Deflection angle of light for an observer and source at finite distance from a rotating global monopole

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By using a method improved with a generalized optical metric, the deflection of light for an observer and source at finite distance from a lens object in a stationary, axisymmetric and asymptotically flat spacetime has been recently discussed [Ono, Ishihara, Asada, Phys. Rev. D **96**, 104037 (2017)]. In this paper, we study a possible extension of this method to an asymptotically nonflat spacetime. We discuss a rotating global monopole. Our result of the deflection angle of light is compared with a recent work on the same spacetime but limited within the asymptotic source and observer [Jusufi *et al.*, Phys. Rev. D **95**, 104012 (2017)], in which they employ another approach proposed by Werner with using the Nazim's osculating Riemannian construction method via the Randers-Finsler metric. We show that the two different methods give the same result in the asymptotically far limit. We obtain also the corrections to the deflection angle due to the finite distance from the rotating global monopole. Near-future observations of Sgr A * will be able to put a bound on the global monopole parameter β as $1-\beta < 10^{-3}$ for a rotating global monopole model, which is interpreted as the bound on the deficit angle $\delta < 8 \times 10^{-4}$ [rad].

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I. INTRODUCTION

Since the experimental confirmation of the theory of general relativity [1] succeeded in 1919 [2], a lot of calculations of the gravitational bending of light have been done not only for black holes [3] but also for other objects such as wormholes and gravitational monopoles [4]. Gibbons and Werner (2008) proposed an alternative way of deriving the deflection angle of light [5]. They assumed that the source and receiver are located at an asymptotic Minkowskian region, and they used the Gauss-Bonnet theorem to a spatial domain described by the optical metric, for which a light ray is described as a spatial curve. Ishihara *et al.* have recently extended Gibbons and Werner's idea in order to investigate finite-distance corrections in the small deflection case (corresponding to a large impact parameter case) [6] and also in the strong deflection limit for which the photon orbits may have the winding number larger than unity [7]. In particular, the asymptotic receiver and source have not been assumed. Our method and Werner's one are limited within asymptotically flat spacetimes.

In this paper, we discuss an extension of our method applied to a rotating global monopole. Due to the existence of a deficit solid angle, the spacetime is not asymptotically flat. A static solution of a global monopole was found in a paper by Barriola and Vilenkin [8]. According to their model, global monopoles are configurations whose energy density decreases with the distance as r^{-2} and whose spacetimes exhibit a solid deficit angle given by $\delta = 8\pi^2\eta^2$, where η is the scale of gauge-symmetry breaking. Recently, global monopoles have been discussed as spacetimes with a cosmological constant, e.g., in [9]. Static spherically symmetric composite global-local monopoles have also been studied [10]. Gravitational lensing in spacetimes with a nonrotating global monopole has been intensively investigated, for instance by Cheng and Man [11] who studied strong gravitational lensing of a Schwarzschild black hole with a solid deficit angle owing to a global monopole. More recently, it has also been proposed that gravitational microlensing by a global monopole may even be used to test Verlinde's emergent gravity theory [12]. As mentioned above, we investigate a possible extension of our method to stationary, axisymmetric spacetimes with a solid deficit angle, especially in order to examine finite-distance corrections to the deflection angle of light. The geometrical setups in the present paper are not those in the optical geometry, in the sense that the photon orbit has a nonvanishing geodesic curvature, though the light ray in the four-dimensional spacetime obeys a null geodesic.

This paper is organized as follows. Section II discusses a generalized optical metric for a rotating global monopole. Section III discusses how to define the deflection angle of light in a stationary, axisymmetric spacetime with the deficit angle. In particular, it is shown that the proposed definition of the deflection angle is also coordinate-invariant by using the Gauss-Bonnet theorem. We discuss also how to compute the gravitational deflection angle of light by the proposed method. Section IV is devoted to the conclusion. Throughout this paper, we use the unit of $G = c = 1$, and the observer may be called the receiver in order to avoid a confusion between r_O and r_0 by using r_R.

II. GENERALIZED OPTICAL METRIC FOR ROTATING GLOBAL MONOPOLE

A. Rotating global monopole

By applying the method of complex coordinate transformation, an extension of the static global monopole solution to a rotating global monopole spacetime was described by Teixeira Filho and Bezerra in Ref. [13].

Its spacetime metric reads

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -\left(1-\frac{2Mr}{r^2+a^2\cos^2\theta}\right)dt^2 + \left[\frac{r^2-a^2\{(1-\beta^2)\sin^2\theta-\cos^2\theta\}}{r^2-2Mr+a^2} - (1-\beta^2)\frac{a^2\sin^2\theta\{2Mr-a^2(1-\sin^4\theta)\}}{(r^2-2Mr+a^2)^2}\right]dr^2 \\ &\quad + \beta^2(r^2+a^2\cos^2\theta)d\theta^2 + \sin^2\theta\frac{[\beta^2r^4+\{1-(1-2\beta^2)\cos^2\theta\}a^2r^2+2Ma^2r\sin^2\theta+a^4\cos^2\theta(\beta^2\cos^2\theta+\sin^2\theta)]}{r^2+a^2\cos^2\theta}d\phi^2 \\ &\quad - \frac{4aMr\sin^2\theta}{r^2+a^2\cos^2\theta}dtd\phi + 2(1-\beta^2)\frac{a\{r^2\sin^2\theta-a^2\cos^2\theta(1+\cos^2\theta)\}}{r^2-2Mr+a^2}drd\phi, \end{aligned} \tag{1}$$

where the coordinates are $-\infty < t < +\infty$, $2M \le r < +\infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. We denote

$$\beta^2 = 1 - 8\pi\eta^2, \tag{2}$$

where η is the scale of a gauge-symmetry breaking.

The rotating global monopole by Eq. (1) is a rotating generalization of the global monopole black hole in Ref. [14]. Here, M denotes the global monopole core mass. The parameter a is the total angular momentum of the global monopole, which gives rise to the Lense-Thirring effect in general relativity, and the parameter β is called the global monopole parameter of the spacetime where β satisfies $0 < \beta \le 1$.

B. Generalized optical metric

By following Ref. [15], we define the generalized optical metric γ_{ij} $(i, j = 1, 2, 3)$ by a relation as

$$dt = \sqrt{\gamma_{ij}dx^i dx^j} + \beta_i dx^i, \tag{3}$$

which is directly obtained by solving the null condition $(ds^2 = 0)$ for dt. Note that γ_{ij} is not the induced metric in the Arnowitt-Deser-Misner (ADM) formalism. We define a three-dimensional space ${}^{(3)}M$ by the generalized optical metric $\gamma_{ij}dx^i dx^j$.

