\beta}\Box R^{\alpha\beta} + 144R_{\alpha\beta;\gamma\delta}R^{\alpha\gamma\beta\delta} + 54\Box\Box R) \\ & + \frac{\xi}{360}(2RR_{\alpha\beta}R^{\alpha\beta} - 5R^3 - 2RR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 12R_{;\alpha}R^{;\alpha} - 22R\Box R - 4R_{;\alpha\beta}R^{\alpha\beta} - 6\Box\Box R) \\ & + \frac{\xi^2}{12}\left(R^3 + R_{;\alpha}R^{;\alpha} + 2R\Box R - \frac{\xi^3}{6}R^3\right). \end{aligned} \tag{19}$$

The above result coincides with those reported in Refs. [26,31]. As we can see, the coincidence limit of the Hadamard-DeWitt coefficients $[a_3]$ is an extremely complicated local expression constructed from the Riemann tensor, their covariant derivatives, and contractions. However, the fact that the above result is valid for a generic spacetime, being static, stationary, or nonstationary, gives rise to the possibility of using it to obtain relatively simple expressions for the regularized one-loop effective action of the quantum massive scalar field in spacetimes with a higher degree of symmetry.

As we can see from the structure of $[a_3]$, it is a local geometric term that depends on the coupling constant ξ and the parameters that describe the geometry of the gravitational background. For the pointlike global monopole spacetime, the only parameter that characterizes the geometry of the manifold is α. Then, it is reasonable to expect that the one-loop effective action, as well as the regularized quantum stress tensor for the massive scalar field in this background, will be functions of ξ, α, and the distance r from the monopole core.

Putting (19) into (17), we can obtain a general expression for the regularized one-loop effective action. However, we can simplify the result using the fact that not all the terms in (19) are independent among themselves. It is possible to show that in four dimensions the following relations hold [16]:

$$\int d^4x\sqrt{-g}\Box\Box R = 0, \tag{20}$$

$$\int d^4x\sqrt{-g}R_{;\alpha\beta}R^{\alpha\beta} = \int d^4x\sqrt{-g}\left(\frac{1}{2}R\Box R\right), \tag{21}$$

$$\int d^4x\sqrt{-g}R_{\alpha\beta;\gamma\delta}R^{\alpha\gamma\beta\delta} = \int d^4x\sqrt{-g}\left(-\frac{1}{4}R\Box R + R_{\alpha\beta}\Box R^{\alpha\beta} - R_{\alpha\beta}R^{\alpha}_{\gamma}R^{\beta\gamma} + R_{\alpha\beta}R_{\gamma\delta}R^{\alpha\gamma\beta\delta}\right), \tag{22}$$

$$\int d^4x\sqrt{-g}R_{;\alpha}R^{;\alpha} = \int d^4x\sqrt{-g}(-R\Box R), \tag{23}$$

$$\int d^4x\sqrt{-g}R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma} = \int d^4x\sqrt{-g}(-R_{\alpha\beta}\Box R^{\alpha\beta}), \tag{24}$$

$$\int d^4x\sqrt{-g}R_{\alpha\beta;\gamma}R^{\alpha\gamma;\beta} = \int d^4x\sqrt{-g}\left(-\frac{1}{4}R\Box R - R_{\alpha\beta}R^{p}_{\alpha\gamma}R^{\beta\gamma} + R_{\alpha\beta}R_{\gamma\delta}R^{\alpha\gamma\beta\delta}\right), \tag{25}$$

$$\begin{aligned}\int d^4x\sqrt{-g}R_{\alpha\beta\gamma\delta;\epsilon}R^{\alpha\beta\gamma\delta;\epsilon} = \int d^4x\sqrt{-g}(&R\Box R - 4R_{\alpha\beta}\Box R^{\alpha\beta} + 4R_{\alpha\beta}R^{\alpha}_{\beta\gamma}R^{\beta\gamma} - 4R_{\alpha\beta}R_{\gamma\delta}R^{\alpha\gamma\beta\delta}\\ &-2R_{\alpha\beta}R^{\alpha}_{\alpha\gamma\delta\epsilon}R^{\beta\gamma\delta\epsilon} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\mu\nu}R^{\gamma\delta}_{\gamma\delta\mu\nu} + 4R_{\alpha\gamma\beta\delta}R^{\alpha\mu\beta}_{\alpha\mu\beta\nu}R^{\gamma\mu\delta\nu}).\end{aligned} \tag{26}$$

Using the above relations, we can show that the one-loop effective action includes only ten geometric terms. The final result is

$$\begin{aligned}S^{(1)}_{\rm ren} = \frac{1}{192\pi^2\mu^2}\int d^4x\sqrt{-g}\Bigg[&\left(\frac{\xi^2}{2} - \frac{\xi}{5} + \frac{1}{56}\right)R\Box R + \frac{1}{140}R_{\alpha\beta}\Box R^{\alpha\beta} - \left(\xi - \frac{1}{6}\right)^3 R^3\\ &+\frac{1}{30}\left(\xi - \frac{1}{6}\right)RR_{\alpha\beta}R^{\alpha\beta} - \frac{8}{945}R_{\alpha\beta}R^{\alpha}{}_{\gamma}R^{\beta\gamma} + \frac{2}{315}R_{\alpha\beta}R_{\gamma\delta}R^{\alpha\gamma\beta\delta} + \frac{1}{1260}R_{\alpha\beta}R^{\alpha}{}_{\gamma\delta\epsilon}R^{\beta\gamma\delta\epsilon}\\ &-\frac{1}{30}\left(\xi - \frac{1}{6}\right)RR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \frac{17}{7560}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\sigma\rho}R^{\gamma\delta}{}_{\sigma\rho} - \frac{1}{270}R_{\alpha\gamma\beta\delta}R^{\alpha}{}_{\sigma}{}^{\beta}{}_{\rho}R^{\gamma\sigma\delta\rho}\Bigg].\end{aligned} \tag{27}$$

