Regular multihorizon black holes in $f(G)$ gravity with nonlinear electrodynamics

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In this work, we study the existence of regular black hole solutions with multihorizons in general relativity and in some alternative theories of gravity. We consider the coupling between the gravitational theory and nonlinear electrodynamics. The coupling generates modifications in the electromagnetic sector. The main objective of this paper is to generalize the solutions already known from general relativity to $f(G)$ theory. First, we correct some misprints of Odintsov and Nojiri's work in order to introduce the formalism that is used in $f(G)$ gravity. In order to satisfy all field equations, the method to find solutions to alternative theories generates different $f(R)$ and $f(G)$ functions for each solution, where only the nonlinear term of $f(G)$ contributes to the field equations. We also analyze the energy conditions, since it is expected that some must be violated to find regular black holes, and using an auxiliary field, we analyze the nonlinearity of the electromagnetic theory.

I. INTRODUCTION

Black holes are one of the most interesting predictions of general relativity [1]. These objects have a region of nonscape where the boundary is a surface which permits the passage only in one direction called the event horizon [2]. The most simple black hole solution is described by the Schwarzschild metric, which is characterized only by its mass [3]. The structure of the Schwarzschild black hole is composed by an event horizon and a singularity in the black hole center [4]. There are several metrics that are more general than the Schwarzschild solution, such as the Reissner-Nordström (electrically charged), Kerr (with rotation), or de Sitter-like solutions (cosmological constant) [5]. The presence of these other parameters can lead to changes in the structure of black holes such as the Cauchy and cosmological horizon [3,5].

Although some solutions have a singularity, this characteristic is not necessary for black holes. Actually, it is possible to find solutions that have an event horizon without singularity; this kind of solution is known as a regular black hole [6]. Bardeen proposed a metric which was later interpreted as a solution to the Einstein equations for nonlinear electrodynamics [7] without the presence of singularities [8]. As in the Reissner-Nordström case, due to the charge, Bardeen's solution presents a Cauchy horizon [9].

Many solutions of regular black holes have arisen since then [10–31], and several studies of their properties have been conducted, such as absorption [32–34], scattering [35,36], quasinormal modes [37–50], thermodynamics [51–61] and even tidal forces [62].

Beyond general relativity, we have the alternatives theories of gravity [63]. The Einstein equations could be obtained from the variational principle if we consider the Einstein-Hilbert action [64]. Modifying this action is a way to find the field equations of these alternative theories [65]. One of the most studied modifications is $f(R)$ theory [63], with R being the curvature scalar, which emerged from Starobinsky's work in 1980, where he inserted an R^2 term into the Einstein-Hilbert action [66,67]. However, we can construct other theories if we consider different curvature invariants in the action. A possible combination of these scalars is the Gauss-Bonnet term G that is a topological invariant in four dimensions [68]. Although G does not make modifications in the field equations, a nonlinear term of G will not be a topological invariant anymore [69–75]. This is what we call $f(G)$ theory, which arises as an alternative to dark matter [68].

As in general relativity, it is possible to find regular solutions in alternative theories [76]. Rodrigues and co-workers generalized the solutions of regular black holes from general relativity to some modified theories of gravity [77–80]. Ghosh *et al.* found a solution to five dimensions when they included a linear term of G in the action [81].

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As we said, due to the presence of charge, rotation, or a cosmological constant, new structures arise in the black holes [82]. So, in the literature, it is possible to find solutions with multiple horizons [83–88]. An example is the Reissner-Nordsröm-de Sitter metric since it has event horizon, Cauchy horizon and cosmological horizon [89]. Actually, if we consider vector-tensor theories, the number of horizons should be much greater than only three. For this type of theory, it is also possible to find metrics with multiple singularities [90]. However, we still have the possibility of constructing regular multihorizon black holes in general relativity and beyond if we relax some energy conditions and assume that the gravitational theory is coupled with nonlinear electrodynamics [91].

This article is organized as follows. In the Sec. II, we construct the formalism for the general relativity theory, considering a spherically symmetric and static source, and we obtain the electromagnetic quantities and the energy conditions associated with the solution. In Sec. III, we use the same formalism and generalize the solutions to $f(R)$ gravity, finding corrections in the electromagnetic sector. In Sec. IV, we construct a regular model, where we generalize the solution with two horizons already known from general relativity to $f(G)$ gravity in four dimensions. In Sec. V, we present our conclusions and discussion. The analytical expressions for $L(P)$ are obtained in Appendix A. As the expressions for the anti–de Sitter example and the solution with three horizons are much too complicated, Appendixes B and C are dedicated to finding the $f(R)$ and $f(G)$ functions that are generated.

II. THE FORMALISM WITH GENERAL RELATIVITY

In this section, we will develop the formalism to study the solutions of regular black holes with multihorizons in the context of general relativity so that we can generalize these results later to the alternative theories of gravity. This procedure was developed in [92] and used to study multihorizon solutions in [91]. The action that describes the nonlinear electromagnetic theory coupled with general relativity is

$$S = \int d^4x \sqrt{-g}[R - 2\kappa^2 L(I)], \qquad (1)$$

where R is the curvature scalar, g is the determinant of the metric, L is the electromagnetic Lagrangian, and I is the electromagnetic scalar defined as $I = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being the Maxwell-Faraday tensor. We will consider a spherically symmetric and static spacetime described by the line element

$$ds^2 = -e^{a(r)}dt^2 + e^{-a(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2)$$

To obtain the electromagnetic and gravitational field equations, we need to vary the Lagrangian (1) with respect

to the gauge potential A_μ and the metric $g_{\mu\nu}$. These equations are

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \kappa^2[L\delta^\mu_\nu - F^{\mu\beta}F_{\beta\nu}\partial_I L], \qquad (3)$$

$$\nabla_\mu(F^{\mu\nu}\partial_I L) = \partial_\mu(\sqrt{-g}F^{\mu\nu}\partial_I L) = 0. \qquad (4)$$

If we consider that the source has only electric charge, the only nonzero component of the Maxwell-Faraday tensor is F^{10}. With this, Eq. (4) becomes

$$\partial_r(r^2\partial_I L F^{01}) = 0. \qquad (5)$$

Integrating this equation, we obtain

$$F^{01} = \frac{q}{r^2}(\partial_I L)^{-1}, \qquad (6)$$

where q is an integration constant that represents the electric charge of the source. Using $I = -\frac{1}{2}(F^{01})^2$ and defining a new variable $X = q\sqrt{-2I}$, we get

$$1 = r^2\partial_X L. \qquad (7)$$

As we can write (3) in the form $G^\mu_\nu = -\kappa^2 T^\mu_\nu$, and identifying $\rho = T^0_0$, $p_r = -T^1_1$, and $p_t = -T^2_2 = -T^3_3$, where ρ is the energy density and p_r and p_t are the radial and tangential pressures, respectively, we find the following equations:

$$p_r = -\rho = L - \frac{X}{r^2}, \qquad p_t = L. \qquad (8)$$

Furthermore, from the components of the Einstein tensor, we also have the following relations:

$$re^a a' + e^a - 1 = -\kappa^2 r^2\rho, \qquad (9)$$

$$re^a a' + e^a - 1 = \kappa^2 r^2 p_r, \qquad (10)$$

$$re^a a'' + re^a a'^2 + 2e^a a' = 2\kappa^2 r p_t. \qquad (11)$$

From these relations and (8), we have

$$\rho = -\frac{1}{\kappa^2 r^2}\frac{d}{dr}[r(e^a - 1)], \qquad L = \frac{X}{r^2} - \rho,$$

$$p_t = -\rho - \frac{r}{2}\rho', \qquad X = -\frac{r^3}{2}\rho'. \qquad (12)$$

Since we have the energy density, in terms of the radial coordinate, it is possible to find an analytical expression for $I(r)$ and invert this function to obtain $r(I)$ and consequently $L(I)$. However, as we are studying electric sources, it is not possible to find the Lagrangian. Actually, in the electric case, is more convenient to work with the auxiliary field $P_{\mu\nu} = (\partial_I L)F_{\mu\nu}$. In Appendix A, we will find analytical expressions for $L(P)$ for the solutions presented here.

It is also important to verify the energy conditions that are given by the following:

(1) null energy condition (NEC) $\rho + p_r \geq 0$ and $\rho + p_t \geq 0$;

(2) weak energy condition (WEC) $\rho \geq 0, \rho + p_r \geq 0$, and $\rho + p_t \geq 0$;

(3) strong energy condition (SEC) $\rho + p_r + 2p_t \geq 0$, $\rho + p_r \geq 0$, and $\rho + p_t \geq 0$;

(4) dominant energy condition (DEC) $\rho \geq 0$, $\rho \pm p_r \geq 0$, and $\rho \pm p_t \geq 0$.

Here we call attention to a mistake in (21) of [91], where p_r is defined. By the condition $p_r = -\rho$, we expect that $p_r = L - \frac{X}{r^2}$; however in (21) of [91], we see $p_r = L + \frac{X}{r^2}$. This modification changes the energy conditions so that they also must be corrected.

In [91], three solutions were analyzed. Here we will review these solutions with the necessary corrections.

A. Example in the Minkowski background

The first model is a solution that behaves asymptotically as Minkowski and is described by

$$e^{a(r)} = 1 - \frac{\alpha r^2}{\beta + r^3} = \frac{(r + r_0)(r - r_1)(r - r_2)}{\beta + r^3}, \quad (13)$$

with

$$\alpha = \frac{r_1^2 + r_2^2 + r_1 r_2}{r_1 + r_2}, \qquad \beta = \frac{r_1^2 r_2^2}{r_1 + r_2}, \qquad r_0 = \frac{r_1 r_2}{r_1 + r_2}. \quad (14)$$

If we take $\alpha = 2m$ and $\beta = 2l^2 m$, we will recover the regular black hole of Hayward [see Eq. (5) in [22]].

Analyzing the regularity of the solution is enough to calculate the Kretschmann scalar $K = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$; however, for future reasons, we will also calculate the curvature scalar and the Gauss-Bonnet invariant $G = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$,

$$R(r) = \frac{6\alpha\beta(2\beta - r^3)}{(\beta + r^3)^3}, \quad (15)$$

$$K(r) = \frac{12\alpha^2(2\beta^4 + r^{12} - 4\beta r^9 + 18\beta^2 r^6 - 2\beta^3 r^3)}{(\beta + r^3)^6}, \quad (16)$$

$$G(r) = \frac{12\alpha^2(2\beta^2 + r^6 - 6\beta r^3)}{(\beta + r^3)^4}. \quad (17)$$

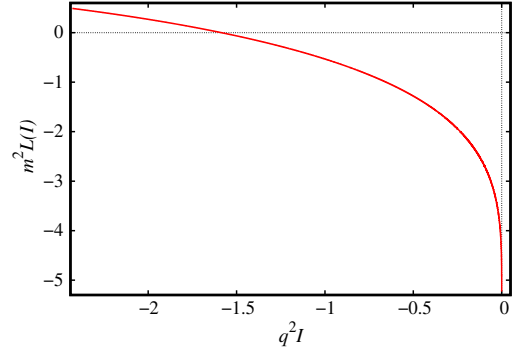We can see that the solution does not present divergences. From (12) we get

FIG. 1. Electromagnetic Lagrangian for the asymptotically flat solution with $\beta = (4/3)m^3$ and $\alpha = (7/3)m$.

$$\rho = -p_r = \frac{3\alpha\beta}{\kappa^2(r^3 + \beta)^2}, \qquad p_t = \frac{3\alpha\beta(2r^3 - \beta)}{\kappa^2(r^3 + \beta)^3},$$

$$X = \frac{9\alpha\beta r^5}{\kappa^2(r^3 + \beta)^3}, \qquad L = \frac{3\alpha\beta(2r^3 - \beta)}{\kappa^2(r^3 + \beta)^3}. \quad (18)$$

The nonlinearity of the electromagnetic theory is described by the graphical representation of $L(I)$ in Fig. 1. Since we have ρ, p_r, and p_t, we obtain the energy conditions

$$\rho \geq 0, \qquad \rho + p_r = 0, \qquad \rho + p_t = \frac{9\alpha\beta r^3}{\kappa^2(r^3 + \beta)^3} \geq 0,$$

$$\rho + p_r + 2p_t = \frac{6\alpha\beta(2r^3 - \beta)}{\kappa^2(r^3 + \beta)^3}, \quad (19)$$

$$\rho - p_r = \frac{6\alpha\beta}{\kappa^2(r^3 + \beta)^2} \geq 0, \qquad \rho - p_t = \frac{3\alpha\beta(2\beta - r^3)}{\kappa^2(r^3 + \beta)^3}. \quad (20)$$

The weak and null energy conditions are always satisfied; however, inside the black hole the strong energy condition is violated, which is normal for regular solutions, and outside the dominant energy condition is violated, which already happened with the Bardeen solution. Now, we highlight that these results present corrections in relation to [91]. From (20), we see that the dominant energy condition is violated for $r > (2\beta)^{1/3}$ and not $r < (2\beta)^{1/3}$ as in (26) of [91]. In [91], we also see that for $r > (\beta/2)^{1/3}$, the relation $\rho - p_r$ is negative, which also violates the dominant energy condition. However, as we corrected the definition of p_r, that problem vanishes.

