Complexity of the Bondi Metric

L. Herrera*
Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca, Salamanca 37007, Spain

A. Di Prisco†
Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas 1050, Venezuela

J. Carot‡
Departament de Física, Universitat Illes Balears, E-07122 Palma de Mallorca, Spain

(Received 3 May 2019; published 18 June 2019)

A recently introduced concept of complexity for relativistic fluids is extended to the vacuum solutions represented by the Bondi metric. A complexity hierarchy is established, ranging from the Minkowski spacetime (the simplest one) to gravitationally radiating systems (the more complex). Particularly interesting is the possibility to differentiate between natural nonradiative (NNRS) and non-natural nonradiative (NNNRS) systems, the latter appearing to be simpler than the former. The relationship between vorticity and the degree of complexity is stressed.

DOI: 10.1103/PhysRevD.99.124028

I. INTRODUCTION

In a recent series of papers we have introduced a new concept of complexity for self-gravitating relativistic fluids, and we have applied it to the spherically symmetric case (both in the static [1] and the dynamic situation [2]) and to the axially symmetric static case [3]. Applications of this concept to other theories of gravity have been proposed in [4,5], while the charged case has been considered in [2] and [6]. Also, applications for some particular cases of cylindrically symmetric fluid distributions may be found in [7].

Our purpose in this paper consists in extending the above-mentioned concept of complexity to vacuum spacetimes. More specifically, we consider the Bondi metric [8], which includes the Minkowski spacetime, the static Weyl metrics, nonradiative nonstatic metrics, and gravitationally radiating metrics. Besides the fact that the Bondi metric covers a vast numbers of spacetimes, it has, among other things, the virtue of providing a clear and precise criterion for the existence of gravitational radiation. Namely, if the news function is zero over a time interval, then there is no radiation over that interval.

In the case of fluid distributions the variable(s) measuring the complexity of the fluid [the complexity factor(s)] appear in the trace-free part of the orthogonal splitting of the electric Riemann tensor [9–12]. In vacuum the Riemann tensor and the Weyl tensor are the same, so we start by calculating the scalar functions defining the electric part of the Weyl tensor for the Bondi metric. Following the results obtained for the fluid case, we consider the scalars defining this tensor as the complexity factors. Next we establish a hierarchy of spacetimes according to their complexity. Particularly appealing is the possibility to discriminate between two classes of spacetimes that depend on time but are not radiative (vanishing of the news function). These two classes are called by Bondi [8] natural and non-natural nonradiative moving systems, they are characterized by different forms of the mass aspect. As we shall see they exhibit different degrees of complexity.

Unfortunately, though, up to the leading order of the complexity factors analyzed here, it is impossible to discriminate between different radiative systems according to their complexity. Higher order terms would be necessary for that purpose, although it is not clear at this point if it is possible to establish such a hierarchy of radiative systems after all.

Finally we emphasize the conspicuous link between vorticity and complexity, and we discuss some open issues in the last section.

II. BONDI'S FORMALISM

The general form of an axially and reflection symmetric asymptotically flat metric given by Bondi [8] is (for the general case see [13])

*lherrera@usal.es
†adiprisc@ciens.ucv.ve
‡jcarot@uib.cat

$$ds^2 = \left(\frac{V}{r}e^{2\beta} - U^2r^2e^{2\gamma}\right)du^2 + 2e^{2\beta}dudr + 2Ur^2e^{2\gamma}dud\theta - r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2), \quad (1)$$

where V, β, U and γ are functions of u, r and θ.

We number the coordinates $x^{0,1,2,3} = u, r, \theta, \phi$, respectively. Here, u is a timelike coordinate ($g_{uu} > 0$) converging to the retarded time as $r \to \infty$. The hypersurfaces $u =$ constant define null surfaces (their normal vectors are null vectors), which, at null infinity ($r \to \infty$), coincides with the Minkowski null light cone open to the future. Here, r is a null coordinate ($g_{rr} = 0$), and θ and ϕ are two angle coordinates (see [8] for details).

Regularity conditions in the neighborhood of the polar axis ($\sin\theta = 0$) imply that as $\sin\theta \to 0$,

$$V, \beta, U/\sin\theta, \gamma/\sin^2\theta, \quad (2)$$

each equals a function of $\cos\theta$ regular on the polar axis.