For the rotating global monopole by Eq. (1), we find the components of the generalized optical metric as

$$\begin{aligned} \gamma_{ij}dx^i dx^j &= \frac{(a^2\cos^2\theta+r^2)}{[a^2+r(r-2M)]^2[a^2\cos^2\theta+r(r-2M)]} \\ &\quad \times [a^4(\beta^2-1)\sin^6\theta + a^2\{a^2+r(r-2M)\}\cos^2\theta + a^2r^2(\beta^2-1)\sin^2\theta + (a^2-2Mr+r^2)r^2]dr^2 \\ &\quad + \frac{\beta^2(a^2\cos^2\theta+r^2)^2}{a^2\cos^2\theta+r(r-2M)}d\theta^2 + \frac{2a(1-\beta^2)[r^2\sin^2\theta-a^2\cos^2\theta(\cos^2\theta+1)]}{[a^2+r(r-2M)](1-\frac{2Mr}{a^2\cos^2\theta+r^2})}drd\phi \\ &\quad + \frac{\sin^2\theta(a^2\cos(2\theta)+a^2+2r^2)^2[a^2(\beta^2-1)\cos(2\theta)+a^2(\beta^2+1)+2\beta^2r(r-2M)]}{8[r(r-2M)+a^2\cos^2\theta]^2}d\phi^2. \end{aligned} \tag{4}$$

We obtain the components of β_i as

$$\beta_i dx^i = -\frac{2aMr\sin^2\theta}{a^2\cos^2\theta+r(r-2M)}d\phi. \tag{5}$$

In the rest of the paper, we focus on the light rays in the equatorial plane, namely $\theta = \pi/2$. Note that the generalized optical metric γ_{ij} does not mean an asymptotically flat space, because there is the deficit angle of spacetime (if $\beta \ne 1$).

III. DEFLECTION ANGLE OF LIGHT BY A ROTATING GLOBAL MONOPOLE

A. Deflection angle of light in asymptotically flat spacetimes

Let us begin this section with briefly summarizing the generalized optical metric method that enables us to calculate the deflection angle of light for a nonasymptotic receiver (denoted as R) and source (denoted as S) [15].

We define the deflection angle of light as [15]

$$\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}. \tag{6}$$

Here, Ψ_R and Ψ_S are angles between the light ray tangent and the radial direction from the lens object, defined in a covariant manner using the generalized optical metric, at the receiver location and the source, respectively. On the other hand, ϕ_{RS} is the coordinate angle between the receiver and source, where the coordinate angle is associated with the rotational Killing vector in the spacetime. If the space under study is Euclidean, this α becomes the deflection angle of the curve. This is consistent with the thin lens approximation in the standard theory of gravitational lensing.

By using the Gauss-Bonnet theorem as [16]

$$\iint_{{}^{\infty}_{R}\square^{\infty}_{S}} K dS + \oint_{\partial T} \kappa_g d\ell + \sum_{i=1}^{n} \Theta_i = 2\pi. \tag{7}$$

Equation (6) can be recast into [15]

$$\alpha = -\iint_{{}^{\infty}_{R}\square^{\infty}_{S}} K dS + \int_S^R \kappa_g d\ell, \tag{8}$$

where K is defined as the Gaussian curvature at some point on the two-dimensional surface, dS denotes the infinitesimal surface element defined with γ_{ij}, ${}^{\infty}_{R}\square^{\infty}_{S}$ denotes a quadrilateral embedded in a curved space with γ_{ij}, κ_g denotes the geodesic curvature of the light ray in this space and $d\ell$ is an arc length defined with the generalized optical metric (see Fig. 2 in Ref. [15]). It is shown by Asada and Kasai that this $d\ell$ for the light ray is an affine parameter [17].

B. Deflection angle of light in spacetimes with a deficit angle

When we consider the deflection angle of light in a spacetime with the deficit angle, we follow Refs. [6,7,15] to use the definition of deflection angle of light as

$$\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}. \tag{9}$$

In the rest of the present paper, we show that the deficit angle contribution to the deflection angle of light can be included.

Note that the surface integral and path integral terms appear in the right-hand side of Eq. (8) if $\beta_i = 0$ (see [6]). However, in the rotating global monopole, Eq. (8) is modified by the deficit angle. Equation (8) is calculated as

$$\iint_{{}^{\infty}_{R}\square^{\infty}_{S}} K dS + \int_{r_\infty}^{R} \tilde{\kappa}_g d\ell + \int_S^{r_\infty} \dot{\kappa}_g d\ell - \int_{C_r} \bar{\kappa}_g d\ell + \int_{C_\infty} \kappa_g d\ell + \Psi_R + (\pi - \Psi_S) + \pi = 2\pi, \tag{10}$$

which is rewritten as

$$\iint_{{}^{\infty}_{R}\square^{\infty}_{S}} K dS + \int_{r_\infty}^{R} \tilde{\kappa}_g d\ell + \int_S^{r_\infty} \dot{\kappa}_g d\ell - \int_{C_r} \bar{\kappa}_g d\ell + \beta\phi_{RS} + \Psi_R - \Psi_S = 0, \tag{11}$$

where $\tilde{\kappa_g}$ is a geodesic curvature along the radial line from the infinity to the receiver, $\dot{\kappa_g}$ is a geodesic curvature along the radial line from the source to the infinity, $\bar{\kappa_g}$ is a geodesic curvature along the light ray from the source to the receiver and κ_g is a geodesic curvature along the path C_∞. The path C_r is a light ray from the receiver to the source in a generalized optical metric, C_∞ is a circular arcsegment of a radius $R \gg r_R, r_S$, and we use $d\ell = \sqrt{1 + \frac{4M}{r}} dr = \{1 + \mathcal{O}(M/r)\} dr$ along the radial line. We shall explain in more detail this calculation in Sec. III D 3. Therefore, the deflection angle of light by the rotating global monopole is rewritten as

$$\alpha = -\iint_{{}^{\infty}_{R}\square^{\infty}_{S}} K dS + \int_R^{r_\infty} \tilde{\kappa}_g d\ell - \int_S^{r_\infty} \dot{\kappa}_g d\ell + \int_{C_r} \bar{\kappa}_g d\ell + (1 - \beta)\phi_{RS}, \tag{12}$$

where we use Eqs. (9) and (11). The deflection angle is also a coordinate-invariant in the spacetimes with deficit angle, because Ψ_R and Ψ_S are obtained by the inner product at a receiver and a source respectively.

We have two ways in order to calculate the deflection angle of light. We shall make detailed calculations of the right-hand side of Eq. (12) and the right-hand side of Eq. (9) below.

