# Unification of dark energy and dark matter from diffusive cosmology

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Generalized ideas of unified dark matter and dark energy in the context of dynamical space time theories with a diffusive transfer of energy are studied. The dynamical space-time theories introduce a vector field whose equation of motion guarantees a conservation of a certain energy momentum tensor, which may be related, but in general is not the same as the gravitational energy momentum tensor. This particular energy momentum tensor is built from a general combination of scalar fields derivatives as the kinetic terms, and possibly potentials for the scalar field. By demanding that the dynamical space vector field be the gradient of a scalar the dynamical space time theory becomes a theory for diffusive interacting dark energy and dark matter. These generalizations produce nonconserved energy momentum tensors instead of conserved energy momentum tensors which lead at the end to a formulation for interacting dark energy and dark matter (DE-DM). We solved analytically the theories and we show that the ΛCDM is a fixed point of these theories at large times. A particular case has asymptotic correspondence to previously studied non-Lagrangian formulations of diffusive exchange between dark energy and dark matter.

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## I. INTRODUCTION

Dark energy and dark matter (DE-DM) constitute most of the observable Universe. Yet the true nature of these two phenomena is still a mystery. One fundamental question with respect to those phenomena is the coincidence problem which is trying to explain the relation between dark energy and dark matter densities. In order to solve this problem, one approach claims that the dark energy is a dynamical entity and hope to exploit solutions of scaling or tracking type to remove dependence on initial conditions. Others left this principle and tried to model the dark energy as a phenomenological fluid which exhibits a particular relation with the scale factor [1], Hubble constant [2] or even the cosmic time itself [3].

Interaction between DM and DE was considered in many cases, such as [4]. Unifications between dark energy and dark matter from an action principle were obtained from scalar fields [5–9] or by other models [10–15] including Galileon cosmology [13] or telleparallel modified theories of gravity [16,17]. Beyond those approaches, a unification of dark energy and dark matter using a new measure of integration (the so-called two measure theories) has been formulated [18–22]. A diffusive interaction between dark energy and dark matter was introduced in [23–28] and was formulated in the context of an action principle based on a generalization of those two measures theories in the context of quintessential scalar fields [24,25].

In recent publications [26], diffusion of energy between dark energy into dark matter was discussed. The models of such type are interesting as an approach to solve the coincidence problem. The basis of those models are considering a non-conserved stress energy tensor $T^{\mu\nu}$ with a source current $j^{\mu}$:

$$\nabla_{\mu} T^{\mu\nu}_{(\mathbf{Dust})} = \gamma^2 j^{\nu} \tag{1}$$

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where $\gamma^2$ is the coupling diffusion coefficient of the fluid. The current $j^\mu$ is a timelike covariant conserved vector field $j^\mu_{;\mu} = 0$ which describes the conservation of the number of particles in the system. Due to the fact that the Einstein tensor is covariantly conserved $\nabla_\mu G^{\mu\nu} = 0$, we have to introduce on the right-hand side of the Einstein tensor a compensating energy momentum tensor, for two diffusive fluids, where:

$$\nabla_\mu T^{\mu\nu}_{(\mathrm{Dust})} = -\nabla_\mu T^{\mu\nu}_{(\Lambda)} = \gamma^2 j^\nu \tag{2}$$

so that the total energy momentum tensor is conserved:

$$\nabla_\mu (T^{\mu\nu}_{(\mathrm{Dust})} + T^{\mu\nu}_{(\Lambda)}) = 0 \tag{3}$$

Such models could originate from irreversible diffusive exchange of energy, or have a Lagrangian origin, by introducing an independent stress energy momentum tensor $T^{\mu\nu}_{(\chi)}$ directly in the Lagrangian. The structure of the paper is as follows: In Sec. II we discuss dynamics of exchange of energies between two diffusive fluids, with two different equation of states. Such a system has a universal model independent behavior. In Sec. III we present the Lagrangian model leading to such an interactive energy momentum tensor. In Sec. IV we discuss solutions for the theory which contains more general combinations for the stress energy momentum tensor $T^{\mu\nu}_{(\chi)}$. In Sec. V we are looking for few asymptotic solutions for the theory. In Sec. VI we discuss a special case of a Lagrangian which corresponds to the diffusive model which has been introduced in Sec. II.

