Quantum metrology in the Kerr metric

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A surprising feature of the Kerr metric is the anisotropy of the speed of light. The angular momentum of a rotating massive object causes co- and counterpropagating light paths to move at faster and slower velocities, respectively, as determined by a far-away clock. Based on this effect we derive the ultimate quantum limits for the measurement of the Kerr rotation parameter a using an interferometric setup. As a possible implementation, we propose a Mach-Zehnder interferometer to measure the "one-way height differential" time effect. We isolate the effect by calibrating to a dark port and rotating the interferometer such that only the direction-dependent Kerr-metric-induced phase term remains. We transform to the zero angular momentum observer (ZAMO) flat metric where the observer sees $c = 1$. We use this metric and the Lorentz transformations to calculate the same Kerr phase shift. We then consider nonstationary observers moving with the planet's rotation, and we find a method for canceling the additional phase from the classical relative motion, thus leaving only the curvature-induced phase.

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I. INTRODUCTION

The noise induced by a measurement device is fundamentally restricted by limits set by quantum mechanics. Quantum metrology is the study of these lower limits for the estimation of physical parameters [1]. Techniques in quantum metrology can assist in developing devices to measure the fundamental interplay between quantum mechanics and general relativity at state-of-the-art precision. A prime example is the detection of gravitational waves from black-hole mergers by LIGO [2].

Recently there have been investigations of how we can exploit quantum resources to measure space-time parameters such as the Schwarzschild radius r_s and the Kerr parameter a in the rotating Kerr metric [3–6]. Quantum communications were shown to be affected by the rotation of Earth [7]. However, more fundamental effects in general relativity induced by the Kerr metric were not analyzed. One interesting feature of the Kerr metric is the anisotropy of the velocity of light (null geodesics). The rotating massive object causes co- and counterpropagating light to move at faster and slower velocities, respectively.

In this paper, we note that there is a phase shift of comoving light beams at different radial positions in the Kerr metric. We use a Mach-Zehnder (MZ) interferometer to probe for this phase. We isolate the effect by calibrating to a dark port and rotating the interferometer and due to the anisotropy of c, only the Kerr phase term remains. From this, we can construct using quantum information techniques lower bounds for the variance of the parameter estimation of a [3,6,8].

Locally, we can find a corotating frame in which the space-time is locally flat ("the zero angular momentum ring-riders") [9]. We find that the locally measured velocity of light is $c = 1$ as expected in the flat metric. If an observer Alice compares the locally measured time with Bob who is a ring-rider at a different radius, there will be a disagreement of simultaneity of events. We also consider nonstationary observers that are moving in the rotational plane of Earth. As expected, we find an additional phase term from rotation and special relativistic time dilation. We find that this term is dominant compared to the Kerr phase. Finally, we compare the magnitude of the Kerr phase on Earth to that achievable by microwave resonator experiments [10].

This paper is organized as follows. We first introduce the full Kerr metric in Sec. II for a rotating black hole. In Sec. II A, we approximate the Kerr metric to first order in angular momentum where the mass quadrupole moment for massive planets or stars is dropped in the weak field limit. In Sec. II B, we solve for the null geodesic to determine the velocity of light in the equatorial plane. We find the anisotropy in c. Next in Sec. II C, we calculate the "height differential effect" which could be detected by a Mach-Zehnder interferometer above a massive planet.

In Sec. III, we determine quantum limits of the Kerr space-time parameter a for the height differential effect. In Sec. III A, we focus on the stationary Mach-Zehnder interferometer in the weak field limit and calculate the phase shift. We comment on how we can calibrate to a dark port and rotate the interferometer to isolate the Kerr phase. We compare the magnitude of the Kerr phase with the Schwarzschild phase for Earth parameters. In Sec. IV, we use the comoving flat metric in which the so-called "ring-rider" measures $c = 1$. In Sec. V, we demonstrate an alternative calculation using Lorentz transformations between stationary and ring-riders to find the phase detected at the output of the MZ interferometer. We also confirm that the "two-way" velocity of light is $c = 1$ as detected by a Michelson interferometer at rest in the Kerr metric. Furthermore, we consider the motion of nonstationary observers on the rotating planet. In Sec. VI, we consider an extremal black hole and we numerically find the full strong field solution of the Kerr phase. Finally, we conclude by commenting on the feasibility of detecting the light anisotropy.

II. KERR ROTATIONAL METRIC

The metric describing the space-time of an axially symmetric rotating massive body is given by the Hartle-Thorne metric, which includes the dimensionless mass quadrupole moment q and the angular momentum j of the massive body [11,12]. The mass quadrupole moment $q = kj^2$, where k is a numerical constant that depends on the structure of the massive body. The Kerr metric for a black hole is obtained from the Hartle-Thorne metric by setting $q = -j^2$ and transforming to Boyer-Lindquist coordinates [13,14].

A rotating black hole tends to drag the space-time with its rotation. The Kerr metric used to describe this space-time includes the Kerr rotation parameter a, which quantifies the amount of space-time drag. The Kerr line element in Boyer-Lindquist coordinates (t, r, θ, ϕ) is [9,11,15]

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_s r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2 \\ &- \frac{2 r_s r a \sin^2\theta}{\Sigma} d\phi dt, \end{aligned} \tag{1}$$

where $\Delta \coloneqq r^2 - r_s r + a^2$, $\Sigma \coloneqq r^2 + a^2\cos^2\theta$, and $a = \frac{J}{Mc}$, where J is the angular momentum of the black hole of mass M. Note that the Schwarzschild radius $r_s = \frac{2GM}{c^2} \equiv 2M$, where we work in natural units for which $c = 1$ and $G = 1$. Compared with the Schwarzschild metric, the cross term $dtd\phi$ introduces a coupling between the motion of the black hole and time, which leads to interesting effects.