B. Example in the anti–de Sitter background

Now we will analyze the model that is asymptotically anti–de Sitter and is described by

$$e^{a(r)} = \frac{(r^2 - r_1^2)(r^2 - r_2^2)}{r_1 r_2(r^2 + r_1 r_2)}. \quad (21)$$

The curvature invariants are

$$R(r) = \frac{2}{r_1 r_2 (r^2 + r_1 r_2)^3}(r^4(r_1^2 - 16r_1 r_2 + r_2^2) + 3r^2 r_1 r_2(r_1^2 - 4r_1 r_2 + r_2^2) + 6r_1^2 r_2^2(r_1^2 + r_1 r_2 + r_2^2) - 6r^6), \quad (22)$$

$$K(r) = \frac{4}{r_1^2 r_2^2 (r^2 + r_1 r_2)^6}(6r^{12} - 2r^{10}(r_1^2 - 16r_1 r_2 + r_2^2) + r^8(r_1^4 - 8r_1^3 r_2 + 72r_1^2 r_2^2 - 8r_1 r_2^3 + r_2^4)$$
$$+ 4r^6 r_1 r_2(r_1^4 - 5r_1^3 r_2 + 18r_1^2 r_2^2 - 5r_1 r_2^3 + r_2^4) + r^4 r_1^2 r_2^2(19r_1^4 + 20r_1^3 r_2 + 92r_1^2 r_2^2 + 20r_1 r_2^3 + 19r_2^4)$$
$$+ 6r^2 r_1^3 r_2^3(r_1^4 - 3r_1^3 r_2 - 2r_1^2 r_2^2 - 3r_1 r_2^3 + r_2^4) + 6r_1^4 r_2^4(r_1^2 + r_1 r_2 + r_2^2)^2), \quad (23)$$

$$G(r) = \frac{8}{r_1^2 r_2^2 (r^2 + r_1 r_2)^4}(3r^8 - r^6(r_1^2 - 10r_1 r_2 + r_2^2) - 2r^4 r_1 r_2(2r_1^2 - 5r_1 r_2 + 2r_2^2)$$
$$- 3r^2 r_1 r_2(r_1^4 + 7r_1^3 r_2 + 8r_1^2 r_2^2 + 7r_1 r_2^3 + r_2^4) + 3r_1^2 r_2^2(r_1^2 + r_1 r_2 + r_2^2)^2), \quad (24)$$

so that the solution is regular in all spacetimes. As in the previous example, from (12) we get

$$\rho(r) = \frac{-3r^4 + r^2(r_1^2 - 4r_1 r_2 + r_2^2) + 3r_1 r_2(r_1^2 + r_1 r_2 + r_2^2)}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^2} = -p_r(r), \quad (25)$$

$$X(r) = \frac{r^4(r_1 + r_2)^2(r^2 + 5r_1 r_2)}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^3}, \quad (26)$$

$$L(r) = \frac{3r^6 + 9r^4 r_1 r_2 + r^2 r_1 r_2(r_1^2 + 11r_1 r_2 + r_2^2) - 3r_1^2 r_2^2(r_1^2 + r_1 r_2 + r_2^2)}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^3} = p_t(r). \quad (27)$$

We can see that as $p_r \neq p_t$ we have the behavior of an anisotropic solution with an equation of state $\rho = -p_r$. We highlight that the expression for $X(r)$ is different from Eq. (29) in [91]. In the infinity of the radial coordinate $L(r)$ and $X(r)$ tend to a constant and the nonlinear dependence of $L(I)$ is represented in Fig. 2. The energy conditions are

$$\rho = \frac{-3r^4 + r^2(r_1^2 - 4r_1 r_2 + r_2^2) + 3r_1 r_2(r_1^2 + r_1 r_2 + r_2^2)}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^2}, \quad \rho + p_r = 0, \quad \rho + p_t = \frac{r^2(r_1 + r_2)^2(r^2 + 5r_1 r_2)}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^3},$$

$$\rho + p_r + 2p_t = \frac{2(3r^6 + 9r^4 r_1 r_2 + r^2 r_1 r_2(r_1^2 + 11r_1 r_2 + r_2^2) - 3r_1^2 r_2^2(r_1^2 + r_1 r_2 + r_2^2))}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^3},$$

$$\rho - p_r = \frac{-6r^4 + 2r^2(r_1^2 - 4r_1 r_2 + r_2^2) + 6r_1 r_2(r_1^2 + r_1 r_2 + r_2^2)}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^2},$$

$$\rho - p_t = \frac{-6r^6 + r^4(r_1^2 - 16r_1 r_2 + r_2^2) + 3r^2 r_1 r_2(r_1^2 - 4r_1 r_2 + r_2^2) + 6r_1^2 r_2^2(r_1^2 + r_1 r_2 + r_2^2)}{\kappa^2 r_1 r_2(r^2 + r_1 r_2)^3}. \quad (28)$$

We can see that the energy density is not always positive, which is not a problem since we are considering a model that is asymptotically anti–de Sitter so that we have the contribution of the negative cosmological constant, which can be regarded as a negative energy density. For points outside the event horizon, WEC and DEC are violated while SEC is violated inside the event horizon.

C. Regular black hole with three horizons

Now we will use the formalism that we developed in the previous section to study the multihorizon solution. We consider the configuration described by

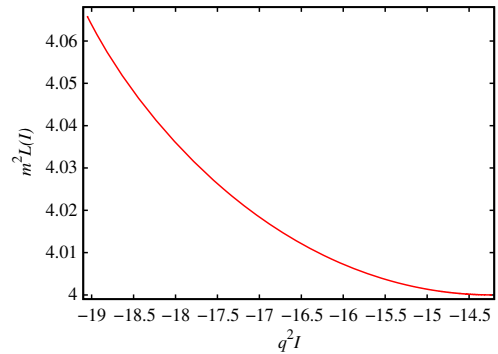$$e^{a(r)} = -\frac{(r^2 - r_1^2)(r^2 - r_2^2)(r^2 - r_3^2)}{l^2(r^2 + \frac{r_1 r_2 r_3}{l})^2}. \quad (29)$$

FIG. 2. Graphical representation of $L(I) \times I$ associated with the asymptotically anti–de Sitter solution with $r_1 = 0.5m$ and $r_2 = 1.5m$.

If we expand this solution near the origin and in the infinity of the radial coordinate, we get

$$e^{a(r)} \approx 1 - r^2 \left(\frac{2l}{\gamma} + \frac{\delta}{\gamma^2} \right) + O(r^3), \quad \text{for } r \to 0, \tag{30}$$

$$e^{a(r)} \approx -\frac{r^2}{l^2} + \frac{l\omega + 2\gamma}{l^3} + O\left(\frac{1}{r^2} \right), \quad \text{for } r \to \infty, \tag{31}$$

with

$$\omega = r_1^2 + r_2^2 + r_3^2, \qquad \gamma = r_1 r_2 r_3, \qquad \delta = r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2. \tag{32}$$

The solution behaves like de Sitter for points inside and far from the event horizon (r_2). The curvature and topological invariants are

$$R(r) = \frac{2}{(lr^2 + \gamma)^4} (l^4 r^6 + 4l^3 r^4 \gamma + l^2 (6r^8 - r^6 \omega + 3\gamma^2) + 2l\gamma(11r^6 - 2r^4 \omega - 3r^2 \delta + 6\gamma^2) + \gamma^2 (28r^4 - 15r^2 \omega + 6\delta)), \tag{33}$$

$$\begin{aligned} K(r) = {}& \frac{4}{(lr^2 + \gamma)^8} (l^8 r^{12} + 8l^7 \gamma r^{10} + 2l^6 (r^6 - \omega r^4 + \delta r^2 + 13\gamma^2) r^8 + 4l^5 \gamma (3r^6 - 3\omega r^4 + 3\delta r^2 + 11\gamma^2) r^6 \\ & + l^4 (6r^{12} - 2\omega r^{10} + \omega^2 r^8 - 2(\delta\omega - 12\gamma^2) r^6 + 14(\delta^2 - 2\omega\gamma^2) r^4 - 48\gamma^2 \delta r^2 + 157\gamma^4) r^4 + 4l^3 \gamma (11r^{12} \\ & - 3\omega r^{10} + (\omega^2 - 4\delta) r^8 + (11\delta\omega + 9\gamma^2) r^6 - (11\delta^2 + 46\omega\gamma^2) r^4 + 47\gamma^2 \delta r^2 + 3\gamma^4) r^2 + \gamma^4 (262r^8 - 230\omega r^6 \\ & + (53\omega^2 + 56\delta) r^4 - 30\delta\omega r^2 + 6\delta^2) + 4l\gamma^3 (55r^{10} + 3\omega r^8 - 3(3\omega^2 + 16\delta) r^6 + 7(3\delta\omega + \gamma^2) r^4 - 3(\delta^2 + 5\omega\gamma^2) r^2 \\ & + 6\gamma^2 \delta) + 2l^2 \gamma^2 (70r^{12} - 28\omega r^{10} + (29\omega^2 + 12\delta) r^8 - (64\delta\omega + 141\gamma^2) r^6 + (34\delta^2 + 31\gamma^2 \omega) r^4 - 9\gamma^2 \delta r^2 + 12\gamma^4)), \end{aligned} \tag{34}$$

$$\begin{aligned} G(r) = {}& \frac{8}{(lr^2 + \gamma)^6} (l^4 r^4 (r^6 + 3r^2 \delta - 10\gamma^2) + 2l^3 r^2 \gamma (3r^6 + 3r^4 \omega - r^2 \delta - 9\gamma^2) + l^2 (3r^{12} - r^{10}\omega - 3r^6(\delta\omega - 7\gamma^2) \\ & + r^4 (5\delta^2 + 16\omega\gamma^2) - 33r^2 \gamma^2 \delta + 12\gamma^4) + 2l\gamma(8r^{10} - 3r^8 \omega - 3r^6(\omega^2 + 2\delta) + r^4(11\delta\omega + 28\gamma^2) \\ & - 3r^2(2\delta^2 + 5\omega\gamma^2) + 6\gamma^2 \delta) + \gamma^2 (33r^8 - 45r^6 \omega + 14r^4(\omega^2 + 2\delta) - 15r^2 \delta\omega + 3\delta^2)). \end{aligned} \tag{35}$$

These functions are complicated; however, it is not difficult to see the regularity at the origin. In infinity of the radial coordinate, we get

$$\lim_{r \to \infty} \{R(r), K(r), G(r)\} = \left\{ \frac{12}{l^2}, \frac{24}{l^4}, \frac{24}{l^4} \right\}, \tag{36}$$

which are viable results since the solution is asymptotically de Sitter. The fluid quantities, $X(r)$, and the electromagnetic Lagrangian are

$$\rho(r) = \frac{1}{\kappa^2 (lr^2 + \gamma)^3} [l^3 r^4 + 3l^2 r^2 \gamma + l(3r^6 - r^4 \omega - r^2 \delta + 6\gamma^2) + \gamma(7r^4 - 5r^2 \omega + 3\delta)] = -p_r(r), \tag{37}$$

$$X(r) = \frac{r^4}{\kappa^2 (lr^2 + \gamma)^4} [l^4 r^4 + 4l^3 r^2 \gamma - l^2 (r^4 \omega + 2r^2 \delta - 15\gamma^2) - 2l\gamma(r^4 + 4r^2 \omega - 5\delta) + \gamma^2 (5\omega - 14r^2)], \tag{38}$$

$$L(r) = \frac{1}{\kappa^2 (lr^2 + \gamma)^4} [l^2 (-(3r^8 + r^4 \delta - 6r^2 \gamma^2)) - 2l\gamma(6r^6 + r^4 \omega - 4r^2 \delta + 3\gamma^2) + \gamma^2 (-21r^4 + 10r^2 \omega - 3\delta)] = p_t(r). \tag{39}$$

We see that ρ in (44) of [91] and X in (45) of [91] differ from what we obtained here in (37) and (38), so it is expected that the Lagrangian L_{BCX} is different from (46) in [91]. The nonlinearity of the electromagnetic theory that generates this metric

is described in Fig. 3. As in the solution before, $L(r)$ and $X(r)$ tend to a constant to $r \to \infty$. Together with (37), the energy conditions are

$$\rho + p_r = 0, \tag{40}$$

$$\rho + p_t = \frac{r^2}{\kappa^2(lr^2 + \gamma)^4}[l^4r^4 + 4l^3r^2\gamma - l^2(r^4\omega + 2r^2\delta - 15\gamma^2) - 2l\gamma(r^4 + 4r^2\omega - 5\delta) + \gamma^2(5\omega - 14r^2)], \tag{41}$$

$$\rho + p_r + 2p_t = -\frac{2}{\kappa^2(lr^2 + \gamma)^4}[l^2(3r^8 + r^4\delta - 6r^2\gamma^2) + 2l\gamma(6r^6 + r^4\omega - 4r^2\delta + 3\gamma^2) - \gamma^2(21r^4 - 10r^2\omega + 3\delta)], \tag{42}$$

$$\rho - p_r = \frac{2}{\kappa^2(lr^2 + \gamma)^3}[l^3r^4 + 3l^2r^2\gamma + l(3r^6 - r^4\omega - r^2\delta + 6\gamma^2) + \gamma(7r^4 - 5r^2\omega + 3\delta)], \tag{43}$$

$$\rho - p_t = \frac{1}{\kappa^2(lr^2 + \gamma)^4}[l^4r^6 + 4l^3r^4\gamma + l^2(6r^8 - r^6\omega + 3r^2\gamma^2) + 2l\gamma(11r^6 - 2r^4\omega - 3r^2\delta + 6\gamma^2)$$
$$+ \gamma^2(28r^4 - 15r^2\omega + 6\delta)]. \tag{44}$$

We can see that there is the possibility for all energy conditions to be violated.