## II. COUPLED DIFFUSIVE FLUIDS

We assume that stress energy momentum tensors are in the form of ideal fluids, where:

$$T^\mu_\nu = \mathbf{Diag}(\rho, -p, -p, -p) \tag{4}$$

where $\rho$ is the energy density and p is the pressure. Then Eqs. (1)–(2) read:

$$\dot{\rho}_{\mathrm{dust}} + 3H(1+\tilde{\omega})\rho_{\mathrm{dust}} = \frac{\gamma^2}{a^3} \tag{5}$$

and

$$\dot{\rho}_\Lambda + 3H(1+\omega)\rho_\Lambda = -\frac{\gamma^2}{a^3}. \tag{6}$$

The diffusion constant $\gamma^2$ is always positive. $\omega$ and $\tilde{\omega}$ denote the ratio of the pressure and the density for the corresponding fluids. In order investigate the behavior of the solution, we introduce the dynamical system method for the equations. The dimensionless quantities for the system are defined as [28]:

$$x = \frac{\rho_{\mathrm{dust}}}{3H^2}, \qquad y = \frac{\rho_\Lambda}{3H^2}, \qquad \delta = \frac{\gamma^2}{a^3 H \rho_{\mathrm{dust}}} \tag{7}$$

where $\delta$ describes the strength of the relative diffusion. From Friedmann equations $x + y = 1$. The complete autonomous system method equations are

$$x' = 6x^2(\tilde{\omega} - \omega) + x(\gamma\delta + 3 + 6\omega + 3\tilde{\omega}) \tag{8a}$$

$$\delta' = \delta(\gamma\delta + 3(x-1)(\omega - \tilde{\omega})). \tag{8b}$$

Table I presents the critical points in the system. In order to determine the stability of the system we have to specify the equations of states. For the case of dark matter and dark energy we can choose two cases: the first on: $\omega = -1, \tilde{\omega} = 0$ and the second one $\omega = 0, \tilde{\omega} = -1$. The phase portrait for both cases is presented in Fig. 1. The case $\omega = 0, \tilde{\omega} = -1$, which represent the exchange of energy from the dark energy into dark matter include a stable point $A(0, -\frac{3}{\gamma})$ which corresponds to dark energy dominant with diffusion effect. However, the second case $\omega = -1$, $\tilde{\omega} = 0$, which represent the exchange of energy from the dark matter into dark energy includes a stable point $C(0,0)$ which corresponds to dark energy dominant with no diffusion effect.

In this model we have chosen $\omega$ and $\tilde{\omega}$ being constants, whereas in general Lagrangian models $\omega$ and $\tilde{\omega}$ are varying in time. However we expect that $\omega$ and $\tilde{\omega}$ can be approximated by constants for large times. In the next sections we investigate more general dynamics on the basis of the action principle.

TABLE I. The properties of the critical points for the exponential potential.

| Name | The point | Eigenvalues | Densities fraction |
|---|---|---|---|
| A | $(0, \frac{3}{\gamma}(\omega - \tilde{\omega}))$ | $3(3\omega+1),\ 3(\omega - \tilde{\omega})$ | 0 |
| B | $\left(\frac{\omega+1/3}{\omega-\tilde{\omega}}, -\frac{3\tilde{\omega}+1}{\gamma}\right)$ | $\frac{1}{2}\left(\pm\sqrt{36\omega^2 - 72\omega\tilde{\omega} + 9(\tilde{\omega}-2)\tilde{\omega} - 3} - 6\omega - 3\tilde{\omega} - 3\right)$ | $-\frac{3\omega+1}{3\tilde{\omega}+1}$ |
| C | (0,0) | $3(\tilde{\omega} - \omega),\ 3(2\omega + \tilde{\omega} + 1)$ | 0 |
| D | $\left(\frac{2\omega+\tilde{\omega}+1}{2(\omega-\tilde{\omega})}, 0\right)$ | $-3(2\omega + \tilde{\omega} + 1),\ \frac{3}{2}(1+3\tilde{\omega})$ | $-\frac{2\omega+\tilde{\omega}+1}{3\tilde{\omega}+1}$ |

## III. A LAGRANGIAN WITH DYNAMICAL SPACE-TIME

### A. Two measures theories

The two measure theory implies other measure of integration in addition to the regular measure of integration in the action $\sqrt{-g}$. The new measure is also a density and a total derivative. A simple example for constructing this measure is by introducing 4 scalar fields $\varphi_a$, where $a = 1$, 2, 3, 4. The measure reads:

$$\Phi = \varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{abcd}\partial_\alpha\varphi_a\partial_\beta\varphi_b\partial_\gamma\varphi_c\partial_\delta\varphi_d. \tag{9}$$

A complete action involving both measures takes the form:

$$S = \int d^4x\Phi\mathcal{L}_1 + \int d^4x\sqrt{-g}\mathcal{L}_2. \tag{10}$$

As a consequence of the variation with respect to the scalar fields $\varphi_a$, under the assumption that $\mathcal{L}_1$ and $\mathcal{L}_2$ are independent of the scalar fields $\varphi_a$, we obtain that:

$$A^\alpha_a\partial_\alpha\mathcal{L}_1 = 0, \tag{11}$$

where $A^\alpha_a = \varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{abcd}\partial_\beta\varphi_b\partial_\gamma\varphi_c\partial_\delta\varphi_d$. Since $\det[A^\alpha_a] \sim \Phi^3$ as one easily see then that for $\Phi \neq 0$, Eq. (11) implies that $\mathcal{L}_1 = M = \text{Const}$. These kind of contributions have been considered in the two measures theories which are of interest in connection with a unified model of dark energy and dark matter [19].