C. Gaussian curvature

For the equatorial case of a rotating global monopole, the Gaussian curvature in the weak field approximation is calculated as

$$
\begin{aligned}
K &= \frac{R_{r\phi r\phi}}{\det\gamma_{ij}^{(2)}} \\
&= \frac{1}{\sqrt{\det\gamma_{ij}^{(2)}}}\left[\frac{\partial}{\partial\phi}\left(\frac{\sqrt{\det\gamma_{ij}^{(2)}}}{\gamma_{rr}^{(2)}}\Gamma^{\phi}_{rr}\right)-\frac{\partial}{\partial r}\left(\frac{\sqrt{\det\gamma_{ij}^{(2)}}}{\gamma_{rr}^{(2)}}\Gamma^{\phi}_{r\phi}\right)\right] \\
&= \left[-\frac{2}{r^3}-\frac{6}{r^5\beta^2}a^2\right]M+\frac{3}{r^4}M^2+\mathcal{O}(M^3/r^5), \qquad (13)
\end{aligned}
$$

where $\gamma_{ij}^{(2)}$ denotes the two-dimensional generalized optical metric in the equatorial plane $\theta=\pi/2$. Here, a and M are dimensional quantities that can be used as book-keeping symbols in iterative calculations under the weak field approximation. As for the first line of Eq. (13), please see e.g., the p. 263 in Ref. [18]. We note that the first term in the second line of Eq. (13) does not contribute because $\Gamma^{\phi}_{rr}=0$. It is not surprising that this Gaussian curvature does not agree with Eq. (26) in Jusufi, *et al.* [19], because their Gaussian curvature describes another surface that is associated with the Randers-Finsler metric different from our optical metric, though the same four-dimensional spacetime is considered by two groups.

In order to perform the surface integral of the Gaussian curvature in Eq. (8), we have to know the boundary shape of the integration domain. In other words, we need to describe the light ray as a function of $r(\phi)$. For the later convenience, we introduce the inverse of r as $u\equiv r^{-1}$. The orbit equation in this case becomes

$$
\begin{aligned}
&[1+2Mu]\left(\frac{du}{d\phi}\right)^2-\left[\frac{2a(1-\beta^2)(b^2u^2-\beta^2)}{b^2}-\frac{4aMu(\beta^2-1)(b^2u^2-2\beta^2)}{b^2}\right]\frac{du}{d\phi} \\
&+\left[\left(\beta^2u^2-\frac{\beta^4}{b^2}\right)+\left\{-\frac{2\beta^4u}{b^2}+\frac{4a\beta^4u}{b^3}\right\}M\right]+\mathcal{O}(a^2u^2,M^2u^2)=0, \qquad (14)
\end{aligned}
$$

where b is the impact parameter of the photon. See e.g., Ref. [15] on how to obtain the photon orbit equation in the axisymmetric and stationary spacetime. The orbit equation is iteratively solved as

$$
\begin{aligned}
u(\phi) &= \frac{\beta}{b}\sin\{\beta\phi+\phi_0(1-\beta)\}+[\beta^2+\beta^2\cos^2\{\beta\phi+\phi_0(1-\beta)\}]\frac{M}{b^2}+\frac{\beta(\beta^2-1)\sin[2\{\beta\phi+\phi_0(1-\beta)\}]}{2b^2}a \\
&+\frac{\beta^2[-4+(-1+\beta^2)\cos\{\beta\phi+\phi_0(1-\beta)\}+(-1+\beta^2)\cos\{3\beta\phi+3\phi_0(1-\beta)\}]}{2b^3}aM+\mathcal{O}(M^2/b^3). \qquad (15)
\end{aligned}
$$

The area element of the equatorial plane dS is

$$
dS=\sqrt{\det\gamma_{ij}^{(2)}}drd\phi=\sqrt{\beta^2r^2+\mathcal{O}(Mr)}drd\phi=\{\beta r+\mathcal{O}(M)\}drd\phi. \qquad (16)
$$

By using Eq. (15) as the iterative solution for the photon orbit, the surface integral of the Gaussian curvature in Eq. (8) is calculated as

$$
\begin{aligned}
-\iint_{{}_R^\infty\square_S^\infty} KdS &= \int_\infty^{r(\phi)}dr\int_{\phi_S}^{\phi_R}d\phi\left(-\frac{2M}{r^3}\right)r\beta+\mathcal{O}(M^3/b^3,a^2M^3/b^5,a^4M^2/b^6) \\
&= \int_0^{u(\phi)}du\int_{\phi_S}^{\phi_R}d\phi(2M\beta)+\mathcal{O}(M^3/b^3,a^2M^3/b^5,a^4M^2/b^6) \\
&= 2M\beta\int_{\phi_S}^{\phi_R}d\phi\left(\frac{\beta}{b}\sin\{\beta\phi+\phi_0(1-\beta)\}+\frac{\beta(\beta^2-1)\sin[2\{\beta\phi+\phi_0(1-\beta)\}]}{2b^2}a\right) \\
&= \frac{2M\beta}{b}\left[\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}+\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}\right]+\frac{aM(1-\beta^2)}{\beta}[{u_R}^2-{u_S}^2]+\mathcal{O}(M^3/b^3), \qquad (17)
\end{aligned}
$$

where u_R and u_S are the inverse of r_R and r_S, respectively, and we used

$$
\sin\{\beta\phi_S+\phi_0(1-\beta)\}=\frac{bu_S}{\beta}+\frac{(1-\beta^2)\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}}{\beta}u_Sa-\frac{\beta(2-\frac{b^2{u_S}^2}{\beta^2})}{b}M+\mathcal{O}(aM/b^2) \qquad (18)
$$

and

$$\sin\{\beta\phi_R + \phi_0(1-\beta)\} = \frac{bu_R}{\beta} - \frac{(1-\beta^2)\sqrt{1-\frac{b^2u_R{}^2}{\beta^2}}}{\beta}u_R a - \frac{\beta(2-\frac{b^2u_R{}^2}{\beta^2})}{b}M + \mathcal{O}(aM/b^2) \tag{19}$$

by Eq. (15) in the last line.

D. Geodesic curvature

1. Light ray in optical metric

The geodesic curvature plays an important role in our calculations of the light deflection, though it is not usually described in standard textbooks on general relativity. Hence, we follow Ref. [15] to briefly explain the geodesic curvature here. The geodesic curvature can be defined in the vector form as (e.g., [20])

$$\kappa_g \equiv \vec{T}' \cdot (\vec{T} \times \vec{N}), \tag{20}$$

where we assume a parametrized curve with a parameter ℓ, $\vec{T}$ is the unit tangent vector for the curve by reparametrizing the curve using its arc length ℓ, $\vec{T}'$ is its derivative with respect to the arc length, and $\vec{N}$ is the unit normal vector for the surface. Equation (20) can be rewritten in the tensor form as

$$\kappa_g = \epsilon_{ijk}N^i a^j e^k, \tag{21}$$

where $\vec{T}$ and $\vec{T}'$ are denoted by e^k and a^j, respectively. Here, the Levi-Civita tensor ϵ_{ijk} is defined by $\epsilon_{ijk} \equiv \sqrt{\gamma}\varepsilon_{ijk}$, where $\gamma \equiv \det(\gamma_{ij})$, and ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = 1$). In the present paper, we use γ_{ij} in the above definitions but not g_{ij}. Note that $a^i \neq 0$ in the three-dimensional optical metric by a nonvanishing g_{0i} [15], even though the light signal follows a geodesic in the four-dimensional spacetime. On the other hand, we emphasize that $a^i = 0$ and thus $\kappa_g = 0$ for the geodesics in the optical metric, because $\beta_i = 0$.