III. REGULAR MULTIHORIZON BLACK HOLES IN $f(R)$ GRAVITY WITH NONLINEAR ELECTRODYNAMICS

Now we will consider a line element written as

$$ds^2 = -e^{a(r)}dt^2 + e^{b(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{45}$$

When we have nonlinear electrodynamics coupled with $f(R)$ gravity, the action is given by

$$S_{f(R)} = \int d^4x \sqrt{-g}[f(R) - 2\kappa^2 L(I)]. \tag{46}$$

If we vary this action with respect to the metric, we get the equations of motion for $f(R)$ gravity, which can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -f_R^{-1}[\kappa^2 T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(f - Rf_R)$$
$$- (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R] \equiv -\kappa^2 T_{\mu\nu}^{(\text{eff})}. \tag{47}$$

Identifying $\mathcal{T}_0^{0(\text{eff})} = \rho^{(\text{eff})}$, $\mathcal{T}_1^{1(\text{eff})} = -p_r^{(\text{eff})}$, and $\mathcal{T}_2^{2(\text{eff})} = \mathcal{T}_3^{3(\text{eff})} = -p_t^{(\text{eff})}$, we can write the energy conditions for $f(R)$ gravity as [93–96]

$$NEC_{1,2}(r) = \rho^{(\text{eff})} + p_{r,t}^{(\text{eff})} \geq 0, \tag{48}$$

$$SEC(r) = \rho^{(\text{eff})} + p_r^{(\text{eff})} + 2p_t^{(\text{eff})} \geq 0, \tag{49}$$

$$WEC_{1,2}(r) = \rho^{(\text{eff})} + p_{r,t}^{(\text{eff})} \geq 0, \tag{50}$$

$$DEC_1(r) = \rho^{(\text{eff})} \geq 0, \tag{51}$$

$$DEC_{2,3}(r) = \rho^{(\text{eff})} - p_{r,t}^{(\text{eff})} \geq 0, \tag{52}$$

where for nonlinear electrodynamics, we have

$$\rho^{(\text{eff})} = -\frac{e^{-b}}{4\kappa^2 r^2 f_R}\left\{4\kappa^2 r^2[F^{10}]^2 e^{a+2b}\partial_I L + 4\kappa^2 r^2 e^b L + 4r^2\frac{d^2 f_R}{dr^2} + (8r - 2r^2 b')\frac{df_R}{dr}\right.$$
$$\left. + [(r^2 a' + 4r)b' + 4e^b - 2r^2 a'' - r^2(a')^2 - 4ra' - 4]f_R + 2r^2 e^b f\right\}, \tag{53}$$

$$p_r^{(\text{eff})} = \frac{e^{-b}}{4\kappa^2 r^2 f_R}\left\{4\kappa^2 r^2[F^{10}]^2 e^{a+2b}\partial_I L + 4\kappa^2 r^2 e^b L + (2r^2 a' + 8r)\frac{df_R}{dr} + [(r^2 a' + 4r)b'\right.$$
$$\left. + 4e^b - 2r^2 a'' - r^2(a')^2 - 4ra' - 4]f_R + 2r^2 e^b f\right\}, \tag{54}$$

$$p_t^{(\text{eff})} = \frac{e^{-b}}{4\kappa^2 r^2 f_R}\left\{4\kappa^2 r^2 e^b L + 4r^2\frac{d^2 f_R}{dr^2} + [2r^2(a'-b')+4r]\frac{df_R}{dr} + [(r^2 a'+4r)b'\right.$$
$$\left. + 4e^b - 2r^2 a'' - r^2(a')^2 - 4ra' - 4]f_R + 2r^2 e^b f\right\}. \tag{55}$$

In view of the identity $WEC_3(r) \equiv DEC_1(r)$, one of the conditions is not written. In [79], the following theorem was proven: Given a solution of Eq. (47) of $f(R)$ gravity described by $S_1 = \{a(r), b(r), f(R), L, F^{10}(r)\}$, if there exists a solution in general relativity $S_2 = \{a(r), b(r), \bar{L}, \bar{F}^{10}(r)\}$, then the energy conditions (48)–(52) are identical for S_1 and S_2. This implies that by taking the models (13), (21), and (29) for $f(R)$ gravity, the energy conditions (48)–(52) are the same as what we have in the general relativity cases.

Even if there are no changes in the energy conditions, due to the coupling with $f(R)$ gravity, we have modifications of the structure of $X(r)$ and L. The components of the field equations for $f(R)$ gravity considering $a(r) = -b(r)$ are

$$\frac{e^a((-ra'-4)f_R' + f_R(ra'' + ra'^2 + 2a') - 2rf_R'')}{r}$$
$$- f(R) - 2\kappa^2\rho = 0, \tag{56}$$

$$\frac{e^a((-ra'-4)f_R' + f_R(ra'' + ra'^2 + 2a'))}{r}$$
$$- f(R) + 2\kappa^2 p_r = 0, \tag{57}$$

$$-\frac{e^a((ra'+1)f_R' + rf_R'')}{r} + \frac{f_R(re^a a' + e^a - 1)}{r^2}$$
$$-\frac{f(R)}{2} + \kappa^2 p_t = 0. \tag{58}$$

In the field equations, we need to pay attention that we are using ρ, p_r, and p_t and not $\rho^{(\text{eff})}$, $p_r^{(\text{eff})}$, and $p_t^{(\text{eff})}$ as in the energy conditions.

Subtracting (56) from (57) we get

$$e^a f_R'' + \kappa^2(\rho + p_r) = 0. \tag{59}$$

To obtain $\rho = -p_r$ with $e^a \neq 0$, we need $f_R'' = 0$, so that f_R is

$$f_R = c_0 + c_1 r, \tag{60}$$

where c_0 and c_1 are integration constants. Since we have $R(r)$, we can invert this function and obtain $r(R)$ and then we find $f_R(R)$. The $f(R)$ function is

$$f(R) = c_0 R + c_1\int r(R)dR, \tag{61}$$

or, in terms of the radial coordinate,

$$f(R) = \int f_R\frac{dR}{dr}dr. \tag{62}$$

Using the models (13), (21), and (29), we can find the expressions for ρ, L, and X in $f(R)$ gravity.

A. Example in the Minkowski background

Considering the model (13), we have the curvature scalar (15). Inverting this function, we get that f_R is

$$f_R(R) = c_0 + c_1\left[\frac{\alpha_1(R)}{R} + \frac{2\alpha\beta}{\alpha_1(R)} - \beta\right]^{1/3}, \tag{63}$$

with

$$\alpha_1(R) = \{\alpha\beta R\sqrt{\beta R(81\beta R - 8\alpha)} - 9\alpha\beta^2 R^2\}^{1/3}.$$

Integrating f_R with respect to the curvature scalar, we get

$$f(R) = c_0 R - \frac{c_1\alpha}{\beta^{2/3}}\left\{2\sqrt{3}\tan^{-1}\left[\frac{1}{\sqrt{3}}\left(1 - 2\sqrt[3]{\frac{2\alpha}{\alpha_1(R)} - 1 + \frac{\alpha_1(R)}{\beta R}}\right)\right] - 2\ln\left(1 + \sqrt[3]{\frac{2\alpha}{\alpha_1(R)} - 1 + \frac{\alpha_1(R)}{\beta R}}\right)\right.$$
$$+ \ln\left(1 - \sqrt[3]{\frac{2\alpha}{\alpha_1(R)} - 1 + \frac{\alpha_1(R)}{\beta R}} + \left(\frac{2\alpha}{\alpha_1(R)} - 1 + \frac{\alpha_1(R)}{\beta R}\right)^{2/3}\right) - \frac{1}{72\,2^{2/3}R^2\alpha^3\beta^3}$$
$$\times \sqrt[3]{\frac{2\alpha_1(R)\beta^2(2\alpha - 9\alpha_1(R))}{R\alpha} - 4\beta^3} - \frac{\alpha_1^5(R)}{R^3\alpha^2}(72R^3\alpha^2\beta^3 + R^2\alpha(32\alpha^2 + 60\alpha_1(R)\alpha + 27\alpha_1^2(R))\beta^2$$
$$\left. - 16R\alpha^2\alpha_1^2(R)\beta + \alpha_1(R)(\alpha_1^3(R) + 9\alpha\beta^2 R^2)(8\alpha + 3\alpha_1(R)))\right\}. \tag{64}$$

If we consider the corrections from $f(R)$ gravity, $\rho(r)$, $X(r)$, and $L(r)$ are

$$\rho(r) = \frac{3\alpha\beta c_0}{\kappa^2(\beta + r^3)^2} + \frac{c_1}{2\kappa^2}\left(\frac{6\alpha r}{\beta + r^3} + \frac{\alpha}{\beta^{2/3}}\left(\ln(\beta^{2/3} + r^2 - \sqrt[3]{\beta}r) - 2\ln(\sqrt[3]{\beta} + r) + 2\sqrt{3}\tan^{-1}\left(\frac{1 - \frac{2r}{\sqrt[3]{\beta}}}{\sqrt{3}}\right)\right)\right.$$
$$\left. - \frac{6\alpha r^4}{(\beta + r^3)^2} - \frac{4}{r}\right),$$
(65)

$$L(r) = \frac{3\alpha\beta c_0(2r^3 - \beta)}{\kappa^2(\beta + r^3)^3} - \frac{c_1}{2\kappa^2}\left(\frac{6\alpha r}{\beta + r^3} + \frac{\alpha}{\beta^{2/3}}\left(\ln(\beta^{2/3} + r^2 - \sqrt[3]{\beta}r) - 2\ln(\sqrt[3]{\beta} + r) + 2\sqrt{3}\tan^{-1}\left(\frac{1 - \frac{2r}{\sqrt[3]{\beta}}}{\sqrt{3}}\right)\right)\right.$$
$$\left. + \frac{18\alpha r^7}{(\beta + r^3)^3} - \frac{27\alpha r^4}{(\beta + r^3)^2} - \frac{2}{r}\right),$$
(66)

$$X(r) = \frac{9\alpha\beta c_0 r^5}{\kappa^2(\beta + r^3)^3} - \frac{c_1 r(2\beta^3 + r^8(2r - 3\alpha) + 3\beta r^5(2r - 7\alpha) + 6\beta^2 r^3)}{2\kappa^2(\beta + r^3)^3}.$$
(67)

It is easy to see that if we take the limit $c_1 \to 0$ and $c_0 \to 1$, we recover the results presented in Sec. II A. From (67), we can see that, different from general relativity, due to the coupling with $f(R)$ gravity, $X(r)$ diverges in infinity.

In Einstein's theory, the energy density from the energy conditions is equal to the electromagnetic energy density. When we consider $f(R)$ gravity, these two quantities are not the same. Here, in the energy conditions (52), we have $\rho^{(\text{eff})}$ which is the same from (18), which is always positive, while the electromagnetic energy density is given by ρ, which has negative contributions due to the coupling with $f(R)$ theory.

With this example, we may realize the modifications that appear in the electromagnetic quantities because of $f(R)$ gravity. It is also possible to apply this formalism to the other examples. However, as the expressions are much more complicated, we will do this in Appendix B, where we obtain analytical expressions for f_R, X, ρ, and L.