When $r_s = 0$, the space-time is flat and reduces to $ds^2 = -dt^2 + \frac{1}{1+\frac{a^2}{r^2}} dr^2 + (r^2 + a^2) d\phi^2$. At first glance, this metric does not seem flat. However, we have used the oblong sphere coordinates $x = \sqrt{r^2 + a^2}\sin\theta\cos\phi$, $y = \sqrt{r^2 + a^2}\sin\theta\sin\phi$, and $z = r\cos\theta$.

A. Approximate Kerr metric for rotating massive bodies

The mass quadrupole moment of a *massive planet* is proportional to the angular momentum squared. Thus, we cannot use the Kerr metric in Eq. (1), where the proportionality constant for *black holes* is $k = -1$. However, in the weak field limit $a \ll r$, we can truncate the Kerr metric to first order in $\frac{a}{r}$. Thus the approximate Kerr metric is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 \\ &+ r^2\sin^2\theta d\phi^2 - \frac{2 r_s a \sin^2\theta}{r} d\phi dt. \end{aligned} \tag{2}$$

This approximate Kerr metric disregards the mass quadrupole moment of the massive body. It is equivalent to the Hartle-Thorne metric with the same first order approximation [16]. When we refer to a massive planet or star, we will use this approximate Kerr metric. We wish next to determine the tangential velocity of light close to the massive object as seen by a far-away observer.

B. Far-away velocity of light

In the equatorial plane (where $\theta = \frac{\pi}{2}$), for the null light geodesic, we set $ds^2 = 0$ and determine the solution for the tangential velocity of light according to Kerr time coordinate t. The Kerr time coordinate corresponds to a clock from the gravitating massive body; hence this is the speed of light inferred by a far-away observer. Using Eq. (2),

$$\begin{aligned} ds^2 = 0 = &-\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 \\ &+ r^2 d\phi^2 - \frac{2 r_s a}{r} d\phi dt. \end{aligned} \tag{3}$$

The tangential distance is $dx = r d\phi$ and the light geodesic solution is

$$0 = -(1 - r_s/r) + \dot{x}^2 - \frac{2 r_s a}{r^2}\dot{x}, \tag{4}$$

where $\dot{x} = \frac{dx}{dt}$. However, if $\frac{a}{r} \ll 1$ and $\frac{r_s}{r} \ll 1$, we have the weak field solution:

$$\begin{aligned} \frac{dx}{dt} &\approx \frac{r_s a}{r^2} \pm \sqrt{1 - \frac{r_s}{r}} \\ &\approx \pm\left(1 - \frac{r_s}{2r} \pm \frac{r_s a}{r^2}\right), \end{aligned} \tag{5}$$

where we have used the Taylor expansion $\sqrt{1-x} \approx 1 - \frac{x}{2}$. We have two solutions representing counter- and corotating light. Notice that, locally, $\frac{dx}{dt}\frac{dt}{d\tau} \approx (1 - \frac{r_s}{2r} + \frac{r_s a}{r^2})(1 + \frac{r_s}{2r}) = 1 + \frac{r_s a}{r^2}$ can exceed 1 for the positive solution. However, we cannot naively use the Schwarzschild coordinate time in this curved metric. Later we will show that there is a locally flat metric where $c = 1$.

C. Height differential effect

Let us consider a stationary observer in the Kerr metric sending comoving beams of light that travel tangentially at velocities $c_1 = 1 - v_1 - \frac{r_s}{2r_1}$ and $c_2 = 1 - v_2 - \frac{r_s}{2r_2}$ at radiuses r_1 and $r_2 = r_1 + h$, where h is the coordinate height. For simplicity we made the weak field approximation and only retained terms from Eq. (5) to first order in $v_{1,2} = \frac{r_s a}{r_{1,2}^2}$. The light travels the distance L with time $t_1 = \frac{L}{c_1}$. Similarly, the second observer measures the travel time $t_2 = \frac{L}{c_2}$. The far-away observer agrees that the length L is the same for both. Thus the time delay to first order is

$$\begin{aligned}\Delta t_r &= \frac{L}{c_1} - \frac{L}{c_2} = L\left(\frac{1}{(1 - v_1 - \frac{r_s}{2r_1})} - \frac{1}{(1 - v_2 - \frac{r_s}{2r_2})}\right)\\ &\approx L\left(r_s a\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)\right) + \frac{Lhr_s}{2r_1 r_2}\\ &\approx \frac{Lr_s ah(2r_1 + h)}{r_1^4(1 + \frac{h}{r_1})^2} + \frac{Lhr_s}{2r_1^2(1 + \frac{h}{r_1})}\\ &\approx \frac{2Lr_s ah}{r_1^3} + \frac{Lhr_s}{2r_1^2},\end{aligned} \tag{6}$$

where we have ignored the cross term $\frac{r_s v_1}{2r_1} - \frac{r_s v_2}{2r_2}$ since it is much smaller and enforced the approximation $h \ll r_1$. This time delay can be incorporated into a Mach-Zehnder interferometric arrangement which can be rotated along its center to measure the phase for $+a$ and $-a$ as will be discussed shortly.

III. QUANTUM LIMITED ESTIMATION OF THE KERR SPACE-TIME PARAMETER

Using these time delays, we want to determine the ultimate bound for estimating the Kerr metric parameter a. The variance of an unbiased estimator is determined by the quantum Cramer-Rao (QCR) bound [8]. In quantum information theory, for M number of independent measurements, the QCR bound for the linear phase estimator ϕ is given by $\langle \Delta\hat{\phi}^2\rangle \geq \frac{1}{MH(\Delta\phi)}$, where $H(\phi)$ is the quantum Fisher information which quantifies the amount of information obtainable about a parameter using the optimal measurement.