IV. SCHWINGER-DEWITT APPROXIMATION FOR $\langle T^{\nu}_{\mu}\rangle_{\rm ren}$

The renormalized quantum stress energy tensor for the massive scalar field nonminimally coupled to a generic spacetime background can be determined from (27) by functional differentiation with respect to the metric tensor:

$$\langle T^{\mu\nu}\rangle_{\rm ren} = \frac{2}{\sqrt{-g}}\frac{\delta S^{(1)}_{\rm ren}}{\delta g_{\mu\nu}}. \tag{28}$$

Because of the identities and relations satisfied by the Riemann tensor, its contractions, and covariant derivatives, there are not unique results for $\langle T^{\mu\nu}\rangle_{\rm ren}$ [11,12,14–16]. However, all the obtained expressions for this quantity must give the same results when applied to specific spacetime backgrounds. Also, all the results must have a covariant divergence equal to zero, which is a fundamental property of the stress energy tensor.

In the following, we use the basis proposed in the beautiful paper of Décanini and Follaci [16], which allows us to obtain an irreducible expression for the renormalized stress energy tensor for the quantum scalar field given by

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{1}{96\pi^2\mu^2}\Bigg[&\left[\xi^2 - \frac{2\xi}{5} + \frac{3}{70}\right](\Box R)_{;\mu\nu} + \frac{1}{10}\left(\xi - \frac{1}{6}\right)R\Box R_{\mu\nu} + \frac{1}{15}\left(\xi - \frac{1}{7}\right)R_{;\alpha(\mu}R^{\alpha}{}_{\nu)}\\ &-\frac{1}{140}\Box\Box R_{\mu\nu} - 6\left(\xi - \frac{1}{6}\right)\left[\xi^2 - \frac{\xi}{3} + \frac{1}{30}\right]RR_{;\mu\nu} - \left(\xi - \frac{1}{6}\right)\left(\xi - \frac{1}{5}\right)(\Box R)R_{\mu\nu}\\ &+\frac{1}{42}R_{\alpha(\mu}\Box R^{\alpha}{}_{\nu)} + \frac{1}{15}\left(\xi - \frac{2}{7}\right)R^{\alpha\beta}R_{\alpha\beta;(\mu\nu)} + \frac{2}{105}R^{\alpha\beta}R_{\alpha(\mu;\nu)\beta} - \frac{1}{70}R^{\alpha\beta}R_{\mu\nu;\alpha\beta}\\ &+\frac{2}{15}\left(\xi - \frac{3}{14}\right)R^{;\alpha\beta}R_{\alpha\mu\beta\nu} - \frac{1}{105}(\Box R^{\alpha\beta})R_{\alpha\mu\beta\nu} + \frac{4}{105}R^{\alpha\beta;\gamma}{}_{(\mu}R_{|\gamma\beta\alpha|\nu)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{35}R^{\alpha}{}_{(\mu}{}^{;\beta\gamma}R_{|\alpha\beta\gamma|\nu)}-\frac{1}{15}\left(\xi-\frac{3}{14}\right)R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta;(\mu\nu)}-6\left(\xi-\frac{1}{4}\right)\left(\xi-\frac{1}{6}\right)^2R_{;\mu}R_{;\nu}\\
&-\frac{1}{5}\left(\xi-\frac{3}{14}\right)R_{;\alpha}R^{\alpha}{}_{(\mu;\nu)}+\frac{1}{5}\left(\xi-\frac{17}{84}\right)R_{;\alpha}R_{\mu\nu}{}^{;\alpha}+\frac{1}{15}\left(\xi-\frac{1}{4}\right)R^{\alpha\beta}{}_{;\mu}R_{\alpha\beta;\nu}\\
&-\frac{1}{210}R^{\alpha}{}_{\mu;\beta}R_{\alpha\nu}{}^{;\beta}+\frac{1}{42}R^{\alpha}{}_{\mu;\beta}R^{\beta}{}_{\nu;\alpha}-\frac{1}{105}R^{\alpha\beta;\gamma}R_{\gamma\beta\alpha(\mu;\nu)}-\frac{1}{70}R^{\alpha\beta;\gamma}R_{\alpha\mu\beta\nu;\gamma}\\
&-\frac{1}{15}\left(\xi-\frac{13}{56}\right)R^{\alpha\beta\gamma\delta}{}_{;\mu}R_{\alpha\beta\gamma\delta;\nu}-\frac{1}{70}R^{\alpha\beta\gamma}{}_{\mu;\delta}R_{\alpha\beta\gamma\nu}{}^{;\delta}+3\left(\xi-\frac{1}{6}\right)^3R^2R_{\mu\nu}\\