IV. REGULAR MULTIHORIZON SOLUTIONS IN $f(G)$ GRAVITY

In this section, we will analyze the possibility of generalizing the solutions for $f(G)$ theory in four dimensions. The action that describes this theory is

$$S_{f(G)} = \int d^4x\sqrt{-g}[R + f(G) - 2\kappa^2 L(I)]. \quad (68)$$

Varying this action with respect to the metric, we get the field equations for $f(G)$ theory, which are written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 8[R_{\mu\alpha\nu\beta} + R_{\alpha\nu}g_{\beta\mu} - R_{\alpha\beta}g_{\nu\mu} - R_{\mu\nu}g_{\beta\alpha} + R_{\mu\beta}g_{\nu\alpha}]\nabla^\alpha\nabla^\beta f_G + (Gf_G - f)g_{\mu\nu} = -\kappa^2 T_{\mu\nu}, \quad (69)$$

where the subscript G denotes the derivation with respect to the Gauss-Bonnet term. To analyze the energy conditions, we rewrite the field equations as

$$G_{\mu\nu} = -\kappa^2 T_{\mu\nu} - 8[R_{\mu\alpha\nu\beta} - R_{\alpha\nu}g_{\beta\mu} - R_{\alpha\beta}g_{\nu\mu} - R_{\mu\nu}g_{\beta\alpha} + R_{\mu\beta}g_{\nu\alpha}]\nabla^\alpha\nabla^\beta f_G - (Gf_G - f)g_{\mu\nu} = -\kappa^2 T^{(\text{eff})}_{\mu\nu}, \quad (70)$$

where the components of $T^{(\text{eff})}_{\mu\nu}$ may be identified by

$$T^{(\text{eff})}_{\mu\nu} = \text{diag}(\rho^{(\text{eff})}(r), -p_r^{(\text{eff})}(r), -p_t^{(\text{eff})}(r), -p_t^{(\text{eff})}(r)), \quad (71)$$

where $\rho^{(\text{eff})}(r)$, $p_r^{(\text{eff})}(r)$, and $p_t^{(\text{eff})}(r)$ are the effective energy density, radial pressure, and tangential pressure, respectively, with contributions from $f(G)$ theory. The energy conditions are given by [75]

$$SEC(r) = \rho^{(\text{eff})} + p_r^{(\text{eff})} + 2p_t^{(\text{eff})} \geq 0, \quad (72)$$

$$WEC_{1,2}(r) = NEC_{1,2}(r) = \rho^{(\text{eff})} + p_{r,t}^{(\text{eff})} \geq 0, \quad (73)$$

$$WEC_3(r) = DEC_1(r) = \rho^{(\text{eff})} \geq 0, \quad (74)$$

$$DEC_{2,3}(r) = \rho^{(\text{eff})} - p_{r,t}^{(\text{eff})} \geq 0. \quad (75)$$

As in the case of $f(R)$ theory, it was shown that due to the fact that the energy conditions are taken on the Einstein tensor, the energy conditions in $f(G)$ theory are the same as in general relativity [80].

The nonzero components of the field equations, to a line element as (45) with $a = -b$, are

$$\frac{e^a(4f_G a'' + a'(4f_G a' - 4f_G' + r) - 8f_G'' + 1) - 4e^{2a}(-3a'f_G' + f_G(a'' + 2a'^2) - 2f_G'') - 1}{r^2} + f(G) + \kappa^2\rho = 0, \quad (76)$$

$$\frac{e^a(4f_G a'' + a'(4f_G a' - 4f_G' + r) + 1) - 4e^{2a}(f_G(a'' + 2a'^2) - 3a'f_G') - 1}{r^2} + f(G) - \kappa^2 p_r = 0, \quad (77)$$

$$\frac{e^a(a''(8e^a(rf_G' - f_G) + 8f_G + r^2) + 2ra'(4e^a f_G'' + 1) + a'^2(8f_G + r^2 - 16e^a(f_G - rf_G'))) }{2r^2} + f(G) - \kappa^2 p_t = 0. \quad (78)$$

Subtracting (76) from (77) we obtain

$$\kappa^2(\rho + p_r) + \frac{8e^a(e^a - 1)f_G''}{r^2} = 0. \quad (79)$$

As $e^a \neq 1$ and $e^a \neq 0$, in general, we need $f_G'' = 0$ to obtain the condition $\rho = -p_r$. Consequently, f_G is

$$f_G = c_0 + c_1 r. \quad (80)$$

As we did in $f(R)$ theory, we need to find an expression for $r = r(G)$ and with that construct the functions $f_G(G)$ and $f(G)$ by
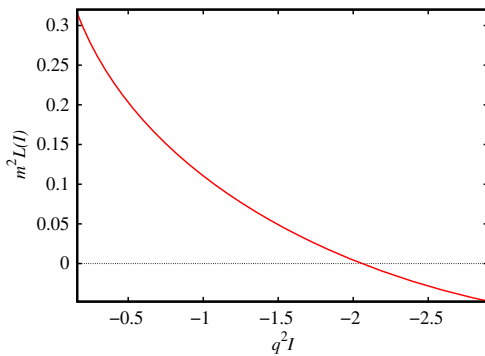
$$f(G) = c_0 G + c_1 \int r(G) dG, \quad (81)$$

or, in terms of the radial coordinate,

$$f(G) = \int f_G \frac{dG}{dr} dr. \quad (82)$$

A. Example in the Minkowski background

In [80], the $f_G(G)$ and $f(G)$ functions were constructed for the models (13), (A8), and (A10) in [80], with $\alpha = 2m$ and $\beta = q^3$, where, due to astrophysical constraints $q^2 \ll m^2$ [97], with m being the Arnowitt-Deser-Misner mass. We expected that the Arnowitt-Deser-Misner mass assumes values of 10^{-17} $M_\odot$ for primordial black holes [98] and up to 10^9 $M_\odot$ for supermassive black holes [99]. In Fig. 4, we plot the $f(G)$ function that generates the solution and we compare to the linear function. Graphically, these functions are different, and for some range of G, the intensity of $c_1 = 0$ is smaller. Even if there are no changes in the energy conditions, we will get corrections for the functions $\rho(r)$, $X(r)$, and L. These functions are given by

FIG. 3. Behavior of the electromagnetic Lagrangian in terms of I associated with the solution with three horizons for $r_1 = 0.5m$, $r_2 = 1.5m$, $r_3 = 16m$, and $l = 20m$.

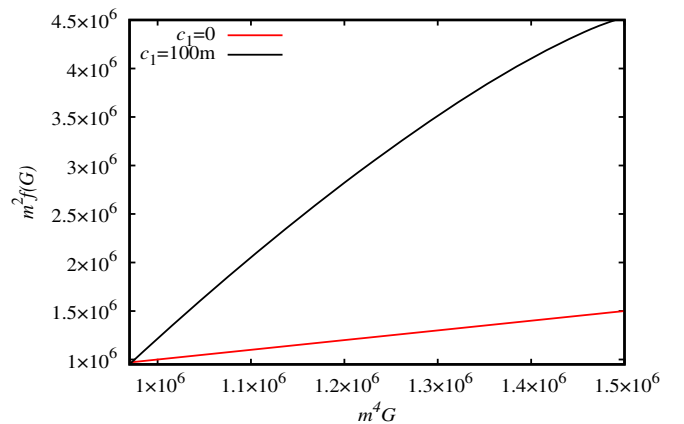FIG. 4. Comparison between the $f(G)$ function, that generates the solution (13), and a linear term of G with $q = 0.1m$.

$$\rho(r) = \frac{3\alpha\beta}{\kappa^2(\beta+r^3)^2} + \frac{\alpha c_1}{3\beta^{5/3}\kappa^2}\left(\frac{12\beta^{2/3}}{r(\beta+r^3)^3}(4\beta^3+\alpha r^8+\beta r^5(5\alpha-2r)+2\beta^2 r^2(r-\alpha))\right.$$

$$\left. -4\alpha\left(\ln(\beta^{2/3}+r^2-\sqrt[3]{\beta}r)-2\ln(\sqrt[3]{\beta}+r)+2\sqrt{3}\tan^{-1}\left(\frac{1-\frac{2r}{\sqrt[3]{\beta}}}{\sqrt{3}}\right)\right)\right) = -p_r, \tag{83}$$

$$L(r) = \frac{3\alpha\beta(2r^3-\beta)}{\kappa^2(\beta+r^3)^3} + \frac{4\alpha c_1}{3\beta^{5/3}\kappa^2}\left(\alpha\left(\ln(\beta^{2/3}+r^2-\sqrt[3]{\beta}r)-2\ln(\sqrt[3]{\beta}+r)+2\sqrt{3}\tan^{-1}\left(\frac{1-\frac{2r}{\sqrt[3]{\beta}}}{\sqrt{3}}\right)\right)\right.$$

$$\left. -\frac{3\beta^{2/3}(2\beta^4+\alpha r^{11}+2\beta r^9-12\beta^2 r^5(r-2\alpha)-2\beta^3 r^2(\alpha+6r))}{r(\beta+r^3)^4}\right) = p_t, \tag{84}$$

$$X(r) = \frac{9\alpha\beta r^5}{\kappa^2(\beta+r^3)^3} + \frac{4\alpha c_1 r(2\beta^3-4r^9+6\alpha r^8+12\beta r^6-21\alpha\beta r^5+18\beta^2 r^3)}{\kappa^2(\beta+r^3)^4}. \tag{85}$$

We realize that the results of general relativity are recovered with $c_1 \to 0$ since c_0 does not appear. The constant c_0 does not appear in the equations due to the fact that it follows the linear term in the Gauss-Bonnet invariant, which in four dimensions does not make modifications in the field equations. Different from $f(R)$ gravity, X is well behaved for $r \to \infty$. From Sec. II, we know that $I = -\frac{1}{2}(F^{10})^2$ and $X = q\sqrt{-2I}$, which implies the electric field and X have the same behavior in relation to the radial coordinate so that F^{10} is well behaved in all spacetimes and goes to zero in the black hole center and in infinity. In Fig. 5, we plot the electromagnetic energy density and pressures as functions of the radial coordinate. The energy density presents a minimal value where the radial pressure has a maximum, while the tangential pressure presents the minimum and maximum. In general relativity, normal to an electrically charged regular solution, we have the behavior of an anisotropic fluid, but near the
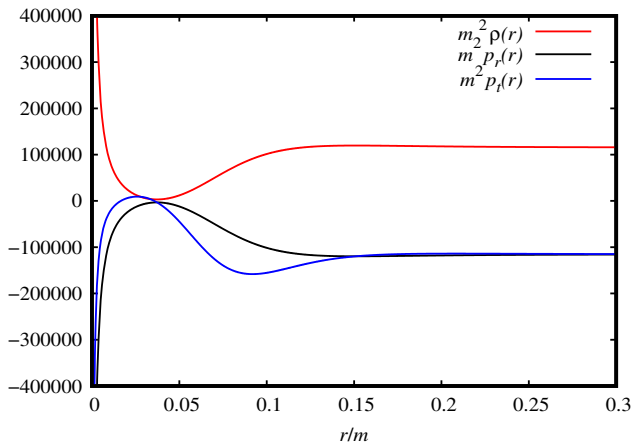
black hole center, it behaves approximately as an isotropic fluid, which is different in $f(G)$ gravity. Expanding these functions near the center, we find

$$\rho(r) \approx \frac{32c_1 m}{\kappa^2 q^3 r} + \frac{2m(-8\pi c_1 m+9\sqrt{3}q^2)}{3\sqrt{3}\kappa^2 q^5} - \frac{80c_1 mr^2}{\kappa^2 q^6}$$
$$+ O(r^3), \tag{86}$$

$$p_r(r) \approx -\frac{32c_1 m}{\kappa^2 q^3 r} + \frac{2m(8\pi c_1 m-9\sqrt{3}q^2)}{3\sqrt{3}\kappa^2 q^5} + \frac{80c_1 mr^2}{\kappa^2 q^6}$$
$$+ O(r^3), \tag{87}$$

$$p_t(r) \approx -\frac{16c_1 m}{\kappa^2 q^3 r} + \frac{2m(8\pi c_1 m-9\sqrt{3}q^2)}{3\sqrt{3}\kappa^2 q^5} + \frac{160c_1 mr^2}{\kappa^2 q^6}$$
$$+ O(r^3). \tag{88}$$

The term that arises from $f(G)$ theory is responsible for the anisotropy in the black hole center. In infinity of the radial coordinate, we get

$$\rho(r) \approx \frac{16\pi c_1 m^2}{\sqrt{3}\kappa^2 q^5} - \frac{16c_1 m}{\kappa^2 r^4} + \frac{192c_1 m^2}{5\kappa^2 r^5} + \frac{6mq^3}{\kappa^2 r^6} + O\left(\frac{1}{r^7}\right), \tag{89}$$

$$p_r(r) \approx -\frac{16\pi c_1 m^2}{\sqrt{3}\kappa^2 q^5} + \frac{16c_1 m}{\kappa^2 r^4} - \frac{192c_1 m^2}{5\kappa^2 r^5} - \frac{6mq^3}{\kappa^2 r^6} + O\left(\frac{1}{r^7}\right), \tag{90}$$

$$p_t(r) \approx -\frac{16\pi c_1 m^2}{\sqrt{3}\kappa^2 q^5} - \frac{16c_1 m}{\kappa^2 r^4} + \frac{288c_1 m^2}{5\kappa^2 r^5} + \frac{12mq^3}{\kappa^2 r^6}$$
$$+ O\left(\frac{1}{r^7}\right). \tag{91}$$

FIG. 5. Behavior of the electromagnetic fluid quantities in terms of the radial coordinate with $c_1 = 1m$ and $q = 0.1m$.
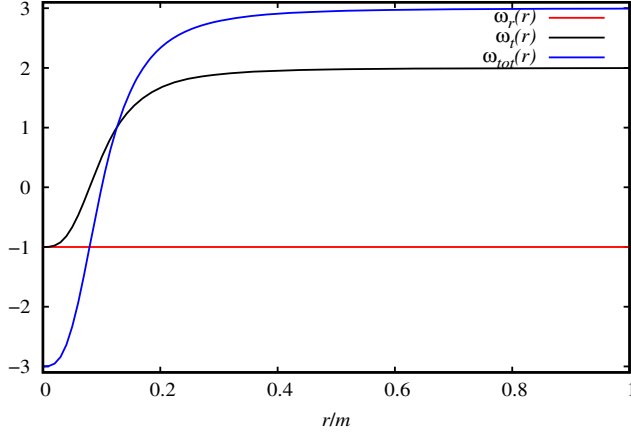
FIG. 6. Parameters of the equations of state in terms of the radial coordinate with $q = 0.1m$.