The four metric functions are assumed to be expanded in series of $1/r$; then, using the field equations, one obtains

$$\gamma = cr^{-1} + \left(C - \frac{1}{6}c^3\right)r^{-3} + \cdots, \quad (3)$$

$$U = -(c_\theta + 2c\cot\theta)r^{-2} + [2N + 3cc_\theta + 4c^2\cot\theta]r^{-3}\ldots, \quad (4)$$

$$V = r - 2M - \left(N_\theta + N\cot\theta - c_\theta^2 - 4cc_\theta\cot\theta - \frac{1}{2}c^2(1 + 8\cot^2\theta)\right)r^{-1} + \cdots, \quad (5)$$

$$\beta = -\frac{1}{4}c^2r^{-2} + \cdots, \quad (6)$$

where c, C, N and M are functions of u and θ satisfying the constraint

$$4C_u = 2c^2c_u + 2cM + N\cot\theta - N_\theta, \quad (7)$$

and letters as subscripts denote derivatives. The three functions c, M and N are further related by the supplementary conditions

$$M_u = -c_u^2 + \frac{1}{2}(c_{\theta\theta} + 3c_\theta\cot\theta - 2c)_u, \quad (8)$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u\cot\theta + c_uc_\theta. \quad (9)$$

In the static case M equals the mass of the system, and Bondi called this the "mass aspect," whereas N and C are closely related to the dipole and quadrupole moments, respectively.

Next, Bondi defines the mass $m(u)$ of the system as

$$m(u) = \frac{1}{2}\int_0^\pi M\sin\theta d\theta, \quad (10)$$

which by virtue of (8) and (2) yields

$$m_u = -\frac{1}{2}\int_0^\pi c_u^2\sin\theta d\theta. \quad (11)$$

Let us now recall the main conclusions emerging from Bondi's approach.

(1) If γ, M and N are known for some $u = a$ (constant) and c_u (the news function) is known for all u in the interval $a \leq u \leq b$, then the system is fully determined in that interval. In other words, whatever happens at the source, leading to changes in the field, it can only do so by affecting c_u and vice versa. In the light of this comment the relationship between the news function and the occurrence of radiation becomes clear.

(2) As it follows from (11), the mass of a system is constant if and only if there is no news.

Now, for an observer at rest in the frame of (1), the four-velocity vector has components

$$V^\alpha = \left(\frac{1}{A}, 0, 0, 0\right), \quad (12)$$

with

$$A \equiv \left(\frac{V}{r}e^{2\beta} - U^2r^2e^{2\gamma}\right)^{1/2}. \quad (13)$$

Next, let us introduce the unit, spacelike vectors $\mathbf{K}, \mathbf{L}, \mathbf{S}$, with components

$$K^\alpha = \left(\frac{1}{A}, -e^{-2\beta}A, 0, 0\right)$$
$$L^\alpha = \left(0, Ure^\gamma e^{-2\beta}, -\frac{e^{-\gamma}}{r}, 0\right), \quad (14)$$

$$S^\alpha = \left(0, 0, 0, -\frac{e^\gamma}{r\sin\theta}\right), \quad (15)$$

or

$$V_\alpha = \left(A, \frac{e^{2\beta}}{A}, \frac{Ur^2e^{2\gamma}}{A}, 0\right), \qquad K_\alpha = \left(0, \frac{e^{2\beta}}{A}, \frac{Ur^2e^{2\gamma}}{A}, 0\right), \quad (16)$$

$$L_\alpha = (0, 0, e^\gamma r, 0), \qquad S_\alpha = (0, 0, 0, e^{-\gamma}r\sin\theta), \quad (17)$$

satisfying the following relations:

$$V_\alpha V^\alpha = -K^\alpha K_\alpha = -L^\alpha L_\alpha = -S^\alpha S_\alpha = 1, \quad (18)$$

$$V_\alpha K^\alpha = V^\alpha L_\alpha = V^\alpha S_\alpha = K^\alpha L_\alpha = K^\alpha S_\alpha = S^\alpha L_\alpha = 0. \quad (19)$$

The unitary vectors V^α, L^α, S^α, K^α form a canonical orthonormal tetrad ($e^{(a)}_\alpha$), such that

$$e^{(0)}_\alpha = V_\alpha, \qquad e^{(1)}_\alpha = K_\alpha, \qquad e^{(2)}_\alpha = L_\alpha, \qquad e^{(3)}_\alpha = S_\alpha,$$

with $a = 0, 1, 2, 3$ (latin indices label different vectors of the tetrad). The dual vector tetrad $e^\alpha_{(a)}$ is easily computed from the condition

$$\eta_{(a)(b)} = g_{\alpha\beta} e^\alpha_{(a)} e^\beta_{(b)},$$

where $\eta_{(a)(b)}$ denotes the Minkowski metric.