&-\frac{2}{15}\left(\xi-\frac{1}{6}\right)RR_{\alpha\mu}R^{\alpha}{}_{\nu}-\frac{1}{30}\left(\xi-\frac{1}{6}\right)R^{\alpha\beta}R_{\alpha\beta}R_{\mu\nu}-\frac{2}{315}R^{\alpha\beta}R_{\alpha\mu}R_{\beta\nu}\\
&+\frac{1}{315}R^{\alpha\gamma}R^{\beta}{}_{r}R_{\alpha\mu\beta\nu}+\frac{1}{315}R^{\alpha\beta}R^{\gamma}{}_{(\mu}R_{|\gamma\beta\alpha|\nu)}+\frac{1}{15}\left(\xi-\frac{1}{6}\right)RR^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\nu}\\
&+\frac{1}{30}\left(\xi-\frac{1}{6}\right)R_{\mu\nu}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-\frac{4}{315}R^{\alpha}{}_{(\mu}R^{\beta\gamma\delta}{}_{|\alpha|}R_{|\beta\gamma\delta|\nu)}-\frac{2}{315}R^{\alpha\beta}R^{\gamma\delta}{}_{p\mu}R_{\gamma\delta\beta\nu}\\
&+\frac{4}{315}R_{\alpha\beta}R^{\alpha\gamma\beta\delta}R_{\gamma\mu\delta\nu}-\frac{1}{315}R_{\alpha\beta}R^{\alpha\gamma\delta}{}_{\mu}R^{\beta}{}_{\gamma\delta\nu}+\frac{2}{315}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\sigma\mu}R_{\gamma\delta}{}^{\sigma}{}_{\nu}\\
&+\frac{4}{63}R^{\alpha\gamma\beta\delta}R^{\sigma}{}_{\alpha\beta\mu}R_{\sigma\gamma\delta\nu}-\frac{2}{315}R^{\alpha\beta\gamma}{}_{\delta}R_{\alpha\beta\gamma\sigma}R^{\delta}{}_{\mu}{}^{\sigma}{}_{\nu}+\frac{1}{15}\left(\xi-\frac{1}{6}\right)RR^{\alpha\beta}R_{\alpha\mu\beta\nu}\\
&+g_{\mu\nu}\Bigg[\left(-\xi^2+\frac{2\xi}{5}-\frac{11}{280}\right)\Box\Box R+6\left(\xi-\frac{1}{6}\right)\left[\xi^2-\frac{\xi}{3}+\frac{1}{40}\right]R\Box R\\
&-\frac{1}{30}\left(\xi-\frac{3}{14}\right)R_{;\alpha\beta}R^{\alpha\beta}-\frac{1}{15}\left(\xi-\frac{5}{28}\right)R_{\alpha\beta}\Box R^{\alpha\beta}+\frac{4}{15}\left(\xi-\frac{1}{7}\right)R_{\alpha\beta;\gamma\delta}R^{\alpha\gamma\beta\delta}\\
&+6\left(\xi^3-\frac{13}{24}\xi^2+\frac{17}{180}\xi-\frac{53}{10080}\right)R_{;\alpha}R^{;\alpha}-\frac{1}{15}\left(\xi-\frac{13}{56}\right)R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma}\\
&-\frac{1}{420}R_{\alpha\beta;\gamma}R^{\alpha\gamma;\beta}+\frac{1}{15}\left(\xi-\frac{19}{112}\right)R_{\alpha\beta\gamma\delta;\sigma}R^{\alpha\beta\gamma\delta;\sigma}-\frac{1}{2}\left(\xi-\frac{1}{6}\right)^3R^3\\
&+\frac{1}{60}\left(\xi-\frac{1}{6}\right)RR_{\alpha\beta}R^{\alpha\beta}+\frac{1}{1890}R_{\alpha\beta}R^{\alpha}{}_{r}R^{\beta\gamma}-\frac{1}{630}R_{\alpha\beta}R_{\gamma\delta}R^{\alpha\gamma\beta\delta}\\
&-\frac{1}{60}\left(\xi-\frac{1}{6}\right)RR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}+\frac{2}{15}\left(\xi-\frac{1}{6}\right)R_{\alpha\beta}R^{\alpha}{}_{\gamma\delta\sigma}R^{\beta\gamma\delta\sigma}\\
&-\frac{1}{15}\left(\xi-\frac{47}{252}\right)R_{\alpha\beta\gamma\delta}R^{\alpha\beta\sigma\rho}R^{\gamma\delta}{}_{\sigma\rho}-\frac{4}{15}\left(\xi-\frac{41}{252}\right)R_{\alpha\gamma\beta\delta}R^{\alpha}{}_{\sigma}{}^{\beta}{}_{\rho}R^{\gamma\sigma\delta\rho}\Bigg]\Bigg].
\end{aligned}
\tag{29}
$$

The above result is a rather complex expression for the renormalized stress energy tensor for a large mass scalar field with arbitrary coupling to gravity in the Schwinger-DeWitt approximation, that is valid for any spacetime [33]. As we can see, the information of the massive scalar field is included in the coefficients accompanying each local geometric term constructed from the Riemann tensor, its covariant derivatives, and contractions.