### B. Dynamical time action

The constraint on the term in the action $\mathcal{L}_2$ as in the two measure theories (10) could be generalized to a covariant conservation of a stress energy momentum tensor $T^{\mu\nu}_{(\chi)}$ which coupled directly in the action [27]:

$$\mathcal{S} = \int d^4x\sqrt{-g}\chi_{\mu;\nu}T^{\mu\nu}_{(\chi)} \tag{12}$$

to a vector field $\chi_\mu$ with its covariant derivatives $\chi_{\mu;\nu} = \partial_\nu\chi_\mu - \Gamma^\lambda_{\mu\nu}\chi_\lambda$. From the variation with respect to the vector field $\chi_\mu$ gives a constraint on the conservation of the stress energy tensor $T^{\mu\nu}_{(\chi)}$.

$$\delta\chi_\mu\colon\ \nabla_\mu T^{\mu\nu}_{(\chi)} = 0. \tag{13}$$

Similarly as the variation with respect to the scalar field $\varphi_a$ in the Lagrangian (10) yields $\partial_\alpha\mathcal{L} = 0$. The correspondence between them is when $T^{\mu\nu}_{(\chi)}$ is taken to be as $T^{\mu\nu}_{(\chi)} = g^{\mu\nu}\mathcal{L}_m$. By introducing the term in the action (12), we get:

$$\begin{aligned}\int d^4x\sqrt{-g}\chi_{\mu;\nu}T^{\mu\nu}_{(\chi)} &= \int d^4x\sqrt{-g}\chi^\lambda_{;\lambda}\mathcal{L}_m \\ &= \int d^4x\partial_\mu(\sqrt{-g}\chi^\mu)\mathcal{L}_m = \int d^4x\Phi\mathcal{L}_m.\end{aligned} \tag{14}$$

Similarly to the variation (10), the variation with respect to the scalar field gives again $\partial_\mu\mathcal{L}_m = 0$. For dynamical time theories, the variation with respect to the dynamical time vector field yields the same constraint.

The name dynamical time theory (DTT) was considered due to the fact the energy density $T^0_0(\chi)$ is the canonically conjugated variable to the dynamical time $\chi^0$:

$$\pi_{\chi^0} = \frac{\partial\mathcal{L}}{\partial\dot{\chi}^0} = T^0_0(\chi) \coloneqq \rho_{(\chi)} \tag{15}$$

where $\rho_{(\chi)}$ is the energy density of the original stress energy tensor.

### C. Dynamical time action with diffusive source

In order to break the conservation of $T^{\mu\nu}_{(\chi)}$ as in the diffusion equation [Eq. (1)], the vector field $\chi_\mu$ should be coupled in a mass like term in the action:

$$\begin{aligned}S_{(\chi,A)} = &\int d^4x\sqrt{-g}\chi_{\mu;\nu}T^{\mu\nu}_{(\chi)} \\ &+ \frac{\kappa}{2}\int d^4x\sqrt{-g}(\chi_\mu + \partial_\mu A)^2\end{aligned} \tag{16}$$

where $A$ is a scalar field different from $\phi$. From a variation with respect to the dynamical space time vector field $\chi_\mu$ we obtain:

$$\nabla_\nu T^{\mu\nu}_{(\chi)} = \kappa(\chi^\mu + \partial^\mu A) = f^\mu, \tag{17}$$

where the current source reads: $f^\mu = \kappa(\chi^\mu + \partial^\mu A)$. From the variation with respect to the new scalar $A$ a covariant conservation of the current indeed emerges:

$$\nabla_\mu f^\mu = \kappa\nabla_\mu(\chi^\mu + \partial^\mu A) = 0. \tag{18}$$

The stress energy tensor $T^{\mu\nu}_{(\chi)}$ is substantially different from stress energy tensor that we all know from Einstein equation which is defined as $\frac{8\pi G}{c^4}T^{\mu\nu}_{(G)} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$. In this case, the stress energy momentum tensor $T^{\mu\nu}_{(\chi)}$ is a diffusive nonconservative stress energy tensor. However, from a variation with respect to the metric, we get the conserved stress energy tensor as in Einstein equation:

$$T^{\mu\nu}_{(G)} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}, \qquad \nabla_\mu T^{\mu\nu}_{(G)} = 0. \tag{19}$$

Using different expressions for $T^{\mu\nu}_{(\chi)}$ which depends on different variables, gives the conditions between the dynamical space time vector field $\chi_\mu$ and the other variables.