For the observer defined by (12) the vorticity vector may be written as (see [14] for details)

$$\omega^\alpha = (0, 0, 0, \omega^\phi). \quad (20)$$

The explicit expressions for ω^ϕ and its absolute value $\Omega \equiv (-\omega_\alpha \omega^\alpha)^{1/2}$ are given in Appendix C.

III. COMPLEXITY FACTORS AND ELECTRIC AND MAGNETIC PARTS OF WEYL TENSOR

As we mentioned in the Introduction, we extend the definition of complexity introduced in [1,2] to the vacuum case; this implies considering the scalars defining the electric Weyl tensor as the complexity factors. Besides the electric part of the Weyl tensor, we also use its magnetic part in the discussion; accordingly, we calculate its corresponding scalars as well.

The electric and magnetic parts of the Weyl tensor, $E_{\alpha\beta}$ and $H_{\alpha\beta}$, respectively, are formed from the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and its dual $\tilde{C}_{\alpha\beta\gamma\delta}$ by contraction with the four-velocity vector given by (12):

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta} V^\gamma V^\delta, \quad (21)$$

$$H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta} V^\gamma V^\delta = \frac{1}{2} \epsilon_{\alpha\gamma\epsilon\delta} C^{\epsilon\delta}_{\beta\rho} V^\gamma V^\rho,$$
$$\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g}\,\eta_{\alpha\beta\gamma\delta}, \quad (22)$$

where $\eta_{\alpha\beta\gamma\delta}$ is the permutation symbol.

Also note that

$$\sqrt{-g} = r^2 \sin\theta e^{2\beta} \approx r^2 \sin\theta \exp\left(-\frac{c^2}{2r^2}\right)$$
$$\approx r^2 \sin\theta + O(1).$$

The electric part of the Weyl tensor has only three independent nonvanishing components, whereas only two components define the magnetic part. Thus, we may write

$$E_{\alpha\beta} = \mathcal{E}_1 (K_\alpha L_\beta + L_\alpha K_\beta) + \mathcal{E}_2 \left(K_\alpha K_\beta + \frac{1}{3} h_{\alpha\beta}\right)$$
$$+ \mathcal{E}_3 \left(L_\alpha L_\beta + \frac{1}{3} h_{\alpha\beta}\right), \quad (23)$$

and

$$H_{\alpha\beta} = H_1 (S_\alpha K_\beta + S_\beta K_\alpha) + H_2 (S_\alpha L_\beta + S_\beta L_\alpha). \quad (24)$$

with $h_{\mu\nu} = g_{\mu\nu} - V_\nu V_\mu$, and

$$\mathcal{E}_1 = L^\alpha K^\beta E_{\alpha\beta}, \quad (25)$$

$$\mathcal{E}_2 = (2K^\alpha K^\beta + L^\alpha L^\beta) E_{\alpha\beta}, \quad (26)$$

$$\mathcal{E}_3 = (2L^\alpha L^\beta + K^\alpha K^\beta) E_{\alpha\beta}. \quad (27)$$

These three scalars will be considered the complexity factors of our solutions.

For the magnetic part we have

$$H_2 = S^\alpha L^\beta H_{\alpha\beta}, \quad (28)$$

$$H_1 = S^\alpha K^\beta H_{\alpha\beta}. \quad (29)$$

Explicit expressions for these scalars are given in Appendixes A and B.

Reference [15] found that if we put $H^\alpha_\beta = 0$ then the field is nonradiative, and up to order $1/r^3$ in γ, the metric is static; the mass, the "dipole" (N) and the "quadrupole" (C) moments correspond to a static situation. However, the time dependence might enter through coefficients of higher order in γ, giving rise to what Bondi calls a "non-natural nonradiative moving system" (NNRS). In this latter case, the system keeps the first three moments independent of time, but allows for time dependence of higher moments. This class of solutions is characterized by $M_\theta = 0$.