The term that is constant in infinity comes from $f(G)$ gravity, and it is also responsible for the isotropy in this region. The effective quantities are the same as in general relativity, Eq. (18), where we have isotropy in the black hole center, and the effective energy density and the pressures are zero in infinity. We may consider that the effective energy density is related to the radial, tangential, and total pressures by equations of state as $p_r^{(\mathrm{eff})} = \omega_r \rho^{(\mathrm{eff})}$,

$p_t^{(\mathrm{eff})} = \omega_t \rho^{(\mathrm{eff})}$, and $(p_r^{(\mathrm{eff})} + 2p_t^{(\mathrm{eff})}) = \omega_{\mathrm{tot}} \rho^{(\mathrm{eff})}$. From ω_r, ω_t, and ω_{tot}, we may extract information about the energy conditions. If the effective energy density is positive, the case $\omega_{r,t} < -1$ implies in the violation of $WEC_{1,2}$. However, to $\omega_{r,t} > 1$ $DEC_{2,3}$ is violated. And to $\omega_{\mathrm{tot}} < -1$ SEC is violated. In Fig. 6, we plot the behavior of these parameters. We may see that $\omega_r = -1$, $-1 < \omega_t < 2$, and $-3 < \omega_{\mathrm{tot}} < 3$ so that DEC_3 is violated outside the event horizon, while SEC is violated inside.

As we have an electric source, it is interesting to use the $H(P)$ formalism, which is a dual description of the same physical problem (for more details, see Appendix A). In this formalism, we do not work with the electric field but with the induction field (A1) [100]. In linear electrodynamics, we have $P^{\mu\nu} = F^{\mu\nu}$, but here they are different by the multiplicative term $\partial_t L$. In the nonlinear case, F^{10} does not behave as Maxwellian; however, if we consider the displacement vector, inserting (6) in (A1), we find

$$P^{10} = \frac{q}{r^2}, \tag{92}$$

which is the same result from the linear electrodynamics. Since we have L as a function of the radial coordinate, we can also find an analytical expression for $L(P)$. From (A2), we find an expression for $r(P)$, and with that, $L(P)$ is given by

$$
\begin{aligned}
L(P) = {} & -\frac{12\sqrt[4]{2}\alpha\beta\sqrt{-P}P(2^{3/4}\beta^2\sqrt{-P}P + \beta(-P)^{3/4}q^{3/2} + \sqrt[4]{2}q^3)}{\kappa^2(\beta(-2P)^{3/4} + q^{3/2})^4} \\
& + \frac{\alpha c_1}{3\beta^{5/3}\kappa^2}\left(4\alpha\left(\ln\left(\beta^{2/3} - \frac{\sqrt[3]{\beta}\sqrt{q}}{\sqrt[4]{-2P}} + \frac{q}{\sqrt{-2P}}\right) - 2\ln\left(\sqrt[3]{\beta} + \frac{\sqrt{q}}{\sqrt[4]{-2P}}\right) + 2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}} - \frac{2^{3/4}\sqrt{q}}{\sqrt[3]{\beta}\sqrt{3}(-P)^{1/4}}\right)\right)\right) \\
& - \frac{12\sqrt[4]{2}\beta^{2/3}\sqrt{-P}}{\sqrt{q}(\beta(-2P)^{3/4} + q^{3/2})^4}\left(-48\sqrt[4]{2}\beta^3 P^2 q^{3/2} + 16\beta^4(-P)^{11/4} + 48\,2^{3/4}\alpha\beta^2(-P)^{3/2}q^{5/2} + 2\,2^{3/4}\beta\sqrt{-P}q^{9/2}\right. \\
& \left. + 24\sqrt{2}\beta^2\sqrt[4]{-P}Pq^3 - 8\sqrt{2}\alpha\beta^3(-P)^{9/4}q + \sqrt[4]{2}\alpha q^{11/2}\right).
\end{aligned}
\tag{93}
$$

With that, the nonlinearity of electromagnetic theory is clear. We comment on the model (21) and (29) with $f(G)$ theory in Appendix C 1. In that Appendix, we obtain the analytical expressions for $f_G(G)$ and the electromagnetic quantities considering (21). For the solution (29), we find the electromagnetic quantities; however, it is not possible to write $f_G(G)$ in a closed form.

V. CONCLUSION

In this work, we analyzed the existence of regular multihorizon black holes in general relativity and in some alternative theories of gravity. The method presented with general relativity was developed earlier in [91]; however, we fixed some misprints which influenced the energy

conditions. The energy density is positive in the model (13); however, due to the presence of the cosmological constant, we can have negative energy density in the models (21) and (29). We realized that the electromagnetic function $X(r)$ goes to zero in $r \to \infty$ for the solution that is asymptotically flat but tends to a constant when we consider the solution that is asymptotically anti–de Sitter and the solution with three horizons. The strong energy condition is violated inside the black hole for all solutions and outside for those that are not asymptotically Minkowski.

In modified theories, in order to satisfy all the field equations, we did not fix the $f(R)$ and $f(G)$. Since we imposed the solutions that we wanted to find, we got the correspondent $f(R)$ and $f(G)$ theories and the modifications in the electromagnetic sector. Because of the

coupling with $f(R)$ gravity, $X(r)$ diverges in infinity of the radial coordinate for all solutions, while in $f(G)$ gravity, it diverges only for those that are not asymptotically flat solutions. This means that in $f(G)$ theory, the electric field is well behaved in all spacetimes for the asymptotically flat solution, while it diverges in $f(R)$ theory. Since f_R and f_G are linear in r, we have a divergence in infinity; however, $f(R)$ and $f(G)$, which are the functions that appear in the action, do not present divergence. Because of the symmetry $a = -b$, the energy conditions will be the same as in general relativity.

Since G is topologically invariant in four dimensions, the authors of [91] needed to consider five dimensions to find modifications in the field equations. Here, as we considered a nonlinear function of G in the action, we did not need to consider high dimensions to find new regular solutions.

As we considered electric sources, we showed the nonlinear behavior of the electromagnetic theory numerically by a parametric plot of $L(I)$. In [91], the authors used auxiliary fields B and C to find an analytical expression for what they called L_{BCX}; however, the nonlinearity of this Lagrangian is not so clear so that we considered using the scalar P that is related to the electromagnetic scalar to represent analytically the electromagnetic Lagrangian in Appendix A.

We also analyzed the consequences of $a \neq -b$ in alternative theories of gravity. As this symmetry modifies the line element, differences in the energy conditions and in the curvature invariants are expected to appear, so some choices can give solutions that are not regular.

ACKNOWLEDGMENTS

APPENDIX A: $H(P)$ FORMALISM

As we already mentioned, for the electric sources it is not possible to construct an analytical closed form of $r(I)$. However, it is still possible to find an expression for the electric Lagrangian in terms of a new scalar $P = P^{\mu\nu}P_{\mu\nu}$, where the field $P_{\mu\nu}$ is defined as

$$P_{\mu\nu} = \partial_I L F_{\mu\nu}. \tag{A1}$$

As the nonzero components of $F_{\mu\nu}$ are proportional to $(\partial_I L)^{-1}$, the scalar P is given by

$$P = -\frac{q^2}{2r^4}. \tag{A2}$$

Using this relation, we can find an expression for $r(P)$. Moreover, we can make the Legendre transformation of L, which results in

$$\mathcal{H} = 2I\partial_I L - L. \tag{A3}$$

For a spherically symmetric line element such as (2), $\mathcal{H}$, in terms of the radial coordinate, is

$$\mathcal{H}(r) = \frac{e^a(ra' + 1) - 1}{\kappa^2 r^2}. \tag{A4}$$

Since we have $\mathcal{H}(r)$ and $r(P)$, from (A1), we may find the function $L(P)$ using an inverse Legendre transformation as

$$L(P) = 2P\partial_P \mathcal{H} - \mathcal{H}. \tag{A5}$$

Considering the models (13), (21), and (29), $L(P)$ is

$$L_1(P) = \frac{24\alpha\beta(-P)^{3/2}(\beta(-P)^{3/4} - \sqrt[4]{2}q^{3/2})}{\kappa^2(2\beta(-P)^{3/4} + \sqrt[4]{2}q^{3/2})^3}, \tag{A6}$$

$$L_2(P) = \frac{24P^2 r_1^2 r_2^2(r_1^2 + r_1 r_2 + r_2^2) - 6\sqrt{-2P}q^3 + 36Pq^2 r_1 r_2 + 4\sqrt{-2P}Pqr_1 r_2(r_1^2 + 11r_1 r_2 + r_2^2)}{\kappa^2\sqrt{-P}r_1 r_2(2\sqrt{-P}r_1 r_2 + \sqrt{2}q)^3}, \tag{A7}$$

$$\begin{aligned}
L_3(P) = -\frac{8\sqrt{-P}}{(\sqrt{2}lq + 2\sqrt{-P}\gamma)^{-5}}&\{(-\sqrt{-2P}q^2 l^4 + 8Pq\gamma l^3 + (\sqrt{-2P}\omega q^2 - 4P\delta q + 30\sqrt{-2P}P\gamma^2)l^2 \\
&+ 2\gamma(\sqrt{-2P}q^2 - 8P\omega q + 10\sqrt{-2P}P\delta)l + 2P\gamma^2(5\sqrt{-2P}\omega - 14q))^2\kappa^2\}^{-1}(q^6(2\sqrt{-2P}P\delta q - 3\sqrt{-2P}q^3 \\
&+ 24P^2\gamma^2)l^{11} + 2Pq^5\gamma(39q^3 + 2\sqrt{-2P}\omega q^2 - 2P\delta q + 96\sqrt{-2P}P\gamma^2)l^{10} - 2q^4(12P\delta q^4 - 3\sqrt{-2P}\omega q^5 \\
&+ \sqrt{-2P}P(2\omega\delta - 291\gamma^2)q^3 - 8P^2(\delta^2 - 5\omega\gamma^2)q^2 + 18\sqrt{2}(-P)^{5/2}\gamma^2\delta q + 1080P^3\gamma^4)l^9 \\
&- 4q^3\gamma(-3\sqrt{-2P}q^6 + 51P\omega q^5 + 2\sqrt{-2P}P(\omega^2 + 13\delta)q^4 - 9P^2(2\delta\omega - 151\gamma^2)q^3 \\
&+ 2\sqrt{2}(-P)^{5/2}(16\delta^2 - 81\gamma^2\omega)q^2 + 526P^3\gamma^2\delta q - 1344\sqrt{2}(-P)^{7/2}\gamma^4)l^8 - q^2(3\sqrt{-2P}\omega^2 q^7 - 17184P^4\gamma^6 \\
&- 24P(\delta\omega - 16\gamma^2)q^6 + 2\sqrt{2}(-P)^{3/2}(\delta\omega^2 + 12\delta^2 - 686\omega\gamma^2)q^5 + 8P^2(2\omega\delta^2 - 19\gamma^2\omega^2 - 14\gamma^2\delta)q^4
\end{aligned}$$

$$+ 4\sqrt{2}(-P)^{5/2}(4\delta^3 + 4445\gamma^4 + 110\gamma^2\delta\omega)q^3 + 32P^3\gamma^2(3\delta^2 - 71\gamma^2\omega)q^2 + 7720\sqrt{2}(-P)^{5/2}P\gamma^4\delta q)l^7$$

$$+ 2q\gamma(-6\sqrt{-2P}\omega q^8 + 3P(21\omega^2 + 8\delta)q^7 + 2\sqrt{-2P}P(\omega^3 + 50(\omega\delta - 3\gamma^2) - 504\gamma^2)q^6$$

$$+ 2P^2(52\delta^2 - 17\omega^2\delta + 2774\gamma^2\omega)q^5 + 8\sqrt{2}(-P)^{5/2}(2\delta^2\omega - 336\gamma^2\delta + 13\gamma^2\omega^2)q^4 + 30504P^4\gamma^4\delta q$$