We have seen that we can measure the phase $\Delta\phi = \omega\Delta t_r$ at different heights, where ω is the central frequency of the probe and Δt_r is given by Eq. (6). The QCR bound for the Kerr rotation parameter is then

$$\frac{\langle \Delta a\rangle}{a} \geq \frac{r_1^3(1 + \frac{h}{r_1})^2}{\omega L a r_s h(2 + \frac{h}{r_1})\sqrt{MH(\Delta\phi)}}. \tag{7}$$

In general $r_1 \gg h$ and therefore the Kerr parameter standard deviation scales as $\langle \Delta a\rangle \gtrapprox \frac{r_1^3}{2\omega L r_s h\sqrt{MN}}$.

A larger height difference h or length L reduces the noise limit. For coherent probe states undergoing linear phase evolution, $H(\phi) = |\alpha|^2 = N$. Therefore, we have the standard quantum noise limit $\propto \frac{1}{\sqrt{N}}$ as expected for coherent probe states. By using nonclassical squeezed states, the noise scales as $\frac{1}{N}$, known as the conventional Heisenberg limit [17–19] or with χ Kerr nonlinearities the noise can scale as $\frac{1}{N^{3/2}}$ [20,21].

A. Mach-Zehnder interferometer

Let us consider a physical system that can detect the discrepancy in the velocity of light from the differential height effect in the Kerr metric. We consider a Mach-Zehnder interferometer (see Fig. 1) that is stationary with respect to the center of mass of a rotating planet. We will work in far-away time coordinates. Although the final implications will be the same, this is an approach where no assumption is made about how the speed of light is measured locally.

The measured phase of the bottom arm of the Mach-Zehnder interferometer is $\Delta\phi_A = \omega\Delta t_A$, where ω is the frequency of light measured locally at the source, Δt_A is the time as seen by a faraway observer, and Φ is a local phase shifter. At $r = r_A$, the far-away time $\Delta t_A = \frac{L}{c_A}$, where c_A is the speed of light as measured by a far-away observer [see Eq. (5)] and L is the arm length also seen by a far-away observer. We have set both arm lengths to be the same. Thus, in the top arm the phase is $\Delta\phi_B = \omega\Delta t_B = \omega\frac{L}{c_B}$.

We assume that $dr = 0$ and the Mach-Zehnder interferometer arms are sufficiently small that the curvature is negligible. The tangential velocity of light depends on R

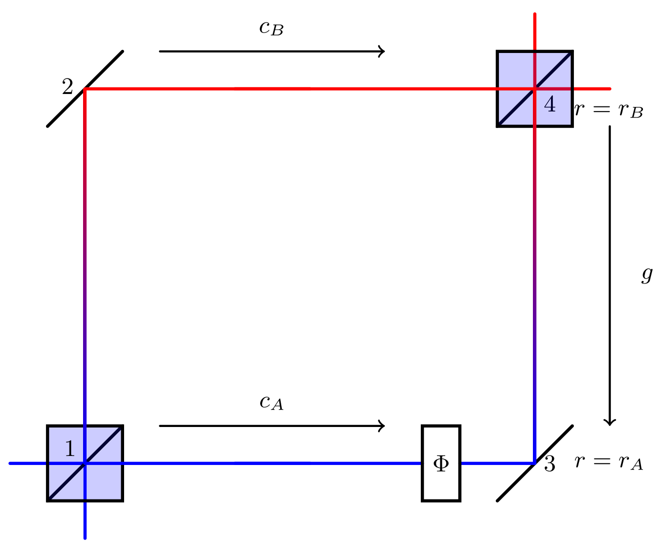

FIG. 1. A Mach-Zehnder interferometer of length L and height h stationary above the rotating planet. Φ is a phase shifter in the bottom arm to calibrate the interferometer to a dark port of zero intensity.

and the sign of a. The solution in the weak field limit is $c' = \frac{dx}{dt} = R\frac{d\phi}{dt} \approx 1 \pm \frac{r_s a}{r^2} - \frac{r_s}{2r}$, where we have chosen the comoving direction such that $c_A \approx 1 - \frac{r_s a}{r_A^2} - \frac{r_s}{2r_A}$ and $c_B \approx 1 - \frac{r_s a}{r_B^2} - \frac{r_s}{2r_B}$. The phase is thus

$$\begin{aligned}\Delta\phi_{MZ} - \Phi &= \omega\left(\frac{L}{c_B} - \frac{L}{c_A}\right)\\ &\approx \omega L\left(\left(1 + \frac{r_s a}{r_B^2} + \frac{r_s}{2r_B} + \frac{r_s^2 a}{r_B^3}\right)\right.\\ &\quad \left. - \left(1 + \frac{r_s a}{r_A^2} + \frac{r_s}{2r_A} + \frac{r_s^2 a}{r_A^3}\right)\right)\\ &\approx \omega L\left(-\frac{r_s a h(2r_A + h)}{r_A^4(1+\frac{h}{r_A})^2} - \frac{h r_s}{2r_A^2(1+\frac{h}{r_A})}\right.\\ &\quad \left. + r_s(\Omega_B - \Omega_A)\right), \end{aligned} \tag{8}$$

where we have used the Taylor expansion $\frac{1}{1-x-y} \approx 1 + x + y + 2xy$. Note that $\Omega_{A,B} = \frac{r_s a}{r_{A,B}^3}$. We have made the approximations $\frac{r_s}{r} \ll 1$, $\frac{a}{r} \ll 1$ and $h \ll r_A$. Note that for the vertical arms, the accumulated phases are equal $\Delta\phi_{12} = \Delta\phi_{34}$, implying that there is no contribution to the total output phase.