### D. Higher derivatives action

A particular case of diffusive energy theories is obtained when $\kappa \to \infty$. In this case, the contribution of the current $f_\mu$ in the equations of motion goes to zero and yields a constraint for the vector field being a gradient of the scalar:

$$f_\mu = \kappa(\chi_\mu + \partial_\mu A) = 0 \Rightarrow \chi_\mu = -\partial_\mu A. \tag{20}$$

For the rest of the paper we use the notation $\chi$ for the scalar field which is coupled to the stress energy momentum tensor and not $A$ due to earlier publications. The theory (16) is reduced to a theory with higher derivatives:

$$\mathcal{S} = -\int d^4x\sqrt{-g}\nabla_\mu\nabla_\nu\chi T^{\mu\nu}_{(\chi)}. \tag{21}$$

The variation with respect to the scalar $A$ gives $\nabla_\mu\nabla_\nu T^{\mu\nu}_{(\chi)} = 0$ which corresponds to the variations (17)–(18). In the following paper we use the reduced theory with higher derivative in the action.

## IV. SCALAR FIELD GRAVITY WITH DIFFUSIVE BEHAVIOR

### A. Dynamical time action with diffusive source

In this section we consider the following action:

$$\mathcal{L} = \frac{1}{2}\mathcal{R} + \chi_{,\mu;\nu}T^{\mu\nu}_{(\chi)} - \frac{1}{2}\phi^{,\mu}\phi_{,\mu} - V(\phi) \tag{22}$$

which contains a scalar field with potential $V(\phi)$. The stress energy momentum tensor $T^{\mu\nu}_{(\chi)}$ is chosen to be

$$T^{\mu\nu}_{(\chi)} = -\frac{\lambda_1}{2}\phi^{,\mu}\phi^{,\nu} - \frac{\lambda_2}{2}g^{\mu\nu}(\phi_{,\alpha}\phi^{,\alpha}) + g^{\mu\nu}U(\phi), \tag{23}$$

where $\lambda_1$ and $\lambda_2$ are arbitrary constants, and $U(\phi)$ is a another potential. In such a case the density and pressure resulting from $T^{\mu\nu}_{(\chi)}$ are

$$\rho_{(\chi)} = (\lambda_1 + \lambda_2)\frac{\dot{\phi}^2}{2} + U(\phi), \tag{24}$$

$$p_{(\chi)} = -\lambda_2\frac{\dot{\phi}^2}{2} - U(\phi). \tag{25}$$

Notice that the starting point was the case of two fluids. But here we discuss the single fluid with a Lagrangian involving two different measures: where the modified measure is generalized by using the dynamical space time vector field $\chi_\mu$.

There are three independent sets of equations of motions: $\chi$, $\phi$ and the metric $g_{\mu\nu}$. The variation with respect to the field $\chi$ yields:

$$\nabla_\mu\nabla_\nu T^{\mu\nu}_{(\chi)} = 0 \tag{26}$$

The variation with respect to the field $\phi$ gives a nonconserved current $j^\mu$:

$$j^\mu = \frac{\lambda_1}{2}(\chi^{,\mu;\nu} + \chi^{,\nu;\mu})\phi_{,\nu} + (1 + \lambda_2\Box\chi)\phi^{,\mu}, \tag{27}$$

with the nonconservation law:

$$\nabla_\mu j^\mu = V'(\phi) - \Box\chi U'(\phi). \tag{28}$$

The Einstein equations derived from the variation with respect to the metric take the form:

$$\begin{aligned} G^{\mu\nu} = {} & g^{\mu\nu}\left(-\chi_{,\alpha;\beta}T^{\alpha\beta}_{(\chi)} + \frac{1}{2}\phi^{,\alpha}\phi_{,\alpha} + V(\phi)\right) \\ & - \phi^{,\mu}\phi^{,\nu} + \chi_{,\alpha;\beta}\frac{\partial T^{\alpha\beta}_{(\chi)}}{g_{\mu\nu}} \\ & + \nabla_\lambda(\chi^{,\mu}T^{\nu\lambda}_{(\chi)} + \chi^{,\nu}T^{\mu\lambda}_{(\chi)} - \chi^{,\lambda}T^{\mu\nu}_{(\chi)}) \end{aligned} \tag{29}$$

where the derivative of the energy momentum tensor $T^{\mu\nu}_{(\chi)}$ with respect to $g_{\mu\nu}$ yields:

$$\begin{aligned} \chi_{,\alpha;\beta}\frac{\partial T^{\alpha\beta}}{\partial g_{\mu\nu}} = {} & -\frac{\lambda_1}{2}\chi^{(,\mu}\phi^{,\nu)}\Box\phi + \left(\frac{\lambda_1}{2} + \lambda_2\right)\phi^{,\mu}\phi^{,\nu}\Box\chi \\ & + \frac{\lambda_1}{2}\chi^{,\gamma;\mu}\phi^{,\nu}\phi_{,\gamma} - \lambda_2\phi^{,\mu;\lambda}\chi^{,\nu}\phi_{,\lambda} - \lambda_2\chi^{,\mu}\phi^{,\gamma;\nu}\phi_{,\nu} \\ & + \frac{\lambda_1}{2}\phi^{,\mu}\chi^{,\gamma;\nu}\phi_{,\gamma} - \frac{\lambda_1}{2}\chi^{(,\nu}\phi^{,\mu);\gamma}\phi_{,\gamma} \\ & + \frac{\lambda_1}{2}\pi^{(,\nu}\phi^{,\mu);\gamma}\chi_{,\gamma} + \chi^{(,\mu}\phi^{,\nu)}U'(\phi). \end{aligned}$$

The expression in the right-hand side of Eq. (29) is the total energy momentum tensor.

### B. Cosmological solution

For the solution we assume homogeneity and isotropy, therefore we solve our theory with a FLRW metric:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right). \tag{30}$$

According to this ansatz the scalar fields are solely functions of time.

Integrating Eq. (26) once, we express it in the form:

$$(\lambda_1+\lambda_2)\dot{\phi}\,\ddot{\phi}+U'(\phi)\dot{\phi}+3H\lambda_1\dot{\phi}^2=\frac{\sigma}{a^3} \tag{31}$$

where $\sigma$ is an integration constant.

For dark energy dynamics we can assume that $U(\phi)=\mathbf{const}$. Then the solution for Eq. (31) is

$$\dot{\phi}^2=\dot{\phi}_{(0)}^2 a^{-\frac{3\lambda_1}{\lambda_1+\lambda_2}}+\frac{\sigma}{\lambda_1+\lambda_2}a^{-\frac{3\lambda_1}{\lambda_1+\lambda_2}}\int_0^t ds a^{-\frac{3\lambda_2}{\lambda_1+\lambda_2}}. \tag{32}$$

In addition for the same theoretical reason we assume that $V(\phi)=\mathbf{Const}$. Then the current conservation law (28) has the solution:

$$\left(\frac{\lambda_1}{2}-\lambda_2\right)\ddot{\chi}+(1-3H\dot{\chi})\lambda_2=\frac{\tilde{\sigma}}{\dot{\phi}a^3} \tag{33}$$

where $\tilde{\sigma}$ is another integration constant. Now from the stress energy momentum tensor the total energy density term is

$$\rho=\frac{3}{2}H(\lambda_1-2\lambda_2)\dot{\chi}\dot{\phi}^2+\frac{1}{2}\dot{\phi}^2(1-2(\lambda_1+\lambda_2)\ddot{\chi}) \\ +\dot{\chi}\,\dot{\phi}\,((\lambda_1+\lambda_2)\ddot{\phi})+V, \tag{34}$$

and the total pressure is

$$p=\frac{1}{2}\dot{\phi}^2-\frac{1}{2}\lambda_1\ddot{\chi}\dot{\phi}^2+\lambda_2\dot{\chi}\,\dot{\phi}\,\ddot{\phi}-V. \tag{35}$$

## V. ASYMPTOTIC SOLUTIONS

We are not able to find the exact solutions for the Einstein Eq. (29) together with the equations for the scalar fields $\chi$ [Eq. (31)] and $\phi$ [Eq. (33)]. So we are looking for asymptotic solutions.

### A. A power law solution

We assume a power law solution for a large time:

$$a\sim t^{\alpha}. \tag{36}$$

Then from Eq. (31) the solution for the scalar field $\phi$ derivative is

$$\dot{\phi}=\sqrt{\frac{2\sigma}{3\alpha(\lambda_1-\lambda_2)+\lambda_1+\lambda_2}}t^{\frac{1}{2}-\frac{3\alpha}{2}}, \tag{37}$$

where $\phi_0$ is an arbitrary integration constant.