As shown first in Ref. [15], Eq. (21) is rewritten in a convenient form as

$$\kappa_g = -\epsilon^{ijk}N_i\beta_{j|k}, \tag{22}$$

where we use $\gamma_{ij}e^ie^j = 1$.

Let us denote the unit normal vector to the equatorial plane as N_p. Therefore, it satisfies $N_p \propto \nabla_p\theta = \delta_p^\theta$, where ∇_p is the covariant derivative associated with γ_{ij}. Hence, N_p is written in a form as $N_p = N_\theta\delta_p^\theta$. By noting that N_p is a unit vector ($N_pN_q\gamma^{pq} = 1$), we obtain $N_\theta = \pm 1/\sqrt{\gamma^{\theta\theta}}$. Therefore, N_p can be expressed as

$$N_p = \frac{1}{\sqrt{\gamma^{\theta\theta}}}\delta_p^\theta, \tag{23}$$

where we choose the upward direction without loss of generality.

For the equatorial case, one can show

$$\epsilon^{\theta pq}\beta_{q|p} = -\frac{1}{\sqrt{\gamma}}\beta_{\phi,r}, \tag{24}$$

where the comma denotes the partial derivative, we use $\epsilon^{\theta r\phi} = -1/\sqrt{\gamma}$ and we note $\beta_{r,\phi} = 0$ owing to the axisymmetry. By using Eqs. (23) and (24), the geodesic curvature of the light ray with the generalized optical metric becomes [15]

$$\kappa_g = -\sqrt{\frac{1}{\gamma\gamma^{\theta\theta}}}\beta_{\phi,r}. \tag{25}$$

For the global monopole case, this is obtained as

$$\overline{\kappa_g} = -\frac{2}{\beta r^3}aM - \frac{2}{\beta r^4}aM^2 + \mathcal{O}(aM^3/r^5). \tag{26}$$

We examine the contribution from the geodesic curvature. This contribution is the path integral along the light ray (from the source to the receiver), which is computed as

$$\begin{aligned}
\int_{C_r}\bar{\kappa}_g d\ell &= -\int_S^R \frac{2}{\beta r^3}aMd\ell + \mathcal{O}(a^3M/b^4) = -\int_{\phi_S}^{\phi_R}\frac{2}{\beta r^3}aM\frac{b}{\cos^2\{\beta\vartheta + \phi_0(1-\beta)\}}d\vartheta + \mathcal{O}(aM^2/b^4)\\
&= -\frac{2}{\beta}aM\int_{\phi_S}^{\phi_R}\left(\frac{\beta\cos\{\beta\vartheta + \phi_0(1-\beta)\}}{b}\right)^3\frac{b}{\cos^2\{\beta\vartheta + \phi_0(1-\beta)\}}d\vartheta + \mathcal{O}(aM^2/b^4)\\
&= -\frac{2\beta^2}{b^2}aM\int_{\phi_S}^{\phi_R}\cos\{\beta\vartheta + \phi_0(1-\beta)\}d\vartheta + \mathcal{O}(aM^2/b^4)\\
&= -\frac{2aM\beta}{b^2}[\sin\{\beta\phi_R + \phi_0(1-\beta)\} - \sin\{\beta\phi_S + \phi_0(1-\beta)\}] + \mathcal{O}(aM^2/b^4)\\
&= -\frac{2aM\beta}{b^2}\left[\sqrt{1-\frac{b^2u_R{}^2}{\beta^2}} + \sqrt{1-\frac{b^2u_S{}^2}{\beta^2}}\right] + \mathcal{O}(aM^2/b^4),
\end{aligned} \tag{27}$$

where we use $d\ell = \frac{\beta^2 r^2}{b} d\vartheta + \mathcal{O}(b^2/r^2, M)$, $u = \frac{\beta \cos\{\beta\vartheta + \phi_0(1-\beta)\}}{b} + \mathcal{O}(a/b, M/b)$. In the last line, we used $\sin\{\beta\phi_R + \phi_0(1-\beta)\} = \sqrt{1 - \frac{b^2 {u_R}^2}{\beta^2}} + \mathcal{O}(au_R, Mu_R)$ and $\sin\{\beta\phi_S + \phi_0(1-\beta)\} = -\sqrt{1 - \frac{b^2 {u_S}^2}{\beta^2}} + \mathcal{O}(au_S, Mu_S)$ by Eq. (15). The sign of the right-hand side of Eq. (27) changes, if the photon orbit is retrograde.

2. Radial lines in the generalized optical metric

The unit tangent vector along a radius line in ${}^{(3)}M$ is $R^i = (R^r, 0, 0)$. On the equatorial plane, from

$$\gamma_{ij} R^i R^j = \gamma_{rr} (R^r)^2 = 1, \tag{28}$$

we obtain

$$R^r = \frac{1}{\sqrt{\gamma_{rr}}}. \tag{29}$$

The acceleration vector a^i along this line is

$$a^i = R^i{}_{|j} R^j. \tag{30}$$

Its explicit form is

$$a^i = \left(\frac{1}{2} \left(\frac{\partial}{\partial r} \frac{1}{\gamma_{rr}} \right) + \frac{\gamma^{rr}}{2\gamma_{rr}} \frac{\partial \gamma_{rr}}{\partial r} + \frac{\gamma^{r\phi}}{\gamma_{rr}} \frac{\partial \gamma_{r\phi}}{\partial r}, 0, \gamma^{\phi\phi} \frac{\partial \gamma_{r\phi}}{\partial r} + \frac{\gamma^{r\phi}}{2} \frac{\partial \gamma_{rr}}{\partial r} \right). \tag{31}$$

Here, the vector a^i $(i = r, \theta, \phi)$ becomes

$$\begin{aligned} a^r &= \frac{2(\beta^2 - 1)^2}{\beta^2 r^4} a^2 M + \mathcal{O}(a^4/r^5), \\ a^\theta &= 0, \\ a^\phi &= \frac{2(\beta^2 - 1)}{\beta^2 r^4} aM + \mathcal{O}(a^3/r^5, aM^2/r^5). \end{aligned} \tag{32}$$

This means that a^i is zero vector in Kerr or Schwarzschild cases ($\beta = 1$).