$$+ 4P^3(366\delta\omega\gamma^2 + 9321\gamma^4 - 68\delta^3)q^3 + 8\sqrt{2}(-P)^{5/2}P\gamma^2(43\gamma^2\omega + 184\delta^2)q^2 - 5760\sqrt{2}(-P)^{9/2}\gamma^6)l^6$$

$$+ 2\gamma^2(-6\sqrt{-2P}q^9 + 240P\omega q^8 + \sqrt{-2P}P(443\omega^2 + 172\delta)q^7 - 24P^2(2\omega^3 - 15\delta\omega - 444\gamma^2)q^6$$

$$+ 2\sqrt{2}(-P)^{5/2}(71\delta\omega^2 + 6138\gamma^2\omega + 208\delta^2)q^5 - 8P^3(198\delta^2\omega - 97\gamma^2\omega^2 - 3070\gamma^2\delta)q^4$$

$$+ 12\sqrt{2}(-P)^{5/2}P(120\delta^3 + 1134\delta\omega\gamma^2 + 4067\gamma^4)q^3 + 32P^4\gamma^2(691\delta^2 + 1101\gamma^2\omega)q^2 + 5160\sqrt{2}(-P)^{9/2}\gamma^4\omega q$$

$$+ 21600P^5\gamma^6)l^5 + 4P\gamma^3(114q^8 + 826\sqrt{-2P}\omega q^7 - P(1207\omega^2 - 308\delta)q^6$$

$$+ 2\sqrt{-2P}P(15\omega^3 + 1094\delta\omega + 6618\gamma^2)q^5 - 2P^2(1031\delta\omega^2 + 168\delta^2 + 6338\gamma^2\omega)q^4$$

$$+ 8\sqrt{2}(-P)^{5/2}(3929\gamma^2\delta + 829\gamma^2\omega^2 + 448\delta^2\omega)q^3 + 4P^3(488\delta^3 - 3693\gamma^4 + 3430\delta\omega\gamma^2)q^2$$

$$+ 120\sqrt{2}(-P)^{5/2}P\gamma^2(56\delta^2 + 115\gamma^2\omega)q + 19800P^4\gamma^4\delta)l^4 + 4P\gamma^4(780\sqrt{-2P}q^7 - 4784P\omega q^6$$

$$+ \sqrt{-2P}P(3720\delta - 881\omega^2)q^5 - 8P^2(188\omega^3 + 1703\delta\omega + 4656\gamma^2)q^4$$

$$+ 2\sqrt{2}(-P)^{5/2}(1883\delta\omega^2 + 4364\delta^2 + 7030\gamma^2\omega)q^3 + 8P^3(82\delta^2\omega - 487\gamma^2\omega^2 - 3990\gamma^2\delta)q^2$$

$$+ 40\sqrt{2}(-P)^{5/2}P(62\delta^3 + 152\gamma^4 + 543\delta\omega\gamma^2)q + 2400P^4\gamma^2(5\delta^2 + 3\gamma^2\omega))l^3 + 8P^2\gamma^5(-2436q^6$$

$$- 2898\sqrt{-2P}\omega q^5 - P(3643\omega^2 + 10544\delta)q^4 + 6\sqrt{-2P}P(25\omega^2 + 290\delta\omega - 1414\gamma^2)q^3$$

$$- 2P^2(3548\delta^2 + 1387\delta\omega^2 + 8574\gamma^2\omega)q^2 + 120\sqrt{2}(-P)^{5/2}(34\delta^2\omega + 37\gamma^2\omega^2 + 49\gamma^2\delta)q$$

$$+ 600P^3\delta(2\delta^2 + 7\gamma^2\omega))l^2 + 8P^2\gamma^6(-3822\sqrt{-2P}q^5 - 2464P\omega q^4 + \sqrt{2}(-P)^{3/2}(1775(\omega^2 - 2\delta) + 10242\delta)q^3$$

$$- 8P^2(150\omega^3 + 1553\delta\omega + 294\gamma^2)q^2 + 210\sqrt{2}(-P)^{5/2}(13\delta\omega^2 + 8\delta^2 + 8\gamma^2\omega)q + 600P^3\omega(2\delta^2 + \omega\gamma^2))l$$

$$+ 16P^3\gamma^7(2058q^4 - 2450\sqrt{-2P}\omega q^3 - 7P(275\omega^2 + 84\delta)q^2 + 10\sqrt{-2P}P\omega(25\omega^2 + 42\delta)q$$

$$+ 150P^2\omega^2\delta)). \tag{A8}$$

With that, the electromagnetic Lagrangian associated with (13), (21), and (29) is not linear in the scalar P and does not behave as Maxwellian for $P \approx 0$.

APPENDIX B: CORRECTIONS FROM $f(R)$ THEORY

In this Appendix, we will write the expressions for $\rho(r)$, $L(r)$, and $X(r)$ with the corrections from $f(R)$ gravity for the models (21) and (29).

1. Example in the anti–de Sitter background

Inverting (22), we find the expression for $r(G)$ and with

$$\alpha_2(R) = 3\sqrt{3}(r_1^2 r_2^2(r_1 + r_2)^4(Rr_1r_2 - 12)^2((2R(54Rr_2^2 - 17)r_2^2 + 15)r_1^4 + 4r_2(117 - 665Rr_2^2)r_1^3$$
$$+ 2r_2^2(8229 - 17Rr_2^2)r_1^2 + 468r_2^3 r_1 + 15r_2^4))^{1/2}, \tag{B1}$$

$$\alpha_3(R) = (9R(6Rr_2^2 - 1)r_2^2 + 4)r_1^4, \tag{B2}$$

$$\alpha_4(R) = (r_1 + r_2)^2(\alpha_3(R) + 2r_2(62 - 657Rr_2^2)r_1^3 + (8016 - 9Rr_2^2)r_2^2 r_1^2 + 124r_2^3 r_1 + 4r_2^4), \tag{B3}$$

$f_R(R)$ is

$$f_R(R) = c_0 + \frac{c_1(\alpha_4(R) - \alpha_2(R))^{-1}}{\sqrt{6}(Rr_1r_2 - 12)}[-2(3Rr_2^2 + 2)\alpha_3(R)r_1^4 - 8r_2(9R(9Rr_2^2(Rr_2^2 - 17) - 55)r_2^2 + 34)r_1^7$$

$$+ (4i(i + \sqrt{3})\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)} - r_2^2(18R(3R(6Rr_2^4 - 414r_2^2 + (1 - i\sqrt{3})\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)})$$

$$+ 6902)r_2^2 + 9i(i + \sqrt{3})R\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)} + 24640))r_1^6 + 4r_2((3R(918Rr_2^2$$

$$+ 9i(i + \sqrt{3})R\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)} - 21364)r_2^2 + 333(1 - i\sqrt{3})R\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)} + 115876)r_2^2$$

$$+ 33i(i + \sqrt{3})\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)})r_1^5 + (-162R^2r_2^8 + 18iR(3(i + \sqrt{3})\sqrt[3]{\alpha_2(R) - 2\alpha_4(R)}R + 6902i)r_2^6$$

$$+ 2(1323(1 - i\sqrt{3})\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)}R + 487888)r_2^4 - 3i\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)}$$

$$\times ((-i + \sqrt{3})R\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)} - 2756(i + \sqrt{3}))r_2^2 + 2(1 + i\sqrt{3})(2\alpha_2(R) - 2\alpha_4(R))^{2/3})r_1^4$$

$$+ 2r_2(1980Rr_2^6 + 2(333(1 - i\sqrt{3})\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)}R + 115876)r_2^4 - i\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)}$$

$$\times (3(-i + \sqrt{3})R\sqrt[3]{\alpha_2(R) - 2\alpha_4(R)} - 8140(i + \sqrt{3}))r_2^2 + 22(1 + i\sqrt{3})(\alpha_2(R) - 2\alpha_4(R))^{2/3})r_1^3$$

$$+ (12Rr_2^8 + (9(1 - i\sqrt{3})R\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)} - 24640)r_2^6 - 3i\sqrt[3]{\alpha_2(R) - 2\alpha_4(R)}$$

$$\times ((-i + \sqrt{3})R\sqrt[3]{\alpha_2(R) - 2\alpha_4(R)} - 2756(i + \sqrt{3}))r_2^4 + 6(\alpha_2(R)R$$

$$+ 14(1 + i\sqrt{3})(2\alpha_2(R) - 2\alpha_4(R))^{2/3})r_2^2 + 4\alpha_2(R))r_1^2 - 4(68r_2^7 + 33(1 - i\sqrt{3})\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)}r_2^5$$

$$- 11i(-i + \sqrt{3})(2\alpha_2(R) - 2\alpha_4(R))^{2/3}r_2^3 + 16\alpha_2(R)r_2)r_1 - 16r_2^8 + 2(1 + i\sqrt{3})r_2^4(2\alpha_2(R) - 2\alpha_4(R))^{2/3}$$

$$+ 4r_2^2\alpha_2(R) + 4i(i + \sqrt{3})r_2^6\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)} + (-\sqrt{3}i + 1)\alpha_2(R)\sqrt[3]{2\alpha_2(R) - 2\alpha_4(R)}]^{1/2}. \tag{B4}$$

Integrating this expression with respect to the curvature scalar, we obtain the $f(R)$ function. The corrections to ρ, L, and X are

$$\rho(r) = \frac{c_0(-3r^4 + r^2(r_1^2 - 4r_1r_2 + r_2^2) + 3r_1r_2(r_1^2 + r_1r_2 + r_2^2))}{\kappa^2r_1r_2(r^2 + r_1r_2)^2} + \frac{c_1}{\kappa^2(r_1r_2)^{3/2}}\left(-3(r_1 + r_2)^2\tan^{-1}\left(\frac{r}{\sqrt{r_1r_2}}\right)\right.$$

$$\left. + \frac{\sqrt{r_1r_2}(r^4(r_1^2 + r_2^2) + r^2r_1r_2(3r_1^2 + 2r_1r_2 + 3r_2^2) - 2r_1^3r_2^3)}{r(r^2 + r_1r_2)^2}\right), \tag{B5}$$

$$L(r) = \frac{c_0(3r^6 + 9r^4r_1r_2 + r^2r_1r_2(r_1^2 + 11r_1r_2 + r_2^2) - 3r_1^2r_2^2(r_1^2 + r_1r_2 + r_2^2))}{\kappa^2r_1r_2(r^2 + r_1r_2)^3}$$

$$+ \frac{c_1}{\kappa^2(r_1r_2)^{3/2}}\left(\frac{\sqrt{r_1r_2}(r^6(r_1^2 + 3r_1r_2 + r_2^2) + r^4r_1r_2(2r_1^2 + 7r_1r_2 + 2r_2^2) - 3r^2r_1^2r_2^2(r_1^2 + r_1r_2 + r_2^2) + r_1^4r_2^4)}{r(r^2 + r_1r_2)^3}\right.$$

$$\left. + 3(r_1 + r_2)^2\tan^{-1}\left(\frac{r}{\sqrt{r_1}\sqrt{r_2}}\right)\right), \tag{B6}$$

$$X(r) = \frac{c_0r^4(r_1 + r_2)^2(r^2 + 5r_1r_2)}{\kappa^2r_1r_2(r^2 + r_1r_2)^3} + \frac{c_1r}{\kappa^2r_1r_2(r^2 + r_1r_2)^3}(r^6(2r_1^2 + 3r_1r_2 + 2r_2^2) + 3r^4r_1r_2(2r_1^2 + 3r_1r_2 + 2r_2^2)$$

$$- 3r^2r_1^3r_2^3 - r_1^4r_2^4). \tag{B7}$$

The electromagnetic energy density tends to a constant in infinity, while $X(r)$ diverges due to the presence of c_1.