A second family of time-dependent nonradiative solutions exists for which $M_\theta \neq 0$. These are called "natural nonradiative moving systems" (NNRSs), and their magnetic Weyl tensor is nonvanishing.

We are now ready to discuss the hierarchy of different spacetimes belonging to the Bondi family, according to their complexity.

IV. HIERARCHY OF COMPLEXITY

The simplest spacetime corresponds to the vanishing of the three complexity factors, and this is just Minkowski spacetime.

Indeed, as it was shown in [15], if we assume $E^\alpha_\beta = 0$ and use regularity conditions, we find that the spacetime must be Minkowski, giving further support to the conjecture that there are no purely magnetic vacuum spacetimes [16].

On the other end (maximal complexity) we have a gravitationally radiating system which requires all three complexity factors to be different from zero.

Indeed, let us assume that $\mathcal{E}_1 = 0$; then it follows at once from (A1) that $c_u = 0$ (otherwise c_u would be a nonregular function of θ on the symmetry axis). Thus $\mathcal{E}_1 = 0$ implies that the system is nonradiative.

If instead we assume that $\mathcal{E}_2 = 0$, then from the first order in (A2) we obtain that $c_{uu} = 0$; this implies that either $c_u = 0$ or $c_u \sim u$. Bondi refers to this latter case as "mass loss without radiative Riemann tensor" and dismisses it as being of little physical significance. As a matter of fact, in this latter case the system would be radiating "forever," which according to (11) requires an unbounded source, incompatible with an asymptotically flat spacetime. Thus, in this case too, we have $c_u = 0$, and the system is nonradiative.

Finally, if we assume $\mathcal{E}_3 = 0$ it follows at once from the first order in (A3) that $c_{uu} = 0$, leading to $c_u = 0$, according to the argument above.

Thus, a radiative system requires all three complexity factors to be nonvanishing, implying a maximal complexity.

In the middle of the two extreme cases we have, on the one hand, the spherically symmetric spacetime (Schwarzschild), characterized by a single complexity factor (the same applies for any static metric), $\mathcal{E}_1 = \mathcal{E}_3 = 0$, and $\mathcal{E}_2 = \frac{3M}{r^3}$. On the other hand, we have the nonstatic nonradiative case.

Let us now analyze in detail this latter case. There are two subclasses in this group of solutions, which using Bondi notation are as follows:

(1) NNRS characterized by $M_\theta \neq 0$.

(2) NNNRS characterized by $M_\theta = 0$.

Let us first consider the NNNRS subcase. Using (A1) we obtain $\mathcal{E}_1 = 0$ (up to order $1/r^3$), while the first nonvanishing terms in $\mathcal{E}_2$ and $\mathcal{E}_3$ are, respectively, $3M$ and 0, where (7), (8), (9), (A2) and (A3) have been used.

Thus, the NNNRS are characterized by only one nonvanishing complexity factor ($\mathcal{E}_2$). Furthermore, as it follows from (C3) the vorticity of the congruence of observers at rest with respect to the frame of (1) vanishes, and the field is purely electric. However, as mentioned before we cannot conclude that the field is static since the u dependence might appear through coefficients of higher order in γ.

Let us now consider the NNRS. In this subcase, using (A1) we obtain $\mathcal{E}_1 = 0$ (up to order $1/r^3$) as for the NNNRS subcase, while the first nonvanishing terms in $\mathcal{E}_2$ and $\mathcal{E}_3$ (up to order $1/r^3$) are, respectively, $3M + \frac{M_{\theta\theta}}{4} - \frac{M_\theta \cot\theta}{4}$ and $\frac{M_{\theta\theta}}{2} - \frac{M_\theta \cot\theta}{2}$.

Also, up to the same order, it follows from (B1) and (B2) that $H_1 = 0$ for both subcases, while the corresponding term in H_2 is (for NNRS)

$$-\frac{1}{4}(M_{\theta\theta} - M_\theta \cot\theta), \qquad (30)$$

which of course vanishes for the NNNRS subcase.

It should be observed that if we assume $\mathcal{E}_3 = 0$ or $H_2 = 0$, then it follows at once from the above that

$$M_{\theta\theta} - M_\theta \cot\theta = 0 \Rightarrow M = a\cos\theta, \quad a = \text{constant}. \qquad (31)$$

But this implies because of (10) that the Bondi mass function of the system vanishes. Therefore, the only physically meaningful NNRS requires $\mathcal{E}_3 \neq 0$, $\Omega \neq 0$ and $H_2 \neq 0$, implying that the complexity is characterized by two complexity factors ($\mathcal{E}_2$, $\mathcal{E}_3$).