We note that on Earth scale the effect of the Kerr rotation parameter is small. If we use the values for the Earth's Schwarzschild radius $r_s = 9$ mm, rotation parameter $a = 3.9$ m, and radius $r_B = 6.37 \times 10^6$ m, and take the area of the interferometer as $A = L \times h = 1\ \mathrm{m}^2$ and the operating frequency of light as $k = 2 \times 10^6\ \mathrm{m}^{-1}$ (with a wavelength of 500 nm), then the order of magnitude of the dominant term for the Kerr rotating effect is

$$|\Delta\phi_{\mathrm{Kerr}}| \approx \frac{2k r_s a L h}{r_B^3} \approx 5 \times 10^{-16}. \tag{9}$$

Conversely, the Schwarzschild time dilation effect is of the order $\Delta\phi_{\mathrm{Schwarzschild}} = \frac{\omega L h r_s}{2 r_A r_B} = 2.2 \times 10^{-10}$. Note that the term $\omega L r_s(\Omega_B - \Omega_A) \approx \omega L \frac{3 r_s a h}{r_B^4} \approx 10^{-22}$ is too small and can be neglected in further calculations.

MZ interferometer calibration. We set the total phase shift $\Delta\phi_{\mathrm{MZ}} = 0$ and thus the phase shifter Φ balances the interferometer to the dark port. Isolating the Kerr phase around the dark port is an optimal strategy for maximizing the signal to noise ratio. We can see in Fig. 2 the phase of the interferometer if it were positioned in the co- and countermoving directions. Thus, we can rotate the Mach-Zehnder interferometer with angle π around its vertical axis and measure the a sign dependence directly. Since only the sign of a changes and Φ stays the same, we have

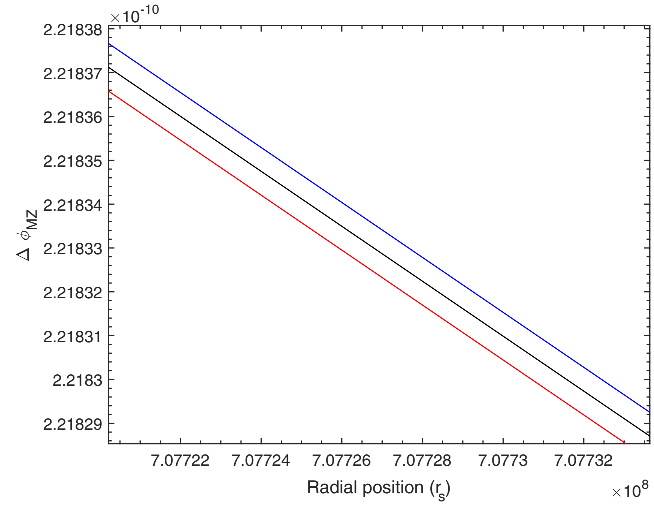

FIG. 2. Measured phase differences of $L = 1$ m and $h = 1$ m Mach-Zehnder interferometer in co- and countermoving directions (blue and red, respectively). The black line is in the Schwarzschild metric with $a = 0$. We use the values for the Earth's Schwarzschild radius $r_s = 9$ mm, rotation parameter $a = 3.9$ m, and the operating frequency of light $\omega = k = 2 \times 10^6\ \mathrm{m}^{-1}$, corresponding to 500 nm measured locally at the source.

$$\begin{aligned}\Delta'\phi_{\mathrm{MZ}} - \Phi &\approx 2\omega L(\Omega_A r_A - \Omega_B r_B - r_s(\Omega_B - \Omega_A))\\ &\approx 2|\Delta\phi_{\mathrm{Kerr}}|. \end{aligned} \tag{10}$$

Therefore, we have a signal which only depends on a. Without the anisotropy of light speed, there would be no signal and the phase would remain a dark port.

IV. ZERO ANGULAR MOMENTUM OBSERVER METRIC

The co- and counterpropagating null light geodesics differ in the Kerr metric. However, locally we expect observers to isotropically measure $c = 1$. It would be useful to transform to a reference frame in which the cross terms $d\phi dt$ vanish and where locally we obtain a flat spacetime metric with $c = 1$ [15]. To determine this transformation, we consider the Killing vectors ∂_t and ∂_ϕ that are responsible for two conserved quantities along the geodesic. These are the energy,

$$E = -k_\mu u^\mu = -g_{t\mu}u^\mu = -p_t = \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau} + \frac{r_s a}{r}\frac{d\phi}{d\tau}, \tag{11}$$

and the angular momentum,

$$\mathcal{L} = g_{\phi\mu}u^\mu = -\frac{r_s a}{r}\frac{dt}{d\tau} + r^2\frac{d\phi}{d\tau}. \tag{12}$$

When we set $\mathcal{L} = 0$, then we have that $\frac{d\phi}{dt} = \frac{r_s a}{r^3}$. Thus there remains an angular motion even with zero angular momentum. The interpretation here is that the rotating space-time drags an object close to the rotating mass, as seen by a far-away observer. If we are corotating in the zero angular momentum reference frame $d\phi' = d\phi_{\mathrm{ring}} + \Omega dt$ with angular velocity $\Omega = \frac{r_s a}{r^3}$, then the metric cross terms $d\phi dt$ cancel out and become

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + r^2 d\phi_{\mathrm{ring}}^2. \tag{13}$$

This is known as the zero angular momentum observer (ZAMO) metric [15]. Taking $dt_{\mathrm{ring}} = \sqrt{1 - \frac{r_{\mathrm{shell}}}{r}} dt$, we have that

$$ds^2 = dt_{\mathrm{ring}}^2 - r^2 d\phi_{\mathrm{ring}}^2, \tag{14}$$

giving a locally flat metric for the ring-riders in which $c = 1$.

We seek the metric in stationary shell coordinates:

$$ds'^2 = dt_s^2 - r_{\mathrm{shell}}^2 d\phi_{\mathrm{ring}}^2, \tag{15}$$

where obviously again $c = 1$ locally.

However, there is a lack of simultaneity between events in the shell metric and events in the ring-rider metric (and hence faraway events). This is the source of the anisotropy of the speed of light. We have from the Lorentz transformation that a spacelike event implies $dt_{\mathrm{ring}} = \gamma(dt_s - v dx_s) = 0$, where $v = \Omega r$ and $dx_s = r_{\mathrm{shell}} d\phi_{\mathrm{ring}}$; thus $dt_s = v r_{\mathrm{shell}} d\phi_{\mathrm{ring}}$.