The fact that we have an analytic result for $\langle T_{\mu\nu}\rangle_{\rm ren}$ valid for a generic gravitational background is very important, because it opens the possibility to study the influence of the quantization of a massive scalar field upon the background spacetime, the so-called backreaction problem. Using the above stress energy tensor as a source in the semiclassical Einstein's equations, we can, in principle, find the quantum corrections to the background metric perturbatively.

In the rest of the paper, we will apply the above obtained formula to the study of the renormalized stress energy tensor for the quantized nonminimally coupled massive scalar field in the spacetime of a pointlike global monopole. As we will show, simple results can be obtained in this case.

V. RENORMALIZED QUANTUM STRESS ENERGY TENSOR FOR MASSIVE SCALAR FIELD IN POINTLIKE GLOBAL MONOPOLE SPACETIME

Using (4) in (29), we obtain, for the temporal component of the renormalized stress energy tensor for the massive scalar field with arbitrary coupling parameter ξ, the very simple result

$$\langle T^t_t\rangle_{\rm ren} = \frac{(\alpha^2-1)}{10080\pi^2\mu^2 r^6}\sum_{k=0}^{3} B_k(\alpha)\xi^k, \qquad (30)$$

where

$$B_0(\alpha) = 101\alpha^2(1+\alpha^2) - 4, \qquad (31)$$

$$B_1 = -504\alpha^4 - 1554\alpha^2 + 42, \qquad (32)$$

$$B_2 = -3150\alpha^4 + 8400\alpha^2 - 210, \qquad (33)$$

and

$$B_3 = 15540\alpha^4 - 15960\alpha^2 + 420. \qquad (34)$$

In Fig. 1, we show the dependence on the coupling constant ξ of the rescaled time component of the renormalized stress energy tensor $\langle T_0\rangle = 96\pi^2\mu^2\langle T^t_t\rangle_{\rm ren}$ for a massive scalar field in the pointlike global monopole spacetime with parameter $1-\alpha^2 = 10^{-5}$ at a fixed distance from the monopole's core.

The first thing that we can observe is that $\langle T_0\rangle$ increases with the increase of the coupling constant until it reach its maximum value at $\xi = 0.2$, becoming positive for values of the coupling constant between $0.17 \le \xi \le 0.23$. For values of the coupling constant outside this interval, the stress energy tensor is negative, decreasing its value for $\xi \ge 0.23$.

The above behavior is similar for all values of the distance r from the monopole's core, as we can see in Fig. 2, where we show the dependence of $\langle T_0\rangle$ as a function of ξ and r. As we can see from the graph, for values of $0.17 \le \xi \le 0.23$, the time component of the stress energy tensor for the massive scalar field is positive and tends to zero at large distances from the monopole's core. This behavior excludes the physical values associated with minimal coupling $\xi = 0$ and conformal coupling $\xi = \frac{1}{6}$.

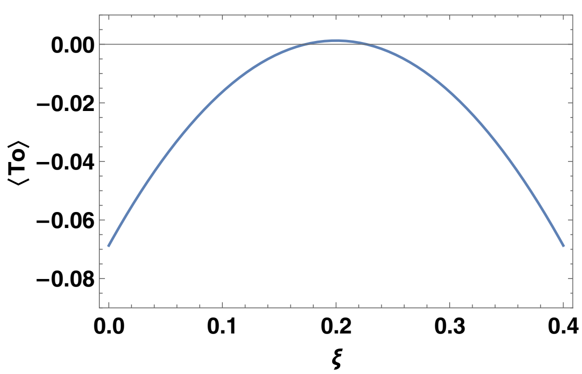

FIG. 1. Dependence on the coupling constant ξ of the rescaled time component of the renormalized stress energy tensor $\langle T_0\rangle = 96\pi^2\mu^2\langle T^t_t\rangle_{\rm ren}$ for a massive scalar field in the pointlike global monopole spacetime. The values of the parameters used in the calculations are $r = \frac{1}{3}$ and $1-\alpha^2 = 10^{-5}$.

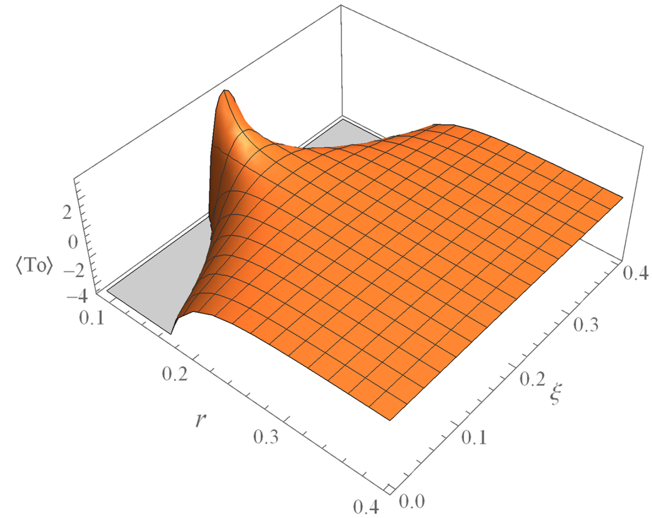

FIG. 2. Dependence on the coupling constant ξ and distance from the monopole's core r of the rescaled time component of the renormalized stress energy tensor $\langle T_0\rangle = 96\pi^2\mu^2\langle T^t_t\rangle_{\rm ren}$ for a massive scalar field in the pointlike global monopole spacetime. The value of the parameter used in the calculations is $1-\alpha^2 = 10^{-5}$.