2. Regular black hole with three horizons

As naturally the expressions for the solution with three horizons are more complicated, we expect that the $f(R)$ function that generates this model is not simple. Choosing

$$\alpha_5(R) = 4l^3 + 3R\gamma l^2 - 4\omega l + 28\gamma, \qquad \alpha_6(R) = -3\gamma l^2 + 6\delta l - 2l\gamma^2R + 15\gamma\omega,$$

$$\alpha_7(R) = 3l^3 - 4R\gamma l^2 - 3\omega l - 54\gamma, \qquad \alpha_8(R) = l^3 + 2R\gamma l^2 - \omega l + 22\gamma,$$

$$\alpha_9(R) = 12\delta + \gamma^2R + 24l\gamma, \tag{B8}$$

the f_R function is

$$
\begin{aligned}
f_R = c_0 + \frac{c_1}{\sqrt{2}} \Bigg\{ &- \frac{\alpha_8}{l(Rl^2+12)} - \Bigg[\frac{\alpha_7(l^3 - \omega l - 2\gamma)}{3l^2(Rl^2+12)^2} + \frac{\sqrt[3]{2}}{3l^2(Rl^2+12)} (4\gamma^3\alpha_5^3 - 36\gamma^3 l^2\alpha_5(Rl^2+12)\alpha_9 \\
&+ 27\gamma^2 l^2(Rl^2+12)\alpha_6^2 + 18l\gamma^2\alpha_5\alpha_8\alpha_6 + 27l^2\gamma^2\alpha_8^2\alpha_9 + (\gamma^3(\gamma(4\gamma\alpha_5^3 - 36l^2\gamma(Rl^2+12)\alpha_5\alpha_9 + 27l^2\alpha_8^2\alpha_9 \\
&+ 18l\alpha_8\alpha_6\alpha_5 + 27l^2(Rl^2+12)\alpha_6^2)^2 - 16(3\gamma l^2(Rl^2+12)\alpha_9 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2)^3))^{1/2})^{1/3} \\
&+ \frac{2\,2^{2/3}\gamma}{3l^2(Rl^2+12)} (3(Rl^2+12)\gamma\alpha_9 l^2 + 3l\alpha_8\alpha_6 + \gamma\alpha_5^2)(4\gamma^3\alpha_5^3 - 36l^2\gamma^3(Rl^2+12)\alpha_5\alpha_9 + 27l^2\gamma^2(Rl^2+12)\alpha_6^2 \\
&+ 18l\gamma^2\alpha_5\alpha_8\alpha_6 + 27l^2\gamma^2\alpha_8^2\alpha_9 + (\gamma^3(\gamma(4\gamma\alpha_5^3 + 18l\alpha_8\alpha_6\alpha_5 + 27l^2\alpha_8^2\alpha_9 + 27l^2(Rl^2+12)\alpha_6^2 \\
&- 36l^2\gamma(Rl^2+12)\alpha_9\alpha_5)^2 - 16(3\gamma(Rl^2+12)\alpha_9 l^2 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2)^3))^{1/2})^{-1/3} \Bigg]^{1/2} + \Bigg[\frac{2\alpha_7(l^3 - \omega l - 2\gamma)}{3l^2(Rl^2+12)^2} \\
&- \frac{\sqrt[3]{2}}{3l^2(Rl^2+12)} (4\gamma^3\alpha_5^3 - 36l^2\gamma^3(Rl^2+12)\alpha_5\alpha_9 + 27l^2(Rl^2+12)\gamma^2\alpha_6^2 + 18l\gamma^2\alpha_5\alpha_8\alpha_6 + 27l^2\gamma^2\alpha_8^2\alpha_9 \\
&+ (\gamma^3(\gamma(18l\alpha_8\alpha_6\alpha_5 - 36l^2(Rl^2+12)\gamma\alpha_9\alpha_5 + 4\gamma\alpha_5^3 + 27l^2(Rl^2+12)\alpha_6^2 + 27l^2\alpha_8^2\alpha_9)^2 \\
&- 16(3(Rl^2+12)\gamma\alpha_9 l^2 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2)^3))^{1/2})^{1/3} - \frac{2\,2^{2/3}\gamma}{3l^2(Rl^2+12)} (3(Rl^2+12)\gamma\alpha_9 l^2 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2) \\
&\times (4\gamma^3\alpha_5^3 + 18l\gamma^2\alpha_5\alpha_8\alpha_6 + 27l^2(Rl^2+12)\gamma^2\alpha_6^2 + 27l^2\gamma^2\alpha_8^2\alpha_9 + (\gamma^3(\gamma(18l\alpha_8\alpha_6\alpha_5 - 36l^2(Rl^2+12)\gamma\alpha_9\alpha_5 \\
&+ 27l^2(Rl^2+12)\alpha_6^2 + 4\gamma\alpha_5^3 + 27l^2\alpha_8^2\alpha_9)^2 - 16(3l^2\gamma(Rl^2+12)\alpha_9 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2)^3))^{1/2})^{-1/3} \\
&- \frac{2}{l^3(Rl^2+12)^3} (-\alpha_8^2 + 2(Rl^2+12)\gamma\alpha_5\alpha_8 + 2l(Rl^2+12)^2\gamma\alpha_6) \Bigg(\frac{\alpha_7(l^3 - \omega l - 2\gamma)}{3l^2(Rl^2+12)^2} + \frac{\sqrt[3]{2}}{3l^2(Rl^2+12)} (4\gamma^3\alpha_5^3 \\
&- 36l^2(Rl^2+12)\gamma^3\alpha_5\alpha_9 + 27l^2(Rl^2+12)\gamma^2\alpha_6^2 + 18l\gamma^2\alpha_5\alpha_6\alpha_8 + 27l^2\gamma^2\alpha_8^2\alpha_9 + (\gamma^3(\gamma(4\gamma\alpha_5^3 + 18l\alpha_8\alpha_6\alpha_5 \\
&- 36l^2(Rl^2+12)\gamma\alpha_9\alpha_5 + 27l^2(Rl^2+12)\alpha_6^2 + 27l^2\alpha_8^2\alpha_9)^2 - 16(3(Rl^2+12)\gamma\alpha_9 l^2 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2)^3))^{1/2})^{1/3} \\
&+ \frac{2\,2^{2/3}\gamma}{3l^2(Rl^2+12)} (3(Rl^2+12)\gamma\alpha_9 l^2 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2)(4\gamma^3\alpha_5^3 - 36l^2(Rl^2+12)\gamma^3\alpha_5\alpha_9 + 27l^2(Rl^2+12)\gamma^2\alpha_6^2 \\
&+ 18l\gamma^2\alpha_5\alpha_8\alpha_6 + 27l^2\gamma^2\alpha_8^2\alpha_9 + (\gamma^3(\gamma(4\gamma\alpha_5^3 + 18l\alpha_8\alpha_6\alpha_5 - 36l^2(Rl^2+12)\gamma\alpha_9\alpha_5 + 27l^2(Rl^2+12)\alpha_6^2 \\
&+ 27l^2\alpha_8^2\alpha_9)^2 - 16(3(Rl^2+12)\gamma\alpha_9 l^2 + 3\alpha_8\alpha_6 l + \gamma\alpha_5^2)^3))^{1/2})^{-1/3} \Bigg)^{-1/2} \Bigg]^{1/2} \Bigg\}^{1/2}.
\end{aligned}
\tag{B9}
$$

The corrections to the electromagnetic quantities are

$$
\begin{aligned}
\rho(r) = &\frac{c_0}{\kappa^2(lr^2+\gamma)^3} (l^3 r^4 + 3l^2 r^2\gamma + l(3r^6 - r^4\omega - r^2\delta + 6\omega^2) + \gamma(7r^4 - 5r^2\omega + 3\delta)) \\
&+ \frac{c_1}{2\kappa^2 l^{5/2}\gamma^{3/2}} \Bigg(-3(3l^3\gamma + l^2\delta - l\gamma\omega - 3\gamma^2)\tan^{-1}\left(\sqrt{\frac{l}{\gamma}}r\right) - \frac{\sqrt{l\gamma}}{r(lr^2+\gamma)^3} (5l^5 r^6\gamma + 3l^4 r^4(r^2\delta + 4\gamma^2) \\
&+ l^3 r^2\gamma(5r^4\omega + 8r^2\delta + 3\gamma^2) + l^2\gamma^2(7r^6 + 16r^4\omega - 3r^2\delta + 4\gamma^2) + 3lr^2\gamma^3(8r^2 + \omega) + 9r^2\gamma^4) \Bigg),
\end{aligned}
\tag{B10}
$$

$$L(r) = -\frac{c_0}{\kappa^2(lr^2+\gamma)^4}(l^2(3r^8 + r^4\delta - 6r^2\gamma^2) + 2l\gamma(6r^6 + r^4\omega - 4r^2\delta + 3\gamma^2) + \gamma^2(21r^4 - 10r^2\omega + 3\delta))$$

$$+ \frac{c_1}{2\kappa^2(lr^2+\gamma)^4}\left(7l^4r^7 + l^3r^5\left(\frac{3r^2}{\gamma}\delta + 25\gamma\right) + l^2r^3(r^4\omega + 11r^2\delta + 39\gamma^2) + \frac{9r\gamma^4}{l^2} + \frac{3r\gamma^3(11r^2+\omega)}{l}\right.$$

$$+ \frac{3(3l^3\gamma + l^2\delta - l\gamma\omega - 3\gamma^2)(lr^2+\gamma)^4\tan^{-1}\left(\sqrt{\frac{l}{\gamma}}r\right)}{l^{5/2}\gamma^{3/2}} + lr\gamma(-r^6 + 5r^4\omega + 29r^2\delta - \gamma^2)$$

$$\left.+ \frac{\gamma^2(-r^6 + 31r^4\omega - 3r^2\delta + 2\gamma^2)}{r}\right), \tag{B11}$$

$$X(r) = \frac{c_0 r^4}{\kappa^2(lr^2+\gamma)^4}(l^4r^4 + 4l^3r^2\gamma + l^2(-r^4\omega - 2r^2\delta + 15\gamma^2) - 2l\gamma(r^4 + 4r^2\omega - 5\delta) + \gamma^2(5\omega - 14r^2))$$

$$+ \frac{c_1 r}{\kappa^2(lr^2+\gamma)^4}(l^4r^8 + 4l^3r^6\gamma - 2l^2(r^8\omega - 6r^4\gamma^2) - 4lr^2\gamma(r^6 + 2r^4\omega - 3r^2\delta + \gamma^2)$$

$$- \gamma^2(16r^6 - 6r^4\omega + \gamma^2)). \tag{B12}$$

We get the same conclusions as the anti–de Sitter model.

APPENDIX C: CORRECTIONS FROM $f(G)$ THEORY

This Appendix is dedicated to the corrections from $f(G)$ gravity in $\rho(r)$, $X(r)$, and $L(r)$.

1. Example in the anti–de Sitter background

From (24), we can find $r(G)$, and using

$$\beta_1(G) = 24 - Gr_1^2 r_2^2, \tag{C1}$$

$$\beta_2(G) = 243Gr_2^4 r_1^{12} + 3024Gr_2^5 r_1^{11} + 108r_2^2(127Gr_2^4 - 68)r_1^{10} + 16r_2^3(1971Gr_2^4 - 5309)r_1^9$$
$$+ 6r_2^4(6867Gr_2^4 - 62192)r_1^8 + 48r_2^5(657Gr_2^4 - 17617)r_1^7 + 4r_2^6(3429Gr_2^4 - 275032)r_1^6$$
$$+ 48r_2^7(63Gr_2^4 - 17617)r_1^5 + 3r_2^8(81Gr_2^4 - 124384)r_1^4 - 84944r_2^9 r_1^3 - 7344r_2^{10}r_1^2, \tag{C2}$$

$$\beta_3(G) = 27G^2 r_2^5(256Gr_2^4 + 81)r_1^{13} + 216G^2 r_2^6(128Gr_2^4 + 267)r_1^{12} + 36Gr_2^3(1152G^2 r_2^8 - 3307Gr_2^4 - 3096)r_1^{11}$$
$$+ 72(384G^3 r_2^{12} - 12857G^2 r_2^8 - 40212Gr_2^4 + 96)r_1^{10} + 18r_2(384G^3 r_2^{12} - 83079G^2 r_2^8 - 383392Gr_2^4 + 88576)r_1^9$$
$$- 8r_2^2(115713G^2 r_2^8 + 624716Gr_2^4 - 4689984)r_1^8 - 4r_2^3(29763G^2 r_2^8 + 523184Gr_2^4 - 35735424)r_1^7$$
$$+ 8r_2^4(7209G^2 r_2^8 - 624716Gr_2^4 + 34942560)r_1^6 + 3r_2^5(729G^2 r_2^8 - 2300352Gr_2^4 + 116347904)r_1^5$$
$$- 96r_2^6(30159Gr_2^4 - 2911880)r_1^4 - 96r_2^7(1161Gr_2^4 - 1488976)r_1^3 + 37519872r_2^8 r_1^2 + 1594368r_2^9 r_1$$
$$+ 6912r_2^{10}, \tag{C3}$$

$$\beta_4(G) = \sqrt[3]{\beta_2(G) + 3\sqrt{3}\sqrt{r_1^3 r_2^3(r_1 + r_2)^8 \beta_3(G)}}, \tag{C4}$$

$$\beta_5(G) = \frac{8\sqrt[3]{2}r_1 r_2 \beta_1(G)(9r_2^2 + 29r_1 r_2 + 9r_1^2 - 9\beta_1(G)r_1 r_2)(r_1 + r_2)^4}{\beta_4(G)}, \tag{C5}$$

$$\beta_6(G) = 3G^2 r_2^5 r_1^7 + 10G^2 r_2^6 r_1^6 + Gr_2^3(3Gr_2^4 - 148)r_1^5 + (8 - 488Gr_2^4)r_1^4 - 4r_2(37Gr_2^4 - 464)r_1^3 + 6000r_2^2 r_1^2$$
$$+ 1856r_2^3 r_1 + 8r_2^4, \tag{C6}$$

we write

$$f_G(G) = c_0 + \frac{c_1}{\sqrt{2\beta_1(G)}} \left\{ 4(r_1 + r_2)^2 - 2r_1 r_2 \beta_1(G) - \sqrt{\frac{2}{3}} [-\beta_5(G) - 6((r_1 + r_2)^2 - \beta_1(G) r_1 r_2)^2 \right.$$

$$- 2r_1 r_2 \beta_1(G)(16(r_1 + r_2)^2 - 3\beta_1(G) r_1 r_2) + 2^{2/3} \beta_1(G)\beta_4(G)]^{1/2} + \left[-\frac{2(1 + 2^{2/3})\beta_5(G)}{3} \right.$$

$$- \frac{8\sqrt{6}\beta_6(G)(r_1 + r_2)^2}{\sqrt{-\beta_5(G) + 6(2(r_1 + r_2)^2 - \beta_1(G) r_1 r_2)^2 + 2r_1 r_2 \beta_1(G)(16(r_1 + r_2)^2 - \beta_1(G) r_1 r_2) - 2^{2/3} \beta_1(G)\beta_4(G)}}$$

$$\left. \left. + 8(2(r_1 + r_2)^2 - \beta_1(G) r_1 r_2)^2 + \frac{8 r_1 r_2}{3}(16(r_1 + r_2)^2 - 3\beta_1(G) r_1 r_2) \right]^{1/2} \right\}^{1/2}. \tag{C7}$$