From the equivalence of the line elements we have

$$ds^2 = ds'^2 - r^2 d\phi_{\mathrm{ring}}^2 = v^2 r_{\mathrm{shell}}^2 d\phi_{\mathrm{ring}}^2 - r_{\mathrm{shell}}^2 d\phi_{\mathrm{ring}}^2. \tag{16}$$

Therefore the ring-rider radius and stationary observer radius are equivalent: $r = r_{\mathrm{shell}}$.

We have redefined the coordinate times of the respective ring-riders as the Schwarzschild time $d\tau = \sqrt{1 - \frac{r_s}{r}} dt$. Between ring-riders, we have the usual Schwarzschild time dilation, as expected. The advantage of the ring-rider frame is that we can use Lorentz transformations to the stationary observer frame to determine the much more significant height differential effect.

V. RING-RIDER PERSPECTIVE

We have previously shown that in the ZAMO flat metric the speed of light is $c = 1$. It is helpful in understanding the physics of our estimation protocols to consider them from the perspective of ring-rider observers. This is also a convenient method to generalize to nonstationary interferometers.

A. Stationary Mach-Zehnder above rotating massive object

Let us consider the Mach-Zehnder interferometer from the reference frames of the ring-riders. The ring-riders are in the flat metric (see Fig. 3). Therefore, for each ring-rider, we can use the Lorentz transformations. We maintain for now the weak field approximations that $\frac{a}{R} \ll 1$ and $\frac{r_s}{r} \ll 1$ such that the Mach-Zehnder interferometer is far enough away from the center of the massive body. Taking into account special relativity, a stationary observer would measure the travel time of light:

$$\begin{aligned} t_1' &= \gamma(t_1 + v_A x_A) = \gamma(L + v_A L) \\ &= \sqrt{\frac{1 + v_A}{1 - v_A}} L \approx (1 + v_A) L, \end{aligned} \tag{17}$$

where $v_A = \Omega_A r_A$ is the relative velocity between the ring-rider and stationary observer at r_A and $t_1 = L$ is the travel time in the ZAMO flat metric. Note that the stationary observer as seen by the ring-rider is traveling in the negative x direction. Similarly, for the ring-rider at R_B, $t_2' = \sqrt{\frac{1+v_B}{1-v_B}} L \approx (1 + v_B) L$, where $v_B = \Omega_B r_B$. For an observer at $r = \infty$, we use the coordinate times of the ZAMO metric. Since the coordinate times are

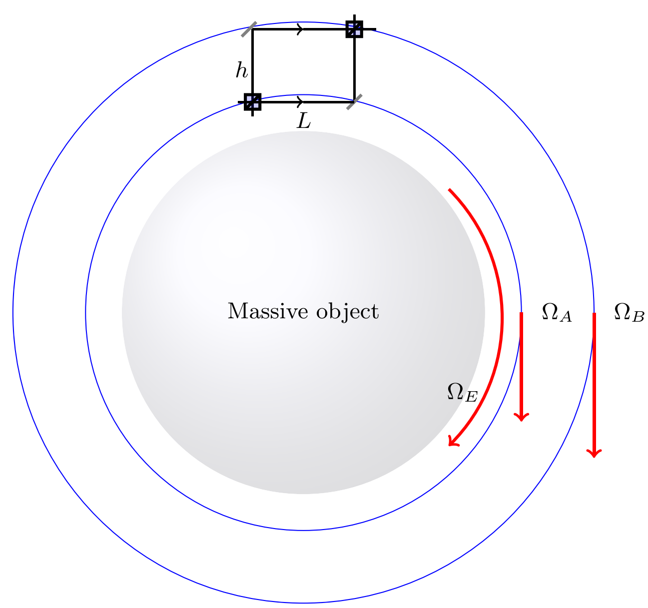

FIG. 3. A Mach-Zehnder interferometer of length L and height h stationary above the rotating massive object in the Kerr metric. Zero angular momentum ring-riders (blue) will have a locally flat space-time with $c = 1$. Their angular frequency as seen from a far-away observer (red) are given by $\Omega_A = \frac{r_s a}{r_A^3}$ and $\Omega_B = \frac{r_s a}{r_B^3}$.

$$t_1'' = \frac{t_1'}{\sqrt{1-\frac{r_s}{r_A}}} \approx \left(1+\frac{r_s}{2r_A}\right)L(1+v_A) \tag{18}$$

and

$$t_2'' = \frac{t_2'}{\sqrt{1-\frac{r_s}{r_B}}} \approx \left(1+\frac{r_s}{2r_B}\right)L(1+v_B), \tag{19}$$

thus the time delay is

$$\begin{aligned}\Delta t = t_2'' - t_1'' &= L\left(\left(1+\frac{r_s}{2r_B}\right)(1+\Omega_B r_B)\right.\\ &\quad\left. - \left(1+\frac{r_s}{2r_A}\right)(1+\Omega_A r_A)\right)\\ &\approx L\left(\Omega_B r_B - \Omega_A r_A - \frac{r_s h}{2r_A r_B} + \frac{r_s}{2}(\Omega_B - \Omega_A)\right).\end{aligned} \tag{20}$$

These calculations are equivalent with using the null geodesics obtained from using the Kerr metric in far-away coordinates in Eq. (8).