For values of the coupling constant outside the above-mentioned interval, the time component of the renormalized stress energy tensor has negative values, again tending to zero as $r \to \infty$. The physical values corresponding to minimal and conformal coupling to gravity are included in this case, as we can see in Fig. 3, when we plot the dependence of the rescaled time component of the renormalized stress energy tensor as a function of r for this value of the coupling constant.

Now substituting (4) in (29), we can obtain the radial and angular components of the renormalized stress energy tensor for the massive scalar field with arbitrary coupling parameter ξ. For the radial component, we obtain the very simple result

$$\langle T^r_r\rangle_{\rm ren} = \frac{(1-\alpha^2)}{10080\pi^2\mu^2 r^6}\sum_{k=0}^{3} Q_k(\alpha)\xi^k, \qquad (35)$$

where

$$Q_0(\alpha) = 4 + 67\alpha^2(1+\alpha^2), \qquad (36)$$

$$Q_1(\alpha) = -[42 + \alpha^2(336\alpha^2 - 966)], \qquad (37)$$

$$Q_2(\alpha) = 210 + \alpha^2(5040 - 1890\alpha^2), \qquad (38)$$

and

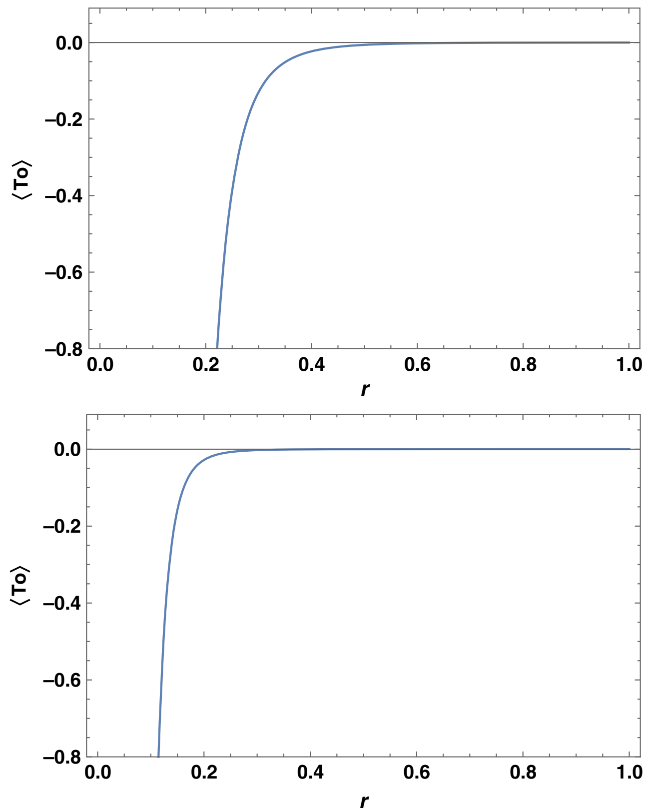

FIG. 3. Dependence on the distance from the monopole's core r of the rescaled time component of the renormalized stress energy tensor $\langle T_0\rangle = 96\pi^2\mu^2\langle T_t^t\rangle_{\text{ren}}$ for a massive scalar field in the pointlike global monopole spacetime. The values of the parameters used in the calculations are $1-\alpha^2 = 10^{-5}$ and $\xi = 0$ (top plot) and $\xi = \frac{1}{6}$ (bottom plot).

$$Q_3(\alpha) = -[420 + \alpha^2(9660\alpha^2 - 9240)]. \tag{39}$$

For the angular components of the renormalized stress energy tensor, we obtain

$$\langle T_\theta^\theta\rangle_{\text{ren}} = \langle T_\varphi^\varphi\rangle_{\text{ren}} = -\frac{1}{2}\langle T_r^r\rangle_{\text{ren}}. \tag{40}$$

In Fig. 4, we show the dependence on the coupling constant ξ of the rescaled radial component of the renormalized stress energy tensor $\langle T_1\rangle = 96\pi^2\mu^2\langle T_r^r\rangle_{\text{ren}}$ for a massive scalar field in the pointlike global monopole spacetime with parameter $1-\alpha^2 = 10^{-5}$ at a fixed distance from the monopole's core.

We can observe that $\langle T_1\rangle$ decreases with the increase of the coupling constant until it reaches its minimum value at $\xi = 0.2$ and then increases its value for $\xi \geq 0.2$. For all values of the coupling constant, this magnitude remains positive, and this behavior is independent of the value of the distance r from the monopole's core, as we can see in Fig. 5, where we show the dependence of $\langle T_1\rangle$ as a function of ξ and r. Also, we observe that $\langle T_1\rangle$ tends to zero at large