The solution for the scalar field $\chi$ is

$$\dot{\chi}=Ct \tag{38}$$

with the constant:

$$C=\frac{2\lambda_2}{-6\alpha\lambda_2+\lambda_1-2\lambda_2}. \tag{39}$$

By inserting the solutions (37) and (38) into Einstein equation we obtain:

$$\rho=\frac{\alpha_1}{a^3}+\frac{\alpha_2 t}{a^3}+V \tag{40}$$

where the constants are

$$\alpha_1=\frac{18\alpha^2\lambda_2(2\lambda_2-\lambda_1)}{2(\lambda_1-2\lambda_2(3\alpha+1))} \tag{41}$$

$$\alpha_2=\frac{(6\alpha+2)\lambda_1\lambda_2+2(3\alpha+1)(\lambda_2-1)\lambda_2+\lambda_1}{2(\lambda_1-2\lambda_2(3\alpha+1))}. \tag{42}$$

We get an asymptotic solution if the potential $V=0$ and the power of the scale factor is one:

$$a\sim t. \tag{43}$$

This solution is the same as the one obtained in the model of Einstein equation with relativistic diffusion exchange of energy [28].

### B. Exponential solution

We insert the exponential solution $a(t)\sim e^{H_0 t}$ in Eq. (32). Then we get:

$$\dot{\phi}^2=\dot{\phi}_0^2 a^{-\frac{3\lambda_1}{\lambda_1+\lambda_2}}-\sigma_0 H_0\frac{\lambda_1+\lambda_2}{3\lambda_2}\frac{1}{a^3} \tag{44}$$

if we impose $\frac{3\lambda_1}{\lambda_1+\lambda_2}>0$. Then from Eq. (33) we obtain the asymptotic solution:

$$\dot{\chi}=\frac{1}{3H_0}+\mathcal{O}\left(\frac{1}{a^3}\right). \tag{45}$$

With those solutions the density is given by:

$$\rho=H_0(3\lambda_2-1)\sigma\frac{\lambda_1+\lambda_2}{6\lambda_2}\frac{1}{a^3}+V \\ +\frac{1}{2}\dot{\phi}_0^2(1-2\lambda_2)a^{-\frac{3\lambda_1}{\lambda_1+\lambda_2}}. \tag{46}$$

This particular solution corresponds to a slowly varying dark energy ($V+\frac{1}{2}\dot{\phi}_0^2(1-2\lambda_2)a^{-\frac{3\lambda_1}{\lambda_1+\lambda_2}}$) approaching a constant value $V$, for $\lambda_1$ and $\lambda_2$ being positive, and $\lambda_1\ll\lambda_2$. In the case of negative $\lambda_1$ but still $|\lambda_1|\ll|\lambda_2|$, we get slowly growing vacuum energy, which corresponds to an asymptotically super accelerating universe.

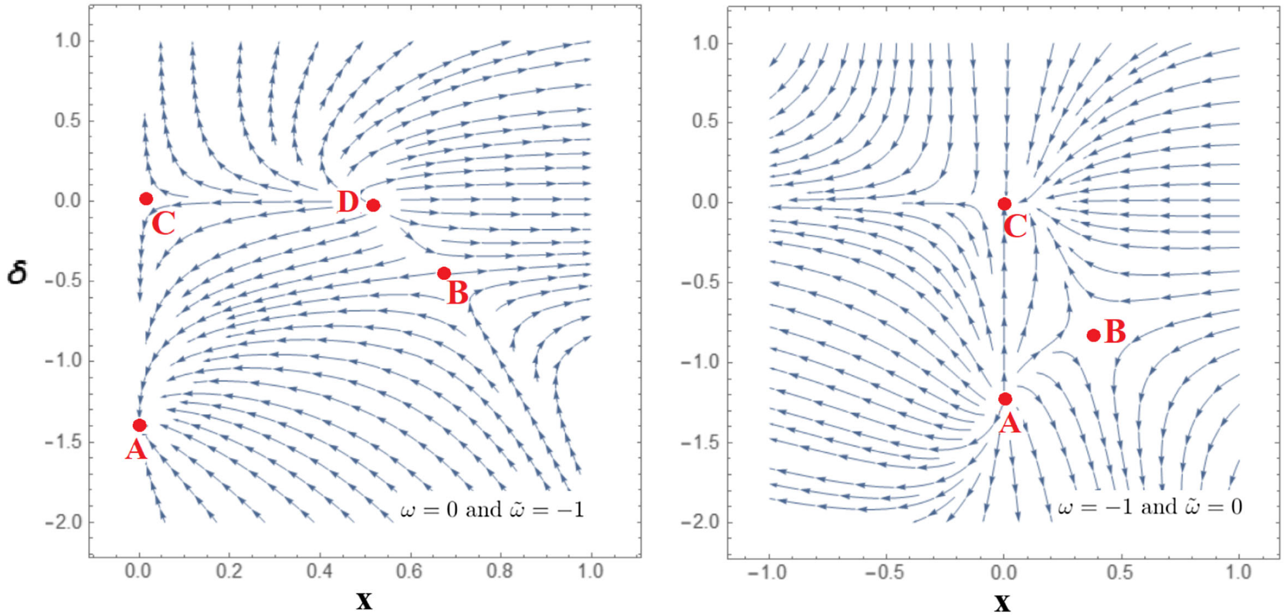


FIG. 1. The phase portrait for the dynamical system method. In the left panel the $\tilde{\omega} = -1$ refers to dark energy and in the right panel the $\omega = -1$ refers to dark energy.