From Eq. (23), we obtain

$$N^i = \left(0, \frac{1}{\sqrt{\gamma_{\theta\theta}}}, 0 \right). \tag{33}$$

By using Eqs. (21), (29), (31) and (33), an explicit form of κ_g is obtained as

$$\kappa_g = \epsilon_{ijk} N^i a^j R^k = \sqrt{\frac{\gamma}{\gamma_{rr}\gamma_{\theta\theta}}} \left(\gamma^{\phi\phi} \frac{\partial \gamma_{r\phi}}{\partial r} + \frac{\gamma^{r\phi}}{2} \frac{\partial \gamma_{rr}}{\partial r} \right). \tag{34}$$

Moreover, by substituting functions of metric γ_{ij} into Eq. (34), we obtain κ_g as

$$\begin{aligned} \kappa_g = &-\sqrt{(r-2M)^2 \{a^2 + r(r-2M)\} \{a^2(\beta^2-1)(2Mr+1) + r^2\}} \\ &\times [a(\beta^2-1) r^2 \{a^4(\beta^2-1)(-8M^2 r + M(3r^2-5) + 2r) + a^2 r \{12(\beta^2-1) M^3 r \\ &- 8(\beta^2-1) M^2 (r^2-1) + Mr\{-6\beta^2 + (\beta^2-1) r^2 + 3\} + \beta^2 r^2\} + 2Mr^3(2M-r)\}] \\ &/[(r-2M)^2 \{a^2 + r(r-2M)\}^2 \{a^2(\beta^2-1)(2Mr+1) + r^2\} \\ &\times \{r^2(a^6(\beta^2-1)(2Mr+1) + a^4 r(-4(\beta^4-1) M^2 r + 2(\beta^4-1) M(r^2-1) + \beta^4 r) \\ &+ 2a^2 \beta^2 r^2 (2M-r)(2(\beta^2-1) M^2 r - (\beta^2-1) M(r^2-1) - r) + \beta^2 r^4 (r-2M)^2)\}^{1/2}]. \end{aligned} \tag{35}$$

This is approximated as

$$\kappa_g = \frac{2(\beta^2-1)}{\beta r^3} aM - \frac{\beta(\beta^2-1)}{r^4} a^3 + \frac{10(\beta^2-1)}{\beta r^4} aM^2 + \mathcal{O}(a^3 M/r^5, aM^3/r^5), \tag{36}$$

where this κ_g vanishes in Kerr or Schwarzschild spacetime ($\beta = 1$), since the acceleration vector a^i becomes 0.

Let us integrate the leading term of κ_g from the source to the infinity,

$$\int_S^{r_\infty} \frac{2(\beta^2-1)}{\beta r^3} aM d\ell = \int_{r_S}^{\infty} \frac{2(\beta^2-1)}{\beta r^3} aM dr = \frac{(\beta^2-1)aM}{\beta} \left[\frac{1}{r^2} \right]_{\infty}^{r_S} = -\frac{(1-\beta^2)aM}{\beta {r_S}^2} + \mathcal{O}(aM^2/{r_S}^3). \tag{37}$$

Similarly, the integral of κ_g from the receiver to the infinity is computed as

$$\int_R^{r_\infty} \frac{2(\beta^2-1)}{\beta r^3} aM d\ell = -\frac{(1-\beta^2)aM}{\beta {r_R}^2} + \mathcal{O}(aM^2/{r_R}^3), \tag{38}$$

where we use $d\ell = \sqrt{1+\frac{4M}{r}}dr = \{1+\mathcal{O}(M/r)\}dr$.

3. Geodesic curvature of circular arcsegment in optical metric

The orbital equation as Eq. (14) can be solved for $\frac{du}{d\phi}$ as

$$\frac{du}{d\phi} = \frac{1}{F_\pm(u)}, \tag{39}$$

where we denote

$$F_+(u) = \frac{1}{\beta\sqrt{{u_0}^2-u^2}} - \left(1-\frac{1}{\beta^2}\right)a + \frac{{u_0}^3-u^3}{\beta({u_0}^2-u^2)^{3/2}}M - 2uaM\left(1-\frac{1}{\beta^2}+\frac{{u_0}^3(u_0-u)}{u\beta^2({u_0}^2-u^2)^{3/2}}\right) + \mathcal{O}(a^2u, M^2u, aM^2u^2, M^3u^2), \tag{40}$$

$$F_-(u) = -\frac{1}{\beta\sqrt{{u_0}^2-u^2}} - \left(1-\frac{1}{\beta^2}\right)a - \frac{{u_0}^3-u^3}{\beta({u_0}^2-u^2)^{3/2}}M - 2uaM\left(1-\frac{1}{\beta^2}-\frac{{u_0}^3(u_0-u)}{u\beta^2({u_0}^2-u^2)^{3/2}}\right) + \mathcal{O}(a^2u, M^2u, aM^2u^2, M^3u^2). \tag{41}$$

For $\phi_0 > \phi > \phi_S$, we use $F_+(u)$, while we use $F_-(u)$ for $\phi_R > \phi > \phi_0$. Here, we use

$$b = \frac{\beta}{u_0} + \beta M - 2u_0 aM + \mathcal{O}(a^2u_0, M^2u_0, aM^2{u_0}^2, M^3{u_0}^2), \tag{42}$$

where u_0 is the inverse of the distance of closest approach.

At $r = r_\infty$ (r_∞ is an infinite constant radius of the circular arc segment), we obtain $d\ell^2 = {r_\infty}^2\beta^2 d\phi^2$, the geodesic curvature $\kappa_g = \frac{1}{r_\infty} + O(M/{r_\infty}^2)$. Let us integrate as

$$\beta\phi_{RS} = \int_S^R \kappa_g dl = \int_S^R \beta d\phi = \beta\int_{\phi_S}^{\phi_R} d\phi = \beta\int_{u_S}^{u_0} F_+(u)du + \beta\int_{u_0}^{u_R} F_-(u)du. \tag{43}$$

$$\begin{aligned}\int F_\pm(u)du &= \int\left\{\pm\frac{1}{\beta\sqrt{{u_0}^2-u^2}} - \left(1-\frac{1}{\beta^2}\right)a \pm \frac{{u_0}^3-u^3}{\beta({u_0}^2-u^2)^{3/2}}M - 2uaM\left(1-\frac{1}{\beta^2}\pm\frac{{u_0}^3(u_0-u)}{u\beta^2({u_0}^2-u^2)^{3/2}}\right)\right\}du \\ &= \pm\frac{1}{\beta}\arcsin\left(\frac{u}{u_0}\right) - \left(1-\frac{1}{\beta^2}\right)au \mp \frac{(2u_0+u)\sqrt{{u_0}^2-u^2}}{\beta(u_0+u)}M + \left\{\left(-1+\frac{1}{\beta^2}\right)u^2 \pm \frac{2{u_0}^2\sqrt{{u_0}^2-u^2}}{\beta^2(u_0+u)}\right\}aM \\ &\quad + \mathcal{O}(M^2/{u_0}^2).\end{aligned} \tag{44}$$

$$\begin{aligned}\phi_{RS} &= \frac{\pi}{\beta} - \frac{1}{\beta}\left\{\arcsin\left(\frac{bu_S}{\beta}\right) + \arcsin\left(\frac{bu_R}{\beta}\right)\right\} - \left(1-\frac{1}{\beta^2}\right)(u_R-u_S)a + \left\{\frac{(2-\frac{b^2{u_R}^2}{\beta^2})}{b\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}} + \frac{(2-\frac{b^2{u_S}^2}{\beta^2})}{b\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}}\right\}M \\ &\quad + \left\{-\left(1-\frac{1}{\beta^2}\right)({u_R}^2-{u_S}^2) - \frac{2}{b^2\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}} - \frac{2}{b^2\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}}\right\}aM + \mathcal{O}(M^2/b^2),\end{aligned} \tag{45}$$

where we use $u_0 = \frac{\beta}{b} + \frac{\beta^2 M}{b^2} - \frac{2\beta^2 aM}{b^3}$. This ϕ_{RS} becomes that for the Kerr case, only if one takes the limit $\beta \to 1$.