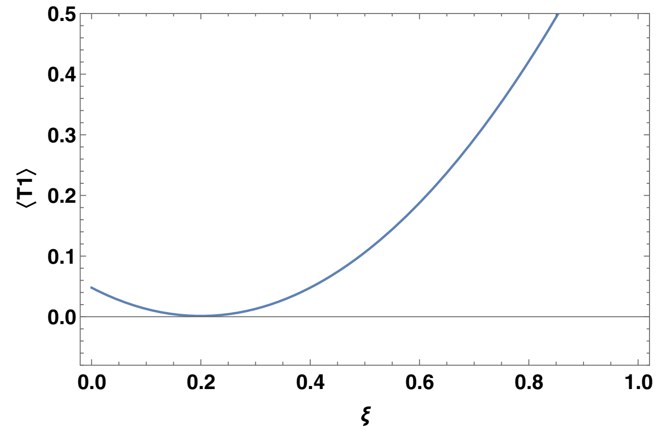

FIG. 4. Dependence on the coupling constant ξ of the rescaled radial component of the renormalized stress energy tensor $\langle T_1\rangle = 96\pi^2\mu^2\langle T_r^r\rangle_{\text{ren}}$ for a massive scalar field in the pointlike global monopole spacetime. The values of the parameters used in the calculations are $r = \frac{1}{3}$ and $1-\alpha^2 = 10^{-5}$.

distances from the monopole's core. For minimal and conformal coupling, Fig. 6 shows this general behavior.

As (40) shows, the angular components of the renormalized stress tensor for the massive scalar field with arbitrary coupling to gravity on a global monopole spacetime is proportional, with opposite sign, to the radial component. For this reason, it is expected that $\langle T_2\rangle$ increases with the increase of the coupling constant until it reach its maximum value at $\xi = 0.2$, and then decreases its value for $\xi \geq 0.2$, and remains negative for all values of the coupling constant, a behavior independent of the distance r from the monopole's core. Also, $\langle T_2\rangle \to 0$ as $r \to \infty$.

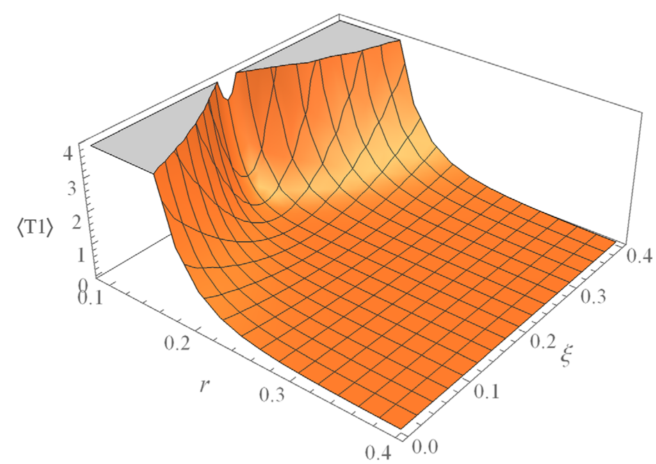

FIG. 5. Dependence on the coupling constant ξ and the distance r from the monopole's core of the rescaled radial component of the renormalized stress energy tensor $\langle T_1\rangle = 96\pi^2\mu^2\langle T_r^r\rangle_{\text{ren}}$ for a massive scalar field in the pointlike global monopole spacetime. The value of the parameter used in the calculations is $1-\alpha^2 = 10^{-5}$.

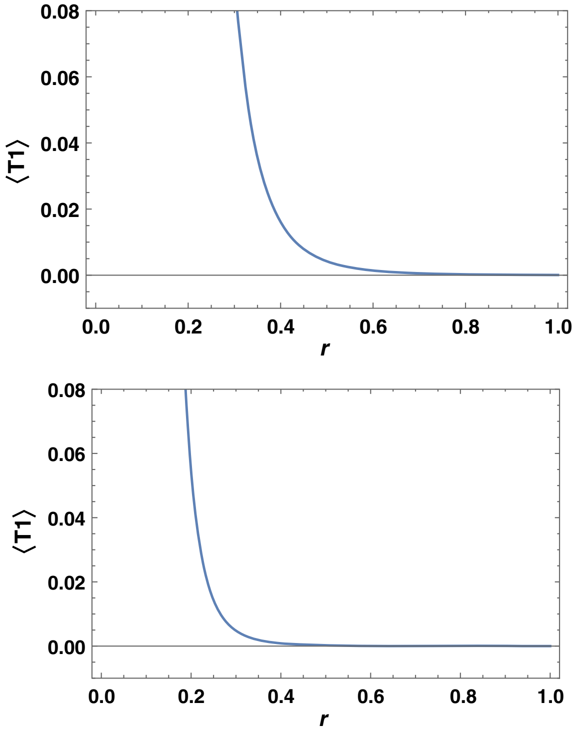

FIG. 6. Dependence on the distance from the monopole's core r of the rescaled radial component of the renormalized stress energy tensor $\langle T_1\rangle = 96\pi^2\mu^2\langle T_r^r\rangle_{\rm ren}$ for a massive scalar field in the pointlike global monopole spacetime. The values of the parameters used in the calculations are $1-\alpha^2 = 10^{-5}$ and $\xi = 0$ (top plot) and $\xi = \frac{1}{6}$ (bottom plot).

If we defined as usual the energy density of the quantum field as

$$\rho = -\langle T_t^t\rangle_{\rm ren}, \tag{41}$$

the above results indicate that, for the massive scalar field in the spacetime of a pointlike global monopole, this magnitude is negative for all values of r if the coupling constant is on the interval $\xi \in [0.17, 0.23]$ and positive for all values of r outside this interval, which includes the minimal and conformally coupled case.