## VI. $\lambda_2 = 0$ CASE

Solution (44) does not make sense for $\lambda_2 = 0$. Therefore this case should be treated separately. This special choice of the energy momentum has been explored by Gao, Kunz, Liddle and Parkison as a unification of dark energy and dark matter [29] without using a Lagrangian formulation. These authors proposed as a unification of dark energy and dark matter:

$$T^{\mu\nu}_{(\chi)} = -\frac{\lambda_1}{2}\phi^{,\mu}\phi^{,\nu} + g^{\mu\nu}U(\phi) \tag{47}$$

as the right-hand side of Einstein tensor. The action that produces asymptotically the same model using dynamical time theories was obtained in Ref. [30]. Here we explore the asymptotic solution with diffusive behavior. Under the assumption that all of the potentials are constant Eq. (31) has the solution:

$$\dot{\phi}^2 = \frac{\dot{\phi}^2_{(0)}}{a^3} + \frac{\sigma}{\lambda_1}\frac{t}{a^3}. \tag{48}$$

Then, the integral of Eq. (33) is

$$\dot{\chi}(t) = \dot{\chi}(0) - \frac{2}{\lambda_1}t + \int dt \frac{2\tilde{\sigma}}{\lambda_1\dot{\phi}a^3} \tag{49}$$

with the asymptotic behavior:

$$\dot{\chi}(t \to \infty) \to -\frac{2t}{\lambda_1}. \tag{50}$$

Notice that this asymptotic behavior is essentially different from the previous cases. Then the total density reads:

$$\rho = V + \frac{\alpha_1}{a^3} + \frac{\alpha_2}{a^{4.5}}, \tag{51}$$

where the coefficients are

$$\alpha_1 = \frac{5\dot{\phi}_0^2\lambda_1 + \lambda_1\sigma\chi_0 + 3\sigma t}{2\lambda_1}, \tag{52}$$

$$\alpha_2 = -\frac{2\tilde{\sigma}}{3\dot{\phi}_0 H_0\lambda_1}(3\dot{\phi}_0^2 H_0\lambda_1 + 3H_0\sigma t + \sigma). \tag{53}$$

Additional symmetry for this case is obtained:

$$\chi \to \chi + ct \tag{54}$$

or in terms of the dynamical time ($\chi^0 \Leftrightarrow \dot{\chi}$)

$$\chi^0 \to \chi^0 + c. \tag{55}$$

In the previous cases $\dot{\chi}$ is asymptotically a constant, equal to $\frac{1}{3H_0}$. In the special case of $\lambda_2 = 0$ there cannot be any particular choice for asymptotic value of $\chi$, because the symmetry will change it to any other arbitrary constant. One can calculate the conserved quantity associated with the symmetry (55) and it is the analogous of particle number.

A remarkable result is the correspondence between the solution (51) and the solutions for the DM-DE interaction system from Sec. II. For $\tilde{\omega} = 0$ the dust density equation yields:

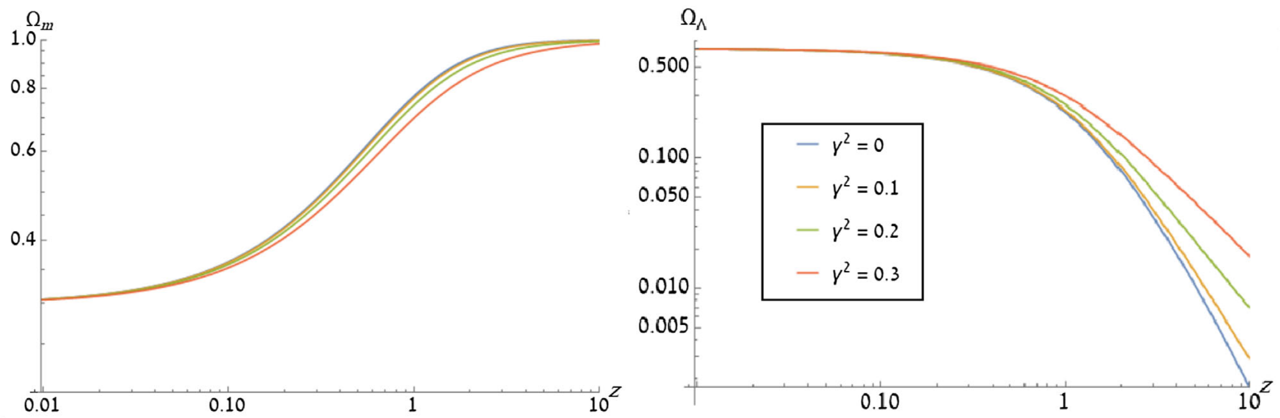


FIG. 2. The numerical solution of the partial densities of the dark energy and dark matter components, for different values of the coupling $\gamma^2$ (which is corresponding to the diffusion constant $\sigma$).