B. Michelson interferometer

Given that the far-away observer sees an anisotropic speed of light it is instructive to ask why a local Michelson interferometer fails to see an effect. A stationary observer sends a light beam tangential to the equator that bounces off a mirror L distance away and returns to the observer. The time delay in this signal arm would be:

$$\begin{aligned}\Delta t_{\mathrm{Signal}} &= \frac{L}{\sqrt{1-\frac{r_s}{r}}}(1+v) + \frac{L}{\sqrt{1-\frac{r_s}{r}}}(1-v)\\ &\approx 2L\left(1+\frac{r_s}{2r}\right).\end{aligned} \tag{21}$$

The reference arm perpendicular to the equator is approximately the Schwarzschild local time as found in Eq. (A3) (see Appendix A). This is the same phase as the signal arm $\Delta t_{\mathrm{Ref}} \approx 2L(1+\frac{r_s}{2r})$. Thus the total phase difference is $\Delta\phi_{\mathrm{Michelson}} = 0$, implying that the speed of light is $c = 1$ locally and isotropic, as expected from the special theory of relativity. From the point of view of the far-away observer, although the speed of light is anisotropic, they find that the "two-way" speed, to the mirror and back, is the same in each direction, leading to no phase shift. It may seem a contradiction with the results of the height differential effect, which requires c to be anisotropic to see a signal in the MZ interferometer. However, this is due to a difference in the amount of space-time dragging at different radial positions in the Kerr metric that the MZ interferometer measures nonlocally.

C. Nonstationary comoving observers on Earth

In an experiment conducted, say, on Earth, the rotation of the nonstationary Earth observers must be taken into account. Our previous calculations have considered only a stationary Mach-Zehnder interferometer with the Earth rotating beneath. However, let us consider the bottom arm of the MZ interferometer on Earth's surface with the tangential velocity $v_A' = \Omega_E r_A - \Omega_A r_A$ and the top arm comoving at $v_B' = \Omega_E r_B - \Omega_B r_B$ with the same angular velocity Ω_E of Earth. This relative velocity between observers introduces an additional time dilation.

Using the Lorentz transformations, a stationary observer would measure the travel time of light at r_A:

$$\begin{aligned}t_1' = \gamma(t_1 + v_A x_A) = \gamma(L + v_A' L) = \sqrt{\frac{1+v_A'}{1-v_A'}}L\\ \approx \left(1+v_A' + \frac{v_A'^2}{2}\right)L.\end{aligned} \tag{22}$$

Similarly, for the moving observer at r_B,

$$t_2' = \sqrt{\frac{1+v_B'}{1-v_B'}}L \approx \left(1+v_B'+\frac{v_B'^2}{2}\right)L. \tag{23}$$

For an observer at $r = \infty$, we use the coordinate times of the ZAMO metric, $t_A'' = \frac{t_1'}{\sqrt{1-\frac{r_s}{r_A}}}$ and $t_B'' = \frac{t_2'}{\sqrt{1-\frac{r_s}{r_B}}}$. Thus,

$$\begin{aligned}\Delta t &= t_B'' - t_A''\\ &= L\left(\left(1+\frac{r_s}{2r_B}\right)\left(1+v_B'+\frac{v_B'^2}{2}\right)\right.\\ &\quad\left. - \left(1+\frac{r_s}{2r_A}\right)\left(1+v_A'+\frac{v_A'^2}{2}\right)\right)\\ &\approx L\left(\frac{r_s h}{2r_A r_B} + v_B' - v_A' + \frac{r_s v_B'}{2r_B} - \frac{r_s v_A'}{2r_A}\right)\\ &\approx \Delta t_{\mathrm{MZ}} + \Omega_E hL + \frac{\Omega_E^2 hL(2r_A+h)}{2},\end{aligned} \tag{24}$$

where we have neglected the terms $(\Omega_A r_A)^2$ and $(\Omega_B r_B)^2$. The term $\Omega_E hL$ is a classical effect due to the relative motion of the observers but the term $\frac{\Omega_E^2 h(2r_A+h)L}{2}$ is the higher order correction due to special relativity. We calibrate the MZ interferometer such that the total phase $\Delta\phi_{MZ} = 0$ and then we rotate it. The only remaining terms in Eq. (24) are linear with the rotation. Thus the new phase is

$$\Delta\phi_{\mathrm{MZ}}' = 2\Delta\phi_{\mathrm{Kerr}} + 2\omega_0\Omega_E hL. \tag{25}$$

The Kerr phase varies inversely with r^3, and thus in principle can be distinguished from the classical effect. However, let us consider uneven lengths of the interferometer such that the classical term cancels. Thus we have $r_A L_A = r_B L_B$, and the Kerr phase is

$$
\begin{aligned}
|\Delta\phi_{\mathrm{Kerr}}| &\approx \frac{\omega_0 L_B r_s a}{r_B^2} - \frac{\omega_0 L_A r_s a}{r_A^2} \\
&= \omega_0 r_s a \left(\frac{L_A r_A}{r_A^3 (1+\frac{h}{r_A})^3} - \frac{L_A}{r_A^2} \right) \\
&\approx \frac{3\omega_0 L_A h r_s a}{r_A^3}. \qquad (26)
\end{aligned}
$$

We note that the vertical phases $\Delta\phi_{12}$ and $\Delta\phi_{34}$ are not equal to each other. However, since we rotate the Mach-Zender interferometer through π, then $\Delta'\phi_{12} = \Delta\phi_{34}$ and $\Delta'\phi_{34} = \Delta\phi_{12}$. Thus the phase difference (given calibration to the dark port before rotation) at the output has no contribution from the phases of the vertical arms.

D. Probing the Kerr phase on Earth using MZ interferometer

An interesting calculation is to estimate how compact an object with Earth mass and spin would need to be such that the Kerr term was dominant over the effect of the spin. The relative velocity term is $|\Delta\phi_{\mathrm{Rotation}}| = \omega_0 L \Omega_E h \approx 5 \times 10^{-7}$ for a fixed interferometer with the angular frequency of the Earth $\Omega_E = \frac{7.2\times10^{-5}}{c}$ Hz. To determine a, we need to isolate it from the dominant effect of Earth's rotation.