E. Jump angles

In the previous section, the unit tangent vector along the radius line in ${}^{(3)}M$ is obtained as

$$R^i = \left(\frac{1}{\sqrt{\gamma_{rr}}}, 0, 0\right), \tag{46}$$

the unit tangential vector along the spatial curve is also obtained as

$$e^i = \xi\left(\frac{dr}{d\phi}, 0, 1\right), \tag{47}$$

where

$$\xi_{+R} = \frac{b}{{r_R}^2\beta^2} - \frac{2b}{{r_R}^3\beta^2}M + \frac{(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}}{{r_R}^2\beta^2}a + \frac{2{r_R}^2\beta^2(1-\frac{b^2}{{r_R}^2\beta^2}) + b^2(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}}{{r_R}^5\beta^4(1-\frac{b^2}{{r_R}^2\beta^2})}aM + \mathcal{O}(M^2/{r_R}^3),$$

$$\xi_{-R} = -\frac{b}{{r_R}^2\beta^2} + \frac{2b}{{r_R}^3\beta^2}M - \frac{(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}}{{r_R}^2\beta^2}a - \frac{2{r_R}^2\beta^2(1-\frac{b^2}{{r_R}^2\beta^2}) + b^2(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}}{{r_R}^5\beta^4(1-\frac{b^2}{{r_R}^2\beta^2})}aM + \mathcal{O}(M^2/{r_R}^3),$$

$$\xi_{+S} = \frac{b}{{r_S}^2\beta^2} - \frac{2b}{{r_S}^3\beta^2}M - \frac{(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}}{{r_S}^2\beta^2}a + \frac{2{r_S}^2\beta^2(1-\frac{b^2}{{r_S}^2\beta^2}) - b^2(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}}{{r_S}^5\beta^4(1-\frac{b^2}{{r_S}^2\beta^2})}aM + \mathcal{O}(M^2/{r_S}^3),$$

$$\xi_{-S} = -\frac{b}{{r_S}^2\beta^2} + \frac{2b}{{r_S}^3\beta^2}M + \frac{(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}}{{r_S}^2\beta^2}a - \frac{2{r_S}^2\beta^2(1-\frac{b^2}{{r_S}^2\beta^2}) - b^2(-1+\beta^2)\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}}{{r_S}^5\beta^4(1-\frac{b^2}{{r_S}^2\beta^2})}aM + \mathcal{O}(M^2/{r_S}^3).$$

Here, ξ_+ means that e^i is the tangent vector of the prograde photon orbit, and ξ_- means that e^i is the tangent vector of the retrograde photon orbit. In addition, the subscripts S and R for $\xi_\pm$ mean from the source to the closest approach and from the receiver to the closest approach, respectively. Therefore, we can define the angle measured from the outgoing radial direction by

$$\begin{aligned}\cos\Psi_R &\equiv \gamma_{ij}e^iR^j = \gamma_{rr}e^rR^r + \gamma_{\phi r}e^\phi R^r \\ &= \sqrt{\gamma_{rr}}\xi_{+R}\frac{dr}{d\phi}\bigg|_+ + \frac{\gamma_{\phi r}}{\sqrt{\gamma_{rr}}}\xi_{+R} \\ &= \sqrt{1-\frac{b^2}{{r_R}^2\beta^2}} + \frac{b^2M}{{r_R}^3\beta^2\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}} + \frac{b(1-\beta^2)}{{r_R}^2\beta^2}a - \frac{2b}{{r_R}^3\beta^2\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}}aM + \mathcal{O}(M^2/{r_R}^2),\end{aligned} \tag{48}$$

$$\begin{aligned}-\cos(\pi-\Psi_S) &\equiv \gamma_{ij}e^iR^j = \gamma_{rr}e^rR^r + \gamma_{\phi r}e^\phi R^r \\ &= \sqrt{\gamma_{rr}}\xi_{+S}\frac{dr}{d\phi}\bigg|_- + \frac{\gamma_{\phi r}}{\sqrt{\gamma_{rr}}}\xi_{+S} \\ &= -\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}} - \frac{b^2M}{{r_S}^3\beta^2\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}} + \frac{b(1-\beta^2)}{{r_S}^2\beta^2}a + \frac{2b}{{r_S}^3\beta^2\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}}aM + \mathcal{O}(M^2/{r_S}^2),\end{aligned} \tag{49}$$

where Eq. (48) is at the receiver position and Eq. (49) is at the source. Therefore, Ψ_R and Ψ_S are obtained as

$$\Psi_R = \arcsin\left(\frac{b}{r_R\beta}\right) - \frac{bM}{{r_R}^2\beta\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}} + \frac{(\beta^2-1)a}{r_R\beta} + \frac{2+(\beta^2-1)\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}}{{r_R}^2\beta\sqrt{1-\frac{b^2}{{r_R}^2\beta^2}}}aM + \mathcal{O}(M^2/{r_R}^2), \tag{50}$$

$$\pi - \Psi_S = \arcsin\left(\frac{b}{r_S\beta}\right) - \frac{bM}{{r_S}^2\beta\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}} - \frac{(\beta^2-1)a}{r_S\beta} + \frac{2-(\beta^2-1)\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}}{{r_S}^2\beta\sqrt{1-\frac{b^2}{{r_S}^2\beta^2}}}aM + \mathcal{O}(M^2/{r_S}^2). \tag{51}$$

F. Deflection angle

By bringing together Eqs. (17), (27), (37), (38), (50) and (51), the deflection angle of light for the prograde case is obtained as

$$\begin{aligned}\alpha_{\mathrm{prog}} = {} & \left(\frac{1}{\beta}-1\right)\pi - \left(\frac{1}{\beta}-1\right)\left\{\arcsin\left(\frac{b^2{u_R}^2}{\beta^2}\right) + \arcsin\left(\frac{b^2{u_S}^2}{\beta^2}\right)\right\} \\ & + \frac{(\beta-1)^2(\beta+1)}{\beta^2}(u_R-u_S)a + \left\{\frac{2\beta-(1+\frac{1}{\beta})b^2{u_R}^2}{b\beta\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}} + \frac{2\beta-(1+\frac{1}{\beta})b^2{u_S}^2}{b\beta\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}}\right\}M \\ & - \left\{\frac{2(\beta-b^2{u_R}^2)}{b^2\beta\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}} + \frac{2(\beta-b^2{u_S}^2)}{b^2\beta\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}} - \frac{(\beta-1)^2(\beta+1)}{\beta^2}({u_R}^2-{u_S}^2)\right\}aM + \mathcal{O}(M^2/b^2).\end{aligned} \tag{52}$$