V. CONCLUSIONS

We have seen so far that the extension of the concept of complexity, adopted for fluids in [1–3], may be extended to the vacuum case without much trouble and provides sensible results.

The three complexity factors corresponding to the three scalars defining the electric part of the Weyl tensor allow us to establish a hierarchy of solutions according to their complexity.

The simplest system (Minkowski) is characterized by the vanishing of all the complexity factors. Next, the static case (including Schwarzschild) is described by a single complexity factor.

The time-dependent nonradiative solutions split into two subgroups depending on the form of the mass aspect M. If $M_\theta = 0$, which corresponds to the NNNRS, the complexity is similar to the static case. Also, in this case, as in the static situation, the vorticity vanishes and the field is purely electric. This result could suggest that in fact NNNRS are just static, and no time dependence appears in the coefficients of higher order in γ. On the contrary, for the NNRS there are two complexity factors, the vorticity is nonvanishing, and the field is not purely electric.

All these results are summarized in Tables I and II. Thus, NNNRS and NNRS are clearly differentiated through their degree of complexity, as measured by the complexity factors considered here.

The fact that radiative systems necessarily decay into NNRS, NNNRS or static systems, since the Bondi mass function must be finite, suggests that higher degrees of complexity might be associated with stronger stability. Of course a proof of this conjecture requires a much more detailed analysis.

TABLE I. Complexity factors for different spacetimes of the Bondi metric.

Complexity factors/Spacetime	Minkowski	Static	NNNRS	NNRS	Radiative
$\mathcal{E}_1$	0	0	0	0	$\mathcal{E}_1^{(n)} \neq 0,\ n \geq 1$
$\mathcal{E}_2$	0	$\mathcal{E}_2^{(3)} = 3M$	$\mathcal{E}_2^{(3)} = 3M$	$\mathcal{E}_2^{(3)} = 3M + \frac{M_{\theta\theta}}{4} - \frac{M_\theta \cot\theta}{4}$	$\mathcal{E}_2^{(n)} \neq 0,\ n \geq 1$
$\mathcal{E}_3$	0	0	0	$\mathcal{E}_3^{(3)} = \frac{1}{2}(M_{\theta\theta} - M_\theta \cot\theta)$	$\mathcal{E}_3^{(n)} \neq 0,\ n \geq 1$

where $\mathcal{E}_{1,2,3}^{(n)}$ are the coefficients of order $\mathcal{O}(r^{-n})$.

TABLE II. The magnetic parts of the Weyl tensor and the vorticity for different spacetimes of the Bondi metric.

Magnetic Weyl; Ω/spacetimes	Minkowski	Static	NNNRS	NNRS	Radiative
H_1	0	0	0	0	$H_1^{(n)} \neq 0,\ n \geq 1$
H_2	0	0	0	$H_2^{(3)} = -\frac{1}{4}(M_{\theta\theta} - M_\theta \cot\theta)$	$H_2^{(n)} \neq 0,\ n \geq 1$
Ω	0	0	0	$\Omega^{(2)} = M_\theta$	$\Omega^{(n)} \neq 0,\ n \geq 1$

It is also worth mentioning the conspicuous link between vorticity and complexity factors. Indeed, vorticity appears only in NNRS and radiative systems, which are the most complex systems, while it is absent in the simplest systems (Minkowski, static, NNNRS). In the radiative case there are contributions at order $\mathcal{O}(r^{-1})$ related to the news function, and at order $\mathcal{O}(r^{-2})$, while for the NNRS there are only contributions at order $\mathcal{O}(r^{-2})$, these describe the effect of the tail of the wave, thereby providing "observational" evidence for the violation of the Huygens's principle, a problem largely discussed in the literature (see for example [8,17–23] and references therein).

We would like to conclude with two questions which, we believe, deserve further attention:

(i) Is it possible to further refine the scheme proposed here so as to discriminate between different radiative systems according to their complexity? Or, in other words, among radiative systems is there a simplest one? Obviously this would require a closer examination of the orders higher than the leading ones in the complexity factors.

(ii) Is it possible to discriminate between different static spacetimes? Again, this would require us to go beyond the orders employed here.