We can vary the position of the interferometer while keeping its size constant. The contribution from the rotation term $\Delta\phi_{\mathrm{Rotation}} \approx \omega_0 \frac{\Omega_E h L}{2}$ is approximately constant. We want to determine at what radial position the Kerr effect becomes dominant. This occurs when $\Delta\phi_{\mathrm{Kerr}} \geq \Delta\phi_{\mathrm{Rotation}}$. Therefore, $\omega_0 L \frac{r_s a h}{r_B^3} = \omega_0 L \Omega_E h$ which implies that $r_B = (\frac{r_s a}{\Omega_E})^{1/3} \approx 5$ km. Note that the condition $\frac{a}{r_A} \ll 1$ is still satisfied. In Fig. 4, we have the same interferometer over a range of positions extending 2 km around the point at which the Kerr phase becomes significant. Clearly an Earth bound measurement is very far from this condition. However, for a compact object such as a neutron star of the same Schwarzschild radius it is possible in principle.

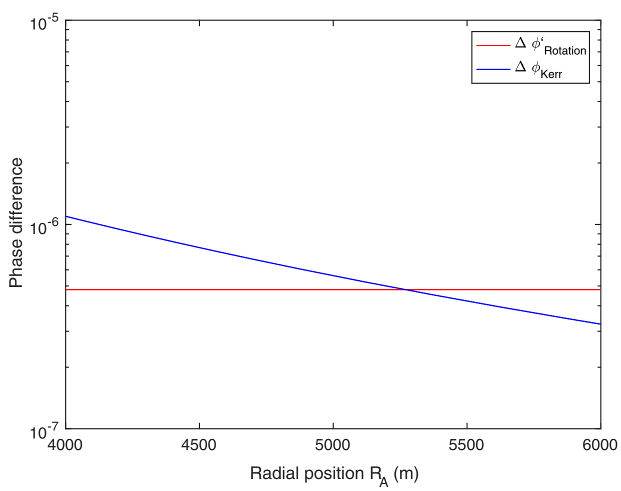

FIG. 4. Measured phase differences of $L = 1$ m and $h = 1$ m Mach-Zehnder interferometer around the radial position at which the Kerr phase (blue) becomes dominant on Earth compared to the phase due to the classical rotation (red).

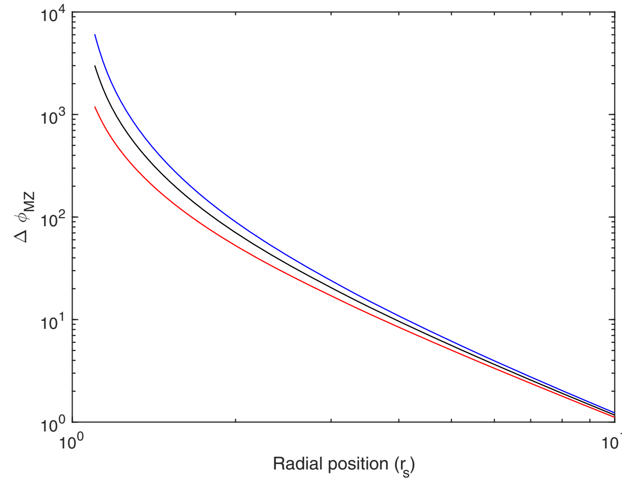

FIG. 5. Measured phase differences of $L = 1$ m and $h = 1$ m Mach-Zehnder interferometer near a black hole of Schwarzschild radius $r_s = 10$ km, angular momentum $a = \frac{r_s}{8}$, and the operating frequency of light $k = 2 \times 10^6$ m^{-1}. Here we have the MZ phases for comoving (red), countermoving (blue), and no rotation $a = 0$ (black).

VI. EXTREMAL BLACK HOLES

To explore the strong field situation, let us now lower our stationary Mach-Zehnder interferometer close to a black hole. We can no longer use the approximations $\frac{a}{r} \ll 1$ and $\frac{r_s}{r} \ll 1$. We must use the full solution of c_A and c_B of the unapproximated Kerr metric as in Eq. (1) and calculated in Appendix B. We note that the Kerr metric is a good description for a collapsed black hole, but not for the exterior metric of neutron stars [22]. We can see in Fig. 5 for a black hole of Schwarzschild radius $r_s = 10$ km and angular momentum $a = \frac{r_s}{8}$, the phase difference for a codirection (red) and counterdirection (blue) Mach Zehnder interferometer. The two directions of the Mach-Zehnder interferometer become increasingly distinguishable as it gets closer to the event horizon at $r = r_s$.

VII. CONCLUSION

We have determined the quantum limits of estimating the Kerr parameter which arises from the anisotropy of the speed of light. We propose a stationary Mach-Zehnder interferometer that can directly measure the Kerr parameter a direction dependence. We identify the flat metric where the ring-rider velocity of light is locally $c = 1$. We find the same Kerr phase using Lorentz transformations between stationary and ring-riders in this ZAMO flat metric. Also, we find that the "two-way" velocity of light is isotropic and $c = 1$ as measured by a Michelson interferometer. However, our Mach-Zehnder interferometer is no longer a dark port after it is rotated by π because of the combined effect of the anisotropy of light and the difference in the amount of space-time dragging in the radial position. On Earth, we have to consider nonstationary observers, which adds an additional classical phase that dominates the Kerr phase. Using a variation on the Mach-Zehnder setup can cancel this additional classical phase with only the Kerr phase remaining.

Recent experiments using microwave resonators have been able to detect the anisotropy of light with a precision of $\Delta c/c \approx 10^{-17}$ [10]. Our Mach-Zehnder interferometer predicts a change in the speed of light due to the Kerr metric of $\Delta c_{\mathrm{Kerr}}/c = \frac{h a r_s}{r^3} \approx 10^{-20}$. In principle, future devices need only increase precision by 3 orders of magnitude to measure the Kerr phase on a small scale Mach-Zehnder interferometer. Using coherent probe states, the noise of the phase is the standard noise limit (SNL) $\Delta\phi \geq \frac{1}{\sqrt{MN}}$. For $M = 10$ GHz measurements [23], this suggests that $N = 10^{22}$–10^{26} per light pulse. This would imply extremely high power, which is one of the current limiting factors to increasing phase sensitivity.