The deflection angle for the retrograde case is

$$\begin{aligned}\alpha_{\mathrm{retro}} = {} & \left(\frac{1}{\beta}-1\right)\pi - \left(\frac{1}{\beta}-1\right)\left\{\arcsin\left(\frac{b^2{u_R}^2}{\beta^2}\right) + \arcsin\left(\frac{b^2{u_S}^2}{\beta^2}\right)\right\} \\ & - \frac{(\beta-1)^2(\beta+1)}{\beta^2}(u_R-u_S)a + \left\{\frac{2\beta-(1+\frac{1}{\beta})b^2{u_R}^2}{b\beta\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}} + \frac{2\beta-(1+\frac{1}{\beta})b^2{u_S}^2}{b\beta\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}}\right\}M \\ & + \left\{\frac{2(\beta-b^2{u_R}^2)}{b^2\beta\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}} + \frac{2(\beta-b^2{u_S}^2)}{b^2\beta\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}} - \frac{(\beta-1)^2(\beta+1)}{\beta^2}({u_R}^2-{u_S}^2)\right\}aM + \mathcal{O}(M^2/b^2).\end{aligned} \tag{53}$$

If $\beta = 1$, Eqs. (52) and (53) agree with the known result for the weak field approximation of the Kerr spacetime in Ref. [15]. For both cases, the source and receiver may be located at finite distance from the monopole. As a matter of course, these results are also obtained by substituting Eqs. (45), (50) and (51) to Eq. (6). Equations (52) and (53) show that the light deflection is affected by a deficit angle.

One can see that, in the limit as $r_R \to \infty$ and $r_S \to \infty$, Eqs. (52) and (53) become

$$\begin{aligned}\alpha_{\mathrm{prog}} \to {} & \left(\frac{1}{\beta}-1\right)\pi + \frac{4M}{b} - \frac{4aM}{b^2} + \mathcal{O}\left(\frac{M^2}{b^2}\right) \\ = {} & \left(\frac{1}{\beta}-1\right)\pi + \frac{4M}{b_K\beta} - \frac{4aM}{(b_K\beta)^2} + \mathcal{O}\left(\frac{M^2}{{b_K}^2}\right) \\ = {} & \left(\frac{1}{\beta}-1\right)\pi + \frac{4M}{b_K} + \frac{16\pi\eta^2M}{b_K} - \frac{4aM}{{b_K}^2} \\ & - \frac{32\pi\eta^2aM}{{b_K}^2} + \mathcal{O}\left(\frac{M^2}{{b_K}^2}\right),\end{aligned} \tag{54}$$

$$\begin{aligned}\alpha_{\mathrm{retro}} \to {} & \left(\frac{1}{\beta}-1\right)\pi + \frac{4M}{b_K} + \frac{16\pi\eta^2M}{b_K} + \frac{4aM}{{b_K}^2} \\ & + \frac{32\pi\eta^2aM}{{b_K}^2} + \mathcal{O}\left(\frac{M^2}{{b_K}^2}\right),\end{aligned} \tag{55}$$

where b_K is a constant of integration in Jusufi, *et al.* [19], we used $b_K = b/\beta$ and $\beta^2 = 1 - 8\pi\eta^2$. These equations coincide with Eq. (53) in [19], in which they are restricted within the asymptotic source and receiver ($r_R \to \infty$ and $r_S \to \infty$). Note that Ref. [19] obtained $\pm\frac{128\pi\eta^2aM}{5{b_K}^2}$ by their method, while a method of the direct integration of the null geodesic gives $\pm\frac{32\pi\eta^2aM}{{b_K}^2}$: The former expression agrees with the latter one but with a different numerical coefficient. Please see Appendix A of Ref. [19], especially Eq. (53) and the last paragraph of the Appendix. According to their comments in the last paragraph, their approximation would need to be modified to recover a correct expression as $\pm\frac{32\pi\eta^2aM}{{b_K}^2}$. Our result as Eqs. (54) and (55) is indeed in agreement with the latter expression. In this sense, our present approach is better than the method in Ref. [19].

IV. POSSIBLE ASTRONOMICAL APPLICATIONS

In this section, we discuss possible astronomical applications. The above calculations discuss the deflection angle of light. In particular, we do not assume that the receiver and the source are located at the infinity. The finite-distance correction to the deflection angle of light, denoted as $\delta\alpha$, is the difference between the asymptotic deflection angle α_∞ and the deflection angle for the finite distance case. It is expressed as

$$\delta\alpha \equiv \alpha_\infty - \alpha. \tag{56}$$

The finite-distance correction to the deflection angle of light is roughly estimated as

$$\begin{aligned}\delta\alpha &\sim \left(\frac{1}{\beta}-1\right)\left\{\arcsin\left(\frac{b^2{u_R}^2}{\beta^2}\right)+\arcsin\left(\frac{b^2{u_S}^2}{\beta^2}\right)\right\}-\frac{(\beta-1)^2(\beta+1)}{\beta^2}(u_R-u_S)a\\&\quad+\left\{\frac{2\beta\left(\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}-1\right)+\left(1+\frac{1}{\beta}\right)b^2{u_R}^2}{b\beta\sqrt{1-\frac{b^2{u_R}^2}{\beta^2}}}+\frac{2\beta\left(\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}-1\right)+\left(1+\frac{1}{\beta}\right)b^2{u_S}^2}{b\beta\sqrt{1-\frac{b^2{u_S}^2}{\beta^2}}}\right\}M+\mathcal{O}(aM/b^2,M^2/b^2)\\&\sim\left(\frac{1}{\beta}-1\right)\left\{\arcsin\left(\frac{b^2{u_R}^2}{\beta^2}\right)+\arcsin\left(\frac{b^2{u_S}^2}{\beta^2}\right)\right\}-\frac{(\beta-1)^2(\beta+1)}{\beta^2}(u_R-u_S)a+\left\{\frac{b{u_R}^2}{\beta}+\frac{b{u_S}^2}{\beta}\right\}M\\&\quad+\mathcal{O}(aM/b^2,M^2/b^2).\end{aligned}\tag{57}$$