ACKNOWLEDGMENTS

This work was partially supported by the Spanish Ministerio de Ciencia e Innovación under Research Project No. FIS2015-65140-P (MINECO/FEDER). L. H. thanks the Departament de Física at the Universitat de les Illes Balears, for financial support and hospitality. A. D. P. acknowledges the Departament de Física at the Universitat de les Illes Balears for hospitality. J. C. gratefully acknowledges the Spanish Ministerio de Economía y Competitividad for financial support through Grant No. FPA2016-76821.

APPENDIX A: COMPLEXITY FACTORS

The complexity factors are:

$$\mathcal{E}_1 = \frac{1}{r^2}(2c_u \cot\theta + c_{\theta u}) + \mathcal{O}(r^{-n}), \quad n \geq 4, \tag{A1}$$

$$\begin{aligned}\mathcal{E}_2 = {} & \frac{1}{r}c_{uu} - \frac{1}{2r^2}\left(c_{\theta\theta u} - 4Mc_{uu} + 2c_u + c_{\theta u}\cot\theta - \frac{4c_u}{\sin^2\theta}\right) \\ & + \frac{1}{r^3}\Bigg[cc_u + 2c_\theta c_{\theta u} + 3M + \frac{\cot\theta}{2}(3c_u c_\theta + 5cc_{\theta u}) - M_u c + \frac{1}{2}M_{\theta\theta} + N_{\theta u} + P_{uu} \\ & - \cot\theta\left(Mc_{\theta u} + \frac{1}{2}M_\theta + N_u - Nc_{uu}\right) - Mc_u\left(1 - \frac{4}{\sin^2\theta}\right) + c_u\left(cc_u + \frac{1}{2}c_{\theta\theta}\right) + c_{uu}(4M^2 + N_\theta) - c_{\theta\theta u}\left(M - \frac{3}{2}c\right)\Bigg] \\ & + \mathcal{O}(r^{-n}), \quad n \geq 4,\end{aligned} \tag{A2}$$

$$\mathcal{E}_3 = \frac{2}{r}c_{uu} - \frac{1}{r^2}\left(c_{\theta\theta u} - 4Mc_{uu} + 2c_u + c_{\theta u}\cot\theta - \frac{4c_u}{\sin^2\theta}\right)$$
$$+\frac{1}{r^3}\Big[-4cc_u + 4c_\theta c_{\theta u} + \cot\theta(3c_u c_\theta + 5cc_{\theta u}) - 2M_u c + M_{\theta\theta} + 2N_{\theta u} + 2P_{uu}$$
$$-\cot\theta(2Mc_{\theta u} + M_\theta + 2N_u - 2Nc_{uu}) - 2Mc_u\left(1 - \frac{4}{\sin^2\theta}\right) + c_u(2cc_u + c_{\theta\theta})$$
$$+2c_{uu}(4M^2 + N_\theta) - c_{\theta\theta u}(2M - 3c)\Big] + \mathcal{O}(r^{-n}), \qquad n \geq 4. \tag{A3}$$

APPENDIX B: MAGNETIC PART OF THE WEYL TENSOR

The magnetic part of the Weyl tensor can be calculated as follows:

$$H_1 = -\frac{1}{r^2}(2c_u\cot\theta + c_{\theta u}) + \mathcal{O}(r^{-n}), \quad n \geq 4, \tag{B1}$$

$$H_2 = -\frac{1}{r}c_{uu} - \frac{1}{r^2}\left[-c_u\left(1 - \frac{2}{\sin^2\theta}\right) - \frac{\cot\theta}{2}c_{\theta u} + 2c_{uu}(M - c) - \frac{1}{2}c_{\theta\theta u}\right]$$
$$-\frac{1}{r^3}\Big\{-Mc_u\left(1 - \frac{4}{\sin^2\theta}\right) - \frac{4cc_u}{\sin^2\theta} + \cot\theta\left[\frac{3}{2}c_u c_\theta - N_u - \frac{1}{2}M_\theta + Nc_{uu} + \left(\frac{7}{2}c - M\right)c_{\theta u}\right] + \left(\frac{5}{2}c - M\right)c_{\theta\theta u}$$
$$+\frac{1}{2}c_{\theta\theta}c_u + 2c_\theta c_{\theta u} + cc_u^2 + \frac{1}{2}M_{\theta\theta} - cM_u + N_{\theta u} + P_{uu} + c_{uu}(4c^2 + 4M^2 - 4Mc + N_\theta)\Big\} + \mathcal{O}(r^{-n}), \quad n \geq 4 \tag{B2}$$

where $P = C - \frac{c^3}{6}$.