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APPENDIX A: PROPER LENGTH PERPENDICULAR TO THE EQUATOR

Let us consider the proper length perpendicular to the equator. The Kerr metric away from the equator is [9]

$$ds^2 = -\left(1 - \frac{r_s r}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \qquad \text{(A1)}$$

where θ is the azimuth in spherical coordinates, and $\Sigma = r^2 + a^2\cos\theta^2$.

Therefore, we set $dt = 0$ and $dr = 0$ and get the proper distance $d\sigma = \sqrt{r^2 + a^2\cos^2\theta}d\theta$. However, for a massive planet, in the weak field limit, we have $d\sigma = r\sqrt{1 + \frac{a^2}{r^2}\cos^2\theta}d\theta \approx r d\theta$. The velocity of light is given by solving the null geodesic for the weak field Kerr metric:

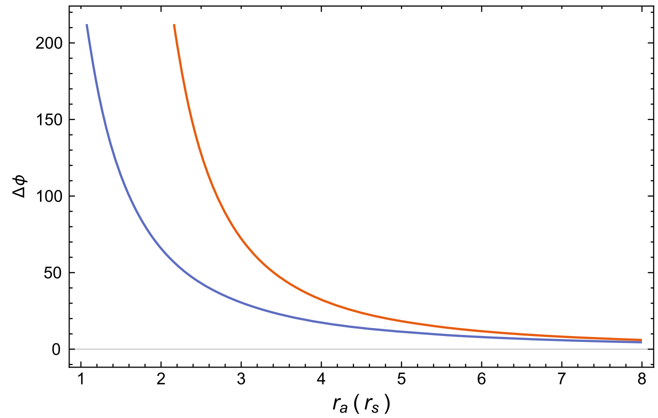

FIG. 6. Difference in exact phase as determined numerically for the full solution of c (red) and weak field approximation (blue). (Note the extremal black hole parameters $r_s = 10$ km, $h' = 10^{-4}$, and $a' = \frac{1}{8}$).

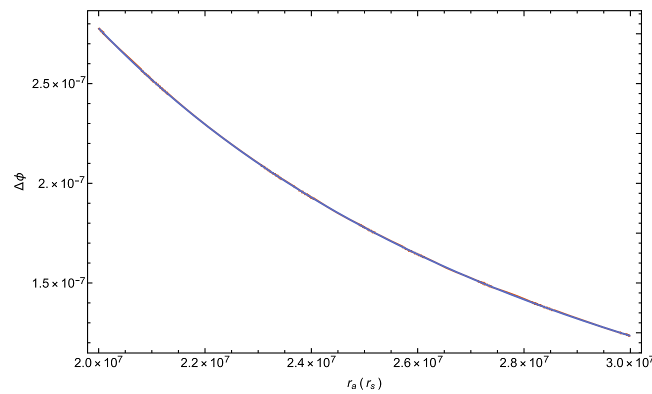

FIG. 7. Difference in exact phase as determined numerically for the full solution of c (red) and weak field approximation (blue) for Earth parameters. (Note that $r_s = 9$ mm, $h' = 111$, and $a' = 433$).

$$\begin{aligned} ds^2 = 0 &= -\left(1 - \frac{r_s r}{r^2 + a^2\cos^2\theta}\right)dt^2 + (r^2 + a^2\cos^2\theta)d\theta^2 \\ &\approx -\left(1 - \frac{r_s}{r}\right)dt^2 + d\sigma^2, \end{aligned} \qquad \text{(A2)}$$

and thus the time traveled by light is

$$\Delta t_{\mathrm{Normal}} = \frac{L}{\frac{d\sigma}{dt}} = \frac{L}{\sqrt{1 - \frac{r_s}{r}}} \approx 2L\left(1 + \frac{r_s}{2r}\right), \qquad \text{(A3)}$$

which is the same as in the Schwarzschild metric.

APPENDIX B: EXTREMAL BLACK HOLES

Let us consider the full solution to the speed of light without any weak field approximations. The phase is therefore

$$\Delta\phi = \omega(t_B - t_A) = kL\left(\frac{1}{c_B} - \frac{1}{c_A}\right), \quad \text{(B1)}$$

where $c_B = \frac{r_s a}{r_B\sqrt{r_B^2 + a^2(1+r_s/r_B)}} \pm \sqrt{\frac{r_s^2 a^2}{r_B^2(r_B^2+a^2(1+r_s/r_B))} + (1 - \frac{r_s}{r_B})}$, using units of r_s, $a \to a' r_s$, $r_A \to r'_A r_s$, and $r_B \to r'_B r_s$. This simplifies to $c_B = \frac{1}{r'_B\sqrt{\frac{r'^2_B}{a'^2} + (1+1/r'_B)}} \pm \sqrt{\frac{1}{r'^2_B((\frac{r'^2_B}{a'^2} + (1+1/r'_B))} + (1 - \frac{1}{r'_B})}$. Let us consider the values of an almost extremal black hole with $r_s = 10\ km$, $a' = \frac{1}{8}$, with $r'_B = r'_A + h'$, where $h' = \frac{1}{10000}$ since $h = 1\ m$. We can see in Fig. 6 the phase difference for the full solution of c (red) and the weak field approximation (blue) for this extremal black hole. The weak field approximation obviously fails near the event horizon. However, for Earth parameters $r_s = 9\ mm$, $h' = 111$ and $a' = 433$ representing $h = 1\ m$ and $a = 3.9\ m$, there is no difference between the exact solution for c and the weak field approximation on the Earth s surface (see Fig. 7).

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