The principal pressures related with the diagonal components of the renormalized stress energy tensor can be defined, in the usual way, by

$$p_r = -\langle T_r^r\rangle_{\rm ren} \tag{42}$$

for the radial pressure and

$$p_\theta = p_\phi = p = \langle T_\theta^\theta\rangle_{\rm ren} = \langle T_\varphi^\varphi\rangle_{\rm ren} \tag{43}$$

for the angular ones.

As we can see from the results discussed above, the radial pressure is negative for all values of the coupling constant, a behavior independent of the distance from the monopole's core. The angular pressures remains negative for all values of ξ and r, too.

The above facts are interesting in relation with possible violations of energy conditions by the quantized massive scalar field in the global monopole background. As all the information of the quantum field in this background is encoded in the components of the stress energy tensor, one can gain better information on the nature of this quantized field by analyzing the fulfillment or not of the various energy conditions that can be considered in this case.

Energy conditions are restrictions that the components of the stress energy tensor of matter fields do satisfy to be in some sense reasonable types of matter and are important in the proofs of various theorems, such as those concerned with singularities, topological censorship, and positivity of mass.

The pointwise energy conditions, in the case of a spherically symmetric spacetime, can be summarized as follows [34,35].

Null energy condition (NEC).—A matter field satisfies the pointwise NEC if, for any null vector k^μ, we have that $T_{\mu\nu}k^\mu k^\nu \geq 0$. In terms of the diagonal components of the stress energy tensor, the above condition is equivalent to the restrictions $\rho - p_r \geq 0$ and $\rho + p \geq 0$.

Weak energy condition (WEC).—The pointwise WEC is satisfied by a matter field if, for any timelike vector V^μ, we have $T_{\mu\nu}V^\mu V^\nu \geq 0$, which is equivalent to the restrictions $\rho + p_i \geq 0$ and $\rho \geq 0$. Then, the WEC is equivalent to the NEC with the constraint $\rho \geq 0$ added.

Strong energy condition (SEC).—A matter field satisfies the SEC if $\rho + p_i \geq 0$ and $\rho + \sum_i p_i \geq 0$, which is equivalent to the NEC with the constraint $\rho - p_r + 2p \geq 0$ added.

Dominant energy condition (DEC).—It is satisfied by a matter field whose locally measured energy density is positive and the energy flux is timelike or null. This is equivalent to the restriction $\rho \geq 0$ and $-\rho \leq p_i \leq \rho$.

As it is difficult to analyze the fulfillment of the above pointwise energy conditions in the case of a quantum massive scalar field on a global monopole spacetime for the case of arbitrary ξ, we will restrict our analysis to the physical cases of minimal and conformal couplings.

Rescaling the energy density and pressures of the quantum massive scalar field as $\varrho = 96\pi^2\mu^2\rho$ and $\mathrm{p}_i = 96\pi^2\mu^2 p_i$, respectively, the results obtained for the renormalized stress energy tensor indicate that, in the minimally coupled case, we have the relations $\varrho > 0$ and

$$\varrho - \mathrm{p}_r = \frac{1.59\times 10^{-4}}{r^6}, \tag{44}$$

$$\varrho + \mathrm{p} = -\frac{3.71\times 10^{-5}}{r^6}, \tag{45}$$

$$\varrho - \mathrm{p}_r + 2\mathrm{p} = -\frac{1.03\times 10^{-4}}{r^6}. \tag{46}$$

Also, in the conformally coupled case we have $\varrho > 0$ and

$$\varrho - \mathrm{p}_r = \frac{4.44 \times 10^{-6}}{r^6}, \quad (47)$$

$$\varrho + \mathrm{p} = -\frac{6.03 \times 10^{-6}}{r^6}, \quad (48)$$

$$\varrho - \mathrm{p}_r + 2\mathrm{p} = -\frac{9.52 \times 10^{-6}}{r^6}. \quad (49)$$

The above results show that, for both minimal and conformally coupled cases, we have $\varrho > 0$, $\varrho - \mathrm{p}_r > 0$, $\varrho + \mathrm{p} < 0$, and $\varrho - \mathrm{p}_r + 2\mathrm{p} < 0$ for all values of r. Then, outside the pointlike global monopole's core, all the pointwise energy conditions are violated by the quantum massive scalar field.

VI. CONCLUSIONS

In this paper, we construct the approximate renormalized stress energy tensor $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$, for a quantum massive scalar field with arbitrary coupling to gravity in the spacetime of a pointlike global monopole. Using the leading term in the Schwinger-DeWitt expansion for the Green's function associated with the Klein-Gordon dynamical operator, we find analytical expressions for the one-loop effective action, as an expansion in powers of the inverse squared mass of the field. Then, by functional differentiation of the effective action with respect to the metric tensor, we find an analytic expression for $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$, valid for a generic spacetime background.

The results obtained for the renormalized stress energy tensor of the quantized massive scalar field in the global monopole background show that, for all values of the distance r from the monopole's core, the quantum massive scalar field violates all the pointwise energy conditions for the minimal and conformally coupled cases.

Our calculations are a sequel of previous work in which, using the Schwinger-DeWitt proper time formalism, we study vacuum polarization of massive fields in a pointlike global monopole's background, constructing the analytic formula for the renormalized vacuum expectation value of the square of the field amplitude $\langle\phi^2\rangle$ [19].