$$\partial_t \rho_{\text{dust}} + 3H\rho_{\text{dust}} = \frac{\gamma^2}{a^3}, \tag{56}$$

with the solution:

$$\rho_{\text{dust}} = \frac{C_1}{a^3} + \frac{\gamma^2 t}{a^3}, \tag{57}$$

where $C_1$ is an integration constant. For interacting dark energy, that satisfies $\omega = -1$, the energy density reads:

$$\partial_t \rho_\Lambda = -\frac{\gamma^2}{a^3}, \tag{58}$$

whereas for $\rho_\Lambda$

$$\rho_\Lambda = C_2 - \gamma^2 \int \frac{dt}{a^3}. \tag{59}$$

The $C_2$ is another integration constant. Asymptotically, the total density gives:

$$\rho = \frac{C_1 + \gamma^2 t}{a^3} + C_2 + \mathcal{O}\left(\frac{1}{a^6}\right), \tag{60}$$

which corresponds to the density (51), and the last term $\frac{\alpha_2}{a^{4.5}}$ becomes negligible. Hence, the integration constants equal to the integration constants from the Lagrangian case:

$$C_1 = \frac{5\dot{\phi}_0^2 + \sigma\chi_0}{2} \tag{61}$$

$$\gamma^2 = \frac{3\sigma}{2\lambda_1}, \qquad C_2 = V. \tag{62}$$

This correspondence does not hold for the whole history of the universe, however asymptotically the models (our Lagrangian model and the previously studied non-Lagrangian models) fit each other for the case $\lambda_2 = 0$ and approach $\Lambda$CDM for late times. Of course that the solutions will have to be studied and this will be a main goal for further investigations.

One can see that both models with exactly the same homogeneous solution where $\tilde{\sigma} = 0$. In this case $\alpha_2 = 0$ [see Eq. (53)] and the corresponding relations between the constants of the models present in Eqs. (61)–(62).

In order to assess the viability of the model, let us see how some physical quantities change versus the redshift (z) for both models. The connection between the cosmic time derivative and the redshift derivative reads:

$$\frac{d}{dt} = -H(z)(z+1)\frac{d}{dz} \tag{63}$$

which is obtained from the dependence of scale factor on the red-shift $a = \frac{1}{z+1}$. The numerical solution of the partial densities for the simplest case appear in Fig. 2. Even this simple case describes a diffusive interaction between dark energy dark matter from an action principle. However, the presence of the coupling constant $\tilde{\sigma}$ yields to additional part ($\sim a^{-4.5}$) which could resolve the singularity problem as discussed in Ref. [30]. But in any case, all the solutions approach $\Lambda$CDM model for the late universe.

## VII. CONCLUSIONS

We have extended the results of our earlier papers concerning the DM-DE interaction in the context of two measures models and the dynamical time theories. The extension consists in a general choice of the conserved non-canonical energy-momentum tensor. The energy momentum tensor is more general than the one proposed by Gao, Kunz, Liddle, and Parkinson [29] as well as the dark energy dark matter unification obtained in the two measures limit, which corresponds to the case where the conserved noncanonical energy-momentum tensor is proportional to the metric tensor [19,20].

The constants $\lambda_1$ and $\lambda_2$ parametrize the more general choice considered here. $\lambda_2 = 0$ corresponds to the case considered by Gao, Kunz, Liddle, and Parkinson in their non-Lagrangian formalism. In our Lagrangian formulation, for this type of energy momentum tensor, as additional shift symmetry for the dynamical time appears and at the same time the dynamical time behaves asymptotically as the cosmic time. Diffusive type is obtained when the dynamical space time vector is taken to be the gradient of a scalar, then instead of a conservation law of the energy momentum introduced in the action, we obtain a nonconservation of this energy momentum tensor of the diffusive type, which leads then to an interacting DE/DM scenario. This formulation of DE-DM have a direct correspondence with the behavior of non Lagrangian formulations of DE/DM interactions only in the case $\lambda_2 = 0$. In the other cases, the asymptotic behavior is different and in particular the dynamical time does not behave as cosmic time asymptotically, in fact as the cosmic time increases, the dynamical time approaches the finite value $\frac{1}{3H}$ in an asymptotically de Sitter space. In all cases we do not need to introduce the dark matter in the initial Lagrangian, it appears dynamically. As a result of the dynamic evolution in our model we obtain an asymptotically ΛCDM solution.

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