The electromagnetic functions are

$$\rho(r) = \frac{-3r^4 + r^2(r_1^2 - 4r_1 r_2 + r_2^2) + 3r_1 r_2(r_1^2 + r_1 r_2 + r_2^2)}{\kappa^2 r_1 r_2 (r^2 + r_1 r_2)^2} + \frac{c_1}{\kappa^2} \left(\frac{2}{r r_1^3 r_2^3 (r^2 + r_1 r_2)^3} (r^6(3r_1^4 + 16 r_1^3 r_2 + 18 r_1^2 r_2^2 \right.$$

$$+ 16 r_1 r_2^3 + 3r_2^4) + 8r^4 r_1 r_2(r_1^2 + r_1 r_2 + r_2^2)(r_1^2 + 5 r_1 r_2 + r_2^2) + r^2 r_1^2 r_2^2(-3r_1^4 + 8r_1^3 r_2 - 2r_1^2 r_2^2 + 8 r_1 r_2^3 - 3r_2^4)$$

$$\left. + 8 r_1^4 r_2^4 (r_1^2 + r_1 r_2 + r_2^2)) + \frac{6(r_1^2 - r_2^2)^2}{r_1^{7/2} r_2^{7/2}} \tan^{-1}\left(\frac{r}{\sqrt{r_1 r_2}} \right) \right), \tag{C8}$$

$$L(r) = \frac{3r^6 + 9r^4 r_1 r_2 + r^2 r_1 r_2(r_1^2 + 11 r_1 r_2 + r_2^2) - 3r_1^2 r_2^2(r_1^2 + r_1 r_2 + r_2^2)}{\kappa^2 r_1 r_2 (r^2 + r_1 r_2)^3} + \frac{c_1}{\kappa^2} \left(\frac{2}{r r_1^3 r_2^3 (r^2 + r_1 r_2)^4} (r^2 r_1^3 r_2^3 (3r_1^4 + 8r_1^3 r_2 \right.$$

$$+ 26 r_1^2 r_2^2 + 8 r_1 r_2^3 + 3r_2^4) - r^8(r_1^2 + r_2^2)(3r_1^2 + 8 r_1 r_2 + 3r_2^2) - r^4 r_1^2 r_2^2(29 r_1^4 + 108 r_1^3 r_2 + 134 r_1^2 r_2^2$$

$$+ 108 r_1 r_2^3 + 29 r_2^4) - r^6 r_1 r_2(11 r_1^4 + 32 r_1^3 r_2 + 26 r_1^2 r_2^2 + 32 r_1 r_2^3 + 11 r_2^4)$$

$$\left. - 4 r_1^5 r_2^5 (r_1^2 + r_1 r_2 + r_2^2)) - \frac{6(r_1^2 - r_2^2)^2}{r_1^{7/2} r_2^{7/2}} \tan^{-1}\left(\frac{r}{\sqrt{r_1 r_2}} \right) \right), \tag{C9}$$

$$X(r) = \frac{r^4(r_1 + r_2)^2(r^4 + 6r^2 r_1 r_2 + 5 r_1^2 r_2^2)}{\kappa^2 r_1 r_2 (r^2 + r_1 r_2)^4} + \frac{8 c_1}{\kappa^2 r_1^2 r_2^2 (r^2 + r_1 r_2)^4} (r^9(2r_1^2 + 3 r_1 r_2 + 2r_2^2) + 4r^7 r_1 r_2(2r_1^2 + 3 r_1 r_2 + 2r_2^2)$$

$$- r^5 r_1 r_2(6r_1^4 + 13 r_1^3 r_2 + 20 r_1^2 r_2^2 + 13 r_1 r_2^3 + 6 r_2^4) + 2r^3 r_1^3 r_2^3(3r_1^2 + 4 r_1 r_2 + 3r_2^2) + r r_1^4 r_2^4 (r_1^2 + r_1 r_2 + r_2^2)). \tag{C10}$$

The functions $\rho(r)$ and $L(r)$ are well behaved while $X(r)$ diverges in infinity. The modifications came from the nonlinear terms of the $f(G)$ function since we do not have the presence of c_0.

2. Regular black hole with three horizons

We are not able to find an analytical form of $r(G)$; however, we still can find the corrections to the electromagnetic theory, which are

$$\rho(r) = \frac{l^3 r^4 + 3l^2 \gamma r^2 + \gamma(7r^4 - 5\omega r^2 + 3\delta) + l(3r^6 - \omega r^4 - \delta r^2 + 6\gamma^2)}{(lr^2 + \gamma)^3 \kappa^2}$$

$$+ \frac{c_1}{\kappa^2} \left(\frac{1}{4l^4 r\gamma^3 (lr^2 + \gamma)^5} [75 l^{10} \gamma^2 r^{10} + 2l^9 \gamma(33\delta r^2 + 175\gamma^2) r^8 + l^8(3(5\delta - 6\omega\gamma^2) r^4 + 308 \gamma^2 \delta r^2 + 640 \gamma^4) r^6 \right.$$

$$+ 2l^7 \gamma((-3\delta\omega - 11\gamma^2) r^6 + 7(5\delta^2 - 6\omega\gamma^2) r^4 + 288 \gamma^2 \delta r^2 + 273 \gamma^4) r^4 + 105 \gamma^8 r^2 + 10 l\gamma^7 (49 r^2 + 3\omega) r^2$$

$$+ l^2 \gamma^6 (896 r^4 + 140 \omega r^2 - 3(\omega^2 + 2\delta)) r^2 + 2l^3 \gamma^5 (395 r^6 + 128 \omega r^4 - 7(\omega^2 + 2\delta) r^2 + 3(\delta\omega - 7\gamma^2)) r^2$$

$$+ l^6 \gamma^2 (3(\omega^2 + 2\delta) r^8 - 4(7\delta\omega + 31\gamma^2) r^6 + 128(\delta^2 - \omega\gamma^2) r^4 + 524 \gamma^2 \delta r^2 + 181 \gamma^4) r^2$$

$$+ l^4 \gamma^4 (151 r^{10} + 500 \omega r^8 - 128(\omega^2 + 2\delta) r^6 + 4(39\delta\omega - \gamma^2) r^4 - 5(3\delta^2 + 22 \gamma^2 \omega) r^2 + 64 \gamma^2 \delta)$$

$$+ 2l^5 \gamma^3 (49 \omega r^{10} + 7(\omega^2 - 2\delta) r^8 - 32(2\delta\omega + 5\gamma^2) r^6 + 3(31\delta^2 + 14 \gamma^2 \omega) r^4 - \gamma^2 \delta r^2 + 64 \gamma^4)]$$

$$\left. + \frac{3(25 \gamma^2 l^6 + 22 \gamma\delta l^5 + (5\delta^2 - 6\gamma^2 \omega) l^4 - 2\gamma(\delta\omega - 7\gamma^2) l^3 + \gamma^2(\omega^2 + 2\delta) l^2 - 10 \gamma^3 \omega l - 35 \gamma^4) \tan^{-1}(\frac{\sqrt{l}r}{\sqrt{\gamma}})}{4l^{9/2} \gamma^{7/2}} \right), \tag{C11}$$

$$L(r) = -\frac{(3r^8 + \delta r^4 - 6\gamma^2 r^2)l^2 + 2\gamma(6r^6 + \omega r^4 - 4\delta r^2 + 3\gamma^2)l + \gamma^2(21r^4 - 10\omega r^2 + 3\delta)}{(lr^2 + \gamma)^4 \kappa^2}$$

$$-\frac{c_1}{\kappa^2}\left(\frac{1}{4l^4 r\gamma^3(lr^2+\gamma)^6}[75l^{11}\gamma^2 r^{12} + l^{10}\gamma(66\delta r^2 + 425\gamma^2)r^{10} + l^9(3(5\delta^2 - 6\omega\gamma^2)r^4 + 374\gamma^2\delta r^2 + 990\gamma^4)r^8\right.$$

$$+ l^8\gamma(-2(3\delta\omega - 5\gamma^2)r^6 + 17(5\delta^2 - 6\omega\gamma^2)r^4 + 852\gamma^2\delta r^2 + 1250\gamma^4)r^6 + l^7\gamma^2(3(\omega^2 + 2\delta)r^8$$

$$- 2(17\delta\omega + 11\gamma^2)r^6 + 6(33\delta^2 - 46\omega\gamma^2)r^4 + 1036\gamma^2\delta r^2 + 919\gamma^4)r^4 + 105\gamma^9 r^2 + 5l\gamma^8(119r^2 + 6\omega)r^2$$

$$+ l^2\gamma^7(1386r^4 + 170\omega r^2 - 3(\omega^2 + 2\delta))r^2 + l^3\gamma^6(1686r^6 + 396\omega r^4 - 17(\omega^2 + 2\delta)r^2 + 6(\delta\omega - 7\gamma^2))r^2$$

$$+ l^6\gamma^3(34\omega r^{10} + 17(\omega^2 + 2\delta)r^8 + 12(3\delta\omega - 11\gamma^2)r^6 + (58\delta^2 - 748\omega\gamma^2)r^4 + 1482\gamma^2\delta r^2 - 267\gamma^4)r^2$$

$$+ l^4\gamma^5(173r^{10} + 1716\omega r^8 - 398(\omega^2 + 2\delta)r^6 + 2(177\omega\delta + 121\gamma^2)r^4 - 3(5\delta^2 + 58\omega\gamma^2)r^2 + 32\gamma^2\delta)$$

$$+ l^5\gamma^4(23r^{12} + 214\omega r^{10} + 270(\omega^2 + 2\delta)r^8 - 4(217\omega\delta + 225\gamma^2)r^6 + 5(111\delta^2 - \omega\gamma^2)r^4 - 258\gamma^2\delta r^2 + 64\gamma^4)]$$

$$\left. + \frac{3(25\gamma^2 l^6 + 22\gamma\delta l^5 + (5\delta^2 - 6\gamma^2\omega)l^4 - 2\gamma(\delta\omega - 7\gamma^2)l^3 + \gamma^2(\omega^2 + 2\delta)l^2 - 10\gamma^3\omega l - 35\gamma^4)\tan^{-1}(\frac{\sqrt{l}r}{\sqrt{\gamma}})}{4l^{9/2}\gamma^{7/2}}\right), \quad \text{(C12)}$$

$$X(r) = \frac{r^4}{\kappa^2(lr^2 + \gamma)^4}(l^4 r^4 + 4l^3 r^2\gamma + l^2(-r^4\omega - 2r^2\delta + 15\gamma^2) - 2l\gamma(r^4 + 4r^2\omega - 5\delta) + \gamma^2(5\omega - 14r^2))$$

$$+ \frac{c_1 r}{(lr^2 + \gamma)^6 \kappa^2}(-8l^4(r^6 - \delta r^2 + 2\gamma^2)r^6 - 16l^3\gamma(3r^6 - \omega r^4 - \delta r^2 + 3\gamma^2)r^4 + 16l^2(\omega r^{10} - 3(\delta\omega + 3\gamma^2)r^6$$

$$+ (4\delta^2 + 11\omega\gamma^2)r^4 - 15\gamma^2\delta r^2 + 9\gamma^4)r^2 + 8\gamma^2(24r^{10} - 30\omega r^8 + 8(\omega^2 + 2\delta)r^6 - 3(2\omega\delta + 3\gamma^2)r^4 + 2\gamma^2\omega r^2$$

$$+ \gamma^2\delta) + 16l\gamma(2r^{12} + 6\omega r^{10} - 6(\omega^2 + 2\delta)r^8 + (14\delta\omega + 9\gamma^2)r^6 - 3(2\delta^2 + 3\gamma^2\omega)r^4 + 5\gamma^2\delta r^2 + \gamma^4)). \quad \text{(C13)}$$

The conclusions are the same as the previous example.

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