The counterpart for the weak-field and slow-rotation Kerr metric is [15]

$$\delta\alpha_{Kerr}\sim(bu_R^2+bu_S^2)M+\mathcal{O}(aM/b^2,M^2/b^2).\tag{58}$$

From Eqs. (57) and (58), the finite correction to the light deflection purely due to the angle deficit $\delta\alpha-\delta\alpha_{Kerr}$ becomes

$$\begin{aligned}\delta\alpha-\delta\alpha_{Kerr}&\sim\left(\frac{1}{\beta}-1\right)\left\{\arcsin\left(\frac{b^2{u_R}^2}{\beta^2}\right)+\arcsin\left(\frac{b^2{u_S}^2}{\beta^2}\right)\right\}-\frac{(\beta-1)^2(\beta+1)}{\beta^2}(u_R-u_S)a+\left(\frac{1}{\beta}-1\right)b({u_R}^2+{u_S}^2)M\\&\quad+\mathcal{O}(aM/b^2,M^2/b^2).\end{aligned}\tag{59}$$

For its simplicity, we consider the mass of the rotating global monopole equals to Sgr A * ($M_{\mathrm{Sgr}}\simeq 4\times10^6\ M_\odot$, $M_\odot$ is the Solar mass), the spin angular momentum of the rotating global monopole is $a=2/3M_{\mathrm{Sgr}}$ and the parameter $\beta=0, 0.999, 359/360$. We assume r_R is the distance from Earth to Sgr A * ($r_R\simeq 8\times10^3$ [pc]). We also assume $b\sim100M$ and $r_s\sim0.1$ pc. As a rough order-of-magnitude estimate under these assumptions, three terms in Eq. (59) become

$$\begin{aligned}&\left(\frac{1}{\beta}-1\right)\left\{\arcsin\left(\frac{b^2{u_R}^2}{\beta^2}\right)+\arcsin\left(\frac{b^2{u_S}^2}{\beta^2}\right)\right\}\\&\quad\sim8\times10^{-3}\left(\frac{1-\beta}{10^{-3}}\right)\left(\frac{b}{100M_{\mathrm{Sgr}}}\right)^2\left(\frac{0.1\ \mathrm{pc}}{r_S}\right)^2[\mathrm{mas}],\end{aligned}\tag{60}$$

$$\begin{aligned}&-\frac{(\beta-1)^2(\beta+1)}{\beta^2}(u_R-u_S)a\\&\quad\sim5\times10^{-4}\left(\frac{1-\beta}{10^{-3}}\right)^2\left(\frac{0.1\ \mathrm{pc}}{r_S}\right)\left(\frac{a}{2M/3}\right)[\mathrm{mas}],\end{aligned}\tag{61}$$

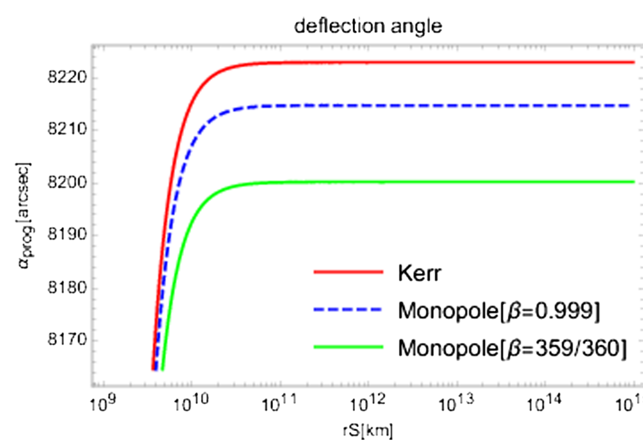

FIG. 1. α_{prog}, where we assume the Sgr A*. The vertical axis denotes the deflection angle of light with the finite-distance correction and the horizontal axis denotes the source distance r_S. The red solid curve, blue dash curve and green dot curve correspond to $\beta=0$ (Kerr spacetime), $\beta=0.999$ and $\beta=359/360$, respectively. The impact parameter is assumed to be $b=10^2M_{\mathrm{Sgr}}$.

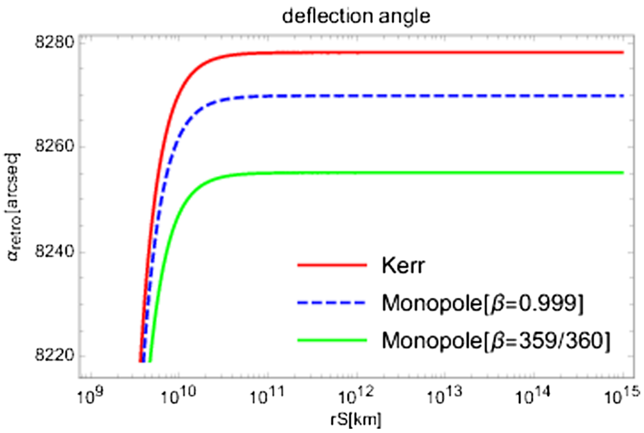

FIG. 2. α_{retro}, where we assume the Sgr A*. The vertical axis denotes the deflection angle of light with the finite-distance correction, and the horizontal axis denotes the source distance r_S. The red solid curve, blue dash curve and green dot curve correspond to $\beta = 0$ (Kerr spacetime), $\beta = 0.999$ and $\beta = 359/360$, respectively. The impact parameter is assumed to be $b = 10^2 M_{\mathrm{Sgr}}$.

$$\left(\frac{1}{\beta} - 1\right) b({u_R}^2 + {u_S}^2) M \sim 8 \times 10^{-5} \left(\frac{1-\beta}{10^{-3}}\right) \left(\frac{b}{100 M_{\mathrm{Sgr}}}\right) \left(\frac{0.1\ \mathrm{pc}}{r_S}\right)^2 \times \left(\frac{M}{4 \times 10^6\ M_\odot}\right) [\mathrm{mas}], \tag{62}$$

respectively. The second and third terms are thus beyond reach of the present technology. On the other hand, the first term is much larger than the second and third ones, and it may be probed by using the present technology, if β is large enough. If present and near-future observations at the level of $\sim 1 \times 10^{-3}$ [mas] find no evidence of the first term, an upper bound on $1 - \beta$ will be placed by Eq. (60) as $1 - \beta < \frac{1}{8} \times 10^{-3} \sim 1 \times 10^{-4}$. For the deficit angle $\delta = 8\pi^2\eta^2 = \pi(1 - \beta^2)$, this bound is interpreted as $\delta \sim 2\pi(1 - \beta) < 8 \times 10^{-4}$ [rad], where we use $1 + \beta \sim 2$ for the small angle deficit. Figure 1 shows the gravitational deflection of light in the prograde orbit for Sgr A *. Figure 2 shows that for the retrograde orbit.

V. CONCLUSION

In the weak field approximation, we have discussed the deflection angle of light for an observer and source at finite distance from a rotating global monopole with a deficit angle. We have shown that both of the Werner's method and the generalized optical metric method give the same deflection angle at the leading order of the weak field approximation, if the receiver and source are at the null infinity. Therefore, our result is a possible extension to asymptotically nonflat spacetimes. We have also found corrections for the deflection angle due to the finite distance from the global monopole. We examined whether near-future observations of Sgr A * can put an upper bound on the deficit angle for a rotating global monopole model. It is left for the future to study higher order terms in the weak field approximation of a rotating global monopole and to examine also the strong deflection limit.

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