As a check of the result presented in this paper, we can use the known fact that, for a conformally coupled scalar field, there exists a relation between the field fluctuation of the scalar field and the trace anomaly of the stress energy tensor. As showed by Anderson in Ref. [36], we have for the trace $\langle T^\nu_\nu\rangle$ the expression

$$\langle T^\nu_\nu\rangle = \frac{[a_2]}{16\pi^2} - \mu^2\langle\phi^2\rangle. \quad (50)$$

In Ref. [19], we obtain for the trace of the renormalized stress tensor of the massive scalar field in the pointlike global monopole spacetime the result

$$\langle T^\nu_\nu\rangle = \frac{25\alpha^6 - 21\alpha^2 - 4}{22680\pi^2\mu^2 r^6}. \quad (51)$$

We can easily show that the above result coincides exactly with the trace of $\langle T^\nu_\mu\rangle_{\mathrm{ren}}$ obtained using the general results presented in this paper, which is an indication of the validity of the results reported here.

An interesting problem to be addressed is the study of the backreaction effects of the quantized massive fields upon the spacetime geometry around the monopole. Using the analytic expressions for the quantum stress tensor reported here, we can solve perturbatively the semiclassical Einstein's field equations. To this specific problem, we will dedicate a future report.

ACKNOWLEDGMENTS

This work has been supported by a TWAS-CONACYT fellowship, under Contract No. FONCICYT/33/2018, that allowed the author to do a sabbatical leave at Departamento de Física Teórica, División de Ciencias e Ingenierías, Universidad de Guanajuato, Campus León. The author also express his gratitude to Professor Oscar Loaiza Brito, for the support during the research stay at his group, where this work was completed.

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] B. S. DeWitt, Phys. Rep. **19**, 295 (1975).
[3] N. D. Birrell and P. C. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[4] J. W. York, Phys. Rev. D **31**, 775 (1985); D. Page, Phys. Rev. D **25**, 1499 (1982).
[5] C. O. Loustó and N. Sanchez, Phys. Lett. B **212**, 411 (1988).
[6] P. Candelas, Phys. Rev. D **21**, 2185 (1980).
[7] L. A. Koffman and V. Sahni, Phys. Lett. **127B**, 197 (1983).
[8] V. P. Frolov and A. I. Zelnikov, Phys. Lett. **115B**, 372 (1982); **123B**, 197 (1983); Phys. Rev. D **29**, 1057 (1984).
[9] I. G. Avramidi, Nucl. Phys. **B355**, 712 (1991); Ph.D. thesis, Moscow State University, arXiv:hep-th/9510140.
[10] P. R. Anderson, W. A. Hisckock, and D. A. Samuel, Phys. Rev. D **51**, 4337 (1995).
[11] J. Matyjasek, Phys. Rev. D **61**, 124019 (2000).
[12] J. Matyjasek, Phys. Rev. D **63**, 084004 (2001).

[13] W. Berej and J. Matyjasek, Acta Phys. Pol. B **34**, 3957 (2003).
[14] O. P. F. Piedra and A. C. M. de Oca, Phys. Rev. D **75**, 107501 (2007).
[15] O. P. F. Piedra and A. C. M. de Oca, Phys. Rev. D **77**, 024044 (2008).
[16] Y. Décanini and A. Folacci, Classical Quantum Gravity **24**, 4777 (2007).
[17] A. Flachi and T. Tanaka, Phys. Rev. D **78**, 064011 (2008).
[18] A. Popov, Phys. Rev. D **70**, 084047 (2004).
[19] O. P. F. Piedra, arXiv:1904.02367.
[20] M. Barriola and A. Vilenkin, Phys. Rev. Lett. **63**, 341 (1989).
[21] W. Hiscock, Classical Quantum Gravity **7**, L235 (1990).
[22] F. D. Mazziteli and C. Lousto, Phys. Rev. D **43**, 468 (1991).
[23] F. C. Carvalho and E. R. Bezerra de Mello, Classical Quantum Gravity **18**, 1637 (2001).
[24] E. R. Bezerra de Mello, V. B. Bezerra, and N. R. Khusnutdinov, Phys. Rev. D **60**, 063506 (1999).
[25] E. R. Bezerra de Mello, Braz. J. Phys. **31**, 211 (2001).
[26] J. Matyjasek, D. Tryniecki, and K. Zwierzchowska, Phys. Rev. D **81**, 124047 (2010).
[27] W. A. Hiscock, Phys. Rev. Lett. **64**, 344 (1990).
[28] P. S. Bennet and S. H. Rhie, Phys. Rev. Lett. **65**, 1709 (1990).
[29] J. Matyjasek and P. Sadurski, Phys. Rev. D **91**, 044027 (2015).
[30] P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975).
[31] A. C. Ottewill and B. Wardell, Phys. Rev. D **84**, 104039 (2011).
[32] J. M. Matin Garcia, Comput. Phys. Commun. **179**, 597 (2008).
[33] A. Belokogne and A. Folacci, Phys. Rev. D **90**, 044045 (2014).
[34] M. Visser, Phys. Rev. D **54**, 5116 (1996).
[35] J. Matyjasek, Acta Phys. Pol. B **34**, 3921 (2003).
[36] P. R. Anderson, Phys. Rev. D **39**, 3785 (1989); **41**, 1152 (1990).