APPENDIX C: VORTICITY

The vorticity is

$$\omega^\phi = -\frac{e^{-2\beta}}{2r^2\sin\theta}\left[2\beta_\theta e^{2\beta} - \frac{2e^{2\beta}A_\theta}{A} - (Ur^2e^{2\gamma})_{,r} + \frac{2Ur^2e^{2\gamma}}{A}A_{,r} + \frac{e^{2\beta}(Ur^2e^{2\gamma})_{,u}}{A^2} - \frac{Ur^2e^{2\gamma}}{A^2}2\beta_u e^{2\beta}\right], \tag{C1}$$

and for the absolute value of ω^α, we get

$$\Omega \equiv (-\omega_\alpha\omega^\alpha)^{1/2} = \frac{e^{-2\beta-\gamma}}{2r}\left[2\beta_\theta e^{2\beta} - 2e^{2\beta}\frac{A_\theta}{A} - (Ur^2e^{2\gamma})_{,r} + 2Ur^2e^{2\gamma}\frac{A_{,r}}{A} + \frac{e^{2\beta}}{A^2}(Ur^2e^{2\gamma})_{,u} - 2\beta_u\frac{e^{2\beta}}{A^2}Ur^2e^{2\gamma}\right]. \tag{C2}$$

Feeding (3)–(6) back into (C2) and keeping only the two leading terms, we obtain

$$\Omega = -\frac{1}{2r}(c_{u\theta} + 2c_u\cot\theta) + \frac{1}{r^2}[M_\theta - M(c_{u\theta} + 2c_u\cot\theta) - cc_{u\theta} + 6cc_u\cot\theta + 2c_u c_\theta]. \tag{C3}$$

[1] L. Herrera, Phys. Rev. D **97**, 044010 (2018).
[2] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **98**, 104059 (2018).
[3] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **99**, 044049 (2019).
[4] G. Abbas and H. Nazar, Eur. Phys. J. C **78**, 510 (2018).
[5] G. Abbas and H. Nazar, Eur. Phys. J. C **78**, 957 (2018).
[6] M. Sharif and I. Butt, Eur. Phys. J. C **78**, 688 (2018).
[7] M. Sharif and I. Butt, Eur. Phys. J. C **78**, 850 (2018).
[8] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. A **269**, 21 (1962).
[9] L. Bel, C.R. Hebd. Seances Acad. Sci. **247**, 1094 (1958).
[10] L. Bel, Cahiers de Physique **16**, 59 (1962); Gen. Relativ. Gravit. **32**, 2047 (2000).
[11] A. G. P. Gómez Lobo, Classical Quantum Gravity **25**, 015006 (2008).
[12] L. Herrera, J. Ospino, A. Di Prisco, E. Fuenmayor, and O. Troconis, Phys. Rev. D **79**, 064025 (2009).
[13] R Sachs, Proc. R. Soc. A **270**, 103 (1962).
[14] L. Herrera and J. L. Hernández-Pastora, Classical Quantum Gravity **17**, 3617 (2000).
[15] L. Herrera, N. O. Santos, and J. Carot, J. Math. Phys. (N.Y.) **47**, 052502 (2006).
[16] W. B. Bonnor, Classical Quantum Gravity **12**, 1483 (1995).
[17] B. S. DeWitt and R. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[18] W. B. Bonnor, in *Proceedings of the Meeting on General Relativity, Florence 1965*, edited by G. Barbera (Comitato Nazionale per le Manifestazione Celebrative, Florence, Italy, 1965), Vol. II.
[19] W. B. Bonnor and M. Rotemberg, Proc. R. Soc. A **289**, 247 (1966).
[20] W. Kundt and E. T. Newman, J. Math. Phys. (N.Y.) **9**, 2193 (1968).
[21] E. T. Newman and R. Penrose, Proc. R. Soc. A **305**, 175 (1968).
[22] W. Couch, R. Torrence, A. Janis, and E. T. Newman, J. Math. Phys. (N.Y.) **9**, 484 (1968).
[23] L. Blanchet and G. Shafer, Classical Quantum Gravity **10**, 2699 (1993).